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The Pricing of Call and Put Options on Foreign Exchange

J. ORLIN GRABBE*

The Wharton School, University of Pennsylvania, Philadelphia PA 19104, USA

This paper derives pricing equations for European puts and calls on foreign exchange. The call and put pricing formulas are unlike the Black–Scholes equations for stock options in that there are two relevant interest rates, interest rates are stochastic, and boundary constraints differ. In addition, it is shown that both American call and put options have values larger than their European counterparts.

This paper develops pricing relationships for European and American call and put options on foreign currency. Foreign exchange (FX) options have features that distinguish them from options on common stock. Consequently, commonly used models for pricing stock options, such as the popular Black–Scholes model, are inadequate for FX options.

Some previous studies have also looked at foreign currency option pricing. Feiger and Jacquillat (1979) attempt to obtain foreign currency option prices by first pricing a currency option bond. They are not able, however, to obtain simple, closed-form solutions by this procedure. Stulz (1982) looks also at currency option bond pricing, but his paper is primarily concerned with the question of default risk on part of a contract, and it is not easy to grasp the fundamentals of foreign currency option pricing in the context of his more general investigation. Black (1976) examines commodity options, and although his results have some relevance if interest rates are non-stochastic, they are not suitably general when the primary focus is foreign currency options.

An *American call option* is a security issued by an individual which gives its owner the right to purchase a given amount of an asset at a stated price (the exercise or striking price) on or before a stated date (the expiration or maturity date). For example, a call option on the British pound might give one the right to purchase £12,500 at 1.70 $/£ on or before the second Saturday in December 1983. A *European call option* is the same as the American call, except that it may be exercised only on the expiration date. In the previous example, only on the second Saturday in December

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1983. An *American put option* is a security issued by an individual which gives its owner the right to sell a given amount of an asset at a stated price on or before a stated date. For example, a put option on the yen might give one the right to sell ¥6,250,000 at 0.004 \$/¥ on or before 13 June. A *European put option* is the same as the American put, except that it may be exercised only on the expiration date. In the preceding example, only on 13 June.

Foreign currency options arise in international finance in three principal contexts. The first is organized trading on an exchange. A number of exchanges have trading in at least one foreign currency option, but presently the FX option market at the Philadelphia Stock Exchange (PHLX) is the broadest. Over the period December 1982–February 1983 the PHLX began trading American call and put options on five foreign currencies—the West German mark, the British pound, the Swiss franc, the Canadian dollar, and the Japanese yen. The exercise price of each option is stated as the US dollar price of a unit of foreign exchange, and the number of foreign currency units is one-half the contract size of the corresponding currency futures contract traded on the International Money Market of the Chicago Mercantile Exchange. Option contract sizes are £12,500; DM62,500; Swiss franc 62,500; ¥6,250,000; and C\$50,000. The expiration dates of the options are set to correspond to the March, June, September, December delivery dates on futures. Futures contracts expire on the second business day prior to the third Wednesday of each of these months, and option contracts expire on the second Saturday of each of these months. For each currency, options are opened with terms to maturity of 3, 6, and 9 months, corresponding to the March, June, September, December cycle. Exercise price intervals are \$0.05 for the £; \$0.02 for the DM, Swiss franc, C\$; and \$0.0002 for the ¥. (If, for example, the spot price of the £ is 1.82 \$/£ when June pounds open for trading, exercise prices are set at 1.80 \$/£ and 1.85 \$/£. If the £ then drops to \$1.80, a new series with exercise prices of 1.75 \$/£ is opened.)

A second type of FX option market is the bank market by which large money-center banks write FX options directly to their corporate customers. This market is largely invisible because banks are currently reluctant to make public any data regarding their activities in this sphere. Finally, one may note the frequent appearance of foreign currency option features on bond contracts in the international bond markets. For example, take the case of a Japanese company which issues \$1000 bonds at par in the Eurobond market. The coupon rate is 12% payable annually in US dollars, and the bonds mature in May 1990. At maturity the bonds may be redeemed, at the owner's discretion, for dollars or for yen at an exchange rate of 0.005 \$/yen. What value would one place on such a bond? Clearly the bond owner will opt for repayment of principal in yen if the spot price of yen is greater than 0.005 \$/yen in May 1990. (For example, if the spot rate is 0.006 \$/yen, she would redeem her bond for 1000/0.005=200,000 yen and then sell the yen for (200,000)(0.006)=\$1200.) Thus the value of this Eurobond can be viewed as the sum of the value of an ordinary \$1000 bond with a 12% coupon, plus the value of a European call option on 200,000 yen, with an exercise price of 0.005 \$/yen, and with an expiration date in May 1990. Thus, provided we can value European options on foreign currency, we can place a value on this Japanese currency-option bond.

The organization of this paper is as follows. Section I sets out notation and assumptions. Section II derives some inequalities and equivalences for puts and calls on foreign exchange. In Section III, some strong distributional assumptions are imposed, and exact pricing equations for European options derived. Section IV looks at American FX options. Section V briefly considers the use of the equations for hedging and speculation, and Section VI concludes the paper.

