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Identities for the Associator in Alternative Algebras

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The associator is an alternating trilinear product for any alternative algebra. We study this trilinear product in three related algebras: the associator in a free alternative algebra, the associator in the Cayley algebra, and the ternary cross product on four-dimensional space. This last example is isomorphic to the ternary subalgebra of the Cayley algebra which is spanned by the non-quaternion basis elements. We determine the identities of degree ≤ 7 satisfied by these three ternary algebras. We discover two new identities in degree 7 satisfied by the associator in every alternative algebra and five new identities in degree 7 satisfied by the associator in the Cayley algebra. For the ternary cross product we recover the ternary derivation identity in degree 5 introduced by Filippov.

Introduction

Let F be a field and let A be a vector space over F. Let $s\colon A \times A \longrightarrow A$ be a bilinear map. We call A, or more precisely the pair (A, s), a (*binary*) *algebra* over F. We write ab instead of $s(a, b)$ to denote the product of two elements $a, b \in A$. We define the *associator* in A by the formula $(a, b, c) = (ab)c - a(bc)$. We say that A is *associative* if $(a, b, c) = 0$ for all $a, b, c \in A$; we say that A is *alternative* if (a, b, c) is an alternating function of the three arguments $a, b, c \in A$. We define the *commutator* in A by the formula $[a, b] = ab - ba$. Given any algebra A we can define a new algebra A^- by using the same vector space A but replacing the original product $s(a, b)$ by the commutator $[a, b]$. We define the *Jacobian* in A by the formula $[a, b, c] = [[a, b], c] + [[b, c], a] + [[c, a], b]$. If A is associative then A^- is a *Lie algebra*, that is, A^- satisfies the Jacobi identity $[a, b, c] = 0$ for all $a, b, c \in A$. If A is alternative then A^- is a *Malcev algebra*, that is, A^- satisfies the Malcev identity $[a, b, [a, c]] = [[a, b, c], a]$ for all $a, b, c \in A$.

A survey of non-associative structures may be found in the article by Kuzmin and Shestakov (1995). A detailed exposition, including a discussion of free algebras, may be found in the book of Zhevlakov *et al.* (1982).

Let $t\colon A \times A \times A \longrightarrow A$ be a trilinear map. We call A, or more precisely the pair (A, t), a *ternary algebra* or *triple system* over F. If $t(a, b, c)$ is an alternating function of its arguments, we call A an *alternating* ternary algebra; the identities defining this are

$$t(a, a, b) = t(a, b, a) = t(b, a, a) = 0, \qquad \text{for all} \quad a, b \in A.$$

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In this case we write the product as (abc) or (a, b, c) instead of $t(a, b, c)$. We are interested in alternating ternary algebras for two reasons:

(1) The associator in any alternative algebra is an alternating function of its three arguments. We obtain an alternating ternary algebra by using the associator as a trilinear product defined on the same underlying vector space.
(2) A natural way to generalize Lie algebras to the ternary case is to start with an alternating ternary algebra (the ternary version of an anticommutative binary algebra) and then find a natural analogue of the Jacobi identity.

In the first case, we are studying identities for alternative algebras which are restricted in the sense that the terms in the identities must be built out of associators. This restriction has the advantage that in each degree the number of possible terms is much smaller than when the original binary product is used. (Identities for alternative algebras, in particular the Cayley algebra, have been studied by Hentzel and Peresi (1997) and Racine (1988).)

In the second case, we are looking for a ternary version of the Jacobi identity. There are many different ways to approach this problem. Some of these are discussed in Kurosh (1969), Baranovich and Burgin (1975), Filippov (1985), Gnedbaye (1995a,b, 1997), Hanlon and Wachs (1995), Bremner (1997, 1998), and Bremner and Hentzel (2000). All the identities presented in this paper can be regarded as candidates for a ternary analogue of the Jacobi identity. (The traditional definition of *Lie triple system* uses a trilinear operation which is alternating only in the first two factors, see Lister (1952).)

In this paper we study the following three related alternating ternary algebras:

(*i*) the associator in a free alternative algebra,
(*ii*) the associator in the Cayley algebra (the Cayley ternary algebra), and
(*iii*) the ternary cross product on four-dimensional space; this is isomorphic to the subsystem of the Cayley ternary algebra which is spanned by the non-quaternion basis elements.

We determine the identities of degree ≤ 7 in each of these systems. It is clear that every identity for system (*i*) is an identity for system (*ii*), and that every identity for system (*ii*) is an identity for system (*iii*). For (*i*) we discover two new identities. For (*ii*) we discover, in addition to the identities in (*i*), five new identities. For (*iii*) we recover the ternary derivation identity introduced by Filippov, which implies all the identities satisfied by the ternary cross product in degrees 5 and 7.

The results in this paper were determined by machine computation over the field with 103 elements. This was necessary in order to be able to store each matrix entry in a single byte. In this way we also avoided integer overflow when computing the row-canonical forms of very large matrices. For this reason the theorems in this paper are stated over fields of characteristic 103. In some cases the results were checked by other computations using the modulus 1009. Since 103 is much larger than the degrees of any of the identities studied in this paper, this restriction on the characteristic is probably not necessary, that is, the Theorems probably do in fact hold in characteristic 0. However, there is a possibility that if these results are interpreted over the rational numbers then we may have missed an identity or included a non-identity. In some cases we have been able to verify directly that our identities hold in characteristic 0.

Preliminaries

OBVIOUS IDENTITIES

Every alternating ternary algebra satisfies certain *obvious identities* in each degree which express the alternating properties of the ternary product. To illustrate this we consider degrees 5 and 7. For a general ternary algebra there are three association types in degree 5:

$$((-,-,-),-,-), \qquad (-,(-,-,-),-), \qquad (-,-,(-,-,-)).$$

For an alternating ternary algebra these 3 types may all be expressed in terms of the single type $((-,-,-),-,-)$. The obvious identities in degree 5 are

$$((abc)de) = -((bac)de) = -((acb)de) = -((abc)ed).$$

Similarly, for a general ternary algebra there are 12 association types in degree 7, but for an alternating ternary algebra these 12 types may all be expressed in terms of the two types

$$((-,-,-),(-,-,-),-) \qquad \text{and} \qquad (((-,-,-),-,-),-,-).$$

Every alternating ternary algebra satisfies the obvious identities in degree 7:

$$\begin{aligned} ((abc)(def)g) &= -((bac)(def)g) = -((acb)(def)g) \\ &= -((abc)(edf)g) = -((abc)(dfe)g) = -((def)(abc)g), \\ (((abc)de)fg) &= -(((bac)de)fg) = -(((acb)de)fg) \\ &= -(((abc)ed)fg) = -(((abc)de)gf). \end{aligned}$$

