Algebra, Proof Theory and Applications for an Intuitionistic Logic of Propositions, Actions and Adjoint Modal Operators

ROY DYCKHOFF, University of St Andrews
MEHRNOOSH SADRZADEH, University of Oxford
JULIEN TRUFFAUT, University of Oxford

We develop a cut-free nested sequent calculus as basis for a proof search procedure for an intuitionistic modal logic of actions and propositions. The actions act on propositions via a dynamic modality (the *weakest precondition* of program logics), whose left adjoint we refer to as "update" (the *strongest postcondition*). The logic has agent-indexed adjoint pairs of epistemic modalities: the left adjoints encode agents' uncertainties and the right adjoints encode their beliefs. The rules for the "update" modality encode learning as a result of discarding uncertainty. We prove admissibility of *Cut*, and hence the soundness and completeness of the logic with respect to an algebraic semantics. We interpret the logic on epistemic scenarios that consist of honest and dishonest communication actions, add assumption rules to encode them, and prove that the calculus with the assumption rules still has the admissibility results. We apply the calculus to encode (and allow reasoning about) the classic epistemic puzzles of *dirty children* (aka "muddy children") and *drinking logicians* and some versions with dishonesty or noise; we also give an application where the actions are movements of a robot rather than announcements.

Categories and Subject Descriptors: F.4.1 [**Mathematical Logic**]: Modal logic, Proof theory

General Terms: Theory, Verification

Additional Key Words and Phrases: Proof Theory, Cut-Admissibility, Algebra, Adjoint Modalities, Epistemic Scenarios, Logics for multi-agent systems, decision procedures.

ACM Reference Format:
Dyckhoff, R., Sadrzadeh, M., Truffaut, J. 2013. Algebra, Proof Theory and Applications for a Logic of Propositions, Actions and Adjoint Modal Operators. ACM Trans. Comput. Logic 666, 0, Article 42 (December 2013), 37 pages.
DOI = 10.1145/0000000.0000000 http://doi.acm.org/10.1145/0000000.0000000

1. INTRODUCTION

Temporal and epistemic logics can express properties of programs [Huth and Ryan 2000] and distributed system protocols [Fagin et al. 1995]. The modalities of epistemic logics encode attitudes such as knowledge and belief, but dynamic changes to these attitudes, after actions such as announcements or navigations, have traditionally been formalised only in a semantic fashion. Dynamic program logics are, on the other hand, developed for the specific purpose of syntactic reasoning about changes of properties. Adding epistemic modalities to dynamic logics led to logics [Baltag and Moss 2004; Baltag et al. 2007] allowing reasoning about belief updates, both syntactically and semantically. But, lacking cut-admissibility, the calculus proposed in [Baltag et al. 2007]

Authors' addresses: R. Dyckhoff, School of Computer Science, St Andrews University; M. Sadrzadeh, School of Electronic Engineering and Computer Science, Queen Mary, University of London; J. Truffaut, Zeebox, London, UK. M. Sadrzadeh acknowledges support by EPSRC grant EP/F042728/1.

DOI 10.1145/0000000.0000000 http://doi.acm.org/10.1145/0000000.0000000

is not a basis for automatic proof search. As always, the presence of a cut rule would ensure infinite non-determinism in proof search, so we have to avoid it; its admissibility is however important for completeness. Likewise, we incorporate weakening and contraction into our logical rules (while ensuring for completeness' sake that they are admissible) rather than having them as primitive, thus reducing the non-determinism even further. Here we develop a cut-free sequent calculus as basis for a proof search procedure for one such logic. The calculus is *nested*, in the sense of calculi studied in, e.g., [Brünnler 2009; Fitting 2012; Goré et al. 2011; Kashima 1994; Moortgat 1995; Restall 2000; Poggiolesi 2010; Straßburger 2013].

The phrase "Dynamic Epistemic Logic" (DEL) refers to a family of logics, developed to reason about information acquisition as a result of communication actions that take place among agents in multi-agent protocols. One example is the logic of public and private announcements of [Baltag and Moss 2004], which extends the public announcement logic of, e.g., [Plaza 2007]. The DEL logical systems are usually presented by means of a Hilbert-style proof system and a relational semantics, whose central notion is an update product between the state and action Kripke models. There has been much activity in the field, extending the domain and applicability of the logics, e.g. to belief revision, and developing semantic automated tools; for references and a comprehensive survey see [van Ditmarsch et al. 2007]. The field has, however, had less activity on the proof-theoretic side: this paper takes steps towards filling this gap, albeit in the intuitionistic case rather than the classical case usual in DEL logics. Restriction to this case is for compatibility with [Baltag et al. 2007]; extension to classical logic is a topic for future research.

The logic comprises a logic Q of actions and an intuitionistic logic M of propositions. The modalities of the logic include the *dynamic modality* of PDL, known as the *weakest precondition*, and its right adjoint, the *strongest postcondition*. We present an algebraic semantics for these logics in terms of a pair of a residuated lattice-monoid of actions and its right lattice of propositions; the completions of these, known as "Quantale-Module pairs", are applied to program semantics in [Abramsky and Vickers 1993]. We endow both Q and M with families of epistemic adjoint operators satisfying distributivity properties over the operations of the action logic, propositional logic, and their dynamic modalities. We present appropriate sequent calculus rules for these modalities; we prove that the calculus admits three cut rules and (for propositions rather than actions) weakening and contraction rules, and that the calculus is sound and complete w.r.t. the algebraic semantics. Finally, following previous work [Baltag et al. 2007], we interpret the epistemic modalities as agents' uncertainties and beliefs, and the distributivity properties as belief update procedures.

To allow encoding of epistemic scenarios, we add "assumption rules" and show that the extended calculus still has the admissibility results. We apply the calculus with assumptions to encode the classic epistemic puzzle of *dirty children*, versions of it with a dishonest father or dishonest children, the puzzle of the *drinking logicians* and a robot navigation example. An early version of the calculus and some example scenarios (with loop-checking and optimisations) have been implemented in [Truffaut 2011].

The sequent calculus of actions and adjoint modalities enriches the calculus of [Moortgat 1995] with the two classical operators $\vee$ and $\wedge$. The rules for the calculus of propositions include those of [Sadrzadeh and Dyckhoff 2010], together with new rules for implication, for the dynamic adjoint modalities, their distributivity interactions with epistemic modalities, and rules for action connectives in propositional contexts. The admissibility theorems are novel contributions. The algebraic semantics is a finite version of [Baltag et al. 2007]; it now has a much simpler completeness proof that does not rely on the lattice completion. Proving admissibility of the cut rules involved hundreds of cases, summarised to 3 major categories in a few pages.

This is a corrected, extended and refined version of the paper published as [Dyckhoff et al. 2012]. We are grateful to various people, especially Lutz Straßburger and two anonymous referees, for comments leading mainly to notational improvements.

2. SYNTAX AND ALGEBRA FOR ACTIONS

We give in this section the syntax and the intended algebraic semantics of our *action logic*; in later sections we will present a cut-free sequent calculus, a soundness theorem and a completeness theorem. Another section will add syntax, algebraic semantics and a sequent calculus for propositions, including those formed by the interaction of actions and propositions in various ways.

2.1. Syntax of Actions

The set Q of *actions* q of the logic is generated over a set $\mathcal{A}$ of *agents* A and a set B of *basic actions* σ by the following grammar:

$$q ::= \bot \mid \top \mid 1 \mid \sigma \mid q \wedge q \mid q \vee q \mid q \bullet q \mid \Box_A\, q \mid \blacklozenge_A q$$

As examples of basic actions σ, we mention navigation, message passing and announcement. The main operations are as in PDL [Abramsky and Vickers 1993]: i.e. $\bot$ is failure, $\top$ is the action that always succeeds, 1 is "skip" (the null action), $\wedge$ is parallel composition, $\vee$ is non-deterministic choice and $\bullet$ is sequential composition. Following [Baltag et al. 2007], we read $\blacklozenge_A q$ as the appearance to agent A of action q and $\Box_A q$ as A's belief that it happens. The appearance $\blacklozenge_A q$ encapsulates all the actions that appear to A as happening when the action q is actually happening. We will see later how to express the public or private nature of announcements.

2.2. Algebra of Actions

Definition 2.1. Let $\mathcal{A}$ be a set, with elements called *agents*. A *lattice monoid with adjoint modalities* (an `LMAM`) *over* $\mathcal{A}$ is both a bounded lattice $(Q, \vee, \wedge, \top, \bot)$ and a unital monoid $(Q, 1, \bullet, \leq)$, where $\bullet$ is associative and preserves joins, with two $\mathcal{A}$-indexed families $\{\blacklozenge_A\}_{A\in\mathcal{A}} \colon Q \to Q$ and $\{\Box_A\}_{A\in\mathcal{A}} \colon Q \to Q$ of order-preserving maps, each $\blacklozenge_A$ being left adjoint to $\Box_A$. We don't impose the distributivity of $\vee$ over $\wedge$ or of $\wedge$ over $\vee$.

