Diagonals of continuous triangular norms

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Abstract

Diagonals of continuous t-norms are studied. The characterization of all functions being diagonals of continuous t-norms is given. To a given diagonal, the class of all continuous t-norms with this diagonal is characterized.

Keywords: Triangular norm, additive generator, diagonal.

1 Introduction

A *triangular norm* (a t-norm for short) is a commutative, associative, non-decreasing function $T\colon [0,1]^2 \to [0,1]$ such that $T(x,1) = x$ for all $x \in [0,1]$. In what follows, we deal only with continuous t-norms, where the usual continuity of real functions is assumed. The basic continuous t-norms are the minimum, $T_{\mathbf{M}}(x,y) = \min(x,y)$, the product, $T_{\mathbf{P}}(x,y) = xy$, and the Łukasiewicz t-norm, $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$.

Definition 1.1 : *For a t-norm T, the mapping $\delta_T\colon [0,1] \to [0,1]$ defined by $\delta_T\colon x \mapsto T(x,x)$, is called the* diagonal *of T.*

The following conditions are necessary for a function to be a diagonal of a continuous t-norm.

Proposition 1.2 : *Let T be a continuous t-norm. Its diagonal is a continuous nondecreasing function $\delta_T\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ such that $\delta_T(x) \le x$ for all $x \in [0,1]$.*

For a function $\delta\colon [0,1] \to [0,1]$, we define $I(\delta) = \{x \in [0,1];\ \delta(x) = x\}$ (=the set of all fixed points of δ). In particular, 0 and 1 are fixed points of all diagonals of t-norms. These are the only fixed points of diagonals of important classes of t-norms.

Definition 1.3 : *A continuous t-norm T is called* Archimedean *if 0 and 1 are the only fixed points of its diagonal, i. e., $I(\delta_T) = \{0,1\}$. If, moreover, δ_T is strictly increasing, then T is called* strict. *A continuous Archimedean t-norm which is not strict is called* nilpotent.

In combination with Proposition 1.2, a continuous t-norm T is Archimedean if and only if $\delta_T(x) < x$ for all $x \in \,]0,1[$.

Problems concerning diagonals of continuous t-norms appeared in several works. Recall, e. g., the famous open problem of Schweizer and Sklar [7] whether the continuity of the diagonal implies the continuity of the underlying t-norm. Mayor and Torrens [6] have characterized the t-norms determined by means of their diagonal, δ, via $T(x,y) = \max(0, \delta(\max(x,y)) - |x-y|)$. Further, Bézivin and Tomás [1] proved that a strict t-norm T is uniquely determined by its diagonal, δ_T, and the values $T(x,y)$ for all $x,y \in [0,1]$ satisfying $x+y = a$ for a fixed value $a \in \,]0,2[$.

*The authors gratefully acknowledge the support of the project no. VS 96049 of the Czech Ministry of Education, COST Action 15, the grants no. 201/97/0437 and 402/96/0414 of the Grant Agency of the Czech Republic and the grant VEGA 1/4064/97.

The first investigation of diagonals of continuous t-norms is due to Kimberling [3], namely for the case of strict t-norms. We recall his result in a slightly modified version in Section 2. Section 3 is devoted to the diagonals of nilpotent and Archimedean t-norms. In Section 4, we characterize the diagonals of general continuous t-norms. In each case, the class of all t-norms with a given diagonal is constructively characterized. Note that the only diagonal of a continuous t-norm having a unique underlying t-norm is the identity, $\delta(x) = x$; in this case, the corresponding t-norm is the minimum t-norm $T_{\mathbf{M}}$ [4, 7].

We shall often use the following representation theorem for continuous Archimedean t-norms (see, e. g., [4, 5, 7]).

