EXPERIMENTING AND PROOF IN MATHEMATICS WITH XCAS

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Xcas groups in the same program a lot of functions which are usually separated: algebra system, spreadsheet, dynamic geometry in the plane and space, programming... The software features and their connections give many possibilities for experimenting and proving in mathematics. We show investigations with Xcas involving three problems, which have been tested with high school students. We propose that the students should not be guided while investigating. The possibility of approaching a problem by various means like spreadsheet, algebra system, graph, geometrical figure, can help the students to build up mathematical knowledge differently from paper-pencil.

We present three examples of practical works to show the interest in using Xcas (http://www-fourier.ujf-grenoble.fr/~parisse/giac_fr.html), for the possibility of changing a function while experimenting or proving in mathematics. The fact that Xcas allows many possibilities involves some complexity in learning the program. It is still less difficult than learning to use several programs. As noticed by Balacheff (1994), implementation of mathematical knowledge in an artifact generates an instrumental distance in comparison with the paper-pencil representations. Why would these paper-pencil representations be better to approach mathematical objects? It is possible that deviation from traditional representations becomes productive in relation to mathematical knowledge (Baron, Guin & Trouche, 2007).

A SEQUENCE OF COMPLEX NUMBERS

Let \((z_n)\) be a sequence of complex numbers so that \(z_0 = 2\) and \(z_{n+1} = ((1+i)/2)z_n\) for all \(n\). We can begin our examination of the sequence by using the formal spreadsheet. The first terms can be represented by points \(A_n\) in the plane in a cartesian coordinate system with the origin \(O\). We obtain a graphic that shows a spiral (figure 1).

Conjectures can be made, like :

- some terms are real numbers ;
- there exists a natural number, \(n_0\), such that all the points \(A_n\) for \(n \geq n_0\) belong to the disk with center \(O\) and radius 0,1 ; the number 0,1 can be replaced by all positive real number, as close to 0 as we want ;
- all the triangles \(OA_nA_{n+1}\) are right-angled and isosceles (as can be seen on the graphic below).
We can continue our investigations with the spreadsheet and find a relationship between the difference $z_{n+1} - z_n$ (the affix of the vector $A_n A_{n+1}$) and $z_{n+1}$ (the affix of the vector $OA_{n+1}$): $z_{n+1} - z_n = i \cdot z_{n+1}$.

We propose that the students should not be guided while investigating the behavior of the sequence. We make them ask questions about mathematical concepts, define their own conjectures, their own proofs or refutations of these conjectures with the object of making them construct their knowledge. The teacher must give scientific status to error. In the first sequence, students work in teams, a second time the teacher can start a scientific debate with the produced conjectures (Legrand, 1993), and a third time students write a report in order to rebuild what has been worked individually. This method is used in several teacher education programs which study the impact of ICT on teaching and learning mathematics (Hitt, 2007).

AN ECONOMIC MODEL

A company manufactures a product. A model is used to describe the operation of the business. Let $x$ be hours per day of labour ($x \in [0 ; 10]$) and let $y$ be hours per day of use of machinery ($y \in [0 ; 12]$). The produced quantity in metric tons is modeled by the two variables function: $f(x, y) = (3 \cdot x \cdot y) / (x + y)$. For each hour the total cost of labour is 4 000 euros and the total cost of use of machinery is 1 000 euros. The company fixed up expenditures per day to 36 000 euros. The problem is to find the highest production of the company per day under these constraints.

The two variables function is represented by a surface $S$ and the constraints by a plane $p$. With two instructions we obtain a figure that we can turn. It nevertheless is not easy for the students to understand these representations.

Figure 1: Right-angled and isosceles triangles in a spiral
If some directions (how to represent $f$, for example…) are given to the students, it is necessary they also may wonder whether a given point (for example, $A (3, 4, 5)$, $B (1, 2, 2)$) belongs or does not belong to the surface. It also should be relevant to add other planes in the figure ($x = 0$, $x = 10$, $y = 0$, $y = 12$) in order to understand the representation. It is important that the students may continue their investigation of the function $f$ and its representation without the teacher’s help to answer the question. When the students are ready to examine the intersection of the surface $S$ and the plane $p$, they may represent the intersection curve in the cartesian coordinate system with $x$-axis (horizontal) and $z$-axis (vertical) as you can see below.

Xcas allows the transition from 3D-geometry to 2D-geometry very easily.

### A QUADRILATERAL IN A PARALLELOGRAM

This problem was investigated by students (14-15 years old) in a math workshop in a highschool (Gandit & al 2007). Let $ABCD$ be a parallelogram. Find points $M$ and $K$ on the sides $[AD]$ and $[BC]$ of $ABCD$ for which the area of the quadrilateral obtained as the intersection of the two triangles $AKD$ and $BMC$ is as great as possible.
Some students begin their investigation and study the case where the parallelogram is a rectangle. They fix one of the variables, define the area of the intersection as a function and draw its graph. Figure 4 shows elements from this research: lines of algebra on the left, a geometrical figure and the graph of the area function, two cursors and windows corresponding with two parameters $k$ and $m$ on the right.

![Figure 4: A screen during the investigation](image)

They also use the spreadsheet and continue numerical investigation.

References


