The Girth of a Design

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August 29, 1999

Abstract

In 1976 Erdös asked about the existence of Steiner triple systems that lack collections of j blocks employing just $j + 2$ points. This has led to the study of anti-Pasch, anti-mitre and 5-sparse Steiner triple systems. Simultaneously generating sets and bases for Steiner triple systems and t-designs have been determined. Combining these ideas, together with the observation that a regular graph is a 1-design, we arrive at a natural definition for the girth of a design. In turn, this provides a natural extension of the search for cages to the universe of all t-designs. We include the results of computational experiments that give an abundance of examples of these new definitions.

1 Introduction

This article discusses two questions concerning the absence of small configurations in designs. One question is old — it was posed by Erdös in 1976. The second question is motivated by recent results about generating sets for designs, and generalizes the search for cages among regular graphs. For an introduction to designs see [4] or [20].

Erdös [8] posed the following question about Steiner triple systems.

Question 1.1 *For any $r \geq 4$ does there exist a $v_0(r)$ so that whenever $v > v_0(r)$ and v is admissible, there exists a Steiner triple system with the property that no set of j blocks employs just $j + 2$ points, for all $2 \leq j \leq r$?*

For this reason, a set of j blocks on $j+2$ points has been called an **Erdös configuration**.

The simplest case occurs when $r = 4$ and the question then asks about the values of v for which there exists a Steiner triple system that lacks the Pasch configuration (also known as a quadrilateral, see Figure 1). The question is still open in this case, though partial results cover most values of v. Erdös' idea has been extended to other avoidance results, such as studying Steiner triple systems that lack a mitre configuration.

This article surveys the current state of Erdös' question in Section 2 and considers the problem of determining generating sets and linear bases for configurations in a Steiner triple system in Section 3. Though it has been used in similar settings for different purposes, in this setting the term **configuration** is used to describe any collection of blocks from a Steiner triple system. Given a Steiner triple system, the set of all configurations that have the same number of blocks can be partitioned according to isomorphism, and then we can ask about the size of each such class. Roughly speaking, a generating set is a set of representatives of some isomorphism classes with the property that if the sizes of their classes are known, then the sizes of all the other isomorphism classes can be easily determined. Generating sets for Steiner triple systems have been found for configurations with seven or fewer blocks [10, 6, 12, 19]. For the smaller numbers of blocks, the generating sets coincide with the configurations to be avoided in Erdös' question. Horak, Phillips, Wallis and Yucas have taken these ideas further and given a characterization of generating sets in Steiner triple systems for configurations of arbitrary size [12]. In turn, this characterization has been generalized to apply not just to Steiner triple systems but to all t-designs [1]. It is this last result that motivates this paper and is the subject of Section 4.

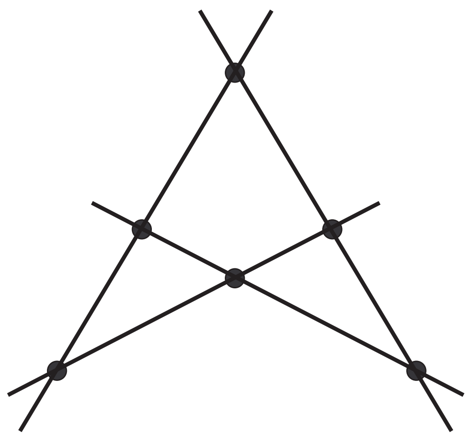

Figure 1. Pasch configuration.

A regular graph can be viewed as a t-design (where $t = 1$) and thus we can specialize and describe generating sets for regular graphs. With such a description, we can mimic one aspect of Erdös' question and ask which regular graphs contain no instances of these generating sets. It happens that the question then becomes one of constructing a regular graph without small cycles. Minimal regular graphs that have fixed girth (a lack of small cycles) are known as cages and have been studied heavily [2, 16, 21]. This connection is detailed in Section 5.

So, for small r, Erdös' original question is partially concerned with the search for Steiner triple systems that avoid elements of a generating set. When this criteria is applied to regular graphs, we arrive at the well-known problem of determining cages. In Erdös' question the problem is to guarantee, beyond a certain order, that for each greater admissible order there is an instance of a Steiner triple system that lacks specific configurations. In the search for cages, the problem is to find the smallest instance which lacks certain configurations. For both problems a study of all the small order cases is often required. With a characterization of generating sets applicable to any t-design, we will argue in Section 5 that it is natural to consider designs that lack elements of a generating set, in addition to considering designs lacking Erdös configurations. In Section 6 we will then present the results of exhaustive computations for small order cases.