Throughout this paper we are interested primarily in non-obvious identities, that is, identities which do not follow from the obvious identities. To be precise, let $S^{(n)}$ denote the free ternary algebra on n generators and let $A^{(n)}$ denote the free alternating ternary algebra on n generators. Let $I^{(n)}$ denote the T-ideal in $S^{(n)}$ defined by the alternating identities, that is, $I^{(n)}$ is the ideal generated by the values of the alternating identities

$$(a,a,b), \qquad (a,b,a), \qquad (b,a,a), \qquad \text{for all} \quad a,b,c \in S^{(n)}.$$

Then by definition $A^{(n)} = S^{(n)}/I^{(n)}$. We call the elements of $I^{(n)}$ the *obvious identities* for the alternating ternary product. We call an element of $S^{(n)} - I^{(n)}$ (or more precisely, a non-zero element of $A^{(n)}$) which is satisfied by some alternating ternary algebra, a *non-obvious identity* for that system.

ALTERNATING SUMS

If I is a multihomogeneous polynomial in an alternating ternary algebra such that the letter x occurs at least twice in each term, then we will use the notation

$$\sum_{\mathrm{alt}(x)} I$$

to denote the alternating sum over the x positions, that is, if there are exactly k occurrences of x in each term of I, then we introduce an ordered list of k letters (which do not already occur in I) and take the alternating sum over the $k!$ permutations of these new letters in the x positions. We call this process *alternating partial linearization.*

Warning: Be careful to distinguish between (axx), which is zero, since it is the value of an alternating ternary product with two equal arguments, and $\sum_{\mathrm{alt}(x)}(axx)$, which is *not* zero, since the alternating sum must be expanded *before* the alternating ternary products are evaluated. An example should make this clear. The expression

$$\sum_{\mathrm{alt}(x)} \{((axx)bx) - 2((bxx)ax)\}$$

indicates that we should fill the x positions in both terms with the new letters c, d, e in all permutations and multiply the coefficient by the sign of the permutation. The previous expression expands to

$$\begin{aligned}&((acd)be) + ((ade)bc) + ((aec)bd) - ((adc)be) - ((ace)bd) - ((aed)bc)\\ &- 2((bcd)ae) - 2((bde)ac) - 2((bec)ad) + 2((bdc)ae) + 2((bce)ad) + 2((bed)ac).\end{aligned}$$

Often the alternating property of the ternary product can be used to simplify the expression. The last expression simplifies to

$$2((acd)be) - 2((ace)bd) + 2((ade)bc) - 4((bcd)ae) + 4((bce)ad) - 4((bde)ac).$$

This notation allows us to write a long identity in a shorter form by using the alternating symmetries of the arguments.

MULTILINEAR TERNARY MONOMIALS

There are $\binom{5}{3} = 10$ multilinear alternating ternary monomials of degree 5:

$$\begin{array}{lllll} ((abc)de), & ((abd)ce), & ((abe)cd), & ((acd)be), & ((ace)bd),\\ ((ade)bc), & ((bcd)ae), & ((bce)ad), & ((bde)ac), & ((cde)ab). \end{array}$$

These 10 monomials form a basis for the S_5-module of all multilinear homogeneous polynomials of degree 5 in a free alternating ternary algebra. The group acts by permuting the letters.

In degree 7, there are $\frac{1}{2}\binom{7}{3,3,1} = 70$ multilinear monomials in the first association type $((abc)(def)g)$, and $\binom{7}{3,2,2} = 210$ multilinear monomials in the second association type $(((abc)de)fg)$. Thus the dimension is 280 for the S_7-module of multilinear identities of degree 7. Any identity in degree 7 can have at most 280 terms when written in multilinear form with the letters in each ternary product put in alphabetical order using the obvious identities.

SKETCH OF THE METHOD

The computational methods (programmed in C and Pascal) used in this paper to study identities were developed originally by Hentzel (1977, 1979, 1998); similar methods (programmed in Maple) have been developed by Bremner (1997, 1998).

Suppose that f is a function with n arguments. It is convenient in this discussion to write the function on the right as $(x_1, x_2, \ldots, x_n)f$. If π is any permutation of n objects, we can first permute $(x_1, x_2, \ldots, x_n)$ by π and then apply f to the rearranged arguments. We introduce a word of caution here. If $\pi = (1, 2, 3)$, then

$$(x_1, x_2, x_3)\pi = (x_3, x_1, x_2) \neq (x_{(1)\pi}, x_{(2)\pi}, x_{(3)\pi}) = (x_2, x_3, x_1).$$

Our permutations apply to positions, not the subscripts of the elements in those positions. The action of the permutation is still defined even when the arguments have no subscripts or when they have inappropriate subscripts or when there are repeated arguments. For example: $(x, y, z)\pi = (z, x, y)$; $(x_7, x_8, x_9)\pi = (x_9, x_7, x_8)$; $(x_1, x_1, x_2)\pi = (x_2, x_1, x_1)$.

In many presentations of the representations of the symmetric group, the action of the group is applied to the entries in a tableau. This is equivalent to applying the action to the subscripts. The representations so generated can be used, but one has to remember to use the representation of π^{-1} when one wants the representation of π. If in the presentation the permutations are written on the left instead of the right, one can use the transpose of the matrices to adjust for the reverse order of composition.

In this discussion we will write $(x_1, x_2, \ldots, x_n)\pi$ to indicate the rearrangement of the objects by the permutation π. If f is a function of n arguments, then $\pi{*}f$ will be the function obtained by permuting the arguments first by π:

$$(x_1, x_2, \ldots, x_n)(\pi{*}f) = ((x_1, x_2, \ldots, x_n)\pi)f.$$

If

$$g = \sum_{\pi \in S_n} c(\pi)\pi$$

is any formal linear combination of the permutations of S_n, then $g{*}f$ is defined by linearity as:

$$\begin{aligned}(x_1, x_2, \ldots, x_n)(g{*}f) &= \sum_{\pi \in S_n} c(\pi)(x_1, x_2, \ldots, x_n)(\pi{*}f) \\ &= \sum_{\pi \in S_n} c(\pi)((x_1, x_2, \ldots, x_n)\pi)f.\end{aligned}$$

The group ring FS_n consists of just such formal sums, and so for each $g \in FS_n$ we have a function $g{*}f$. (The structure of the group ring FS_n has been described in the classic works of Alfred Young; see Rutherford (1948) for an exposition based closely on Young's original work, and James and Kerber (1984), especially Chapter 3, for a more modern presentation.)