I. Notation and Assumptions

Throughout this paper it will be assumed that contracts are default-free. In addition we will assume the absence of transaction costs, taxes, exchange controls, or similar factors. Trading takes place in continuous time. Finally, there exist discount bonds at which each currency may be borrowed or lent. The interest parity theorem is assumed for some results.

The symbols used will follow Smith's (1976) notation, except for some minor variations that are tailored to the foreign exchange market. The notation is:

$S(t)$ –the spot domestic currency price of a unit of foreign exchange at time t.
$F(t,T)$ –the forward domestic currency price of a unit of foreign exchange, for a contract made at time t and which matures at time $t+T$.
T –the time until expiration.
$C(t)$ –the domestic currency price at time t of an American call option written on one unit of foreign exchange.
$C^*(t)$ –the foreign currency price at time t of an American call option written on one unit of domestic currency.
$c(t)$ –the domestic currency price at time t of a European call option written on one unit of foreign exchange.
$c^*(t)$ –the foreign currency price at time t of a European call option written on one unit of domestic currency.
$P(t)$ –the domestic currency price at time t of an American put option written on one unit of foreign currency.
$P^*(t)$ –the foreign currency price at time t of an American put option written on one unit of domestic currency.
$p(t)$ –the domestic currency price at time t of a European put option written on one unit of foreign currency.
$p^*(t)$ –the foreign currency price at time t of a European put option written on one unit of domestic currency.
$B(t,T)$ –the domestic currency price of a pure discount bond which pays one unit of domestic currency at time $t+T$.
$B^*(t,T)$ –the foreign currency price of a pure discount bond which pays one unit of foreign exchange at time $t+T$.
X –the domestic currency exercise price of an option on foreign currency.
X^* –the foreign currency exercise price of an option on domestic currency.

II. Some Basic Relationships

II.A

Consider the following two portfolio strategies undertaken at time t when the spot exchange rate is $S(t)$:

Strategy A: 1. Purchase for $c(S(t),X,t,T)$ a European call option, with an exercise price of X and which expires in T units of time, on one unit of foreign exchange.

2. Purchase X domestic currency discount bonds, which mature in T units of time, at the current price $B(t,T)$.
   Total Domestic Currency Investment: $c+XB$.

Strategy B: 1. Purchase one foreign discount bond, which matures in T units of time, at a domestic currency price of $S(t)B^*(t,T)$.

At time $t+T$, the spot exchange rate $S(t+T)$ will either be less than X, or greater than or equal to X. The bonds will have values, in their respective currencies, of $B(t+T, 0)=1$, $B^*(t+T, 0)=1$. The call option will have a value $c(S(t+T), X, T, 0)=\max(0, S(t+T)-X)$. Therefore:

	Value of Strategy A	Value of Strategy B
$S(t+T) < X$	X	$S(t+T)$
$S(t+T) \geqslant X$	$S(t+T)$	$S(t+T)$

In either case, the payoff to strategy A will always be as good or better than strategy B. Hence the cost of A must, in economic equilibrium, be at least as great as that of B. Thus $c+XB \geqslant SB^*$ or $c \geqslant SB^*-XB$. Since an American call must be at least as valuable as its European counterpart (because it has all the same features plus the additional one that it can be exercised at any time), we get

$$\langle 1\rangle \qquad C(S(t),X,t,T) \geqslant c(S(t),X,t,T) \geqslant S(t)\,B^*(t,T)-XB(t,T)$$

For example, if one-year Eurodollar deposits have an interest rate of 11.11% ($B(t, 1)=1/1.1111=0.9$), one-year Euro-Swiss franc deposits have an interest rate of 5.26% ($B^*(t, 1)=1/1.0526=0.95$), and the spot dollar price of Swiss francs is $S(t)=0.55$ \$/Swiss fr., then a 12-month American option on one Swiss franc with an exercise price of $X=\$0.50$ will have a value

$$C(0.55,0.50,t,1) \geqslant c(0.55,0.50,t,1) \geqslant (0.55)(0.95)-(0.50)(0.9) = \$0.0725$$

Since an American option can be exercised at any time, and must have a value at least as large as its immediate exercise value, we get the stronger inequality

$$\langle 2\rangle \qquad C(S(t),X,t,T) \geqslant \max(0,S(t)-X,S(t)\,B^*(t,T)-XB(t,T))$$

II.B

Using the notation of Section I, we may write the Interest Parity Theorem as

$$\langle 3\rangle \qquad F(t,T) = S(t)\frac{B^*(t,T)}{B(t,T)}$$

Substituting for $S(t)B^*(t,T)$ in equation ⟨1⟩, we obtain the relation

$$\langle 4\rangle \qquad C(S(t),X,t,T) \geqslant c(S(t),X,t,T) \geqslant B(t,T)[F(t,T)-X]$$

The call option on one unit of foreign exchange must have a value at least as great as the discounted difference between the forward exchange rate and the exercise price. This assumes that interest parity holds. (Hence, in practical application $B(t,T)$ and $B^*(t,T)$ should be thought of as Eurocurrency securities, rather than as treasury bills.) The intuition is clear if we consider the case $F>X$. An owner of a European call can sell foreign currency forward for $F(t,T)$ even though the purchase price will be X. Thus such an option has a value at least as large as this difference, $F-X$, once it is discounted to the present.