2 Avoidance Results in Steiner Triple Systems

A **Steiner triple system** on v points is a pair $(V, \mathcal{B})$ where V is a set of v elements (called points or vertices) and $\mathcal{B}$ is a set of 3-element subsets of V (called blocks or lines) with the property that every 2-element subset of V is a subset of exactly one block in $\mathcal{B}$.

Returning to Erdös' question (Question 1.1) in the case when $r = 4$, we will use the term **anti-Pasch** for a Steiner triple system that lacks a Pasch configuration. The complete spectrum of admissible values of v for which there exists an anti-Pasch configuration is still not determined. It is known that when $v = 13$ there are no anti-Pasch Steiner triple systems and when $v = 15$ there is an anti-Pasch Steiner triple system. For $v \geq 19$, computational experiments indicate there is an abundance of anti-Pasch Steiner triple systems [11, 15]. Thus it is conjectured that $v_0(4) = 13$.

Results of Brouwer [3] and Doyen [7] provide the existence of anti-Pasch Steiner triple systems for each $v \equiv 3 \bmod 6$. When $v \equiv 1 \bmod 6$, the results are more complicated, but the results of several authors provide the existence of anti-Pasch configurations in almost all of the cases (see [9] for more details). For the remainder of these cases, the existence question has not been settled.

The two Erdös configurations with 5 blocks are known as the mia and mitre (see Figure 2 and Figure 3). Since the mia configuration can be obtained from the Pasch configuration by adding a line, attention has focused on Steiner triple systems lacking a mitre configuration, which are known as **anti-mitre**. Progress on the question of the existence of anti-mitre Steiner triple systems has been made by Colburn, Mendelsohn, Rosa and Siran [5].

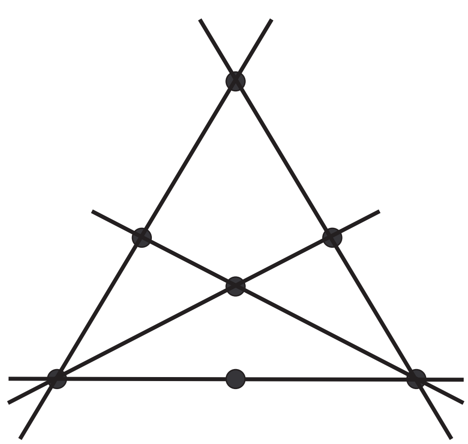

Figure 2. Mia configuration.

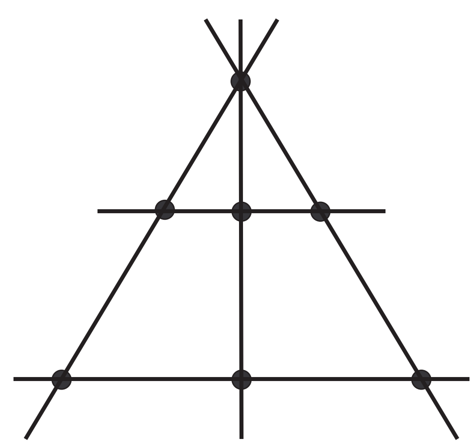

Figure 3. Mitre configuration.

However, Erdös question in the case when $r = 5$ asks about Steiner triple systems which simultaneously lack instances of the Pasch, mitre and mia configurations. Since the absence of a Pasch configuration implies the absence of a mia configuration, we are left determining the existence of Steiner triple systems that are simultaneously anti-Pasch and anti-mitre.

Such systems have been called 5-sparse, while more generally a system that has no Erdös configurations on j blocks, for all $2 \leq j \leq r$, has been called **r-sparse**. The elements of the first known infinite class of 5-sparse systems are all Netto systems, and no example of a 6-sparse system is known [5]. A direct product construction [13] yields a further infinite class of 5-sparse systems.

3 Generating Sets for Steiner Triple Systems

We begin this section with an example first reported by Grannell, Griggs and Mendelsohn [10].

Given a Steiner triple system on v points, consider subsets of size 4 from the set of blocks. Each of these is referred to as a 4-line configuration, and will be one of 16 types (according to isomorphism). If we construct the set of all such 4-line configurations from the design, we can then group them into 16 isomorphism classes, and consider the size of each class. For five of these classes the sizes are functions of v, and will not vary among different designs with the same number of points. For example, the 4-star is the configuration with exactly one point common to all 4 blocks, which are otherwise pairwise disjoint (see Figure 4). The size of the class for this configuration is $v(v-1)(v-3)(v-5)/(2^4\,4!)$.