If one takes any bijection of the set FS_n to any other set Q, then one could say that for any $q \in Q$, there is a function $q{*}f$. The set Q we shall use is the isomorphic image of FS_n as a direct sum of matrices, and the bijection is actually an isomorphism of associative algebras. Let $s = s_n$ denote the number of partitions of the positive integer n, and denote the partitions by λ_j for $1 \le j \le s$. Any partition λ_j of n determines an irreducible representation $R_j = R_{\lambda_j}$ of S_n of dimension $d_j = d_{\lambda_j}$. Let $M_d = M_d(F)$ denote the complete $d \times d$ matrix ring over F. The group ring FS_n is isomorphic to Q, the direct sum of the matrix rings M_{d_j} for $1 \le j \le s$:

$$Q = M_{d_1}^{(1)} \oplus M_{d_2}^{(2)} \oplus \cdots \oplus M_{d_s}^{(s)}.$$

There is one matrix ring summand for each partition of n where the matrix sizes equal the dimensions of the corresponding irreducible representations of S_n. Each of these matrix rings is a two-sided ideal in FS_n, and the columns of these matrix rings are minimal left ideals (simple left S_n-submodules) in FS_n. The matrix ring corresponding to partition λ_j is the isotypic component of FS_n corresponding to λ_j, that is, it is the sum of the simple left submodules of FS_n isomorphic to R_j.

If E^i_{jk} is the jk matrix unit of the ith summand, it is associated with some element g^i_{jk} of the group ring. If

$$\sum_{i,j,k} c^i_{jk} E^i_{jk}$$

is any element of the matrix direct sum, it must correspond to the group ring element

$$g = \sum_{i,j,k} c^i_{jk} g^i_{jk}.$$

It is clear that if one could construct the elements g^i_{jk} it would be easy to generate the group ring element corresponding with any fill of the matrix direct sum.

There is no really nice correspondence between the group ring and the matrix direct sum. The units of the matrix direct sum are the E^i_{jk}. The "natural units" of the group ring are usually expressed as

$$e^i_j S^i_{jk}$$

which are derived from the Young's tableaux. Here e^i_j is the idempotent corresponding to the jth tableau for the ith partition of n, and S^i_{jk} is the permutation which sends the kth tableau into the jth tableau. The basic construction is that for the jth tableau of partition i we have

$$e^i_j = \sum_{p,q} \operatorname{sgn}(q) pq,$$

where the p are the permutations which leave the rows fixed as sets, and the q are the permutations which leave the columns fixed as sets.

The map that sends E^i_{jk} to $e^i_j S^i_{jk}$ is one-to-one and linear. It is not a ring isomorphism, but is close to one. It is essentially an isomorphism of FS_n-modules, which is all we really need to be able to identify the left action of FS_n with row operations on matrices. If one insists that the mapping preserve multiplication exactly, then the map from the direct sum of matrices to FS_n involves summing over the $n!$ elements of the symmetric group. This produces an element g which is generally too big to even look at. If one is willing to work with a map that is not exactly a representation, then although the element g^i_{jk} still involves a substantial number of non-zero terms, the coefficients are easily expressed as "alternating sums" which can be displayed and manipulated by hand.

If $(A, +, *)$ is a non-associative algebra over F, and f is a non-associative polynomial over F, we say that $(x_1, x_2, \ldots, x_n)f$ is an identity for A if whenever the indeterminates are replaced by elements of A, the resulting expression evaluates to zero. It is obvious that if F is an identity, then $\pi{*}f$ is an identity for any $\pi \in S_n$ and $g{*}f$ is an identity for any $g \in FS_n$.

In our applications f is usually not itself an identity. We generally know, however, of a particular h in FS_n so that $h{*}f$ is an identity. The analysis asks questions such as the following: *If $h{*}f$ is an identity and g is some element in FS_n, is $g{*}f$ an identity?* The question can be posed more generally: *What are all the identities that $h{*}f$ implies?* From the definition it follows that $\pi{*}(\sigma{*}f) = (\pi{*}\sigma){*}f$. It also follows that for all $g, h \in FS_n$ one has $g{*}(h{*}f) = (g{*}h){*}f$. This shows that if $h{*}f$ is an identity, then any $x{*}f$ is also an identity for any element x of FS_n which lies in the left ideal generated by h.

Viewing FS_n as a direct sum of matrix rings, any left ideal of FS_n decomposes into a direct sum of left ideals from the components. Since a component is a $d \times d$ matrix ring, the problem of describing left ideals of FS_n reduces to the problem of describing left ideals of complete matrix rings. In matrix rings, left multiplications correspond to row operations. A matrix K is a left multiple of a matrix H, i.e. $K = XH$, if and only if $\mathrm{RowSpace}(K) \subseteq \mathrm{RowSpace}(H)$.

Suppose that $h{*}f$ is an identity where h is in FS_n and f is a non-associative polynomial. In general the element h will be non-zero in several of the matrix ring summands. We say that the identity $h{*}f$ exists in those summands which are non-zero. If h is non-zero in only one summand and in that one summand has rank 1, then we describe $h{*}f$ as an irreducible identity. If h is non-zero in several summands, but in each of the non-zero summands the matrix has rank 1, then h can be decomposed uniquely into irreducible identities. If the rank of h in one of the summands is greater than 1, we can still decompose h into irreducible identities but the decomposition is not unique. A simple way to do the decomposition is to use the rows of the row-canonical form of the representation of h.

In general, identities $g{*}f$ and $h{*}f$ are equivalent if and only if in each representation, $\mathrm{RowSpace}(g) = \mathrm{RowSpace}(h)$. Identity $h{*}f$ implies identity $g{*}f$ if and only if, in each representation, $\mathrm{RowSpace}(g) \subseteq \mathrm{RowSpace}(h)$.

One can extend this theory to several functions $f_1, f_2, \ldots, f_t$. Now an identity is of the form

$$g_1{*}f_1 + g_2{*}f_2 + \cdots + g_t{*}f_t.$$

Instead of using one copy of the group ring, one considers elements

$$(g_1, g_2, \ldots, g_t)$$

in the direct sum of t copies of FS_n. Instead of working with left ideals, we work with left submodules of

$$\bigoplus_{i=1}^{t} FS_n^{(i)}.$$

The identities implied by

$$g_1{*}f_1 + g_2{*}f_2 + \cdots + g_t{*}f_t$$

are now the elements of the form

$$g_1'{*}f_1 + g_2'{*}f_2 + \cdots + g_t'{*}f_t$$

where the t-tuples $(g_1', g_2', \ldots, g_t')$ are in the left submodule generated by $(g_1, g_2, \ldots, g_t)$.