PROPOSITION 2.2. *In any* `LMAM` Q *over* $\mathcal{A}$*, the following hold, for all* $q, q', q'' \in Q$ *and all* $A \in \mathcal{A}$*:*

0	$q \bullet \bot = \bot$	*0*	$\bot \bullet q = \bot$
1	$q \bullet (q' \vee q'') = (q \bullet q') \vee (q \bullet q'')$	*1*	$(q' \vee q'') \bullet q = (q' \bullet q) \vee (q'' \bullet q)$
2	$q \bullet 1 = q$	*2*	$1 \bullet q = q$
3	$\blacklozenge_A q \leq \blacklozenge_A q'$ *if* $q \leq q'$	*4*	$\Box_A q \leq \Box_A q'$ *if* $q \leq q'$
5	$\blacklozenge_A q \leq q'$ *iff* $q \leq \Box_A q'$		
6	$\blacklozenge_A(q \vee q') = \blacklozenge_A q \vee \blacklozenge_A q'$	*7*	$\Box_A(q \wedge q') = \Box_A q \wedge \Box_A q'$
8	$\blacklozenge_A(q \wedge q') \leq \blacklozenge_A q \wedge \blacklozenge_A q'$	*9*	$\Box_A q \vee \Box_A q' \leq \Box_A(q \vee q')$
10	$\blacklozenge_A \bot = \bot$	*11*	$\Box_A \top = \top$
12	$q \bullet (q' \wedge q'') \leq (q \bullet q') \wedge (q \bullet q'')$	*13*	$(q' \wedge q'') \bullet q \leq (q' \bullet q) \wedge (q'' \bullet q)$
14	$\blacklozenge_A \Box_A q \leq q$	*15*	$q \leq \Box_A \blacklozenge_A q$

PROOF. Routine. □

Definition 2.3. An `LMAM` Q over $\mathcal{A}$ is *multiplicative* whenever $\blacklozenge_A$ satisfies the following, for all $q, q' \in Q$ and all $A \in \mathcal{A}$:

$$\blacklozenge_A(q \bullet q') \quad \leq \quad \blacklozenge_A q \bullet \blacklozenge_A q' \tag{16}$$

$$\blacklozenge_A 1 \quad \leq \quad 1 \tag{17}$$

PROPOSITION 2.4. *In any multiplicative* LMAM Q *over* $\mathcal{A}$*, the following hold, for all* $q, q' \in Q$ *and all* $A \in \mathcal{A}$*:*

$$\Box_A q \bullet \Box_A q' \;\leq\; \Box_A (q \bullet q') \tag{18}$$
$$1 \;\leq\; \Box_A 1 \tag{19}$$

PROOF. Routine. □

The relation $\leq$ is the information order, as defined in [Abramsky and Vickers 1993], with $q \leq q'$ meaning that q is at least as deterministic as q'. Inequalities (16) and (17) encode a form of learning, that the appearance of a sequential composition is learnt from the sequential composition of the appearances.

2.3. Interpretations of Actions

Let Q be a multiplicative LMAM over $\mathcal{A}$. An *interpretation* of the action logic (over $\mathcal{A}$, and with set B of basic actions) *in* Q is a map $[\![-]\!]\colon \mathsf{B} \to Q$. The meanings of more complex actions are obtained by induction on the structure of the actions:

$$[\![q_1 \vee q_2]\!] = [\![q_1]\!] \vee [\![q_2]\!], \qquad [\![q_1 \wedge q_2]\!] = [\![q_1]\!] \wedge [\![q_2]\!], \qquad [\![q_1 \bullet q_2]\!] = [\![q_1]\!] \bullet [\![q_2]\!],$$
$$[\![\blacklozenge_A q]\!] = \blacklozenge_A [\![q]\!], \qquad [\![\Box_A q]\!] = \Box_A [\![q]\!],$$
$$[\![\top]\!] = \top, \qquad [\![\bot]\!] = \bot, \qquad [\![1]\!] = 1\,.$$

3. SEQUENT CALCULUS FOR ACTIONS

3.1. Preliminaries

Rather than using sets, lists or multi-sets as contexts in our sequents, we have "nested" contexts, in the sense mentioned and referenced in the introduction. Thus, we have *action items* S and *action contexts* Θ generated by the following syntax:

$$S ::= q \mid \Theta^A \qquad \Theta ::= \langle\rangle \mid S :: \Theta$$

where lists will be written in a simpler syntax, e.g. $\langle S_1, S_2\rangle$, or just S_1, S_2, rather than $S_1 :: S_2 :: \langle\rangle$, and Θ^A will be interpreted as $\blacklozenge_A(\bigodot \Theta)$, for $\bigodot \Theta$ the *composition* of the interpretations of the elements in Θ, e.g. $\bigodot\langle q_1, q_2\rangle = q_1 \bullet q_2$ and $\bigodot\langle\rangle = 1$.

Thus, *action contexts* (abbreviated to *a-contexts*) are finite lists of action items, where *action items* (abbreviated to *a-items*) are either actions or agent-annotated a-contexts. The term *context* is **not** used in the sense of "having a hole waiting to be filled" but is a short name for the antecedent of a sequent: when later we discuss propositions, "context" is the traditional name for the situation in which the succedent proposition is asserted.

The use of lists rather than sets (or multi-sets) reflects the non-commutativity (and non-idempotence) of the composition operation on actions. Lists may be empty. The concatenation of two lists is indicated by a comma, as in Θ, Θ' or (treating an action item S as a one element list) as in Θ, S or S, Θ. Similarly: Θ, S, Θ' indicates a typical list of which S is a member.

If one of the items inside an a-context is replaced by a "hole" [ ], we have an *a-context-with-an-a-hole*. More precisely, we have the notions of *a-context-with-an-a-hole* Σ and *a-item-with-an-a-hole* R, defined using mutual recursion as follows:

$$\Sigma ::= \Theta, R, \Theta' \qquad R ::= [\,] \mid \Sigma^A$$

So an a-context-with-an-a-hole is an a-context (i.e. a list of a-items) except for the replacement of an a-item by an *a-item-with-an-a-hole*, i.e. either a hole or an agent-annotated a-context-with-an-a-hole. To emphasise that an a-context-with-an-a-hole is not an a-context, we use Σ for the former and Θ for contexts; similarly for a-items-with-an-a-hole R and a-items J.

Given an a-context-with-an-a-hole Σ and an a-context Θ, the result $\Sigma[\Theta]$ of *applying* the first to the second, i.e. replacing the hole $[\,]$ in Σ by Θ, is an a-context, defined recursively (together with the application of an a-item-with-an-a-hole to an a-context, to form an a-context) as follows:

$$(\Theta', R, \Theta'')[\Theta] = \Theta', R[\Theta], \Theta'' \qquad ([\,])[\Theta] = \Theta \qquad (\Sigma^A)[\Theta] = (\Sigma[\Theta])^A$$

The last of these looks more like an a-item; but that just forms a one element a-context. Note that $\Sigma[\,]$ is an a-context-with-an-a-hole Σ applied to the empty a-context, and so is an a-context (i.e. without a hole). The attentive reader will notice that we sometimes use the notation $\Sigma[\Theta_1][\Theta_2]$, for an a-context with two holes applied to two a-contexts; we leave the tedious details of this terminology to the reader.

Given a-contexts-with-an-a-hole Σ', Σ, and an a-item-with-an-a-hole R, the *combinations* $\Sigma' \circ \Sigma$ and $R \circ \Sigma$ are defined to be a-contexts with holes, as follows, by mutual recursion on the structures of Σ' and R:

$$(\Theta, R, \Theta') \circ \Sigma = \Theta, (R \circ \Sigma), \Theta' \qquad ([\,]) \circ \Sigma = \Sigma \qquad (\Sigma''^A) \circ \Sigma = (\Sigma'' \circ \Sigma)^A.$$

The last of these looks more like an a-item-with-an-a-hole; but that just forms a one element a-context-with-an-a-hole.

LEMMA 3.1. *Given a-contexts-with-an-a-hole Σ', Σ, an item-with-an-a-hole R and an a-context Θ, the following hold:*

$$(\Sigma' \circ \Sigma)[\Theta] = \Sigma'[\Sigma[\Theta]] \qquad (R \circ \Sigma)[\Theta] = R[\Sigma[\Theta]].$$

3.2. Sequents and their Interpretations

Sequents consist of an action context Θ (on the left), a turnstile and an action q (on the right). On the left, it is convenient to omit the list constructors, e.g. we write $1, \sigma, \bot \vdash \sigma'$ rather than $\langle 1, \sigma, \bot \rangle \vdash \sigma'$. The empty list is written $\langle\,\rangle$ or even omitted.

The meanings of a-items and of a-contexts, given an interpretation, are obtained by mutual induction on their structure:

$$\begin{aligned} [\![q]\!] &= \text{as above} \\ [\![\Theta^A]\!] &= \blacklozenge_A [\![\Theta]\!] \\ [\![J_1, \cdots, J_n]\!] &= [\![J_1]\!] \bullet \cdots \bullet [\![J_n]\!] \\ [\![\langle\,\rangle]\!] &= 1. \end{aligned}$$

We can see here also the notion of the *formula equivalent* of an a-context Θ or of an a-item J, by ignoring the interpretation of actions and focusing on the conversion of contexts and items to actions.

Note that, since $\bullet$ is associative (but not necessarily commutative), the meaning of an a-context Θ depends on its presentation as a list of items in a particular order.

LEMMA 3.2. *Let Θ, Θ' be a-contexts. Then*

$$[\![\Theta, \Theta']\!] = [\![\Theta]\!] \bullet [\![\Theta']\!].$$

PROOF. Routine use of associativity of $\bullet$. □

A sequent $\Theta \vdash q$ is *true* in an interpretation $[\![-]\!]$ in Q iff $[\![\Theta]\!] \leq [\![q]\!]$; it is *true* in Q iff true in all interpretations in Q, and it is *valid* iff true in every multiplicative `LMAM` Q.

LEMMA 3.3. *Let Θ, Θ' be a-contexts with $[\![\Theta]\!] \leq [\![\Theta']\!]$ and Σ an a-context-with-an-a-hole. Then*

$$[\![\Sigma[\Theta]]\!] \leq [\![\Sigma[\Theta']]\!].$$

PROOF. Routine induction on the structure of Σ (using also a similar result for action items-with-an-a-hole). For example, the induction step for dealing with $\blacklozenge_A$ is as follows: suppose $[\![\Theta]\!] \leq [\![\Theta']\!]$; then (by (3) of Proposition 2.2) $\blacklozenge_A[\![\Theta]\!] \leq \blacklozenge_A[\![\Theta']\!]$, and so $[\![\Theta^A]\!] \leq [\![\Theta'^A]\!]$ follows. □

This lemma will be useful in showing that all the initial sequents of the following sequent calculus are valid and that the rules preserve truth in an LMAM.