The remaining 11 classes will have variable sizes, depending on the triple system itself. However, to determine these sizes, it is sufficient to know the size of just one class, the one represented by the Pasch configuration. The sizes for the 10 other configurations can be determined from linear equations using v, and the size of the class represented by the Pasch configuration. For example, consider the configuration where the 4 blocks form 2 parallel pairs. Within these pairs, the 2 blocks are disjoint, but any other pair of blocks has a unique point in common (see Figure 5). Then, if P denotes the number of Pasch configurations and S denotes the number of square configurations, we have

$$S = v(v-1)(v-3)(v-8)/8 + 3P.$$

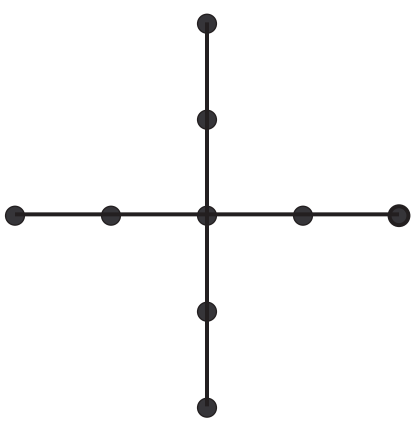

Figure 4. A 4-star.

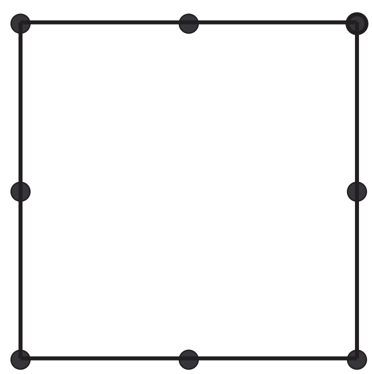

Figure 5. A square.

So if a Steiner triple system is anti-Pasch, then because we know that there are zero Pasch configurations, we can determine the size of each class of 4-line configurations simply as functions of v. In particular, any two anti-Pasch Steiner triple systems on the same number of points will have an identical number of 4-line configurations of each type. Also, depending on the sign of the coefficient of the number of Pasch configurations, the equations in [10] show that anti-Pasch Steiner triple systems are extremal with regard to the size of other classes.

Suppose that for Steiner triple systems we have a set of configurations, G, and for any other configuration outside of G, with m or fewer blocks, we can write a formula for the size of its class that is a linear combination of the sizes of the classes in G where the coefficients are functions of v, but otherwise are independent of any other property of the Steiner triple system. Then we say that G is a **generating set for the m-line configurations**. It is convenient to always include the empty configuration (no blocks) in any generating set, and thereby avoid the distinction used by other authors between "constant" and "variable" configurations. Finally, we call a generating set a **basis** if no proper subset is also a generating set.

For a Steiner triple system, the isomorphism class of any configuration with 3 or fewer blocks has a size that is just a function of v [10]. To illustrate the definition of a basis, the results above can be rephrased by saying that the set of configurations containing the empty configuration and the Pasch configuration forms a basis for the 4-line configurations of a Steiner triple system.

With the appearance of this interesting result, it was natural to search for bases for configurations with greater numbers of lines. Danziger, Mendelsohn, Grannell and Griggs [6] found a basis for the 5-line configurations of a Steiner triple system. It has three elements, adding the mitre configuration to the basis for the 4-line configurations. The mitre (see Figure 3) has 5 blocks on 7 points, and thus is an Erdös configuration (as is the Pasch configuration).

Horak, Phillips, Wallis and Yucas [12] found a basis for the 6-line configurations of a Steiner triple system, adding 5 new configurations to the basis for the 5-line configurations. However, not all of these are Erdös configurations. In [10] it was conjectured that the empty configuration together with the relevant Erdös configurations would form a generating set. So the situation for 5-line configurations shows this conjecture to be false. This begs the question: should avoidance results for Steiner triple systems (anti-Pasch, anti-mitre) be viewed as questions about Steiner triple systems that lack Erdös configurations, or as Steiner triple systems that lack the non-trivial elements of bases? We will return to this question once we have more evidence to support an answer.

In [12], the following characterization of a generating set for the m-line configurations of a Steiner triple system is given. The **degree** of a point in a configuration is the number of blocks of which it is a member.

Theorem 3.1 ([12], Theorem 1) *The set of all configurations with m or fewer blocks, where each point occurs with degree two or more, forms a generating set for the m-line configurations of a Steiner triple system.*

Note that this theorem has been reworded slightly to allow the empty configuration to be a member of the generating set. Urland [19] used this characterization to enumerate the 27 configurations in a generating set for the 7-line configurations, and then was able to show through extensive computations that this generating set is also a basis. However, to establish in general that the generating set described in Theorem 3.1 is a basis would appear to be a difficult problem, despite the evidence that it is so.

4 Generating Sets for Designs

We now describe the generalization of the previous ideas to arbitrary designs.