One standard set of functions is the association types of degree n, that is, the distinct well-formed bracketings of n factors with $n-1$ pairs of brackets. The number of these association types is the Catalan number

$$t_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

We denote the $t = t_n$ distinct association types in degree n by α_i for $1 \le i \le t$. (If one were studying a commutative case, one could use a different function for each of the association types which are inequivalent using commutativity).

Given any multilinear polynomial $(x_1, x_2, \ldots, x_n)f$ of degree n in n indeterminates, we first sort the terms of f by association type. Thus we can write $f = f_1 + \cdots + f_t$. For $1 \le i \le t$ every term in f_i has the association type α_i, so the terms in f_i are distinguished only by the order of the factors. Since the factors are just a permutation of the indeterminates $x_1, x_2, \ldots, x_n$, we can identify each term in f_i with an element of the symmetric group S_n acting on $1, 2, \ldots, n$. Hence f_i can be identified with a linear combination of the elements of S_n, that is, with an element g_i of the group ring FS_n. Thus the original identity f can be identified with an element of

$$M = FS_n^{(1)} \oplus FS_n^{(2)} \oplus \cdots \oplus FS_n^{(t)},$$

the direct sum of t copies of FS_n. Then M is a module over the group ring and the set of all identities we seek is a submodule of M.

If we fix a partition λ of n, then projecting onto the corresponding matrix ring we see that each element of the group ring FS_n corresponds to a matrix of size $d_\lambda \times d_\lambda$. Combining the t association types we put together (horizontally) the t matrices of size $d_\lambda \times d_\lambda$ to obtain a matrix of size $d_\lambda \times td_\lambda$. Thus the component for partition λ of the original multilinear polynomial f can be represented as a matrix of size $d_\lambda \times td_\lambda$. Stacking the matrices (vertically) for a number k of identities $f^{(1)}, \ldots, f^{(k)}$ gives a matrix of size $kd_\lambda \times td_\lambda$. Row operations on this matrix correspond to left multiplication in the group ring, and so the row-canonical form of this matrix corresponds to a set of independent module generators for the submodule of M generated by the identities $f^{(1)}, \ldots, f^{(k)}$. Since the rank of this matrix can be no greater than td_λ, the number of independent generators is at most td_λ.

Since we can identify multilinear identities $(x_1, x_2, \ldots, x_n)f$ of degree n in n indeterminates with elements of the FS_n-module M, we can use the language of representation theory to describe identities. In particular an irreducible identity is one that generates a simple submodule (irreducible subrepresentation) of M. An irreducible identity f must lie in one isotypic component of M, that is, there must exist a partition λ of n such that f lies in the sum of the t matrix rings corresponding to λ.

This discussion involved permutations of the arguments in functions of several variables. Such substitutions leave the number of variables in the function fixed. If one replaces one of the arguments of f by a product, then the number of arguments increases by one. In fact any multilinear identity $(x_1, x_2, \ldots, x_n)f$ can be lifted to an identity of degree $n+1$ in $n+2$ ways: we can replace one of the arguments x_i by a product $x_i x_{n+1}$ or we can multiply f on the left or right by x_{n+1}.

Identities for a Free Alternative Algebra

Theorem 1. *Over a field of characteristic* 103*, there are no non-obvious identities in degree* 5 *for the ternary associator product in a free alternative algebra. The following two identities in degree* 7 *are irreducible. Together with the obvious identities in degree* 7*, they generate all the identities of degree* 7 *satisfied by the ternary associator product in a free alternative algebra:*

$$\begin{aligned} F_1 = &-((abc)(abd)c) + ((abc)(acd)b) - ((abc)(bcd)a) - (((abc)ab)cd) \\ &+ (((abc)ac)bd) - (((abc)ad)bc) - (((abc)bc)ad) + (((abc)bd)ac) \\ &- (((abc)cd)ab) \end{aligned}$$

$$\begin{aligned} F_2 = \sum_{alt(x)} \{ &-((abx)(bxx)a) + ((abx)(axx)b) + 2((abx)(abx)x) \\ &+ 2(((abx)ax)bx) - 2(((abx)bx)ax) + (((xxx)ab)ab) \\ &- (((abx)ab)xx) + 2(((abx)xx)ab) + 3(((axx)bx)ab) \\ &- 3(((bxx)ax)ab) \}. \end{aligned}$$

Identity F_1 *has four variables and corresponds to partition* 2221*. Identity* F_2 *has five variables and corresponds to partition* 22 111*.*

Proof. (By Computer) We first consider degree 5. The two alternative identities imply $2 \cdot 5 \cdot 6 = 60$ identities of degree 5: there are five ways to lift degree 3 to degree 4, and then six ways to lift degree 4 to degree 5. The Catalan number in degree 5 is 14, so we have 14 association types for these binary polynomials in a free alternative algebra. There is one more identity, which expands the monomial $((abc)de)$ by regarding the alternating ternary products as associators:

$$((abc)de) = (((ab)c)d)e - ((a(bc))d)e - ((ab)c)(de) + (a(bc))(de).$$

This gives one more (ternary) association type. We have 61 identities and 15 association types. Our goal is to find all identities expressible in terms of these 15 types. (We do not use the other two ternary association types in degree 5, since these may be converted into our one chosen type using the obvious identities.)

The submodule generated by these 61 identities gives all the identities of degree 5 which hold for the binary product in every alternative algebra. The submodule of these identities, which are zero in all but the last association type, will be all the identities expressible only with associators. If we also compute the submodule generated by the obvious identities, and compare them with the set of all identities, the non-obvious identities will be the new ones that appear.

All of these computations are done in the isomorphic image of the group ring FS_5 as a direct sum of matrix rings. For each partition of five (that is, each irreducible representation) we compute the matrix of size $61d \times 15d$ where d is the dimension of the representation. After reducing the matrix to row-canonical form, the leading ones which occur within the last association type give the rows which contain the identities we seek; that is, the identities expressible in terms of the associator.

We checked all the representations and found that all the identities which exist are consequences of the obvious identities; hence there are no new identities in degree 5.