3.3. Sequent Calculus

We have the following initial sequents (in which σ is restricted to being a basic action):

$$\frac{}{\vdash 1}\ 1R \qquad \frac{}{\sigma \vdash \sigma}\ Id \qquad \frac{}{\Sigma[\bot] \vdash q}\ \bot L \qquad \frac{}{\Theta \vdash \top}\ \top R$$

The rules for the lattice operations, composition and the modalities are:

$$\frac{\Sigma[\;] \vdash q}{\Sigma[1] \vdash q}\ 1L$$

$$\frac{\Sigma[q_i] \vdash q}{\Sigma[q_1 \wedge q_2] \vdash q}\ \wedge L_i \qquad \frac{\Theta \vdash q_1 \quad \Theta \vdash q_2}{\Theta \vdash q_1 \wedge q_2}\ \wedge R$$

$$\frac{\Sigma[q_1] \vdash q \quad \Sigma[q_2] \vdash q}{\Sigma[q_1 \vee q_2] \vdash q}\ \vee L \qquad \frac{\Theta \vdash q_1}{\Theta \vdash q_1 \vee q_2}\ \vee R_1 \qquad \frac{\Theta \vdash q_2}{\Theta \vdash q_1 \vee q_2}\ \vee R_2$$

$$\frac{\Sigma[q_1, q_2] \vdash q}{\Sigma[q_1 \bullet q_2] \vdash q}\ \bullet L \qquad \frac{\Theta_1 \vdash q_1 \quad \Theta_2 \vdash q_2}{\Theta_1, \Theta_2 \vdash q_1 \bullet q_2}\ \bullet R$$

$$\frac{\Sigma[q^A] \vdash q'}{\Sigma[\blacklozenge_A q] \vdash q'}\ \blacklozenge_A L \qquad \frac{\Theta \vdash q}{\Theta^A \vdash \blacklozenge_A q}\ \blacklozenge_A R$$

$$\frac{\Sigma[q] \vdash q'}{\Sigma[(\Box_A q)^A] \vdash q'}\ \Box_A L \qquad \frac{\Theta^A \vdash q}{\Theta \vdash \Box_A q}\ \Box_A R$$

We also have *structural* rules, i.e. rules involving none of the algebraic or modal operators but rearranging the structure in the antecedent:

$$\frac{\Sigma[\Theta^A, \Theta'^A] \vdash q}{\Sigma[(\Theta, \Theta')^A] \vdash q}\ Dist \qquad \frac{\Sigma[\;] \vdash q}{\Sigma[\langle\;\rangle^A] \vdash q}\ Unit$$

Clearly, these are motivated by the two multiplicativity axioms (Definition 2.3), i.e. that $\blacklozenge_A(q \bullet q') \leq \blacklozenge_A q \bullet \blacklozenge_A q'$ and $\blacklozenge_A 1 \leq 1$. There is no need here for rules such as *Contraction* and *Weakening*; in this calculus of actions, they are not even admissible, but when we deal later with calculi where they are needed we incorporate them into the logical rules in a standard way.

Various notational abbreviations are in use here, such as $\Sigma[\;]$ meaning $\Sigma[\langle\;\rangle]$, $\Sigma[q]$ meaning $\Sigma[\langle q \rangle]$, $\Sigma[q_1, q_2]$ meaning $\Sigma[\langle q_1, q_2 \rangle]$ and q^A meaning $\langle q \rangle^A$. In fact, every rule that is not a right rule has a conclusion in which the antecedent appears to be some Σ applied to an a-item; but in fact that Σ is applied to a one item context. For example, $Unit$ (if we move from conclusion to premiss) removes an a-item $\langle\rangle^A$ from a subcontext of the antecedent, just as $1L$ removes an a-item 1. In examples, where the empty list is an antecedent, it is omitted.

The two indicated occurrences of σ in the *Id* rule are *principal*. Each right rule has its conclusion's succedent as its *principal action* (but in this context we will always call it a *formula*); in addition, the $\blacklozenge_A R$ rule has Θ^A as a *principal item*. Each left rule has a *principal item*; these are as usual.

The *size* $s(q)$ *of an action* q can be defined in various ways. Ideally we would define it in such a way that we can show that root-first proof search is depth-bounded; this is left for later work. Meanwhile, we let

$$s(\sigma) = s(1) = s(\top) = s(\bot) = 1;$$

$$s(q_1 \wedge q_2) = s(q_1 \vee q_2) = s(q_1) + s(q_2);$$

$$s(q_1 \bullet q_2) = s(q_1) + 1 + s(q_2);$$

$$s(\blacklozenge_A q) = s(\Box_A q) = s(q) + 1.$$

As an example of a derivation, we show that a sequence of $\blacklozenge_A$ operations preserves composition and conjunction in one direction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,Id}{q^B \vdash \blacklozenge_B q}\,\blacklozenge_B R}{(q^B)^A \vdash \blacklozenge_A\blacklozenge_B q}\,\blacklozenge_A R \quad \dfrac{\dfrac{\dfrac{}{q' \vdash q'}\,Id}{q'^B \vdash \blacklozenge_B q'}\,\blacklozenge_B R}{(q'^B)^A \vdash \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_A R}{(q^B)^A, (q'^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\bullet R}{(q^B, q'^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,Dist}{((q, q')^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,Dist}{((q \bullet q')^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\bullet L}{(((q \bullet q') \wedge q'')^B)^A \vdash \blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q'}\,\wedge L \quad \dfrac{\dfrac{\dfrac{\dfrac{}{q'' \vdash q''}\,Id}{q''^B \vdash \blacklozenge_B q''}\,\blacklozenge_B R}{(q''^B)^A \vdash \blacklozenge_A\blacklozenge_B q''}\,\blacklozenge_A R}{(((q \bullet q') \wedge q'')^B)^A \vdash \blacklozenge_A\blacklozenge_B q''}\,\wedge L}{(((q \bullet q') \wedge q'')^B)^A \vdash (\blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q') \wedge \blacklozenge_A\blacklozenge_B q''}\,\wedge R}{(\blacklozenge_B((q \bullet q') \wedge q''))^A \vdash (\blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q') \wedge \blacklozenge_A\blacklozenge_B q''}\,\blacklozenge_B L}{\blacklozenge_A\blacklozenge_B((q \bullet q') \wedge q'') \vdash (\blacklozenge_A\blacklozenge_B q \bullet \blacklozenge_A\blacklozenge_B q') \wedge \blacklozenge_A\blacklozenge_B q''}\,\blacklozenge_A L$$

As a standard check on the rules, we show the following:

LEMMA 3.4. *For every action q, the sequent $q \vdash q$ is derivable.*

PROOF. By induction on the size of q. In case q is a basic action, or $\bot$, or $\top$, the sequent $q \vdash q$ is already initial; and for $q = 1$, the sequent $1 \vdash 1$ follows from the initial sequent $\vdash 1$ by one step of $1L$. For compound q, consider the cases. Meet, join and composition are routine. Suppose q is $\blacklozenge_A q'$; by inductive hypothesis, we can derive $q' \vdash q'$, and by $\blacklozenge_A R$ we can derive $q'^A \vdash \blacklozenge_A q'$, whence $\blacklozenge_A q' \vdash \blacklozenge_A q'$ by $\blacklozenge_A L$.

Now suppose q is $\Box_A q'$. By inductive hypothesis, we can derive $q' \vdash q'$, and by $\Box_A L$ we get $(\Box_A q')^A \vdash q'$; from this we obtain $q \vdash q$ by $\Box_A R$. $\Box$

In proofs and examples below, we shall allow the use of Id in the form $q \vdash q$ even where the action q is not basic.

LEMMA 3.5. *The $1L$, $\vee L$, $\bullet L$, $\blacklozenge_A L$, $\wedge R$ and $\Box_A R$ rules are invertible.*

PROOF. Induction on the height of the derivation of the premiss. $\Box$

LEMMA 3.6. *The rules $\top L^-$ and $\bot R^-$ are admissible:*

$$\frac{\Sigma[\top] \vdash q}{\Sigma[\Theta] \vdash q}\ \top L^- \qquad \frac{\Theta \vdash \bot}{\Sigma[\Theta] \vdash q}\ \bot R^-$$

PROOF. Induction on the height of the derivation of the premiss. □

PROPOSITION 3.7. *Let Θ (resp. J) be an a-context (resp. an a-item) and $[\![\Theta]\!]$ (resp. $[\![J]\!]$) be the* formula equivalent *given in the definition in Section 3.2 of their interpretations. Then $\Theta \vdash [\![\Theta]\!]$ and $J \vdash [\![J]\!]$ are derivable.*

PROOF. Routine induction on the structure of Θ and J, using $1R$ and Lemma 3.4 for the base cases (where Θ is empty and J is an action) and $\bullet R$ and $\blacklozenge_A R$ for the induction steps. □

3.4. Admissibility of *Cut*

THEOREM 3.8. *The* Cut *rule is admissible*

$$\frac{\Theta \vdash q \quad \Sigma'[q] \vdash q'}{\Sigma'[\Theta] \vdash q'}\ Cut$$

PROOF. Strong induction on the rank of the cut, where the *rank* is given by the pair (size of cut formula q, sum of heights of derivations of premisses).

There are of course many strategies for eliminating cuts. We give just one of them, following also the pattern that premisses of cuts are reduced until cut-free before the cut itself is reduced. We adopt the strategy of pushing cuts into the first premiss rather than the second in cases where there is a choice.