A t**-**(v, k, λ) design is a pair $(V, \mathcal{B})$ where V is a set of v elements (called points or vertices) and $\mathcal{B}$ is a set of k-element subsets of V (called blocks or lines) with the property that every t-element subset of V is a subset of exactly λ blocks from $\mathcal{B}$. As with Steiner triple systems, we can use the term m**-line configuration** to refer to a subset of $\mathcal{B}$ with m blocks. If we partition the set of all m-line configurations of a particular design according to isomorphism, we can again consider the size of each class. As before, a generating set will be a collection of configurations whose sizes are employed in linear combinations that yield the sizes of any other class. Only now the coefficients in these formulas are allowed to be expressions that involve λ in addition to v.

Definition 4.1 ([1], Definition 3.1) *A **generator** is a configuration in a t-design where each block has more than t points of degree two or more.*

Theorem 4.2 ([1], Theorem 3.2) *The set of all generators with m or fewer blocks (including the empty configuration) for a t-design forms a generating set for the m-line configurations of a t-design.*

Notice how this theorem implies Theorem 3.1 in the case of Steiner triple systems, since a Steiner triple system has $k - t = 1$. While a design can have several different generating sets, in the remainder when we refer to a generating set for a design, we will mean the one described in this result.

We now turn our attention to applying this result to the special case of regular graphs.

5 Generating Sets for Regular Graphs

A regular graph of degree r on n vertices can be construed as a t-design with parameters 1-$(n, 2, r)$. Specializing, the elements of a generating set are then configurations (edge-induced subgraphs) where each block (edge) has strictly more than one point (vertex) of degree two or more. As was the case with Steiner triple systems, because $k - t = 1$, *every* vertex of the induced subgraph must have degree two or more. Therefore these subgraphs lack any vertices of degree one, and cannot be trees or forests, and thus must contain cycles.

Which regular graphs lack non-trivial members of a generating set for the m-line configurations? Each cycle on m or fewer edges will be a member of the generating set (since all the vertices have degree exactly two), so such a graph will not have any cycles with m or fewer edges.

Conversely, suppose we have a graph that has no cycles with m or fewer edges. Each non-trivial element of a generating set for the m-line configurations has m or fewer edges and contains a cycle. Since the graph has no cycles with m or fewer edges, it cannot contain any non-trivial elements of a generating set.

So we have established the following key theorem.

Theorem 5.1 *A regular graph has no cycles of length m or less if and only if it has no non-trivial elements of the generating set for m-line configurations (which are edge-induced subgraphs).*

The **girth** of a graph is the length of its shortest cycle. This prompts the following definition, which is the *raison d'etre* for this article.

Definition 5.2 *A t-(v, k, λ) design has* ***girth*** *m if it has no non-trivial configurations from the generating set for the $(m-1)$-line configurations and has at least one configuration from the generating set for the m-line configurations.*

As an illustration of this definition, we can now refer to a 5-sparse Steiner triple system as being a design of girth 6 (or greater).

An (r, g)**-cage** is a regular graph with degree r and girth g having the fewest number of vertices. Much attention has been given to the search for these graphs, in part because they are often very interesting graphs for other purposes. See [2, 16, 21] for more details.

With a definition of the girth of a design we can formulate an analogous definition of a cage for designs.

Definition 5.3 *A (t, k, λ, g)-**cage** is a design with specified values of t, k, λ, and girth g that has the fewest number of points.*

We can now compare Erdös question (Question 1.1) with that of determining (t, k, λ, g)-cages. In the former, we wish to avoid Erdös configurations in Steiner triple systems and find the smallest number of points so that for any particular greater number, there will always be an example of at least one Steiner triple system that lacks the specified configurations. The latter asks about designs that avoid generating sets, and desires the smallest number of points for the existence of a single example of a design that does so. Both questions require that the small order cases be understood entirely.

A search for $(2, 3, 1, r)$-cages coincides with Erdös' question in the search for anti-Pasch and anti-mitre Steiner triple systems when $r = 4, 5$ (respectively), but diverges when $r \geq 6$, with Erdös question being the less restrictive. The search for $(1, 2, r, g)$-cages coincides exactly with the search for cages among regular graphs. It is this last correspondence and the natural properties of generating sets (which we believe are also bases), which have motivated the above definitions. One could also modify Erdös question slightly and generalize it, asking for an order $v(t, k, \lambda, r)$ so that for every admissible $v > v(t, k, \lambda, r)$ there exists a t-(v, k, λ) design that lacks every element of the generating set for the r-line configurations. Given the difficulty in finding just $v(2, 3, 1, 4)$ (the case of anti-Pasch Steiner triple systems) this is likely to be a difficult question in general.