Now we consider identities of degree 7. There are 132 binary association types. There are two association types for an alternating ternary product, so we have a total of 134 association types in degree 7. There are $2 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 3360$ liftings of the alternative identities to degree 7. (These 3360 identities are FS_7-module generators for the multilinear subspace of the homogeneous component of degree 7 of the T-ideal generated by the alternative identities in the free non-associative algebra on seven generators.) There are also two identities expressing the two ternary association types in terms of the binary alternative product by expanding the ternary product as the associator:

$$\begin{aligned} ((abc)(def)g) = {} & (((ab)c)((de)f))g - ((a(bc))((de)f))g \\ & - (((ab)c)(d(ef)))g + ((a(bc))(d(ef)))g \\ & - ((ab)c)(((de)f)g) + (a(bc))(((de)f)g) \\ & + ((ab)c)((d(ef))g) - (a(bc))((d(ef))g), \\ (((abc)de)fg) = {} & (((((ab)c)d)e)f)g - ((((a(bc))d)e)f)g \\ & - ((((ab)c)(de))f)g + (((a(bc))(de))f)g \\ & - ((((ab)c)d)e)(fg) + (((a(bc))d)e)(fg) \\ & + (((ab)c)(de))(fg) - ((a(bc))(de))(fg). \end{aligned}$$

Altogether we have 3362 identities and 134 association types. Since the largest irreducible representation of S_7 has dimension 35, the largest matrix will have $35 \cdot 3362 = 117670$ rows and $35 \cdot 134 = 4690$ columns.

In the table, column 1 gives the partition labeling an irreducible representation of S_7. Column 2 gives the dimension of the representation. All of the entries in the remaining columns are ranks of modules over the group ring FS_7 as explained in the Preliminaries. Since there are only two ternary association types, all the ranks in the table are less than or equal to twice the dimension of the representation. Column 3 gives the dimension of the subspace of obvious identities. Column 4 gives the dimension of the subspace of the alternative identities expressible using only the associators: this is always at least as large as the dimension in column 3. The column labeled Cayley associator will be discussed in the next section:

Partition	Representation dimension	Obvious identities	Free alternative associator	Cayley associator
7	1	2	2	2
61	6	12	12	12
52	14	28	28	28
511	15	30	30	30
43	14	28	28	28
421	35	70	70	70
4111	20	40	40	40
331	21	41	41	41
322	21	41	41	41
3211	35	67	67	67
31 111	15	29	29	29
2221	14	25	26	26
22 111	14	24	25	26
211 111	6	9	9	11
1 111 111	1	0	0	2

Notice that the dimension of the space of obvious identities equals twice the module dimension until we reach partition 331: in the first seven representations, the obvious identities generate everything possible. It is only possible for new (non-obvious) identities to correspond to partitions at or below the line labeled 331 in the table. The dimension of the space of free alternative identities always lies between the dimension of the space of obvious identities and twice the dimension of the representation. The dimension of the space of free alternative identities is greater than the dimension of the space of obvious identities only for partitions 2221 and 22 111. There is a single new irreducible identity in each of these two representations. The new identities appear as additional rows when the row-canonical form of the matrix for free alternative identities is compared to the row-canonical form of the matrix of only the obvious identities.

In partition 2221 the dimension of the space of obvious identities is 25 while the dimension of the space of free alternative associator identities is 26. The row-canonical form of the obvious identities has 25 (non-zero) rows while the row-canonical form of the free alternative associator identities has 26 (non-zero) rows. It suffices to pick one of the 26 rows which is not in the row space of the 25 rows.

In this particular case this was easy to do. The obvious identities are generated by identities which are non-zero in only one type; the row-canonical form preserves the separation of the obvious identities by type. One needs only to choose a row from the free alternative identities which has non-zeros entries in both types.

It is convenient for the next step to choose an identity which has as few non-zero coefficients as possible. We looked for a linear combination of the 26 identities which had non-zero coefficients under both types and which was not composed of two obvious identities, one from each type. The row we created has 28 entries (14 entries for each association type) only three of which are non-zero; the entry in position 14 is 1, and the entries in positions 24 and 28 are both -1. We need to determine the group ring element corresponding to this matrix row; this will give us the (multilinear form of) the new identity we are looking for. We generated all 5040 permutations of a, b, c, d, e, f, g using the Maple command *combinat*[*permute*]. We then considered the 14×14 matrices corresponding to the component of the group ring of S_7 labeled by the partition 2221. To each permutation in S_7 we assigned a coefficient equal to the 10–14 entry of the matrix representing the inverse permutation; this gives the group ring element corresponding to the elementary matrix with 1 in position 14–10 and 0 elsewhere. Similarly we found the group ring element corresponding to the elementary matrix with 1 in position 14–14 and 0 elsewhere. Combining these results we obtain the element of the direct sum of two copies of the group ring which corresponds to the 14×28 matrix in which row 14 is the row described earlier. This double group ring element gives the new identity we are looking for.

We checked this new identity using the same C programs, and verified that it is an identity for the associator in every alternative algebra and that it does not follow from the obvious identities.

The new identity is multilinear, but since it occurs in representation 2221, the theory of Young's tableaux and the corresponding idempotents in the group ring of the symmetric group shows that there must be three pairs of variables such that the identity is symmetric in the variables in each pair, that is, the identity must be the linearization of a polynomial in *aabbccd*. We therefore replaced d by a, e by b, and f by c. We then straightened the terms using the obvious identities, collected and sorted the terms, and removed the terms with coefficient 0. This simplification resulted in the much more compact identity called F_1 in the statement of the Theorem. We verified this identity using Jacobs' non-associative algebra system Albert, see Jacobs.

We also expanded each of the associators as commutators using the formula

$$6(a, b, c) = [[ab]c] + [[bc]a] + [[ca]b].$$

This holds in any alternative algebra since the alternating sum of the associators equals six times an associator and also equals the Jacobi identity. We used Albert to verify that the result is an identity that holds in every Malcev algebra. (If it had not held, it would have been the first known S-identity for Malcev algebras.)