Theorem 1.4 : *Let T be a continuous Archimedean t-norm. There is an* additive generator *of T, i. e., a continuous strictly decreasing function $f\colon [0,1] \to [0,\infty]$ such that*

$$\forall x, y \in [0,1] : T(x,y) = f^{-1}(\min(f(0), f(x) + f(y))).$$

(Notice that the condition $T(x,1) = x$ implies that $f(1) = 0$.) Conversely, each continuous strictly decreasing function $f\colon [0,1] \to [0,\infty]$ satisfying $f(1) = 0$ is an additive generator of a unique t-norm.

The additive generator f of a t-norm T is determined uniquely up to a positive multiplicative constant. The value $f(0)$ is finite if and only if T is nilpotent; in this case, the function $f/f(0)\colon [0,1] \xrightarrow{\text{onto}} [0,1]$ is uniquely determined and it is called the normed additive generator *of T.*

If f is an additive generator of a continuous Archimedean t-norm T, then the diagonal δ_T satisfies the functional equation $f \circ \delta_T = \min(f(0), 2f)$.

Corollary 1.5 : *The diagonal δ_T of a continuous Archimedean t-norm T is strictly increasing on $\delta_T^{-1}(]0,1])$. In particular, it is strictly increasing on $[0,1] \setminus \delta_T^{-1}(I(\delta_T))$.*

(The latter corollary is important only for nilpotent t-norms; strict t-norms satisfy a stronger condition by their definition.

Example 1.6 : 1. For the product, $T_{\mathbf{P}}$, a corresponding additive generator is, e. g., $f(x) = -\log x$. The corresponding diagonal is $\delta_{T_{\mathbf{P}}}(x) = x^2$ and it is evident that $-\log x^2 = 2\,(-\log x)$.

2. Let $f(x) = \frac{1}{x} - 1$. Then f is an additive generator of so called *Hamacher product*, $T_{\mathbf{H}}$, where

$$T_{\mathbf{H}}(x,y) = \begin{cases} \frac{xy}{x+y-xy} & \text{for } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

(see [2]). The corresponding diagonal is $\delta_{T_{\mathbf{H}}}(x) = \frac{x}{2-x}$.

Example 1.7 : 1. The Łukasiewicz t-norm $T_{\mathbf{L}}$ is nilpotent. Its normed additive generator is $f(x) = 1 - x$ and the diagonal is $\delta_{T_{\mathbf{L}}}(x) = \max(0, 2x-1)$.

2. Let $f(x) = (1-x)^2$. Then the corresponding t-norm, $T_{\mathbf{Y}(2)}$, is the (nilpotent) Yager t-norm with parameter 2 (see [8]),

$$T_{\mathbf{Y}(2)}(x,y) = 1 - \sqrt{\min(1, (1-x)^2 + (1-y)^2)},$$

$$\delta_{T_{\mathbf{Y}(2)}}(x) = \max(0, 1 - \sqrt{2}\,(1-x)).$$

Proposition 1.8 : *Let f be an additive generator of a continuous Archimedean t-norm T. The function $g\colon [0,1] \to [0,1]$ defined by*

$$g\colon x \mapsto \exp(-f(x))$$

is strictly increasing, continuous, and satisfies

$$\forall x, y \in [0,1] : T(x,y) = g^{-1}(\max(g(0), g(x) \cdot g(y))).$$

It is called the multiplicative generator *of T. If T is strict, then $g(0) = 0$, g is an automorphism (=increasing bijection) of $[0,1]$ and*

$$\forall x, y \in [0,1] : T(x,y) = g^{-1}(g(x) \cdot g(y)).$$

Remark 1.9 : The ambiguity in the choice of an additive generator causes an ambiguity of a multiplicative generator. For each $r > 0$,

$$g^r : x \mapsto g(x)^r$$

is a multiplicative generator of the same t-norm.

The following proposition shows that many t-norms have diagonals which are "close to $\delta_{T_{\mathbf{P}}}$ near the boundary points".