II.C

Here we derive a relationship between the prices of European calls and European puts. Consider the following two portfolio strategies:

Strategy A: 1. Buy, at a domestic currency price of $p(S(t),X,t,T)$, a put option on one unit of foreign currency, with exercise price X.

Strategy B: 1. Issue a foreign-currency-denominated discount bond at $B^*(t,T)$ and sell the foreign currency for $S(t)\,B^*(t,T)$.

2. Buy X domestic discount bonds at a price of $B(t)$ each, for a total domestic currency amount of $XB(t)$.
3. Buy, at a domestic currency price of $c(S(t),X,t,T)$, a European call option on one unit of foreign currency, with an exercise price of X.

Total domestic currency investment: $c-SB^*+XB$.

At time $t+T$ the spot exchange rate $S(t+T)$ will be such that either $S(t+T)<X$ or $S(t+T)\geqslant X$. In each case, since $p(S(t+T),X,T,0)=\max(0,X-S(t+T))$, the strategies will have the payoffs:

	Value of Strategy A	Value of Strategy B
$S(t+T) < X$	$X-S(t+T)$	$X-S(t+T)$
$S(t+T) \geqslant X$	0	0

Since each strategy gives the same payoff, each must cost the same in equilibrium. Hence

$$\langle 5\rangle \qquad p(S(t),X,t,T) = c(S(t),X,t,T)-S(t)\,B^*(t,T)+XB(t,T)$$

Thus the price of a European put is totally determined by the price of the corresponding European call, the spot exchange rate, and the prices of discount bonds denominated in the two currencies. Equation $\langle 5\rangle$ is the put-to-call conversion equation for European FX options.

II.D

Using the Interest Parity Theorem, equation $\langle 3\rangle$, and substituting into equation $\langle 5\rangle$, we obtain

$$\langle 6\rangle \qquad p(S(t),X,t,T) = c(S(t),X,t,T)+B(t,T)\,[X-F(t,T)]$$

The price of a European put differs from the price of the corresponding call by a factor which represents the discounted difference between the exercise price and the forward exchange rate.

II.E

A call option on a foreign currency, written at an exercise price in terms of the domestic currency, is a put option on the domestic currency, written at an exercise price in terms of the foreign currency. For example, take the PHLX call option on

62,500 DM and suppose that the exercise price is 0.40 \$/DM. This option gives one the right to buy 62,500 DM for (62,500)(0.40) = \$25,000. But at the same time it gives the right to sell \$25,000 for DM62,500. Thus it is a put option on \$25,000 with an exercise price of 1/0.40 = 2.5 DM/\$. This is a simple consequence of the fact an exchange rate has two sides. Hence, whether viewed as a call or a put, a contract must have the same domestic currency value.

For American option contracts, this observation implies[1]

$$\langle 7 \rangle \qquad C(S(t),X,t,T) = S(t)XP^*(1/S(t),1/X,t,T)$$

(For example, if C is the value of an American call on 1 DM with an exercise price $X = \$0.40$, then the same contract is a put option on \$0.40 with an exercise price of 1 DM. Hence it has the value of 40/100 of $P^*(1/S(t),2.5,t,T)$, where P^* is the DM value of a put option on \$1 with an exercise price of 2.5 DM. But the dollar value of $0.40P^* = XP^*$ is just SXP^*, which must be equal to C.)

For domestic-currency-valued puts on foreign exchange, we have

$$\langle 8 \rangle \qquad P(S(t),X,t,T) = S(t)XC^*(1/S(t),1/X,t,T)$$

Analogous equations hold for European options. Notice that equations ⟨7⟩ and ⟨8⟩ imply that if we obtain a pricing equation for American call options on foreign exchange, then we immediately get a pricing equation for American put options. We can take either currency as the 'domestic' currency, and then using ⟨7⟩ or ⟨8⟩ translate the call equation for the 'domestic' currency into a put equation on the second currency.[2]

III. Pricing Equations for European Call and Put Options

Here we derive exact pricing equations for European calls and puts. The initial relationships apply to both American and European FX options. Once through the preliminaries, we will give the European solutions, and defer to the following section continued discussion of American options.

Equation ⟨1⟩ suggests that the value of an American call C (or European call c) will be a function of $S(t)B^*(t,T)$, $B(t,T)$, X, T. The first assumption, then, is that C has the general functional form $C = C(S(t)B^*(t,T), B(t,T), X, T)$. Such a function would be subject to the boundary conditions

$$\langle 9 \rangle \qquad C(S(t+T),1,X,0) = \max(0, S(t+T) - X)$$

$$\langle 10 \rangle \qquad C(0,B(t,T),X,T) = 0$$

$$\langle 11 \rangle \qquad C(S(t)B^*(t,T),\ B(t,T),X,T) \geqslant \max(0, S(t) - X)$$

The first boundary condition is the terminal value of the call option, which has to be the greater of zero or the exercise value. The second boundary condition says that when the value of spot exchange is zero, the option to buy it has a zero value. The third boundary condition says the value of an American call can never be less than its immediate exercise value. (This third boundary condition does not necessarily apply in the case of a European call option.)