(1) We begin with the cases of the second premiss being an initial sequent, i.e. conclusion by a zero-premiss rule:
  (a) If the rule is $1R$, no cut is possible.
  (b) If the rule is Id, then the first premiss of the cut is already the conclusion.
  (c) If the rule is $\bot L$, then the conclusion follows either by $\bot L$ or by using Lemma 3.6.
  (d) If the rule is $\top R$, then the conclusion follows by $\top R$.
(2) Next, if the first premiss is an initial sequent, we have the following cases:
  (a) If the cut-formula is 1, we permute the cut into the second premiss.
  (b) If the rule is Id, the second premiss is already the conclusion.
  (c) If the rule is $\bot L$, the conclusion of the cut is an instance of $\bot L$.
  (d) If the rule is $\top R$, we use Lemma 3.6 to infer the conclusion from the second premiss.
(3) Otherwise, if the cut-formula is principal in both premisses, we have a *principal cut*, and these are dealt with as follows:
  (a) The cut-formula is of the form 1: the cut disappears, i.e. the premiss of the second premiss is identical with the conclusion.
  (b) The cut-formula is of the form $q \wedge q'$: routine.
  (c) The cut-formula is of the form $q \vee q'$: routine.
  (d) The cut-formula is of the form $q \bullet q'$:

$$\frac{\dfrac{\Theta \vdash q \quad \Theta' \vdash q'}{\Theta, \Theta' \vdash q \bullet q'}\ \bullet R \qquad \dfrac{\Sigma[q, q'] \vdash q''}{\Sigma[q \bullet q'] \vdash q''}\ \bullet L}{\Sigma[\Theta, \Theta'] \vdash q''}\ Cut$$

transforms to (for example)

$$\dfrac{\Theta' \vdash q' \quad \dfrac{\Theta \vdash q \quad \Sigma[q,q'] \vdash q''}{\Sigma[\Theta,q'] \vdash q''}\,Cut}{\Sigma[\Theta,\Theta'] \vdash q''}\,Cut$$

(e) The cut-formula is of the form $\blacklozenge_A q''$:

$$\dfrac{\dfrac{\Theta \vdash q''}{\Theta^A \vdash \blacklozenge_A q''}\,\blacklozenge_A R \quad \dfrac{\Sigma'[q''^A] \vdash q'}{\Sigma'[\blacklozenge_A q''] \vdash q'}\,\blacklozenge_A L}{\Sigma'[\Theta^A] \vdash q'}\,Cut$$

transforms to

$$\dfrac{\Theta \vdash q'' \quad \Sigma'[q''^A] \vdash q'}{\Sigma'[\Theta^A] \vdash q'}\,Cut$$

(f) The cut-formula is of the form $\Box_A q''$:

$$\dfrac{\dfrac{\Theta^A \vdash q''}{\Theta \vdash \Box_A q''}\,\Box_A R \quad \dfrac{\Sigma'[q''] \vdash q'}{\Sigma'[(\Box_A q'')^A] \vdash q'}\,\Box_A L}{\Sigma'[\Theta^A] \vdash q'}\,Cut$$

transforms to

$$\dfrac{\Theta^A \vdash q'' \quad \Sigma'[q''] \vdash q'}{\Sigma'[\Theta^A] \vdash q'}\,Cut$$

(4) Otherwise, if the cut-formula is non-principal in the first premiss the cut is permuted into the first premiss; for example:

$$\dfrac{\dfrac{\Sigma[q] \vdash q'}{\Sigma[(\Box_A q)^A] \vdash q'}\,\Box_A L \quad \Sigma'[q'] \vdash q''}{\Sigma'[\Sigma[(\Box_A q)^A]] \vdash q''}\,Cut$$

transforms to

$$\dfrac{\dfrac{\Sigma[q] \vdash q' \quad \Sigma'[q'] \vdash q''}{\Sigma'[\Sigma[q]] \vdash q''}\,Cut}{\Sigma'[\Sigma[(\Box_A q)^A]] \vdash q''}\,\Box_A L$$

(5) Otherwise, if it is principal in the first premiss it must be non-principal in the second premiss; the cut is permuted into the second premiss; for example,

$$\dfrac{\dfrac{\Theta \vdash q''}{\Theta^A \vdash \blacklozenge_A q''}\,\blacklozenge_A R \quad \dfrac{\Sigma'[\blacklozenge_A q''][q_1, q_2] \vdash q'}{\Sigma'[\blacklozenge_A q''][q_1 \bullet q_2] \vdash q'}\,\bullet L}{\Sigma'[\Theta^A][q_1 \bullet q_2] \vdash q'}\,Cut$$

transforms to

$$\dfrac{\dfrac{\dfrac{\Theta \vdash q''}{\Theta^A \vdash \blacklozenge_A q''}\,\blacklozenge_A R \quad \Sigma'[\blacklozenge_A q''][q_1, q_2] \vdash q'}{\Sigma'[\Theta^A][q_1, q_2] \vdash q'}\,Cut}{\Sigma'[\Theta^A][q_1 \bullet q_2] \vdash q'}\,\bullet L$$

□

4. SYNTAX AND ALGEBRA FOR PROPOSITIONS

4.1. Syntax of propositions

Given sets $\mathcal{A}$ of agents A and B of basic actions σ, we have as above an action logic with a set Q of actions q. Now let C be a set of *basic propositions* p; the set M of *propositions* m is generated by the following grammar:

$$m ::= \bot \mid \top \mid p \mid m \wedge m \mid m \vee m \mid m \to m \mid \Box_A\, m \mid \blacklozenge_A m \mid m \cdot q \mid [q]m$$

Here the last two binary connectives are mixed action-proposition connectives: the operator $[q]\_$ is the *dynamic modality* operator and $[q]m$ is read as "after action q proposition m holds"; $\_ \cdot q$ is (as we shall see) its left adjoint, called *update*. $\blacklozenge_A$ is the left adjoint of $\Box_A$; the former is the uncertainty of an agent about a proposition, the latter is his belief; $\Box_A m$ is read as 'A believes that m'.

4.2. Algebra of propositions

Definition 4.1. Let $\mathcal{A}$ be a set, with elements called *agents*. A HAAM (Heyting Algebra with Adjoint Modalities) over $\mathcal{A}$ is a Heyting algebra $(M, \wedge, \vee, \to, \top, \bot)$ with two $\mathcal{A}$-indexed families $\{\blacklozenge_A\}_{A\in\mathcal{A}} \colon M \to M$ and $\{\Box_A\}_{A\in\mathcal{A}} \colon M \to M$ of order-preserving maps, with each $\blacklozenge_A$ left adjoint to $\Box_A$, i.e. $\blacklozenge_A(m) \leq m'$ iff $m \leq \Box_A(m')$.

By the addition of intuitionistic implication, this extends and supersedes the work in [Sadrzadeh and Dyckhoff 2010] on DLAMs, i.e. Distributive Lattices with Adjoint Modalities.

Definition 4.2. A multiplicative LMAM Q *acts on* a HAAM M (with the same sets of agents) if there is a pointwise order-preserving map $\_ \cdot \_ \colon M \times Q \to M$ with a right adjoint $[\_]\_ \colon Q \times M \to M$, s.t. the following hold:

$$m \cdot (q \bullet q') \;=\; (m \cdot q) \cdot q' \tag{20}$$
$$m \cdot 1 \;\leq\; m \tag{21}$$
$$\blacklozenge_A(m \cdot q) \;\leq\; \blacklozenge_A m \cdot \blacklozenge_A q \tag{22}$$

It is part of the definition that the following hold, for all $m, m' \in M$ and all $q, q' \in Q$:

$$q \leq q' \quad \text{implies} \quad m \cdot q \leq m \cdot q' \text{ and } [q']m \leq [q]m \tag{23}$$
$$m \leq m' \quad \text{implies} \quad m \cdot q \leq m' \cdot q \text{ and } [q]m \leq [q]m' \tag{24}$$
$$m \cdot q \leq m' \quad \text{iff} \quad m \leq [q]m' \tag{25}$$

Intuitively, each m is a proposition; the relation $\leq$ is the logical entailment, and $\wedge, \vee, \to$ are conjunction, disjunction and implication. The dynamic connectives $[q]m$ and $m \cdot q$ are like the weakest precondition and strongest postcondition of PDL. As for the epistemic modalities, again following [Baltag et al. 2007], we read $\blacklozenge_A m$ as the appearance to agent A of a proposition m, and $\Box_A m$ as his belief that m holds. The appearance $\blacklozenge_A m$ encapsulates all the propositions that appear to A as true when the proposition m is actually true. Equalities (20) and (21) are almost the axioms of a Quantale-Module [Abramsky and Vickers 1993]; the difference is that the latter include $m \leq m \cdot 1$ (which we do not need) and the distribution of $\cdot$ over arbitrary joins (which exist both in a quantale and in a module, but not necessarily here).

Inequality (22) encodes a form of learning: that the appearance to A of an updated proposition $m \cdot q$ is learnt from the update of the appearance to A of m by the appearance to A of q.

PROPOSITION 4.3. *Whenever a multiplicative* LMAM Q *acts on a* HAAM M*, the following hold, for all* $m, m' \in M$*, all* $q, q' \in Q$*, and all* $A \in \mathcal{A}$*:*

26	$(m \vee m') \cdot q = (m \cdot q) \vee (m' \cdot q)$	*27*	$(m \wedge m') \cdot q \leq (m \cdot q) \wedge (m' \cdot q)$
28	$[q]\,(m \wedge m') = [q]\,m \wedge [q]\,m'$	*29*	$[q]\,m \vee [q]\,m' \leq [q]\,(m \vee m')$
30	$\bot \cdot q \;=\; \bot$	*31*	$[q]\top \;=\; \top$
32	$([q]m) \cdot q \leq m$	*33*	$m \leq [q](m \cdot q)$
34	$[q \bullet q']\,m = [q]\,[q']\,m$	*35*	$m \leq [1]\,m$
36	$\Box_A m \cdot \Box_A q \leq \Box_A (m \cdot q)$		

PROOF. Routine. □

It should be noted that (21) differs very slightly from the definition in the paper's preliminary version [Dyckhoff et al. 2012]; that included the condition that $m \leq m \cdot 1$. Incorporating this is a major (but insignificant) challenge, and appears to require a rule that allows more non-determinism in proof search. That version also included in (35) the incorrect consequence that $[1]m \leq m$.