We then repeated the same steps for representation type 22 111. This gives us a second new identity (in five variables) for the associator in every alternative algebra, called F_2 in the statement of the theorem. This identity was also rewritten in terms of commutators and checked by C programs against the Malcev identity: this identity also holds in the free Malcev algebra. This completes the proof. □

Identities for the Cayley Ternary Algebra

THE CAYLEY ALGEBRA

The only simple finite-dimensional non-associative alternative algebras over any field F are forms of the eight-dimensional Cayley algebra. In terms of the basis 1, i, j, k, ℓ, m, n, p the multiplication table for the non-unit elements is

	i	j	k	ℓ	m	n	p
i	-1	k	$-j$	m	$-\ell$	$-p$	n
j	$-k$	-1	i	n	p	$-\ell$	$-m$
k	j	$-i$	-1	p	$-n$	m	$-\ell$
ℓ	$-m$	$-n$	$-p$	-1	i	j	k
m	ℓ	$-p$	n	$-i$	-1	$-k$	j
n	p	ℓ	$-m$	$-j$	k	-1	$-i$
p	$-n$	m	ℓ	$-k$	$-j$	i	-1

(This is taken from Jacobson (1974, p. 426), with $c_1 = c_2 = c_3 = -1$, but note the misprint in the case $i_7 i_3$, which is pk in our notation; cf. Kleinfeld (1963, p. 137)).

We now compute the multiplication table for the Cayley ternary algebra: the Cayley algebra using the associator as the operation. The associator is alternating and any associator with an argument 1 is zero. Since the quaternion subalgebra is associative, we only need to consider associators which contain three distinct factors, none of which is 1, in alphabetical order, with at least one factor from ℓ, m, n, p. There are 12 associators with two factors from i, j, k and one factor from ℓ, m, n, p:

$$(i,j,\ell) = 2p, \quad (i,j,m) = -2n, \quad (i,j,n) = 2m, \quad (i,j,p) = -2\ell,$$
$$(i,k,\ell) = -2n, \quad (i,k,m) = -2p, \quad (i,k,n) = 2\ell \quad (i,k,p) = 2m,$$
$$(j,k,\ell) = 2m, \quad (j,k,m) = -2\ell, \quad (j,k,n) = -2p, \quad (j,k,p) = 2n.$$

There are 18 associators with one factor from i, j, k and two factors from ℓ, m, n, p:

$$(i,\ell,m) = 0, \quad (i,\ell,n) = -2k, \quad (i,\ell,p) = 2j, \quad (i,m,n) = -2j,$$
$$(i,m,p) = -2k, \quad (i,n,p) = 0, \quad (j,\ell,m) = 2k, \quad (j,\ell,n) = 0,$$
$$(j,\ell,p) = -2i, \quad (j,m,n) = 2i, \quad (j,m,p) = 0, \quad (j,n,p) = -2k,$$
$$(k,\ell,m) = -2j, \quad (k,\ell,n) = 2i, \quad (k,\ell,p) = 0, \quad (k,m,n) = 0,$$
$$(k,m,p) = 2i, \quad (k,n,p) = 2j.$$

There are four associators with all three factors from ℓ, m, n, p:

$$(\ell,m,n) = -2p, \quad (\ell,m,p) = 2n, \quad (\ell,n,p) = -2m, \quad (m,n,p) = 2\ell.$$

The value of an associator in the Cayley ternary algebra always lies in the span of the non-unit basis elements; also, an associator which has a scalar argument is zero. Therefore the Cayley ternary algebra is the direct sum of two ideals: the trivial ideal of scalars and the ideal spanned by the non-unit basis elements. After factoring out the scalars we are left with a seven-dimensional alternating ternary algebra which contains all the information about the associator in the Cayley algebra.

The Cayley algebra has a grading by the group $(\mathbb{Z}_2)^3$. The degrees of the standard basis elements are

$$\begin{aligned} &d(1) = (0,0,0), \quad d(i) = (0,1,1), \quad d(j) = (1,0,1), \quad d(k) = (1,1,0), \\ &d(\ell) = (0,0,1), \quad d(m) = (0,1,0), \quad d(n) = (1,0,0), \quad d(p) = (1,1,1). \end{aligned}$$

This grading satisfies the usual property that $d(xy) = d(x) + d(y)$.

For computational purposes one of the most efficient ways to represent the Cayley algebra is by the Zorn vector–matrix algebra. We consider 2×2 matrices in which the diagonal entries are complex numbers, and the off-diagonal entries are vectors in $\mathbb{C}^3$. The product is defined by the rule

$$\begin{pmatrix} a & B \\ C & d \end{pmatrix} \begin{pmatrix} e & F \\ G & h \end{pmatrix} = \begin{pmatrix} ae + B \cdot G & aF + hB - C \times G \\ eC + dG + B \times F & C \cdot F + dh \end{pmatrix}.$$

Here $\cdot$ and $\times$ represent the usual dot and cross product. The eight-dimensional complex vector space consisting of these matrices with this product is isomorphic to the split complex Cayley algebra, see Paige (1963, p. 180). If we write these matrices as eight-dimensional row vectors via

$$\begin{pmatrix} a & B \\ C & d \end{pmatrix} \longmapsto (a, b_1, b_2, b_3, c_1, c_2, c_3, d)$$

then an isomorphism is given by

$$\begin{aligned} 1 &= (1,0,0,0,0,0,0,1) & i &= (0,-1,0,0,1,0,0,0) \\ j &= (0,0,-1,0,0,1,0,0) & k &= (0,0,0,-1,0,0,1,0) \\ l &= \big(\sqrt{-1},0,0,0,0,0,0,-\sqrt{-1}\big) & m &= \big(0,\sqrt{-1},0,0,\sqrt{-1},0,0,0\big) \\ n &= \big(0,0,\sqrt{-1},0,0,\sqrt{-1},0,0\big) & p &= \big(0,0,0,\sqrt{-1},0,0,\sqrt{-1},0\big). \end{aligned}$$

Theorem 2. *There are no non-obvious identities for the associator in the Cayley algebra in degree* 5. *Over a field of characteristic* 103, *the following five irreducible identities in degree* 7, *together with the identities* F_1 *and* F_2 *from the previous section and the obvious identities, generate all the identities of degree* ≤ 7 *satisfied by the associator in the Cayley algebra:*

$$\begin{aligned} C_1 = \sum_{\mathrm{alt}(x)} \{ &((abx)(axx)b) - ((abx)(bxx)a) + 2((abx)(abx)x) \\ &+ 2(((abx)ax)bx) - 2(((abx)bx)ax) - (((axx)bx)ab) \\ &+ (((bxx)ax)ab) - 2(((abx)xx)ab) + 3(((abx)ab)xx) \\ &+ (((xxx)ab)ab) + 4(((axx)ab)bx) - 4(((bxx)ab)ax)\} \\ C_2 = \sum_{\mathrm{alt}(x)} \{ &6((axx)(xxx)a) - 2(((axx)ax)xx) + 2(((xxx)ax)ax) \\ &- 5(((axx)xx)ax)\} \\ C_3 = \sum_{\mathrm{alt}(x)} \{ &(((axx)xx)ax) + 2(((xxx)ax)ax) + 4(((axx)ax)xx)\} \end{aligned}$$

$$C_4 = \sum_{\mathrm{alt}(x)} ((xxx)(xxx)x)$$

$$C_5 = \sum_{\mathrm{alt}(x)} (((xxx)xx)xx)$$

Identity C_1 has five variables and corresponds to partition 22 111*. Identities C_2 and C_3 have six variables and correspond to partition* 211 111*. Identities C_4 and C_5 have seven variables and correspond to partition* 1 111 111*; these two identities hold over a field of any characteristic.*