Proposition 1.10 : *Let T be a strict t-norm such that its multiplicative generator g has nonzero finite derivatives at 0, 1. (We assume unilateral derivatives from the domain, i. e., the right derivative at 0 and the left derivative at 1.) Then the derivatives of its diagonal at the boundary points are $\delta'_T(0+) = 0$, $\delta'_T(1-) = 2$.*

PROOF: : Derivative at 0+: Suppose that $g'(0+) = r \in \,]0, \infty[$, i. e.

$$\lim_{x \to 0+} \frac{g(x)}{x} = r.$$

For the inverse g^{-1} we have

$$\lim_{x \to 0+} \frac{g^{-1}(x)}{x} = \frac{1}{r}.$$

The derivative of the diagonal is

$$\delta'_T(0+) = \lim_{x \to 0+} \frac{\delta_T(x)}{x} = \lim_{x \to 0+} \frac{g^{-1}(g(x)^2)}{x} = \lim_{x \to 0+} \frac{g^{-1}(g(x)^2)}{g(x)^2} \cdot \frac{g(x)}{x} \cdot g(x).$$

(We denote the square of $g(x)$ by $g(x)^2$, not by $g^2(x)$, in order to avoid a confusion with $g(g(x))$.) The first factor converges to $\frac{1}{r}$, the second to r, and the third to 0, so $\delta'_T(0+) = 0$.

Derivative at 1−: Suppose that $g'(1-) = r \in \,]0, \infty[$, i. e.

$$\lim_{x \to 1-} \frac{g(x) - 1}{x - 1} = r.$$

For the inverse g^{-1} we have

$$\lim_{x \to 1-} \frac{g^{-1}(x) - 1}{x - 1} = \frac{1}{r}.$$

The derivative of the diagonal is

$$\delta'_T(1-) = \lim_{x \to 1-} \frac{\delta_T(x) - 1}{x - 1} = \lim_{x \to 1-} \frac{g^{-1}(g(x)^2) - 1}{x - 1} = \lim_{x \to 1-} \frac{g^{-1}(g(x)^2) - 1}{g(x)^2 - 1} \cdot \frac{g(x) - 1}{x - 1} \cdot (g(x) + 1).$$

The first factor converges to $\frac{1}{r}$, the second to r, and the third to 2, so $\delta'_T(1-) = 2$. □

Corollary 1.11 : *Under the assumptions of Proposition 1.10, the diagonal δ_T converges to $\delta_{T_{\mathbf{P}}}$ at the boundary points in the following sense:*

$$\lim_{x \to 0+} \frac{\delta_T(x) - \delta_{T_{\mathbf{P}}}(x)}{x} = 0,$$

$$\lim_{x \to 1-} \frac{\delta_T(x) - \delta_{T_{\mathbf{P}}}(x)}{1 - x} = 0.$$

The latter corollary suggests that the diagonals of strict t-norms should be "similar" to the diagonal $\delta_{T_{\mathbf{P}}}: x \mapsto x^2$ of the product t-norm. It is therefore surprising that no such relation holds in general. In the next section, we shall show that the diagonals of strict t-norms may be much more general.

2 Diagonals of strict t-norms

We start the characterization of diagonals with the special case of strict t-norms. Proposition 1.2 and Definition 1.3 give necessary conditions for a function to be a diagonal of a strict t-norm. We may reformulate them using the fact that a strictly increasing continuous surjection is an (order) automorphism.

Proposition 2.1 : *Let T be a strict t-norm. Its diagonal, δ_T, is an automorphism of $[0,1]$ such that $\delta_T(x) < x$ for all $x \in \,]0,1[$.*

We shall prove that the necessary conditions from the latter proposition are also sufficient. Let δ be an automorphism of $[0,1]$ such that $\delta(x) < x$ for all $x \in \,]0,1[$. We shall construct an additive generator, f, of a strict t-norm with the given diagonal δ (see also [3]).