4.3. Interpretation of propositions

Let Q be a multiplicative LMAM acting on a HAAM M and $[\![-]\!]$ an interpretation of the set of actions of the action logic (over a set B of basic actions) in Q, as defined in the previous subsection. An *interpretation* of the set of propositions of the propositional logic (over a set C of basic propositions) *in* M is given by a map $[\![-]\!] \colon \mathsf{C} \to M$; extension to an interpretation of propositions is obtained by induction on the structure of the propositions:

$$\begin{aligned}
[\![m_1 \vee m_2]\!] &= [\![m_1]\!] \vee [\![m_2]\!], & [\![m_1 \wedge m_2]\!] &= [\![m_1]\!] \wedge [\![m_2]\!],\\
[\![m_1 \to m_2]\!] &= [\![m_1]\!] \to [\![m_2]\!], & &\\
[\![\blacklozenge_A (m)]\!] &= \blacklozenge_A ([\![m]\!]), & [\![\Box_A m]\!] &= \Box_A [\![m]\!],\\
[\![\top]\!] &= \top, & [\![\bot]\!] &= \bot,\\
[\![m \cdot q]\!] &= [\![m]\!] \cdot [\![q]\!], & [\![[q]m]\!] &= [[\![q]\!]]\,[\![m]\!]\,.
\end{aligned}$$

Note that we overload the semantic operators $[\![-]\!]$ to serve both for interpretations of actions and for interpretations of propositions; the correct usage can always be determined without difficulty.

5. SEQUENT CALCULUS FOR PROPOSITIONS

5.1. Preliminaries

As in the action logic, we have *propositional contexts* Γ and *propositional items* I (abbreviated to *p–contexts* and *p-items*), generated by the following grammar:

$$\Gamma ::= I \; multiset \qquad\qquad I ::= m \mid \Gamma^A \mid \Gamma \dagger \Theta$$

where Γ^A will be interpreted as $\blacklozenge_A(\bigwedge \Gamma)$, for $\bigwedge \Gamma$ the conjunction of the interpretations of elements in Γ, and $\Gamma \dagger \Theta$ as $(\bigwedge \Gamma) \cdot \bigodot \Theta$, for $\bigodot \Theta$ the composition of the interpretations of elements in Θ. $\Gamma \dagger \Theta$ was in our earlier work written as Γ^Θ; we have followed a referee's suggestion to use an infix binary connective to emphasise that the connective is a structural one. But we have kept Γ^A because A (the name of an agent) is not decomposable into a structure.

The *sizes* of propositions, contexts and items are defined as follows:

$$s(p) = s(\top) = s(\bot) = 1;$$

$$s(m_1 \wedge m_2) = s(m_1 \vee m_2) = s(m_1 \to m_2) = s(m_1) + 1 + s(m_2);$$

$$s(\blacklozenge_A m) = s(\Box_A m) = s(m) + 2;$$

$$s(\Gamma^A) = s(\Gamma \dagger \Theta) = s(\Gamma) + 1;$$

with $s(\Gamma)$ = the sum of the sizes of the items in Γ.

Note that, in contrast to the syntax for action contexts, the *p-contexts* are (finite) *multi-sets* of items, allowing the role of the *Contraction* rule to be made explicit. (It is nevertheless the case that one can prove $q \vdash q \wedge q$ in the action calculus.)

The union of two multi-sets is indicated by a comma, as in Γ, Γ' or (treating an item I as a one element multiset) as in Γ, I. A *p-item* I can either be a proposition m or be a p-context Γ annotated by an agent A, as in [Sadrzadeh and Dyckhoff 2010]; but it can also be a p-context Γ annotated by an a-context Θ.

To express the rules correctly, we need, as in Section 3, some notion of p-context (or p-item) with a hole. There are now two kinds of hole: one for propositions and one for actions, both represented by $[\,]$; we use the notations Δ for a *p-context-with-a-p-hole*, J for a *p-item-with-a-p-hole*, Λ for a *p-context-with-an-a-hole* and K for a *p-item-with-an a-hole*, defined, using mutual recursion, as follows:

$$\Delta ::= \Gamma, J \qquad J ::= [\,] \mid \Delta^A \mid \Delta \dagger \Theta \qquad \Lambda ::= \Gamma, K \qquad K ::= \Gamma \dagger \Sigma \mid \Lambda \dagger \Theta$$

in which we recall from Section 3 that Σ indicates an a-context-with-an-a-hole. As before, we leave to the reader the task of organising the notation for contexts with two holes.

We can now define various applications of something with an appropriate hole to a p-context Γ or an a-context Θ, constructing p-contexts:

$$(\Gamma', J)[\Gamma] = \Gamma', J[\Gamma]$$

$$([\,])[\Gamma] = \Gamma \qquad (\Delta^A)[\Gamma] = \Delta[\Gamma]^A \qquad (\Delta \dagger \Theta)[\Gamma] = \Delta[\Gamma] \dagger \Theta$$

$$(\Gamma', K)[\Theta] = \Gamma', K[\Theta]$$

$$(\Gamma' \dagger \Sigma)[\Theta] = \Gamma' \dagger (\Sigma[\Theta]) \qquad (\Lambda \dagger \Theta')[\Theta] = \Lambda[\Theta] \dagger \Theta'$$

Given p-contexts-with-a-p-hole Δ', Δ, and a p-item-with-a-p-hole J, the *combinations* $\Delta' \circ \Delta$ and $J \circ \Delta$ are defined as follows by mutual recursion on Δ' and J, giving in each case a p-context-with-a-p-hole:

$$(\Gamma, J) \circ \Delta = \Gamma, (J \circ \Delta)$$

$$([\,]) \circ \Delta = \Delta \qquad (\Delta''^A) \circ \Delta = (\Delta'' \circ \Delta)^A \qquad (\Delta'' \dagger \Theta) \circ \Delta = (\Delta'' \circ \Delta) \dagger \Theta$$

And, likewise, given a p-context-with-an-a-hole Λ, a p-item-with-an-a-hole K, and an a-context-with-an-a-hole Σ, the *combinations* $\Lambda \circ \Sigma$ and $K \circ \Sigma$ are defined by mutual recursion on Λ and K, giving in each case a p-context-with-an-a-hole:

$$(\Gamma, K) \circ \Sigma = \Gamma, (K \circ \Sigma) \qquad (\Gamma \dagger \Sigma') \circ \Sigma = \Gamma \dagger (\Sigma' \circ \Sigma) \qquad (\Lambda \dagger \Theta) \circ \Sigma = (\Lambda \circ \Sigma) \dagger \Theta$$

LEMMA 5.1. *Given p-contexts-with-a-p-hole Δ', Δ, a p-item-with-a-p-hole J and a p-context Γ, the following hold:*

$$(\Delta' \circ \Delta)[\Gamma] = \Delta'[\Delta[\Gamma]] \qquad (J \circ \Delta)[\Gamma] = J[\Delta[\Gamma]]$$

LEMMA 5.2. *Given a p-context-with-an-a-hole Λ, an a-context-with-an-a-hole Σ, a p-item-with-an-a-hole K and an a-context Θ, the following hold:*

$$(\Lambda \circ \Sigma)[\Theta] = \Lambda[\Sigma[\Theta]] \qquad (K \circ \Sigma)[\Theta] = K[\Sigma[\Theta]]$$

5.2. Sequents and their Interpretations

Sequents are now of the form $\Gamma \vdash m$. Given the meanings of action items and contexts in a multiplicative `LMAM` Q as defined in Subsection 3.2, and of propositions in an `HAAM`

M acted on by Q as defined in Subsection 4.3, the meanings of propositional items and contexts are obtained by mutual induction on their structure:

$$\begin{aligned}
[\![m]\!] &= \text{as before}\\
[\![\Gamma^A]\!] &= \blacklozenge_A([\![\Gamma]\!])\\
[\![\Gamma \dagger \Theta]\!] &= [\![\Gamma]\!] \cdot [\![\Theta]\!]\\
[\![I_1, \cdots, I_n]\!] &= [\![I_1]\!] \wedge \cdots \wedge [\![I_n]\!]\\
[\![\emptyset]\!] &= \top
\end{aligned}$$

in the course of which we can (again) see the notion of the *formula equivalent* of a p-context or a p-item by ignoring the interpretation of propositions and focusing on the conversion of contexts and items to propositions. A sequent $\Gamma \vdash m$ is *true* in an interpretation $[\![-]\!]$ in M iff $[\![\Gamma]\!] \leq [\![m]\!]$; it is *true* in M iff true in all interpretations in M, and it is *valid* iff true in every `HAAM`.

LEMMA 5.3. *Let* Γ, Γ' *be p-contexts with* $[\![\Gamma]\!] \leq [\![\Gamma']\!]$ *and* Δ *a p-context-with-a-p-hole. Then*

$$[\![\Delta[\Gamma]]\!] \leq [\![\Delta[\Gamma']]\!].$$

PROOF. Routine induction on the structure of Δ (using also a similar result for propositional items-with-a-p-hole). □

LEMMA 5.4. *Let* Θ, Θ' *be a-contexts with* $[\![\Theta]\!] \leq [\![\Theta']\!]$ *and* Λ *a p-context-with-an-a-hole. Then*

$$[\![\Lambda[\Theta]]\!] \leq [\![\Lambda[\Theta']]\!].$$

PROOF. Routine induction on the structure of Λ (using also a similar result for propositional items-with-an-a-hole). □

5.3. Sequent Calculus

Rules from the action calculus. Our sequent calculus for propositions must first incorporate all the initial sequents and all the fifteen rules of the calculus for the action logic.