The second assumption has to do with the dynamics of S, B^*, and B. Let dx, dy, dz denote standardized Wiener processes with unit instantaneous variances and correlation matrix

$$\begin{bmatrix} 1 & \rho_{SB^*} & \rho_{SB} \\ \rho_{SB^*} & 1 & \rho_{B^*B} \\ \rho_{SB} & \rho_{B^*B} & 1 \end{bmatrix} \mathrm{d}t$$

where $\rho_{ij}=\rho_{ij}(t,T)$ may be a known function of time (t) and the term to maturity of the bond (T). Then S, B^*, B are assumed to follow the diffusion processes

$$\frac{\mathrm{d}S}{S} = \mu_S(t)\,\mathrm{d}t+\sigma_S(t)\,\mathrm{d}x$$

$$\frac{\mathrm{d}B^*}{B^*} = \mu_{B^*}(t,T)\,\mathrm{d}t+\sigma_{B^*}(t,T)\,\mathrm{d}y$$

$$\frac{\mathrm{d}B}{B} = \mu_B(t,T)\,\mathrm{d}t+\sigma_B(t,T)\,\mathrm{d}z$$

On the basis of these assumptions we can define new variables $\mathrm{d}G$, $\mathrm{d}w$, as

$$\frac{\mathrm{d}G}{G} = \frac{\mathrm{d}(SB^*)}{SB^*} = (\mu_S+\mu_{B^*}+\rho_{SB^*}\sigma_S\sigma_{B^*})\,\mathrm{d}t+\sigma_S\,\mathrm{d}x+\sigma_{B^*}\,\mathrm{d}y$$
$$\equiv \mu_G(t,T)\,\mathrm{d}t+\sigma_G(t,T)\,\mathrm{d}w$$

and write the correlation marix of $\mathrm{d}w$, $\mathrm{d}z$ as

$$\begin{bmatrix} 1 & \rho_{GB} \\ \rho_{GB} & 1 \end{bmatrix} \mathrm{d}t \tag{10}$$

where $\rho_{GB}=\rho_{GB}(t,T)$.

We know from equation ⟨1⟩ that the derived call option value must satisfy the condition $C(t)\geqslant S(t)B^*(t,T)-XB(t,T)$. But that does not ensure that the boundary constraint in ⟨11⟩ will not be violated, since $S(t)B^*(t,T)-XB(t,T)<S-X$ for sufficiently large S. Thus we have to take explicit account of ⟨11⟩ in deriving the American pricing equation.

It is plausible to assume that the call option dynamics will be different away from the boundary than on the boundary. Thus we first consider the case that either $C(t)>S-X>0$, or $C(t)>0>S-X$. We will assume that in this region, where the constraint ⟨11⟩ is not binding, that $C(t)$ is everywhere twice continuously differentiable. In order to derive the call option pricing equation for this case, we form a zero-wealth portfolio, the return to which is non-stochastic, or riskless. In economic equilibrium, the return to such a portfolio must be zero.

Now applying Ito's lemma to the function $C(SB^*,B,X,T)=C(G,B,X,T)$, we get the option dynamics, for $C>S-X>0$ or $C>0>S-X$,

$$\mathrm{d}C = \frac{\partial C}{\partial G}\mathrm{d}G+\frac{\partial C}{\partial B}\mathrm{d}B+\frac{\partial C}{\partial T}\mathrm{d}T$$
$$-\tfrac{1}{2}\left(\frac{\partial^2 C}{\partial G^2}G^2\sigma_G^2+2\frac{\partial^2 C}{\partial G\,\partial B}GB\rho_{GB}\sigma_G\sigma_B+\frac{\partial^2 C}{\partial B^2}B^2\sigma_B^2\right)\mathrm{d}T$$
$$= \frac{\partial C}{\partial G}\mathrm{d}G+\frac{\partial C}{\partial B}\mathrm{d}B+\frac{\partial C}{\partial T}\mathrm{d}T-\tfrac{1}{2}\phi\,\mathrm{d}T$$

where ϕ represents the elements involving second derivatives, and the relation $\mathrm{d}t=-\mathrm{d}T$ has been used.

Let V be a portfolio composed of one option, b units of G, and e units of B:

$$V = C + bG + eB$$

The dynamics of this portfolio are

$$\mathrm{d}V = \mathrm{d}C + b\,\mathrm{d}G + e\,\mathrm{d}B$$

Choose b, e such that $b = -\frac{\partial C}{\partial G}$, $e = -\frac{\partial C}{\partial B}$.