We denote by id the identity on $[0,1]$ and by Z the set of all integers. We define functions δ^n, $n \in Z$, recursively by

$$\delta^n = \begin{cases} \mathrm{id} & \text{if} \;\; n = 0, \\ \delta \circ \delta^{n-1} & \text{if} \;\; n > 0, \\ \delta^{-1} \circ \delta^{n+1} & \text{if} \;\; n < 0. \end{cases}$$

We start the construction of f at an arbitrary point $s \in \,]0,1[$, and we put $f(s) = 1/2$. (We may take for $f(s)$ an arbitrary positive real number. This ambiguity corresponds to the fact that positive multiples of f generate the same t-norm.) We form a sequence $(\delta^n(s))_{n \in Z}$. It is strictly decreasing. As 0 and 1 are the only fixed points of δ (as well as of δ^{-1}), we obtain

$$\lim_{n \to +\infty} \delta^n(s) = 0,$$

$$\lim_{n \to -\infty} \delta^n(s) = 1.$$

Because of the required functional equation $f \circ \delta = 2f$, we must define f for all $\delta^n(s)$, $n \in Z$, by

$$f(\delta^n(s)) = 2^n f(s) = 2^{n-1}.$$

These are the values of f which are determined by the choice of s and $f(s)$ and by the diagonal δ. The values of f at other points are restricted only by the monotony of f and by the functional equation $f \circ \delta = 2f$, so we have some freedom in their choice. Let $\varphi: [\delta(s), s] \to [1/2, 1]$ be an antiisomorphism (=decreasing bijection). For each $x \in \,]\delta(s), s[$, we define $f(x) = \varphi(x)$. For all $n \in Z$, the mapping δ^{-n} maps isomorphicly $]\delta^{n+1}(s), \delta^n(s)[$ onto $]\delta(s), s[$. For $n \in Z$ and $x \in \,]\delta^{n+1}(s), \delta^n(s)[$, we define

$$f(x) = 2^n \, \varphi(\delta^{-n}(x)).$$

The function f defined on $]0,1[$ this way is strictly decreasing and continuous, because $\varphi \circ \delta^{-n}$ is an antiisomorphism of $[\delta^{n+1}(s), \delta^n(s)]$ onto $[1/2,1]$, and $2^n\, \varphi\circ\delta^{-n}$ is an antiisomorphism of $[\delta^{n+1}(s), \delta^n(s)]$ onto $[2^{n-1}, 2^n] = [f(\delta^n(s)), f(\delta^{n+1}(s))]$.

Thus we have f defined for all elements of $]0,1[$. It remains to define $f(0) = +\infty$, $f(1) = 0$, and verify the continuity. As

$$\lim_{n\to+\infty} f(\delta^n(s)) = \lim_{n\to+\infty} 2^{n-1} = +\infty = f(0),$$

$$\lim_{n\to+\infty} f(\delta^{-n}(s)) = \lim_{n\to+\infty} 2^{-n-1} = 0 = f(1),$$

the continuity and monotony of f on $[0,1]$ are verified, and f is an additive generator of some t-norm, $T_{s,\varphi}$. (The indices of the t-norm refer to the chosen value s and an antiisomorphism $\varphi\colon [\delta(s), s] \to [1/2,1]$ which, together with the given function δ, determine the t-norm uniquely.) According to the definition of f,

$$T_{s,\varphi}(x,x) = f^{-1}(2f(x)) = f^{-1}(f(\delta(x))) = \delta(x)$$

for all $x \in [0,1]$, so δ is the diagonal of $T_{s,\varphi}$.

The choice of the antiisomorphism φ leads to infinitely many different t-norms with the same diagonal. Indeed, let $\varphi_1, \varphi_2\colon [\delta(s), s] \to [1/2,1]$ be two different antiisomorphisms and let f_1, f_2 be the corresponding additive generators obtained by the above construction. There is a $y \in\,]\delta(s), s[$ such that $\varphi_1(y) \neq \varphi_2(y)$. Then $f_1(y) \neq f_2(y)$ and $f_1(s) = 1/2 = f_2(s)$, so f_2/f_1 is not a constant function and f_1, f_2 generate different strict t-norms (Theorem 1.4).