Propositional variants. We further include the "propositional variants" of the L rules ($\bot L, 1L, \wedge L, \vee L, \bullet L, \blacklozenge_L, \Box L, Dist$ and $Unit$) of the action logic, obtained by replacing any Σ by Λ and the succedent action q by a proposition m; they are as follows:

$$\frac{}{\Lambda[\bot] \vdash m}\ \bot L \qquad \frac{\Lambda[\,] \vdash m}{\Lambda[1] \vdash m}\ 1L \qquad \frac{\Lambda[q_i] \vdash m}{\Lambda[q_1 \wedge q_2] \vdash m}\ \wedge L_i$$

$$\frac{\Lambda[q_1] \vdash m \quad \Lambda[q_2] \vdash m}{\Lambda[q_1 \vee q_2] \vdash m}\ \vee L \qquad \frac{\Lambda[q_1, q_2] \vdash m}{\Lambda[q_1 \bullet q_2] \vdash m}\ \bullet L \qquad \frac{\Lambda[q^A] \vdash m}{\Lambda[\blacklozenge_A q] \vdash m}\ \blacklozenge_A L$$

$$\frac{\Lambda[q] \vdash m}{\Lambda[(\Box_A q)^A] \vdash m}\ \Box_A L \qquad \frac{\Lambda[\Theta^A, \Theta'^A] \vdash m}{\Lambda[(\Theta, \Theta')^A] \vdash m}\ Dist \qquad \frac{\Lambda[\,] \vdash m}{\Lambda[\langle\,\rangle^A] \vdash m}\ Unit$$

Main rules. These are the following:

(1) Initial sequents. In these, p is restricted to being basic:

$$\frac{}{\Gamma, p \vdash p}\ Id \qquad \frac{}{\Delta[\bot] \vdash m}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R$$

(2) Rules for the Heyting algebra operations and the modal operators.

$$\frac{\Delta[m_1, m_2] \vdash m}{\Delta[m_1 \wedge m_2] \vdash m}\ \wedge L \qquad \frac{\Gamma \vdash m_1 \quad \Gamma \vdash m_2}{\Gamma \vdash m_1 \wedge m_2}\ \wedge R$$

$$\frac{\Delta[m_1] \vdash m \quad \Delta[m_2] \vdash m}{\Delta[m_1 \vee m_2] \vdash m}\ \vee L \qquad \frac{\Gamma \vdash m_1}{\Gamma \vdash m_1 \vee m_2}\ \vee R_1 \qquad \frac{\Gamma \vdash m_2}{\Gamma \vdash m_1 \vee m_2}\ \vee R_2$$

$$\frac{\Gamma, m_1 \rightarrow m_2 \vdash m_1 \quad \Delta[\Gamma, m_2] \vdash m}{\Delta[\Gamma, m_1 \rightarrow m_2] \vdash m}\ \rightarrow L \qquad \frac{\Gamma, m_1 \vdash m_2}{\Gamma \vdash m_1 \rightarrow m_2}\ \rightarrow R$$

$$\frac{\Delta[m^A] \vdash m'}{\Delta[\blacklozenge_A m] \vdash m'}\ \blacklozenge_A L \qquad \frac{\Gamma \vdash m}{\Gamma', \Gamma^A \vdash \blacklozenge_A m}\ \blacklozenge_A R$$

$$\frac{\Delta[(\Box_A m, \Gamma)^A, m] \vdash m'}{\Delta[(\Box_A m, \Gamma)^A] \vdash m'}\ \Box_A L \qquad \frac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A m}\ \Box_A R$$

(3) Rules for the dynamic operators.

$$\frac{\Delta[m \dagger q] \vdash m'}{\Delta[m \cdot q] \vdash m'}\ \cdot L \qquad \frac{\Gamma \vdash m \quad \Theta \vdash q}{\Gamma', \Gamma \dagger \Theta \vdash m \cdot q}\ \cdot R$$

$$\frac{\Theta \vdash q \quad \Delta[([q]m, \Gamma) \dagger \Theta, m] \vdash m'}{\Delta[([q]m, \Gamma) \dagger \Theta] \vdash m'}\ DyL \qquad \frac{\Gamma \dagger q \vdash m}{\Gamma \vdash [q]m}\ DyR$$

(4) Structural rules.

$$\frac{\Delta[(\Gamma', \Gamma \dagger \Theta)^A, (\Gamma^A) \dagger (\Theta^A)] \vdash m}{\Delta[(\Gamma', \Gamma \dagger \Theta)^A] \vdash m}\ DyDist \qquad \frac{\Delta[\Gamma \dagger \langle\rangle, \Gamma] \vdash m}{\Delta[\Gamma \dagger \langle\rangle] \vdash m}\ On$$

$$\frac{\Delta[(\Gamma', \Gamma \dagger \Theta) \dagger \Theta', \Gamma \dagger (\Theta, \Theta')] \vdash m}{\Delta[(\Gamma', \Gamma \dagger \Theta) \dagger \Theta'] \vdash m}\ Split \qquad \frac{\Delta[\Gamma \dagger (\Theta, \Theta'), (\Gamma \dagger \Theta) \dagger \Theta'] \vdash m}{\Delta[\Gamma \dagger (\Theta, \Theta')] \vdash m}\ Merge$$

The Γ' in $\blacklozenge_A R$, $\cdot R$, *DyDist* and *Split* is to ensure admissibility of *Weakening*. The rule On augments (if used root-first) a context Γ with the p-item $\Gamma \dagger \langle\rangle$. (Without this rule, one cannot prove $m \cdot 1 \vdash m$.)

Note that, for example, $\blacklozenge_A L$ is included in three forms in the two calculi:

$$\frac{\Sigma[q^A] \vdash q'}{\Sigma[\blacklozenge_A q] \vdash q'}\ \blacklozenge_A L \qquad \frac{\Lambda[q^A] \vdash m}{\Lambda[\blacklozenge_A q] \vdash m}\ \blacklozenge_A L \qquad \frac{\Delta[m^A] \vdash m'}{\Delta[\blacklozenge_A m] \vdash m'}\ \blacklozenge_A L$$

of which the first is from the action logic, the second is one of the propositional variants and the third is one of the main rules. We leave it to the context to disambiguate the provenance of the rule.

As in the action logic, the two indicated occurrences of p in the *Id* rule are *principal* and each right rule has its conclusion's succedent as its *principal formula*. But in addition, the $\blacklozenge_A R$ rule has Γ^A as its *principal item* and Γ' (which is there to ensure admissibility of *Weakening*) as its *parameter*. Similarly, the $\cdot R$ rule has $\Gamma \dagger \Theta$ as its *principal item* and Γ' as its *parameter*.

Each left rule has a *principal item*; these are as usual, except that the $\Box_A L$ rule has the proposition $\Box_A m$ *principal* as well as the principal item $(\Box_A m, \Gamma)^A$, and similarly for DyL. Also, note that the $\Box_A L$ (similarly DyL, $DyDist$, On, $Merge$ and $Split$) rule duplicates the principal item in the conclusion into the premiss (which is to make *Contraction* admissible); in examples, we may omit this duplicated item for simplicity. Some parentheses are to clarify the scope of the annotations and will be dropped when there is no ambiguity.

When I is an item and As is a non-empty list of agents, we define I^{As} recursively so that I^{AB} is just $(I^A)^B$, etc, where the parentheses create a single element multiset from an item. A similar notion defines Θ^{As} where Θ is an action context.

LEMMA 5.5. *For any p-context* Γ *and proposition* m*, the sequent* $\Gamma, m \vdash m$ *is derivable.*

PROOF. Routine induction on the size of m. □

LEMMA 5.6. *The* Weakening *rule* Wk *is admissible:*

$$\frac{\Delta[\Gamma] \vdash m}{\Delta[\Gamma, \Gamma'] \vdash m}\ Wk$$

PROOF. Routine induction on the height of the derivation: see [Sadrzadeh and Dyckhoff 2010] for some of the cases. □

In view of the number of cases, we spell out the proof of invertibility of most of the "main" rules in detail. Some of the other rules are invertible, but we don't need their invertibility in the proof of admissibility of contraction.

LEMMA 5.7. *All the main rules are invertible, except for* $\vee R$*,* $\to L$ *(so far as the first premiss is concerned),* $\blacklozenge_A R$ *and* $\cdot R$*.*

PROOF.

$\wedge L$. Routine, i.e. induction on derivation height. Apart from $\wedge L$, the only rules that can introduce a proposition $m_1 \wedge m_2$ into the antecedent are the initial sequents, $\blacklozenge_A R$ and $\cdot R$. Wherever this is done, the two propositions m_1 and m_2 can be introduced instead.

$\wedge R$. Routine, i.e. induction on derivation height. Apart from $\bot L$ (easily handled), $\wedge R$ is the only rule that can introduce $m_1 \wedge m_2$ into the succedent.

$\vee L$. Similar to $\wedge L$.

$\vee R$. Easy counterexample: we don't have a derivation of $\bot \vee \top \vdash \bot$ but we can derive $\bot \vee \top \vdash \bot \vee \top$.

$\to L$. Routine (as in intuitionistic logic).

$\to R$. Similar to $\wedge R$.

$\blacklozenge_A L$. Similar to $\wedge L$.

$\blacklozenge_A R$. Easy counterexample: we can't derive $\top \vdash \bot$ but we can derive $\bot, \top^A \vdash \blacklozenge_A \bot$. For a more complex example with Γ' empty, consider any model (for which see later) where $\blacklozenge_A$ is the trivial operator that takes its argument to $\bot$.

$\Box_A L$. Using the admissibility of *Weakening*.