6 Examples of the Girth of a Design

In this section we list combinations of our own computational experiments and previously known results to determine various cages. Chiefly, we rely on known results, and the tables of small 2-designs from [14] (as corrected) for which we have exhaustively classified all small configurations into isomorphism classes. Note that we have not considered designs with repeated blocks, since they will automatically have girth 2.

Three trends should be observed from studying these examples.

- Smaller values of $k - t$ are associated with larger girth, since in these cases generators are "harder" to construct (thus explaining the greater girths for regular graphs and Steiner triple systems).
- Smaller values of λ are associated with larger girth.
- (t, k, λ, g)-cages typically have rich automorphism groups and are often transitive on the points.

6.1 1-$(n, 2, r)$ Designs (Regular Graphs)

Any (r, g)-cage (as defined for regular graphs) is automatically a $(1, 2, r, g)$-cage (as defined for designs) by reason of Theorem 5.1. See [2, 16, 21] for examples.

6.2 2-$(v, 3, 1)$ Designs (Steiner Triple Systems)

6.2.1 $(2, 3, 1, 4)$-cage

The finite projective plane of order 3, which has 7 points and 7 blocks, has no non-trivial elements of the generating set for 3-line configurations (no Steiner triple system does) and it has 7 configurations isomorphic to the Pasch configuration. Its automorphism group is transitive of order 168.

6.2.2 $(2, 3, 1, 5)$-cage

The unique Steiner triple system on 9 points [14, Table 1.15] is anti-Pasch, and contains 36 instances of the mitre configuration. Its automorphism group is transitive of order 432.

6.2.3 $(2, 3, 1, 6)$-cage

The two Steiner triple systems on 13 points [14, Table 1.20] each contain instances of Pasch configurations, and so have girth 4. The 80 Steiner triple systems on 15 points [14, Table 1.21] all contain Pasch configurations, with one exception [14, Table 1.22]. This exceptional Steiner triple system has 30 instances of the mitre configuration, giving it girth 5. The Netto triple system on 19 points has been shown to be anti-Pasch and anti-mitre [5] and is therefore an instance of a $(2, 3, 1, 6)$-cage, though it is not known if it is unique.

6.3 2-Designs with $k = 3$ and $\lambda > 1$

6.3.1 $(2, 3, 2, 4)$-cage

The unique 2-$(6, 3, 2)$ design [14, Table 1.11] lacks any element of the generating set for 3-line configurations. Notice that this is not the triviality that it is for Steiner triple systems, since in the case where $\lambda = 2$ a configuration such as C_1 (whose point-block incidence graph is depicted in Figure 6) is an element of the generating set for the 3-line configurations. However, this design does have 5 instances of the Pasch configuration, in addition to 30 instances of the generator C_2 with 4 blocks, whose point-block incidence graph is depicted in Figure 7. Its automorphism group is transitive of order 60.

6.3.2 $(2, 3, 2, 5)$-cage

The unique 2-$(7, 3, 2)$ design [14, Table 1.12] has girth 4 since it contains 14 instances of the Pasch configuration and 21 instances of configuration C_2. The thirteen 2-$(9, 3, 2)$ designs [14, Table 1.16] have girth 3 or 4, so the $(2, 3, 2, 5)$-cage will have 10 or more points.

6.3.3 $(2, 3, 3, 3)$-cage

The unique 2-$(5, 3, 3)$ design is complete and has girth 3, since it contains 20 instances of the configuration C_1 with 3 blocks, whose block-incidence graph is depicted in Figure 6. Its automorphism group is transitive of order 120.

6.3.4 $(2, 3, 3, 4)$-cage

The unique 2-$(7, 3, 3)$ design [14, Table 1.13] has girth 3 since it contains 14 instances of the configuration C_1. So the $(2, 3, 3, 4)$-cage will have 9 or more points.

6.4 2-Designs with $k = 4$

6.4.1 $(2, 4, 1, 4)$-cage

The unique 2-$(13, 4, 1)$ design [14, Table 1.19] has girth 4, since it has no elements of the generating set for 3-line configurations (such as configuration C_3) and it has 234 instances of the configuration with 4 blocks, C_8, whose point-block incidence graph is pictured in Figure 13. Its automorphism group is transitive of order 5 616.

6.4.2 $(2, 4, 1, 5)$-cage

The unique 2-$(16, 4, 1)$ design [14, Table 1.24] has girth 4, since it has 240 instances of configuration C_8. The eighteen 2-$(25, 4, 1)$ designs [14, Table 1.27] [17] each has girth 4, and in each case the lone obstacle to greater girth is instances of configuration C_8. So a $(2, 4, 1, 5)$-cage has 28 or more points.