In contrast to the latter discussion, the choice of the starting point s has no influence on the resulting t-norm in the following sense: For any strict t-norm $T_{s,\varphi}$ constructed by the above procedure and for an arbitrarily chosen starting point $s^* \in\,]0,1[$, we can always find an antiisomorphism $\varphi^*\colon [\delta(s^*), s^*] \to [1/2,1]$ such that $T_{s,\varphi} = T_{s^*,\varphi^*}$. We summarize the above results.

Theorem 2.2 : *Let δ be an automorphism of $[0,1]$ such that $\delta(x) < x$ for all $x \in\,]0,1[$. Let $s \in\,]0,1[$ be a chosen point. The class $\mathcal{T}_\delta$ of all strict t-norms with the diagonal δ is given by*

$$\mathcal{T}_\delta = \{T_{s,\varphi};\ \ \varphi\colon [\delta(s), s] \to [1/2,1] \text{ is an antiisomorphism}\}.$$

Corollary 2.3 : *The necessary conditions of Proposition 2.1 for a function to be a diagonal of a strict t-norm are also sufficient.*

Example 2.4 : Let $\delta(x) = x^2$, $s = 1/\sqrt{2}$. Then $\delta(s) = 1/2$ and $\delta^n(s) = 2^{-2^{n-1}}$, $n \in Z$.

1. We define $\varphi_1\colon [1/2, 1/\sqrt{2}] \to [1/2,1]$ by $\varphi_1(x) = -\log_2 x$. Then the corresponding additive generator, f_1, is given by $f_1(x) = -\log_2 x$, $x \in [0,1]$, and $T_{s,\varphi_1} = T_{\mathbf{P}}$.

2. We define $\varphi_2\colon [1/2, 1/\sqrt{2}] \to [1/2,1]$ by

$$\varphi_2(x) = \frac{1}{4x^2}.$$

Then the values of the corresponding additive generator, f_2, for $x \in\,]2^{-2^n}, 2^{-2^{n-1}}[$, $n \in Z$, are

$$f_2(x) = \frac{2^{n-2}}{x^{2^{1-n}}}.$$

The corresponding strict t-norm, T_{s,φ_2}, satisfies

$$T_{s,\varphi_2}(0.4, 0.5) = f_2^{-1}(2.25) = \frac{16}{81} \neq \frac{16}{80} = T_{\mathbf{P}}(0.4, 0.5),$$

so $T_{s,\varphi_2} \neq T_{\mathbf{P}}$.

3 Diagonals of nilpotent and Archimedean t-norms

Now we shall characterize the diagonals of nilpotent and Archimedean t-norms. Necessary conditions are given by the following proposition (a consequence of Proposition 1.2 and Definition 1.1).

Proposition 3.1 : *Let T be an Archimedean t-norm. Its diagonal δ_T satisfies the conditions of Proposition 1.2 and*

(N1) $I(\delta_T) = \{0, 1\}$,

(N2) *δ_T is strictly increasing on $\delta_T^{-1}(]0,1[)$.*

As a consequence of Corollary 1.5, there is an $s \in [0,1[$ such that $\delta_T(x) = 0$ for all $x \in [0,s]$, and δ_T is strictly increasing on $[s,1]$. The value s is given by $s = f^{-1}(1/2)$, where f is the normed additive generator of T, and s is zero (resp. positive) if and only if T is strict (resp. nilpotent).

Again, we shall show that the necessary conditions from the latter proposition are also sufficient. The case of a strict t-norm was solved in the preceding section, now we shall modify the construction for a nilpotent t-norm.

Let $\delta: [0,1] \to [0,1]$ be a continuous nondecreasing function satisfying the conditions of Proposition 1.2 and (N1), (N2). Due to continuity, there is a maximal s satisfying $\delta(s) = 0$. We shall construct a normed additive generator f of an Archimedean t-norm T with diagonal δ. For $s = 0$ we obtain a strict t-norm by Theorem 2.2. Suppose that $s \in]0,1[$. By a modification of the construction from the preceding section we shall construct a nilpotent t-norm T with diagonal δ.