$\Box_A R$. Similar to $\wedge R$.

$\cdot L$. Similar to $\wedge L$.

$\cdot R$. Easy counterexample: similar to $\blacklozenge_A R$.

DyL. (For the second premiss) Using the admissibility of *Weakening*.

DyR. Similar to $\wedge R$.

$DyDist$. Using the admissibility of *Weakening*.

On. Using the admissibility of *Weakening*.

$Merge$. Using the admissibility of *Weakening*.

$Split$. Using the admissibility of *Weakening*.

□

Some of these arguments are by admissibility of *Weakening*, where, in order to ensure admissibility of *Contraction*, the principal item has been duplicated into the premiss. In appropriate proofs (such as object-level proofs) we will (for clarity and to save space) often omit this duplication; e.g. we will present as an instance of $DyDist$ something which is really an instance of $DyDist$ together with a Weakening.

LEMMA 5.8. *The* Contraction *rule $Contr$ is admissible:*

$$\frac{\Delta[\Gamma, \Gamma] \vdash m}{\Delta[\Gamma] \vdash m} \; Contr$$

PROOF. This follows (by induction on the number of p-items) from the contractibility of a single duplicated p-item I rather than an entire multiset; this was done in [Sadrzadeh and Dyckhoff 2010], using strong induction on the size of the p-item and a subsidiary induction on the derivation height, together with case analysis (i.e. consideration of the last step of the derivation of the premiss) and some inversions. The same argument still works. Note that we are only interested in rules where the principal item is a p-item rather than an a-item; in particular, our case analysis doesn't need to consider the rules of the action logic or the "propositional variants". The crucial ingredients are these:

(1) Where the item being contracted is not principal, or a left rule duplicates its principal item into a premiss, the argument uses induction on derivation height;
(2) Where the item being contracted is principal, but the rule does not duplicate its principal item into a premiss, each new item in the premiss is of smaller size than the principal item and the rule is invertible—as in $\wedge L$, $\vee L$, the second premiss of $\rightarrow L$, $\blacklozenge_A L$ and $\cdot L$—the argument uses induction on the size of the item.

For example, if the last step of the derivation is

$$\frac{\Delta[m_1 \wedge m_2, m_1, m_2] \vdash m}{\Delta[m_1 \wedge m_2, m_1 \wedge m_2] \vdash m} \; \wedge L$$

then, by invertibility of $\wedge L$ applied to the premiss, $\Delta[m_1, m_2, m_1, m_2] \vdash m$ is derivable and two applications of the inductive hypothesis (the size having been reduced) yield the derivability of $\Delta[m_1, m_2] \vdash m$, from which that of $\Delta[m_1 \wedge m_2] \vdash m$ follows by $\wedge L$. □

PROPOSITION 5.9. *The following generalised and simplified version of* DyDist *is admissible, where* As *is a non-empty list of agents:*

$$\frac{\Delta[(\Gamma^{As}) \dagger \Theta^{As}] \vdash m}{\Delta[(\Gamma', \Gamma \dagger \Theta)^{As}] \vdash m} \; DyDist$$

PROOF. The base case (where the length of *As* is 1) is just the simplified $DyDist$ rule. The general case is handled by induction, using admissibility of *Weakening* and *Contraction*. □

In examples, we may use this generalised version for simplicity.

As an example of a derivation we show that a sequence of $\blacklozenge_A$s preserves an information update by a composite action as follows (in which we use a superfix BA to indicate first an annotation by B and then by A):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{}{q \vdash q}\,Id}{m \dagger q \vdash m \cdot q}\,{\cdot}R}{(m \dagger q)^{B} \vdash \blacklozenge_B(m \cdot q)}\,\blacklozenge_B R}{(m \dagger q)^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q)}\,\blacklozenge_A R \quad \dfrac{\dfrac{\dfrac{}{q' \vdash q'}\,Id}{q'^{B} \vdash \blacklozenge_B q'}\,\blacklozenge_B R}{q'^{BA} \vdash \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_A R}{((m \dagger q)^{BA}) \dagger q'^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,{\cdot}R}{((m \dagger q) \dagger q')^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,DyDist}{(m \dagger (q \bullet q'))^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\bullet L, Merge}{(m \dagger (q \bullet q'))^{BA} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,{\cdot}L}{\blacklozenge_B(m \cdot (q \bullet q'))^{A} \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_B L}{\blacklozenge_A\blacklozenge_B(m \cdot (q \bullet q')) \vdash \blacklozenge_A\blacklozenge_B(m \cdot q) \cdot \blacklozenge_A\blacklozenge_B q'}\,\blacklozenge_A L$$

But we can also have the following (also sound) form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id}{m^{B} \vdash \blacklozenge_B m}\,\blacklozenge_B R}{m^{BA} \vdash \blacklozenge_A\blacklozenge_B m}\,\blacklozenge_A R \quad \dfrac{\dfrac{\dfrac{}{q \bullet q' \vdash q \bullet q'}\,Id}{(q \bullet q')^{B} \vdash \blacklozenge_B(q \bullet q')}\,\blacklozenge_B R}{(q \bullet q')^{BA} \vdash \blacklozenge_A\blacklozenge_B(q \bullet q')}\,\blacklozenge_A R}{(m^{BA}) \dagger (q \bullet q')^{BA} \vdash \blacklozenge_A\blacklozenge_B m \cdot \blacklozenge_A\blacklozenge_B(q \bullet q')}\,{\cdot}R}{(m \dagger (q \bullet q'))^{BA} \vdash \blacklozenge_A\blacklozenge_B m \cdot \blacklozenge_A\blacklozenge_B(q \bullet q')}\,DyDist}{(m \cdot (q \bullet q'))^{BA} \vdash \blacklozenge_A\blacklozenge_B m \cdot \blacklozenge_A\blacklozenge_B(q \bullet q')}\,{\cdot}L}{\blacklozenge_B(m \cdot (q \bullet q'))^{A} \vdash \blacklozenge_A\blacklozenge_B m \cdot \blacklozenge_A\blacklozenge_B(q \bullet q')}\,\blacklozenge_B L}{\blacklozenge_A\blacklozenge_B(m \cdot (q \bullet q')) \vdash \blacklozenge_A\blacklozenge_B m \cdot \blacklozenge_A\blacklozenge_B(q \bullet q')}\,\blacklozenge_A L$$

LEMMA 5.10. *The rules $\bot R^-$ and $\top L^-$ are admissible:*

$$\dfrac{\Gamma \vdash \bot}{\Delta[\Gamma] \vdash m}\,\bot R^- \qquad \dfrac{\Theta \vdash \bot}{\Lambda[\Theta] \vdash m}\,\bot R^- \qquad \dfrac{\Delta[\top] \vdash m}{\Delta[\Gamma] \vdash m}\,\top L^- \qquad \dfrac{\Lambda[\top] \vdash m}{\Lambda[\Theta] \vdash m}\,\top L^-$$

PROOF. Induction on the height of the derivation of the premiss. □

PROPOSITION 5.11. *Let Γ (resp. I) be a p-context (resp. a p-item) and $[\![\Gamma]\!]$ (resp. $[\![I]\!]$) be the* formula equivalent *given in the definition in Section 5.2 of their interpretations. Then $\Gamma \vdash [\![\Gamma]\!]$ and $I \vdash [\![I]\!]$ are derivable.*

PROOF. Routine induction on the structure of Γ and I, using $\top R$ and Lemma 5.5 for the base cases (where Γ is empty and I is a proposition) and $\wedge R$, Proposition 3.7, $\cdot R$ and $\blacklozenge_A R$ for the induction steps. □

5.4. Admissibility of DyCut

THEOREM 5.12. *The following* DyCut *rule is admissible:*

$$\dfrac{\Theta \vdash q \quad \Lambda[q] \vdash m}{\Lambda[\Theta] \vdash m}\,DyCut$$

PROOF. Strong induction on the rank of the cut, where the *rank* is given by the pair: (size of cut formula q, sum of heights of derivations of premisses). We follow the same pattern as our earlier proof.

(1) We begin with the cases of the second premiss being derived by a zero-premiss rule: these are the same cases as before, using in some cases the propositional variants of the earlier rules.
(2) If the cut-formula q is principal in both premisses, we have a *principal cut*, and these are dealt with as before, since the first premiss is the conclusion of a rule from the action calculus and the second is the conclusion of a propositional variant of a rule from the action calculus.
(3) Otherwise, if the first premiss is by a zero-premiss rule, i.e. one of $1R$, Id, $\bot L$ and $\top R$, we proceed exactly as before.
(4) Otherwise, if the cut-formula q is non-principal in the first premiss the cut is permuted into the first premiss, exactly as before.
(5) Otherwise, if it is principal in the first premiss and therefore non-principal in the second premiss, the cut is permuted into the second premiss; of the many examples we present the two most complex:
   (a) The second premiss is derived by $\cdot R$, which for convenience we recall as

$$\frac{\Gamma \vdash m' \quad \Theta' \vdash q'}{\Gamma', \Gamma \dagger \Theta' \vdash m' \cdot q'} \cdot R$$

so we present its three cases:

— The cut formula q is in Θ', i.e. Θ' is of the form $\Sigma'[q]$:

$$\frac{\Theta \vdash q \quad \dfrac{\Gamma \vdash m' \quad \Sigma'[q] \vdash q'}{\Gamma', \Gamma \dagger (\Sigma'[q]) \vdash m' \cdot q'} \cdot R}{\Gamma', \Gamma \dagger (\Sigma'[\Theta]) \vdash m' \cdot q'} \, DyCut$$

transforms (using the admissibility of *Cut* from the action calculus) to

$$\frac{\Gamma \vdash m' \quad \dfrac{\Theta \vdash q \quad \Sigma'[q] \vdash q'}{\Sigma'[\Theta] \vdash q'} \, Cut}{\Gamma', \Gamma \dagger (\Sigma'[\Theta]) \vdash m' \cdot q'} \cdot R$$