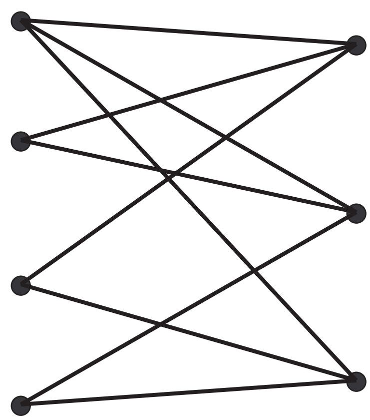

Figure 6. Incidence graph of C_1.

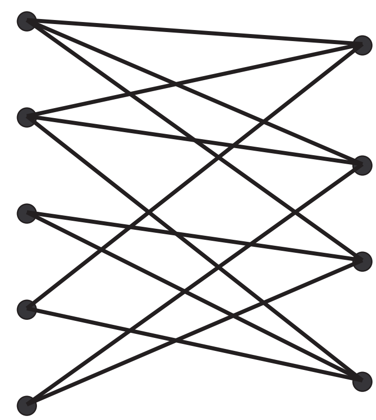

Figure 7. Incidence graph of C_2.

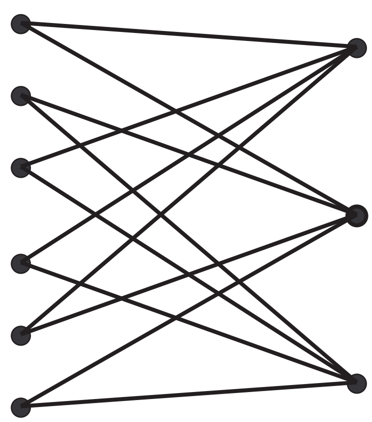

Figure 8. Incidence graph of C_3.

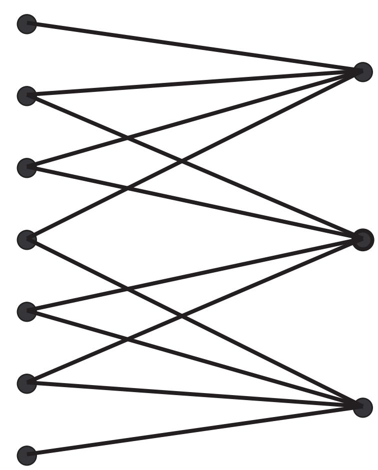

Figure 9. Incidence graph of C_4.

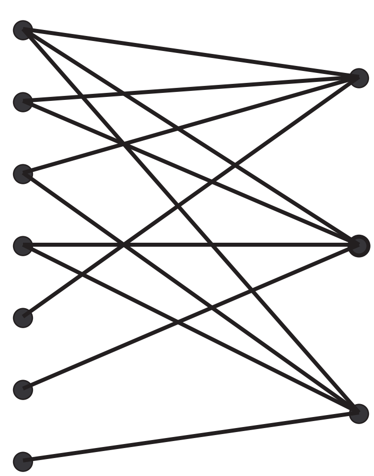

Figure 10. Incidence graph of C_5.

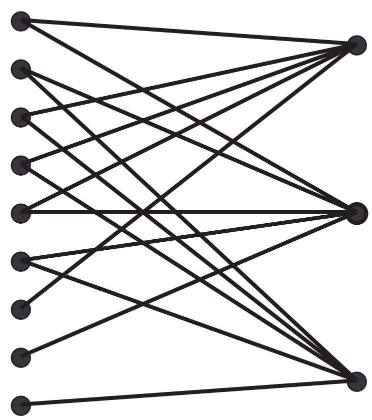

Figure 11. Incidence graph of C_6.

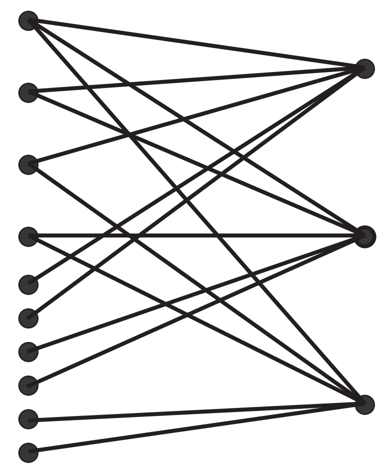

Figure 12. Incidence graph of C_7.

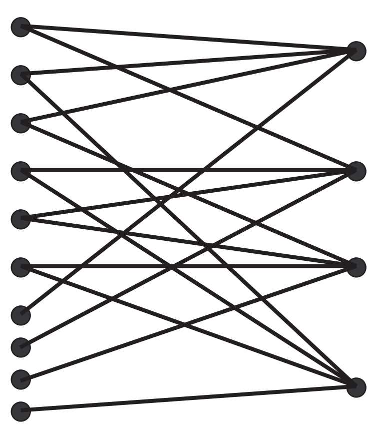

Figure 13. Incidence graph of C_8.