The sequence $(\delta^n(s))_{n=1,0,-1,\ldots}$ is strictly increasing and

$$\lim_{n\to-\infty} \delta^n(s) = 1.$$

It is easy to see that each normed generator f of a nilpotent t-norm T with diagonal δ has to satisfy $f(\delta^n(s)) = 2^{n-1}$, $n = 1, 0, -1, -2, \ldots$. Let $\varphi: [0,s] \to [1/2, 1]$ be an antiisomorphism. Repeating the ideas from Section 2, it is enough to put $f(1) = 0$ and

$$f(x) = 2^n\, \varphi(\delta^{-n}(x))$$

whenever $x \in]\delta^{n+1}(s), \delta^n(s)[$, $n = 0, -1, -2, \ldots$. We obtain a normed additive generator, f, of a t-norm T_φ with diagonal δ. (In this case the value s was given by the properties of δ, not arbitrarily, so we index the t-norm only by the antiisomorphism φ.)

Theorem 3.2 : *Let $\delta: [0,1] \to [0,1]$ be a function satisfying the conditions of Proposition 1.2 and* (N1), (N2). *Suppose that $s = \sup\{x;\ \delta(x) = 0\} > 0$. Then the class $\mathcal{T}_\delta$ of all nilpotent t-norms with diagonal δ is given by*

$$\mathcal{T}_\delta = \{T_\varphi;\ \varphi: [0,s] \to [1/2,1] \text{ is an antiisomorphism}\}.$$

Corollary 3.3 : *The necessary conditions of Proposition 3.1 for a function to be a diagonal of an Archimedean t-norm are also sufficient.*

Example 3.4 : Let $\delta(x) = \max(0, 2x-1)$. Then $s = 1/2$, $\delta^n(s) = 1 - 2^{n-1}$, $n = -1, -2, \ldots$.

1. We define $\varphi_1: [0, 1/2] \to [1/2, 1]$ by $\varphi_1(x) = 1 - x$. Then the corresponding normed additive generator f_1 is given by $f_1(x) = 1 - x$, and $T_{\varphi_1} = T_{\mathbf{L}}$.

2. We define $\varphi_2: [0, 1/2] \to [1/2, 1]$ by $\varphi_2(x) = 2^{-2x}$. Then the corresponding normed additive generator f_2 is given by

$$f_2(x) = 2^{2^{1-n}(1-x)+n-2}$$

whenever $x \in [1 - 2^n, 1 - 2^{n-1}]$, $n = 0, -1, -2, \ldots$. For the corresponding t-norm T_{φ_2} we obtain

$$T_{\varphi_2}(0.5, 0.75) = 1 - \frac{1}{2}\log_2 3 \neq 0.25 = T_{\mathbf{L}}(0.5, 0.75)$$

so $T_{\varphi_2} \neq T_{\mathbf{L}}$.

4 Diagonals of continuous t-norms

All continuous t-norms were characterized by Ling [5] (see also [4, 7]).

Theorem 4.1 : *A mapping $T: [0,1]^2 \to [0,1]$ is a continuous t-norm if and only if there is a disjoint system $(]\alpha_k, \beta_k[)_{k\in K}$ of open subintervals of $[0,1]$ and a system $(f_k)_{k\in K}$ of continuous strictly decreasing functions $f_k: [\alpha_k, \beta_k] \to [0,\infty]$, $f_k(\beta_k) = 0$, $k \in K$, such that*

$$T(x,y) = \begin{cases} f_k^{-1}(\min(f_k(\alpha_k), f_k(x) + f_k(y)) & \textit{if } (x,y) \in [\alpha_k, \beta_k]^2 \textit{ for some } k \in K, \\ \min(x,y) & \textit{otherwise.} \end{cases}$$

Let T be a continuous t-norm and δ_T its diagonal. If $\delta_T(x) = \delta_T(y)$ for some $x < y$, then x, y are contained in an interval $[\alpha_k, \beta_k[$ of the representation of T with $f_k(\alpha_k)$ finite, and hence $\delta_T(x)$ is a fixed point of δ_T. We obtain the following necessary conditions for diagonals of continuous t-norms.