Then

$$\mathrm{d}V = \left(\frac{\partial C}{\partial T} - \tfrac{1}{2}\phi\right)\mathrm{d}T$$

If the portfolio V uses no wealth, then in equilibrium it should yield a zero return. That is, if

$$\langle 12\rangle \qquad V = C - \frac{\partial C}{\partial G}G - \frac{\partial C}{\partial B}B = 0$$

then $\mathrm{d}V = 0$, which in turn implies that

$$\langle 13\rangle \qquad \frac{\partial C}{\partial T} = \tfrac{1}{2}\phi$$

For the American call option, we look for a function $C(G,B,X,T)$ that solves equations ⟨12⟩ and ⟨13⟩, and is also subject to the boundary conditions ⟨9⟩–⟨11⟩. For the European call option, $c(G,B,X,T)$, condition ⟨11⟩ may be omitted.

It may be verified by direct substitution that a solution to the European call is

$$\langle 14\rangle \qquad c(t) = S(t)B^*(t,T)N(d_1) - XB(t,T)N(d_2)$$

where $N(d)$ is the standard normal distribution

$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\,\mathrm{d}x$$

and

$$d_1 = \frac{\ln(SB^*/XB) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(SB^*/XB) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$\sigma^2 = \int_0^T \frac{1}{T}[\sigma_G^2(t+T-u,u) + \sigma_B^2(t+T-u,u)$$
$$-2\rho_{GB}(t+T-u,u)\,\sigma_G(t+T-u,u)\cdot\sigma_B(t+T-u,u)]\,du$$

The equation for pricing European calls on foreign exchange, ⟨14⟩, differs from the Black–Scholes formula for pricing European options on common stock in three respects. First, there are two interest rates, not one. These interest rates are represented in the current prices of discount bonds $B(t,T)$, $B^*(t,T)$. In the Black–Scholes model, money (a bond) yields interest, but not stock. Hence interest at the domestic rate is foregone if a call option is purchased, and there is no possibility of receiving interest if the option is exercised. For an option on foreign exchange, however, the situation is different. One acquires foreign exchange if the option is exercised, and one may then receive interest at the foreign interest rate. Hence both the foreign and domestic interest rates form part of the expected return on an FX call. Second, the Black–Scholes model assumes a constant interest rate, and hence explicitly excludes covariation between movements in stock prices and interest rate movements. While this may be a reasonable simplification for the stock market, it is not appropriate for the foreign exchange market, where interest rate movements induce co-movements in spot and forward exchange rates.[3] Third, the asymmetry between interest payments on money (positive interest) and stock (zero interest, ignoring dividends) means that an American call option on stock is always worth more alive than dead, so that it will not be exercised prematurely. Thus American call options on stock have the same price as their European counterparts. American calls on foreign currency, however, have values strictly greater than European calls. This is shown in the following section.

The European call option formula may be rewritten, using the Interest Parity relation $S(t)B^*(t,T) = F(t,T)B(t,T)$, as

$$\langle 15 \rangle \qquad c(t) = B(t,T)[F(t,T)N(d_1) - XN(d_2)]$$

where

$$d_1 = \frac{\ln(F/X) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F/X) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

and

$$\sigma^2 = \int_0^T \frac{1}{T}\sigma_F^2(t+T-u,u)\,du,$$

where $\sigma_F^2(t,T)$ is the instantaneous variance of $dF(t,T)/F$. (Note that σ^2 here is exactly the same variable as in equation ⟨14⟩, as may be verified by applying Ito's lemma to the interest parity relation ⟨3⟩.)

What is remarkable about equation ⟨15⟩ is the disappearance of the price of spot exchange, which is the underlying asset on which the option is written. This results because, given the current price of domestic currency discount bonds B, all of the relevant information concerning both the spot exchange rate and the foreign currency discount bond price that is necessary for option pricing is already reflected in the forward rate. (That the forward and spot rates are not independent follows, of course, from the Interest Parity Theorem.)

Using the put to call conversion equation ⟨5⟩, we get that the value of the European put on 1 unit of foreign currency is

$$\langle 16 \rangle \qquad p(t) = XB(t,T)N(d_1^*) - S(t)B^*(t,T)N(d_2^*)$$

where (referring to equation ⟨14⟩) $d_1^* = -d_2$ and $d_2^* = -d_1$.

Equation ⟨16⟩ may be rewritten in terms of the forward rate as

$$p(t) = B(t,T)[XN(d_1^*) - FN(d_2^*)] \tag{17}$$

where (referring now to equation ⟨15⟩) $d_1^* = -d_2$ and $d_2^* = -d_1$.