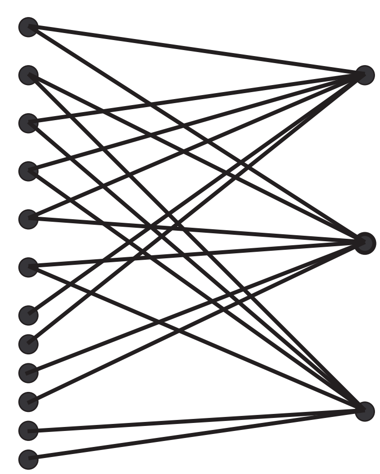

Figure 14. Incidence graph of C_9.

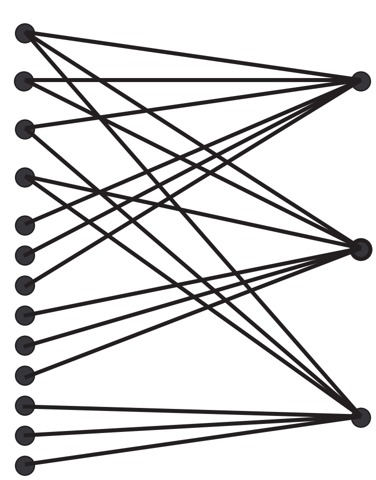

Figure 15. Incidence graph of C_{10}.

6.4.3 $(2, 4, 2, 3)$-cages

The three 2-$(10, 4, 2)$ designs [14, Table 1.18] all have girth 3. Since there are no smaller 2-$(10, 4, 2)$ designs, each is a $(2, 4, 2, 3)$-cage. The obstacles to greater girth are the configurations with three blocks, C_3, C_4 and C_5 whose point-block incidence graphs are depicted in Figures 8, 9, 10.

For these three designs, one has an automorphism group of order 24 and has 3 instances of C_3, 36 instances of C_4 and 12 instances C_5. The second one has an automorphism group of order 48 and has 7 instances of C_3, 24 instances of C_4 and 8 instances C_5. The third has a transitive automorphism group of order 720 and just 15 instances of C_3. Notice that as the size of the automorphism groups increase, the obstacles to attaining greater girth generally become fewer.

6.4.4 $(2, 4, 3, 2)$-cages

The four 2-$(8, 4, 3)$ designs [14, Table 1.14] all have girth 2 since each has at least one pair of blocks with three points in common.

6.4.5 $(2, 4, 3, 3)$-cage

The eleven 2-$(9, 4, 3)$ designs [14, Table 1.17] all have girth 2, with the exception of one design with girth 3. This exceptional design contains 12 instances of C_3, 108 instances of C_4 and 216 instances of C_5. Its automorphism group is transitive with order 144.

6.5 2-Designs with $k > 4$

6.5.1 2-$(11, 5, 2)$ Designs

The unique 2-$(11, 5, 2)$ design has girth 3, since it has 55 instances of configuration C_6 and 110 instances of configuration C_7. The point-block incidence graphs of these configurations on 3 blocks are depicted in Figures 11 and 12. Interestingly, together these 165 configurations account for all of the possible configurations with 3 blocks from this design. Its automorphism group is transitive of order 660.

6.5.2 2-$(16, 6, 2)$ Designs

Each of the three 2-$(16, 6, 2)$ designs has girth 3. Remarkably, each has exactly 240 instances of configuration C_9 and 320 instances of configuration C_{10}. The point-block incidence graphs of these configurations on 3 blocks are depicted in Figures 14 and 15. Each of these designs has a transitive automorphism group, and their orders are 384, 768 and 11 520.

6.5.3 2-$(15, 7, 3)$ Designs

The three 2-$(15, 7, 3)$ designs each has girth 2. For each of these designs, every pair of blocks has three points in common.

6.5.4 2-(19, 9, 4) Designs

The six 2-(19, 9, 4) designs each has girth 2. For each of these designs, every pair of blocks has four points in common.

6.6 Designs with large t

Since examples of designs with large girth seem to have small values of λ and large, transitive, automorphism groups, we analyzed several designs associated with the extended binary Golay code (see [18] for details on their construction). These have larger values of t, $\lambda = 1$, interesting automorphism groups, and in one case $k - t = 1$. Here we describe the results of computing their girths.

6.6.1 A 5-(12, 6, 1) Design

This design has girth 3, with 1 980 instances of C_{11} and 880 instances of C_{12} (Figures 16 and 17).