Proof. (By Computer) To determine the identities of degree 5, we first define a matrix of size 18×10 and initialize it to zero. Given five elements of the Cayley algebra (each of these elements is a vector with eight components) we substitute them for the letters *a*–*e* in the 10 multilinear monomials, and then evaluate the monomials by interpreting the ternary product as the associator. The result is an ordered list of 10 vectors with eight components. We insert these vectors vertically in the last eight rows of the matrix. Each of these eight rows gives a relation on the coefficients for the 10 monomials which must be satisfied by any identity for the associator in the Cayley algebra. (Each of the eight rows corresponds to the coefficients for one of the basis elements in the Cayley algebra.) We then compute the row-canonical form of the 18×10 matrix. The rank of the matrix can be no more than 10, so the last eight rows must be zero. We repeatedly generate five random eight-vectors (with components from one to 100) and produce eight new rows which are copied into the bottom eight rows of the matrix, and compute the row-canonical form using rational arithmetic in Maple. The rank soon reaches 10, which implies that there are no non-obvious identities.

We next consider identities of degree 7. The method is essentially the same as in degree 5, except that now we have 280 multilinear monomials. We first define a matrix of size 288×280 and initialize it to zero. During each of the first few iterations the rank increases by seven (not eight, since any associator is in the span of the non-unit basis elements). The rank stabilized at 224 after 32 iterations, and did not increase again; a total of 100 iterations were performed. This implies that there are at least 224 linearly independent relations that any identity must satisfy, and so the module of identities has dimension at most 56.

This computation was started using random numbers from one to 100 and rational arithmetic for the row-canonical form. However the calculations went very slowly; it turned out that each component of the evaluated multilinear ternary monomials had about 12 digits, and the matrix entries in the middle of the row-canonical form computation were rational numbers with 100-digit numerators and denominators. Maple can handle arbitrarily large integers, but not without a loss of speed. Therefore we decided to use random numbers from one to 10 and use arithmetic modulo 1009 for the row-canonical form. There is a small chance that using modular arithmetic will cause some information to be lost; for example in the case of a matrix row in which all the entries (in rational arithmetic) are multiples of the modulus. Even if this does occur, it does not create new, false relations on the coefficients of an identity: all the relations we do produce are valid. So the conclusions of the previous paragraph are still correct.

These computations were repeated in C using the modulus 103. From the 224 relations for the Cayley identities we obtain 56 identities by computing the nullspace of the relations. Running these identities through the same C programs described in the previous section, we obtained the dimensions given in the column labeled Cayley associator in the table given there.

Each of the two new identities for the associator in a free alternative algebra generates (after linearization) a 14-dimensional subspace of the 280-dimensional space; these subspaces are irreducible representations of S_7 labeled by partitions 2221 and 22 111. We checked that this 28-dimensional space of free alternative identities is a subspace of the 56-dimensional space of Cayley identities. We then found the orthogonal complement of the space of free identities in the space of Cayley identities with respect to the usual scalar product. (That is, the scalar product determined by the condition that the 280 monomials are an orthonormal basis.) The 28 basis vectors for this complement span the space of Cayley identities that are not implied by the free alternative identities. This space of non-free Cayley identities is also an S_7-module which decomposes as the direct sum of five irreducible representations: one for partition 22 111, and two for partition 211 111, and two for partition 1 111 111.

Generators for these irreducible submodules were found by the same method used in the previous section: from the matrix row representing an identity we computed the double group ring element which gives the identity. These identities are the linearizations of the identities in the Theorem.

We checked identities C_1, C_2 and C_3 by setting the indeterminates equal to random vectors and evaluating the identity over the integers. In each case the result was the zero vector. This provides some evidence of the validity of these identities in characteristic 0.

We have the following direct proof of identity C_4; the same argument applies to identity C_5. Let I be the alternating function of seven vectors in F^7 defined by the formula

$$I(a,b,c,d,e,f,g) = \sum_{\mathrm{alt}(x)} ((xxx)(xxx)x).$$

The value of I is a vector in F^7. Each of the seven components is an alternating scalar function of seven vectors in F^7 and hence must be a scalar multiple of the determinant. Thus there exist scalars t_i for $1 \le i \le 7$ such that

$$I(a,b,c,d,e,f,g) = \det(a,b,c,d,e,f,g)(t_1,t_2,t_3,t_4,t_5,t_6,t_7).$$

If we take a–g to be the standard basis vectors in F^7 then the determinant equals 1. Therefore

$$I(i,j,k,\ell,m,n,p) = (t_1,t_2,t_3,t_4,t_5,t_6,t_7).$$

The degree of $I(i,j,k,\ell,m,n,p)$ using the $(\mathbb{Z}_2)^3$-grading on the Cayley algebra equals the sum of all non-zero elements of the grading group, which is $(0,0,0)$. Therefore the value of I on these basis elements must be a scalar. But each term in the definition of I is an associator, and the only scalar value that an associator in the Cayley algebra can take is 0. Therefore $t_i = 0$ for $1 \le i \le 7$, which completes the proof. □

Identities for the Ternary Cross Product

THE TERNARY CROSS PRODUCT

One of the simplest and most natural examples of an alternating ternary algebra is the *ternary cross product* on a four-dimensional vector space A over F. Let L, M, N, P

be a basis of A, and let $X_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})$ for $1 \leq i \leq 3$ be three vectors in A expressed in coordinates with respect to this basis. We define the ternary cross product by a generalization of the usual determinant definition of the familiar three-dimensional cross product:

$$[X_1, X_2, X_3] = \begin{vmatrix} L & M & N & P \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \end{vmatrix}.$$

This is clearly an alternating trilinear function of X_1, X_2, X_3. There are only four nonzero products of basis vectors:

$$[L, M, N] = -P, \qquad [L, M, P] = N, \qquad [L, N, P] = -M, \qquad [M, N, P] = L.$$

For $F = \mathbb{R}$ the ternary cross product is a ternary analogue of the compact simple Lie algebra $su(2)$.