TABLE 1. European call values*

	Months			Months			Months		
	3	6	9	3	6	9	3	6	9
				$\sigma=0.10$					
B	0.97	0.94	0.91	0.98	0.96	0.94	0.99	0.98	0.97
B^*	0.97	0.94	0.91	0.97	0.94	0.91	0.97	0.94	0.91
1.55	6.07	6.94	7.55	4.99	5.07	4.98	4.03	3.55	3.08
1.60	3.10	4.24	5.03	2.38	2.88	3.07	1.78	1.86	1.74
1.65	1.30	2.36	3.15	0.92	1.47	1.77	0.63	0.88	0.92
B	0.97	0.94	0.91	0.98	0.96	0.94	0.99	0.98	0.97
B^*	0.98	0.96	0.94	0.98	0.96	0.94	0.98	0.96	0.94
1.55	7.31	9.29	11.03	6.13	7.09	7.80	5.05	5.21	5.22
1.60	3.98	6.08	7.87	3.13	4.33	5.19	2.41	2.96	3.22
1.65	1.80	3.64	5.32	1.31	2.41	3.26	0.93	1.52	1.86
B	0.97	0.94	0.91	0.98	0.96	0.94	0.99	0.98	0.97
B^*	0.99	0.98	0.97	0.99	0.98	0.97	0.99	0.98	0.97
1.55	8.64	11.93	15.02	7.37	9.44	11.27	6.19	7.23	8.05
1.60	4.98	8.26	11.34	4.01	6.16	8.02	3.16	4.42	5.36
1.65	2.42	5.30	8.18	1.81	3.69	5.41	1.32	2.46	3.36
				$\sigma=0.20$					
B	0.97	0.94	0.91	0.98	0.96	0.94	0.99	0.98	0.97
B^*	0.97	0.94	0.91	0.97	0.94	0.91	0.97	0.94	0.91
1.55	8.82	10.90	12.33	7.91	9.26	10.00	7.07	7.79	8.01
1.60	6.19	8.48	10.05	5.45	7.06	7.99	4.78	5.83	6.27
1.65	4.16	6.46	8.09	3.59	5.28	6.31	3.09	4.27	4.85
B	0.97	0.94	0.91	0.98	0.96	0.94	0.99	0.98	0.97
B^*	0.98	0.96	0.94	0.98	0.96	0.94	0.98	0.96	0.94
1.55	9.88	12.97	15.41	8.91	11.14	12.74	8.00	9.49	10.40
1.60	7.05	10.26	12.79	6.25	8.66	10.38	5.52	7.24	8.32
1.65	4.82	7.97	10.50	4.20	6.60	8.36	3.64	5.42	6.57
B	0.97	0.94	0.91	0.98	0.96	0.94	0.99	0.98	0.97
B^*	0.99	0.98	0.97	0.99	0.98	0.97	0.99	0.98	0.97
1.55	11.00	15.20	18.81	9.97	13.20	15.81	9.00	11.37	13.14
1.60	7.98	12.23	15.87	7.12	10.44	13.12	6.32	8.84	10.71
1.65	5.56	9.67	13.25	4.87	8.11	10.76	4.24	6.74	8.62

* Entries in the table are prices, in US cents, of a European call option on one British pound, with the designated standard deviation rate (σ), strike price ($1.55, $1.60, $1.65), time to maturity (3, 6, 9 months), and domestic (B) and foreign (B^*) bond prices, when the spot rate is $1.60 per pound.

Table 1 shows values, in US cents, for a European call option on one British pound when the spot rate is 1.60 \$/£. Values are given for $\sigma=0.10, 0.20$; $T=3, 6, 9$ months; and $X=\$1.55, \$1.60, \$1.65$. B and B^*, with maturities of 3, 6, 9 months, are members of the set {(0.97 0.94 0.91), (0.98 0.96 0.94), (0.99 0.98 0.97)}. The values for B, B^* imply the term structures of interest rates are upward sloping in all cases.

Diagonal cells in the table illustrate that a lower level of interest rates yields higher option values when the term structure is the same in both countries. Cells below the diagonal show option values are an increasing function of the positive difference of domestic over foreign interest rates. Cells above the diagonal not only demonstrate that option prices are a decreasing function of the positive difference of foreign over domestic interest rates, but that the option can have an ambiguous time derivative. For $\sigma=0.10$, $X=\$1.55$, $B=(0.99\ 0.98\ 0.97)$, $B^*=(0.97\ 0.94\ 0.91)$, for example, the value of the option decreases from 4.03 cents for a three-month term to maturity to 3.08 cents for a nine-month term to maturity.

These relations can be seen more exactly from an inspection of partial derivatives in equation ⟨14⟩. The option price is decreasing with respect to domestic bond prices,

$$\langle 18\rangle \qquad \frac{\partial c}{\partial B} = -XN(d_2) < 0$$

increasing with respect to foreign bond prices,

$$\langle 19\rangle \qquad \frac{\partial c}{\partial B^*} = SN(d_1) > 0$$

increasing with respect to the domestic currency value of a foreign bond ($G=SB^*$),

$$\langle 20\rangle \qquad \frac{\partial c}{\partial G} = N(d_1) > 0$$

decreasing with respect to exercise price,

$$\langle 21\rangle \qquad \frac{\partial c}{\partial X} = -BN(d_2) < 0$$

and increasing with respect to the spot or forward rate (where $F=SB^*/B$),

$$\langle 22\rangle \qquad \frac{\partial c}{\partial S} = B^*N(d_1) > 0$$

$$\langle 23\rangle \qquad \frac{\partial c}{\partial F} = BN(d_1) > 0$$

The partial derivative with respect to term to maturity has to be interpreted with care:

$$\langle 24\rangle \qquad \frac{\partial c}{\partial T} = \frac{SB^*N'(d_1)}{2\sigma\sqrt{T}}[\sigma_G^2 - 2\rho_{GB}\sigma_G\sigma_B + \sigma_B^2] > 0$$