Proposition 4.2 : *Let T be a continuous t-norm. Its diagonal δ_T fulfils the conditions of Proposition 1.2 and*

δ_T *is strictly increasing on* $[0,1] \setminus \delta_T^{-1}(I(\delta_T))$.

Again, we shall prove that the necessary conditions of Proposition 4.2 are also sufficient. We start from a continuous function $\delta: [0,1] \to [0,1]$ fulfilling the properties of Proposition 4.2. The continuity of δ ensures that the set $I(\delta)$ is closed. Consequently, its complement $[0,1] \setminus I(\delta)$ can be written as a disjoint union of a system $(]\alpha_k, \beta_k[)_{k\in K}$ of open subintervals of $[0,1]$. If $K = \emptyset$, then only $T_{\mathbf{M}}$ has diagonal δ. Assume that K is nonempty. The construction of a continuous t-norm T with diagonal δ can be done in two steps:

1. For arbitrary $k \in K$, put $\delta_k: [0,1] \to [0,1]$,

$$\delta_k(x) = \frac{\delta(\alpha_k + (\beta_k - \alpha_k)\, x) - \alpha_k}{\beta_k - \alpha_k}.$$

Then δ_k satisfies the conditions of Corollary 3.3 and it is the diagonal of an Archimedean t-norm, T_k.

2. We define the t-norm T as follows:

$$T(x,y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) \cdot T_k(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k}) & \text{if } (x,y) \in [\alpha_k, \beta_k]^2 \text{ for some } k \in K, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

It is immediate that $T(x,x) = \delta(x)$ for all $x \in [0,1]$. Based on the preceding results, we obtain the following characterization of diagonals of continuous t-norms T.

Theorem 4.3 : *The conjunction of the following conditions is necessary and sufficient for a function $\delta: [0,1] \to [0,1]$ to be a diagonal of a continuous t-norm:*

δ *is a continuous, nondecreasing surjection,*

$\forall x \in [0,1] : \delta(x) \le x,$

δ *is strictly increasing on* $[0,1] \setminus \delta^{-1}(I(\delta))$, *where* $I(\delta) = \{x \in [0,1];\ \delta(x) = x\}$.

The characterization of all continuous t-norms T with given diagonal δ follows directly from the preceding results. In each interval $]\alpha_k, \beta_k[$, we construct the function f_k according to Theorem 2.2 if δ is strictly increasing on $[\alpha_k, \beta_k]$, and according to Theorem 3.2 otherwise (with the corresponding choice of a point s_k and an antiisomorphism $\varphi_k: [\delta(s_k), s_k] \to [1/2, 1]$). This procedure allows to describe the collection of all continuous T-norms with the diagonal δ. For strict (resp. Archimedean) t-norms, Theorem 4.3 with Definition 1.3 give Corollaries 2.3 and 3.3 as special cases.

References

[1] Bézivin, J.P., Tomás, M.S.: On the determination of strict t-norms on some diagonal segments. *Aequat. Math.* **45** (1993), 239–245.

[2] Hamacher, H.: *Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien.* Rita G. Fischer Verlag, Frankfurt, 1978.

[3] Kimberling, C.: On a class of associative functions. *Publ. Math. Debrecen* **20** (1973), 21–39.

[4] Klement, E.P., Mesiar, R., Pap, E.: *Triangular Norms.* Manuscript in preparation.

[5] Ling, C.M.: Representation of associative functions. *Publ. Math. Debrecen* **12** (1965), 189–212.

[6] Mayor, G., Torrens, J.: On a family of t-norms. *Fuzzy Sets Syst.* **41** (1991), 161–166.

[7] Schweizer, B., Sklar, A.: *Probabilistic Metric Spaces.* North-Holland, New York, 1983.

[8] Yager, R.: On a general class of fuzzy connectives. *Fuzzy Sets Syst.* **4** (1980), 235–242.

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