6.6.2 A 4-(23, 7, 1) Design

This design has girth 3, with 212 520 instances of C_{13} (Figure 18).

6.6.3 A 5-(24, 8, 1) Design

This design also has girth 3, with 35 240 instances of C_{14} and 2 550 240 instances of C_{15} (Figures 19 and 20).

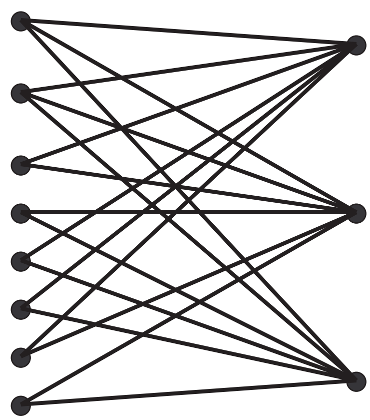

Figure 16. Incidence graph of C_{11}.

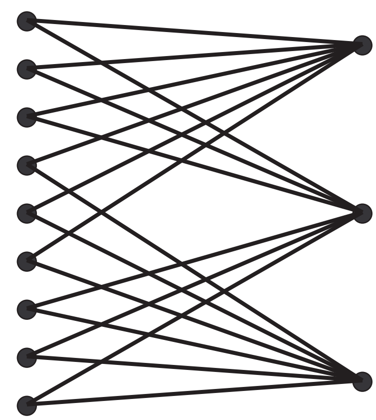

Figure 17. Incidence graph of C_{12}.

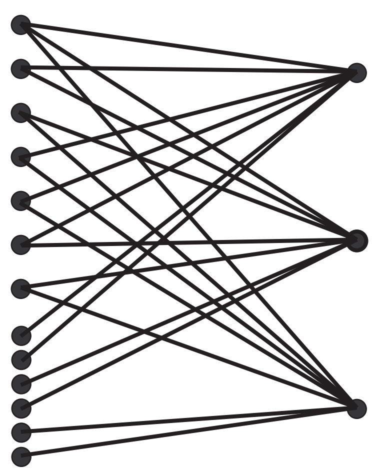

Figure 18. Incidence graph of C_{13}.

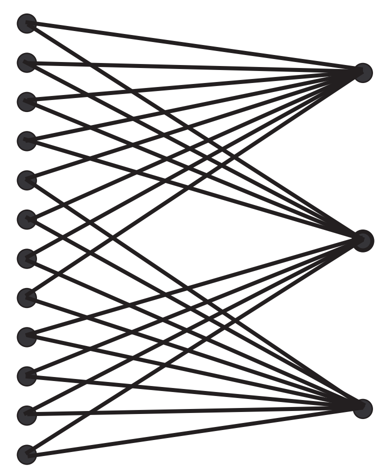

Figure 19. Incidence graph of C_{14}.

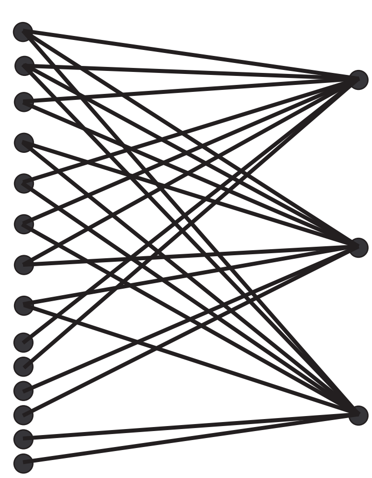

Figure 20. Incidence graph of C_{15}.

7 Questions

We end with several open questions and directions for further work.

1. Find a Steiner triple system with girth 7.

2. Find any non-trivial design, that is not a regular graph or a Steiner triple system, with girth 5 or greater. Or prove that this is impossible.

3. Extend any of the results in the previous section. For a fixed combination of t, k and λ find a cage for the next larger girth.

4. Perhaps new techniques developed searching for (t, k, λ, g)-cages in the more general setting of designs will specialize to the case of regular graphs and yield improvements in the search for (r, g)-cages.

5. The Moore bound is a simple function of r and g that provides a lower bound on the number of vertices of an (r, g)-cage. Any (r, g)-cage that meets this bound is called a Moore graph.

   Is there an analagous function of t, k, λ and g that provides a lower bound for the order of (t, k, λ, g)-cages? Does this bound coincide with the Moore bound in the case of regular graphs? Are there any "Moore designs"?

6. Many of the generators that actually occur as obstacles to greater girth (C_1 through C_{15}) are very similar. In several cases, the removal of vertices of degree 1 yield identical stuctures. For example, compare C_8 with the Pasch configuration. Or compare C_5, C_7 and C_{10} with C_1. Are there more fundamental building blocks for configurations in a design than generators?

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