The fact that $\partial c/\partial T > 0$ does not mean the option price c strictly increases with term to maturity. On the contrary, by the relation $\partial c/\partial T = \frac{1}{2}\phi$ of equation ⟨13⟩, the incremental change in the option value, dc, is independent of this partial derivative. We have

$$\langle 25\rangle \quad \begin{aligned} dc &= \frac{\partial c}{\partial G}dG+\frac{\partial c}{\partial B}dB+\frac{\partial c}{\partial T}dT-\tfrac{1}{2}\phi\, dT \\ &= N(d_1)\,dG-XN(d_2)\,dB \end{aligned}$$

so that in fact $dc>0$ if $dG>X(N(d_2))/(N(d_1))\,dB$. If incremental foreign interest is too great (dG too large for a fixed S) relative to incremental domestic interest (relative to dB), then the European option price will decrease with an increment in term to maturity (recalling that $dT=-dt$).

Finally, defining σ as $\sigma\equiv[\sigma^2T]^{\frac{1}{2}}/\sqrt{T}$, we have

$$\langle 26\rangle \quad \frac{\partial c}{\partial \sigma} = \sqrt{T}\, SB^*N'(d_1) > 0$$

The option price is an increasing function of the average instantaneous standard deviation rate of the forward rate, where the time average is taken over the interval $(t,t+T)$ for forward contracts maturing at time $t+T$.

The above partial derivatives were derived from equation ⟨14⟩. An additional result is obtained if we take the partial derivative with respect to the domestic bond price in equation ⟨15⟩:

$$\langle 27\rangle \quad \left.\frac{\partial c}{\partial B}\right|_{F=\bar{F}} = \frac{c}{B} > 0$$

In equation ⟨18⟩ the spot rate was held constant, while in equation ⟨27⟩ the forward rate is held constant. The derivative in equation ⟨27⟩ is positive because if the domestic bond price rises (the domestic interest rate falls) with the forward rate held constant, then from the interest parity relation we know that the domestic currency price of a foreign bond ($G-SB^*$) rises, which increases the option value by equation ⟨20⟩.

IV. American Options

If $c(t)$ is the price of a European FX call expiring at $t+T$, then the price $C(t)$ of an American call expiring at $t+T$ satisfies the inequality

$$C(t) \geqq \max\,[S(t)-X,\, c(t)]$$

A pricing equation for the European call $c(t)$ was derived in the previous section. The American price will be strictly greater than this European price only if the additional constraint ⟨11⟩ is binding. That is,

$$C(t) > c(t) \text{ iff } \underset{\tau\varepsilon(t,t+T)}{\text{Prob}}\,[S(\tau)-X > c(\tau)] > 0$$

But it is clear that $S(\tau)-X>c(\tau)$ for sufficiently large S. To see this, note that as $S\to\infty$, $c(\tau)=SB^*N(d_1)-XBN(d_2)\to SB^*-XB$. But $S-X>SB^*-XB$ for $S>X(1-B)/(1-B^*)$. Thus we know that $C(t)>c(t)$.

Now an exercised option always has a value of $S-X$. Thus, if an American option is rationally exercised prematurely when the exchange rate is S_e, we must have the relation

$$\langle 28\rangle \quad C(S_e(t),t) = S_e(t)-\mathrm{X}$$

But we know there is a positive probability of premature exercise, because if an American option were always exercised only at expiration it would be indistinguishable from, and hence priced the same as, a European option. But $C(t) > c(t)$, so it follows that an $S_e(t)$ always exists such that for $S(t) > S_e(t)$ the call will be worth more exercised than held. The American call therefore satisfies ⟨12⟩ and ⟨13⟩ for $0 \leqq S \leqq S_e$, and has as additional boundary conditions ⟨28⟩ and ⟨9⟩–⟨10⟩.

Using equation ⟨7⟩, we see that an American put will be exercised prior to expiration for a sufficiently low value of S (relative to X), namely for $S < S'_e(t)$, where $S'_e(t)$ satisfies the relation

$$P(S'_e, t) = X - S'_e(t)$$

At present, a simple solution for $C(t)$ (or $P(t)$) is not known. However, approximate solutions may be found numerically by following procedures similar to those used to calculate American put prices for stock options. Geske and Johnson (1982) compare the efficiency of three approaches to the latter problem—namely numerical integration, a binomial approximation, and a type of polynomial approximation to an infinite series solution. The reader is referred to their paper for discussion and references.

V. Use of the Pricing Formulas for Hedging or Speculation

There are a wide variety of hedging relationships and speculative strategies that may be based on the option pricing formulas derived in Section III. For the purpose of illustration, we will look at one hedging example and one speculative strategy.