Comparing the four ternary products displayed with the four associators involving ℓ, m, n, p in the Cayley ternary algebra, we see that the span of L, M, N, P is isomorphic to a subsystem of the Cayley ternary algebra. The correspondence

$$L \mapsto \frac{1}{\sqrt{2}}\ell, \qquad M \mapsto \frac{1}{\sqrt{2}}m, \qquad N \mapsto \frac{1}{\sqrt{2}}n, \qquad P \mapsto \frac{1}{\sqrt{2}}p,$$

is an isomorphism from the ternary cross product to this subsystem.

Theorem 3. *Over a field of characteristic* 0*, the following reducible identity in degree* 5*, together with the obvious identities in degrees* 5 *and* 7*, generates all the identities of degree* ≤ 7 *satisfied by the ternary cross product:*

$$((abc)de) - ((abd)ce) + ((abe)cd) - ((cde)ab).$$

This identity is equivalent to the two irreducible identities

$$((abc)ad) + ((acd)ab) + ((adb)ac), \qquad \sum_{\text{alt}(x)} ((xxx)xx).$$

The first identity says that the binary algebra with product $[x, y] = (a, x, y)$ *for any fixed* a *is a Lie algebra. The second identity is the alternating sum of the* 10 *degree-*5 *monomials.*

Proof. (By Computer) We first consider the identities of degree 5. Consider a general linear combination of the 10 multilinear monomials. Given any assignment of the basis vectors L, M, N, P to the indeterminates a, b, c, d, e we can evaluate this linear combination; the result is a scalar multiple of a basis vector. By the graded property of the Cayley algebra, this basis vector depends only on the number of times each of L, M, N, P occurs in the assignment. In this way we obtain $4^5 = 1024$ linear relations on the coefficients in the general linear combination of the monomials. We create a matrix in which the 10 columns are labeled by the monomials and the 1024 rows express the linear relations. We find all the identities by computing the row-canonical form of this matrix. Using Maple we found that the nullspace of this matrix is generated (as an S_5-module) by the reducible identity in the Theorem. This identity can also be written as

$$(ab(cde)) = ((abc)de) + (c(abd)e) + (cd(abe)),$$

and so it is called the *ternary derivation identity*, see the paper of Filippov (1985).

As shown in Bremner (1997), the S_5-module with basis consisting of the 10 multilinear alternating ternary monomials of degree 5 decomposes as the direct sum of the simple S_5-modules labeled by the partitions 221, 2111 and 11 111. The submodule generated by the ternary derivation identity is the sum of the simple submodules corresponding to 2111 and 11 111. The two irreducible identities in the Theorem generate (respectively) these two simple submodules. Therefore the ternary derivation identity is reducible, and is equivalent to the two irreducible identities, in the sense that a ternary algebra satisfies the ternary derivation identity if and only if it satisfies the two irreducible identities.

We next consider the identities of degree 7. We want to determine if there are any new identities for the ternary cross product in degree 7; that is, identities which are not implied by the ternary derivation identity.

We first determine the submodule of identities in degree 7 which are implied by the ternary derivation identity

$$I(a,b,c,d,e) = ((abc)de) - ((abd)ce) + ((abe)cd) - ((cde)ab).$$

Since $I(a,b,c,d,e)$ alternates in a, b and also in c, d, e there are three inequivalent ways to lift I to degree 7:

$$(I(a,b,c,d,e),f,g), \qquad I((a,b,c),d,e,f,g), \qquad I(a,b,(c,d,e),f,g).$$

These three polynomials are generators of the S_7-submodule of identities in degree 7 which follow from the derivation identity in degree 5. To compute the dimension of this submodule, we need to apply the 5040 permutations of the seven letters to each of the three liftings; altogether this gives a matrix of size $15\,120 \times 280$ for which we need to compute the row-canonical form. We can greatly reduce the number of rows as follows: for lifting 1, we consider only the permutations which are coset representatives with respect to the subgroup $S_2 \times S_3 \times S_2$ acting on a, b and c, d, e and f, g. (If we apply any permutation in this subgroup to lifting 1 we obtain the same identity up to a sign.) So we only need to consider permutations which are inequivalent with respect to this subgroup: this means that the letters in positions 1, 2 and 3, 4, 5 and 6, 7 are in alphabetical order. This gives a total of $5040/24 = 210$ rows which span the submodule generated by lifting 1. Applying the same procedure to the other two liftings, we obtain 140 rows for lifting 2, and 210 rows for lifting 3. Altogether this gives a matrix of size 560×280, a matrix 1/27 the size of the original matrix. The row-canonical form of this matrix was computed using a Maple program; however it was necessary to write a new procedure different from the predefined *linalg*[*rref*] procedure, since the latter procedure does not work efficiently for large matrices. The result: the matrix has rank 224, which is therefore the dimension of the submodule of identities in degree 7 which follow from the ternary derivation identity in degree 5.

The second step is to determine the S_7-module of all identities of degree 7 satisfied by the ternary cross product. We used the same random-vector method as in the previous section, now with a matrix of size 284×280. At the beginning of this procedure, the rank of the matrix increased by four after each iteration. The rank stabilized at 56 after iteration 14; altogether 100 iterations were performed. This implies that there are at least 56 linearly independent relations that an identity must satisfy. From this it follows that there are a maximum of $280 - 56 = 224$ identities for the ternary cross product in degree 7. Since all of these identities are accounted for by the liftings of the derivation identity, it follows that there are no new identities in degree 7. This completes the proof. □

Concluding Remark

A semi-simple alternative algebra is a subdirect sum of associative and Cayley–Dickson algebras. Thus any element of a free alternative algebra which is zero in every associative and Cayley–Dickson algebra must be in the radical, see Zhevlakov *et al.* (1982, p. 271, Corollary 1). Our identities C_1–C_5 are then elements of the radical of the free alternative algebra.

An element in the commutative center of the free alternative algebra must evaluate to a multiple of the identity element in a Cayley–Dickson algebra, since in a Cayley–Dickson algebra the commutative center consists only of the scalars. Furthermore, commuting elements built out of associators map to zero in a Cayley–Dickson algebra, since in a Cayley–Dickson algebra the value of an associator is in the span of the non-unit basis elements. We checked our identities to see if any of them were in the commutative center of the free alternative algebra. We found that C_1, $5C_2 + 7C_3$ and C_4 are elements of the commutative center of the free alternative algebra. (Compare the results in *An Alternative Identity of Degree* 5 by Hentzel and Kleinfeld, to appear in Journal of Algebra.)

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