V.A. Hedging the Domestic Currency Value of a Discount Foreign Bond

We will only consider the case of a long position in a foreign-currency discount bond, since the case of borrowing is symmetrical. The domestic-currency value of the foreign-currency bond is $G = S(t) B^*(t, T)$. Then for a European call option c, the hedge is that previously described in equation ⟨12⟩. Equation ⟨12⟩ may be rewritten as

$$G + \left[\frac{\partial c}{\partial B} \Big/ \frac{\partial c}{\partial G} \right] B - \left[1 \Big/ \frac{\partial c}{\partial G} \right] c = 0$$

Thus for each domestic-currency unit of the foreign-currency bond held long, the hedge is effected by buying $[(\partial c/\partial B)/(\partial c/\partial G)]$units of the domestic currency bond and writing $[1/(\partial c/\partial G)]$ call options. Such a hedge would yield a change in wealth of zero, if the hedge were continually adjusted to reflect any change in the state variables.

A similar hedge may be formed with the European put option p. First, verify that

$$p - \frac{\partial p}{\partial G} G - \frac{\partial p}{\partial B} B = 0$$

Then for each domestic-currency unit of the foreign-currency bond held long, the hedge is brought about through the purchase of $[(\partial p/\partial B)/(\partial p/\partial G)]$ units of the domestic currency bond and writing $[1/(\partial p/\partial G)]$ put options.

V.B. The Mutual Consistency of Option Prices

The option pricing formulas depend on six variables, five of which are observable. The variance rate involved in each formula must be estimated from past data in order to obtain a dollar figure for the price of the option. But it is not necessary to know the variance rate in order to price options relative to each other.

To price options relative to a given option, say a call option with exercise price X_1 and time to expiration T_1, take the five observable variables and the current market price c_1 of the given option, and solve the call option equation backwards to obtain the implied variance rate σ_1^2. Then, assuming that σ_1^2 is the correct variance rate for the currency in question, use σ_1^2 with the five observable variables to price all other call options with a different exercise price or, if the variance rate is assumed constant, with a different term to maturity, and also to price all put options on the same currency. The prices obtained may be lower than or greater than the market prices of the other options. If, for example, a three-month option is used to calculate σ_1^2, and using σ_1^2 to price the corresponding six-month option yields a theoretical value larger than the market price of the six-month option, then we can say that six-month options are 'underpriced' relative to three-month options, or that three-month options are 'overpriced' relative to six-month options. The strategy, then, would be to buy six-month options and to write or sell three-month options. (The existence of transactions costs allows, of course, a certain amount of non-profitable inconsistency in implied variance rates.) The profitability of such a strategy would depend on the relation between the assumptions employed in the derivation of the pricing formula and the actual variables that determine market prices, as well as whether the variance rate is really constant.

VI. Conclusion

This paper has explored a set of inequality-equality constraints on rational pricing of foreign currency options, and has developed exact pricing equations for European puts and calls when interest rates are stochastic. The assumption that relevant variables follow diffusion processes allows us to set up a riskless hedge that uses no wealth, and which therefore must have a zero return in equilibrium. The construction of this hedge yields a partial differential equation whose solution is the European call option value. The put option equations are obtained immediately from the call equations through a put-to-call conversion equation that holds for FX options. Finally, it was shown that for sufficiently high values (low values) of the spot rate relative to the exercise price, American calls (puts) will be exercised prior to maturity. Hence (for positive interest rates) American FX options have values strictly greater than European FX options.

Notes

1. By contrast to the example, equations ⟨7⟩ and ⟨8⟩ are not purely identities, since use is also made of the fact that the option price is first degree homogeneous in the price of the underlying asset and the exercise price:

$$C(NS(t),NX,t,T) = NC(S(t),X,t,T), \text{ for } N > 0$$

and similarly for the put option. Since two call options on 1 DM, each with an exercise price of X, give me the same privileges as a single option on 2 DM with an exercise price of $2X$, first degree homogeneity ought to hold for rationally-priced foreign exchange options just as it does for stock options. See Merton (1973), Theorem 6, p. 147, and Theorem 9, p. 149, for discussion of the stock option case.

2. The same is true with stock options, since a call on a share of stock with an exercise price in terms of money is a put option on money with an exercise price in terms of stock. The symmetry breaks down, however, in that money (a bond) bears interest while the stock does not. Thus while the American call option on money gives us an American put option on stock, we are brought no closer to our goal if we do not have a pricing formula for the American call on money.
3. The FX call equation here is closer to Merton's (1973) stochastic interest rate version of the Black–Scholes model. The FX model, however, is complicated by the presence of two correlated interest rates. If both interest rates were constant, so that B_1 and B^* could be rewritten as $B=e^{-rT}$, $B^*=e^{-r^*T}$, then equation ⟨14⟩ becomes identical to the solution for the value of a call option on a stock with a constant proportional dividend rate. See Merton (1973), p. 171, footnote 62, for the latter solution.

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