— The cut formula q is in Γ, i.e. Γ is of the form $\Lambda[q]$:

$$\frac{\Theta \vdash q \quad \dfrac{\Lambda[q] \vdash m' \quad \Theta' \vdash q'}{\Gamma', \Lambda[q] \dagger \Theta' \vdash m' \cdot q'} \cdot R}{\Gamma', \Lambda[\Theta] \dagger \Theta' \vdash m' \cdot q'} \, DyCut$$

transforms to

$$\frac{\dfrac{\Theta \vdash q \quad \Lambda[q] \vdash m'}{\Lambda[\Theta] \vdash m'} \, DyCut \quad \Theta' \vdash q'}{\Gamma', \Lambda[\Theta] \dagger \Theta' \vdash m' \cdot q'} \cdot R$$

— The cut formula q is in Γ', i.e. Γ' is of the form $\Lambda'[q]$:

$$\frac{\Theta \vdash q \quad \dfrac{\Gamma \vdash m' \quad \Theta' \vdash q'}{\Lambda'[q], \Gamma \dagger \Theta' \vdash m' \cdot q'} \cdot R}{\Lambda'[\Theta], \Gamma \dagger \Theta' \vdash m' \cdot q'} \, DyCut$$

which transforms to

$$\frac{\Gamma \vdash m' \quad \Theta' \vdash q'}{\Lambda'[\Theta], \Gamma \dagger \Theta' \vdash m' \cdot q'} \cdot R$$

(b) The second premiss is derived by DyL, which for convenience we recall as

$$\frac{\Theta' \vdash q' \quad \Delta[([q']m', \Gamma) \dagger \Theta', m'] \vdash m}{\Delta[([q']m', \Gamma) \dagger \Theta'] \vdash m}\ DyL$$

The various cases are as follows:

— The cut-formula q occurs in Δ : routine.

— The cut-formula q occurs in Γ : routine

— The cut-formula q occurs in $\Theta' = \Sigma[q]$: this requires two separate cuts—a *Cut* and a *DyCut*, both on the cut-formula q but of lower rank. The details are as follows, with $\Lambda[] = \Delta[\,([q']m', \Gamma) \dagger (\Sigma[])\,]$:

$$\frac{\Theta \vdash q \quad \dfrac{\Sigma[q] \vdash q' \quad \Delta[([q']m', \Gamma) \dagger \Sigma[q], m'] \vdash m}{\Delta[([q']m', \Gamma) \dagger \Sigma[q]] \vdash m}\ DyL}{\Delta[([q']m', \Gamma) \dagger \Sigma[\Theta]] \vdash m}\ DyCut$$

is transformed to

$$\frac{\dfrac{\Theta \vdash q \quad \Sigma[q] \vdash q'}{\Sigma[\Theta] \vdash q'}\ Cut \quad \dfrac{\Theta \vdash q \quad \Delta[([q']m', \Gamma) \dagger \Sigma[q], m'] \vdash m}{\Delta[([q']m', \Gamma) \dagger \Sigma[\Theta], m'] \vdash m}\ Cut}{\Delta[([q']m', \Gamma) \dagger \Sigma[\Theta]] \vdash m}\ DyL$$

□

5.5. Admissibility of *PrCut*

THEOREM 5.13. *The following* PrCut *rule is admissible:*

$$\frac{\Gamma \vdash m \quad \Delta[m] \vdash m'}{\Delta[\Gamma] \vdash m'}\ PrCut$$

PROOF. Strong induction on the rank of the cut, where the *rank* is given as before. A proof for the fragment without dynamic operators is presented in [Sadrzadeh and Dyckhoff 2010]; we include here the necessary modifications. At the same time we present the structure of the proof in a simplified fashion, as in Section 2, and avoid using invertibility lemmas to reduce all cuts to principal cuts.

Our proof deals with many cases, since there are the following rules to consider: 18 main rules and 9 propositional variants. (The rules of the action calculus cannot appear just above cuts of the $PrCut$ form, but are available for use when we show the result of a cut-reduction step.) However, the propositional variants cannot appear in principal cuts, because their principal formulae are actions rather than propositions.

(1) We begin with the cases of the second premiss being an initial sequent, i.e. by a zero-premiss rule:
   (a) If the rule is Id, then there are two sub-cases: in one case, a weakened form of the first premiss of the cut is already the conclusion and, in the other case, the conclusion is an instance of Id.
   (b) If the rule is one of the two forms of $\bot L$, then the conclusion follows by Lemma 5.10 or by the same form of $\bot L$, according to whether the cut-formula is principal in the $\bot L$ step or not.
   (c) If the rule is $\top R$, then the conclusion follows by $\top R$.
(2) if the cut-formula is principal in both premisses, we have a *principal cut*, and these are dealt with as follows:
   (a) The cut-formula is of the form $m \wedge m'$: routine.

(b) The cut-formula is of the form $m \vee m'$: routine.
(c) The cut-formula is of the form $m \to m'$:

$$\dfrac{\dfrac{\Gamma', m \vdash m'}{\Gamma' \vdash m \to m'}\to R \quad \dfrac{\Gamma, m \to m' \vdash m \quad \Delta[\Gamma, m'] \vdash m''}{\Delta[\Gamma, m \to m'] \vdash m''}\to L}{\Delta[\Gamma, \Gamma'] \vdash m''}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma', m \vdash m'}{\Gamma' \vdash m \to m'}\to R \quad \Gamma, m \to m' \vdash m}{\Gamma, \Gamma' \vdash m}\ PrCut \quad \dfrac{\Gamma', m \vdash m' \quad \Delta[\Gamma, m'] \vdash m''}{\Delta[\Gamma, \Gamma', m] \vdash m''}\ PrCut}{\Delta[\Gamma, \Gamma, \Gamma', \Gamma'] \vdash m''}\ PrCut}{\Delta[\Gamma, \Gamma'] \vdash m''}\ Contr^*.$$

(d) The cut-formula is of the form $\blacklozenge_A q''$: this is dealt with rather like case (10) in [Sadrzadeh and Dyckhoff 2010]—the cut is reduced to a cut on a smaller formula, followed by a Weakening step.
(e) The cut-formula is of the form $\Box_A q''$: this is dealt with as in case (11(h)) in [Sadrzadeh and Dyckhoff 2010]—the cut is reduced to a cut on the same formula with lesser height, a cut on a smaller formula, a Weakening step and a Contraction step.
(f) The cut-formula is of the form $m' \cdot q'$:

$$\dfrac{\dfrac{\Gamma \vdash m' \qquad \Theta \vdash q'}{\Gamma', \Gamma \dagger \Theta \vdash m' \cdot q'}\cdot R \quad \dfrac{\Delta[m' \dagger q'] \vdash m''}{\Delta[m' \cdot q'] \vdash m''}\cdot L}{\Delta[\Gamma', \Gamma \dagger \Theta] \vdash m''}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\Theta \vdash q' \quad \dfrac{\Gamma \vdash m' \quad \Delta[m' \dagger q'] \vdash m''}{\Delta[\Gamma \dagger q'] \vdash m''}\ PrCut}{\Delta[\Gamma \dagger \Theta] \vdash m''}\ DyCut}{\Delta[\Gamma', \Gamma \dagger \Theta] \vdash m''}\ Wk.$$

(g) The cut-formula is of the form $[q]m$:

$$\dfrac{\dfrac{\Gamma \dagger q \vdash m}{\Gamma \vdash [q]m}\ DyR \quad \dfrac{\Theta \vdash q \quad \Delta[([q]m, \Gamma') \dagger \Theta, m] \vdash m'}{\Delta[([q]m, \Gamma') \dagger \Theta] \vdash m'}\ DyL}{\Delta[(\Gamma, \Gamma') \dagger \Theta] \vdash m'}\ PrCut$$

transforms to

$$\dfrac{\dfrac{\dfrac{\dfrac{\Theta \vdash q \quad \Gamma \dagger q \vdash m}{\Gamma \dagger \Theta \vdash m}\ DyCut \quad \dfrac{\Gamma \vdash [q]m \quad \Delta[([q]m, \Gamma') \dagger \Theta, m] \vdash m'}{\Delta[(\Gamma, \Gamma') \dagger \Theta, m] \vdash m'}\ PrCut}{\Delta'[(\Gamma, \Gamma') \dagger \Theta, \Gamma \dagger \Theta] \vdash m'}\ PrCut}{\Delta[(\Gamma, \Gamma') \dagger \Theta, (\Gamma, \Gamma') \dagger \Theta] \vdash m'}\ Wk}{\Delta[(\Gamma, \Gamma') \dagger \Theta] \vdash m'}\ Contr.$$

(3) Otherwise, if the first premiss is by a zero-premiss rule, we have the following cases:
(a) If the rule is Id, we can Weaken the second premiss to obtain the conclusion.
(b) If the rule is $\bot L$, the conclusion of the cut is an instance of $\bot L$.
(c) If the rule is $\top R$, we infer the conclusion from the second premiss by Lemma 5.10.

(4) Otherwise, if the cut-formula is non-principal in the first premiss, the cut is permuted into the first premiss; for example:

$$\dfrac{\dfrac{\Delta[\,(\Gamma',\Gamma\dagger\Theta)\dagger\Theta',\Gamma\dagger(\Theta,\Theta')\,]\vdash m}{\Delta[\,(\Gamma',\Gamma\dagger\Theta)\dagger\Theta'\,]\vdash m}\;Split \qquad \Delta'[m]\vdash m'}{\Delta'[\,\Delta[\,(\Gamma',\Gamma^{\Theta})\dagger\Theta'\,]\,]\vdash m'}\;PrCut$$

transforms to

$$\dfrac{\dfrac{\Delta[\,(\Gamma',\Gamma\dagger\Theta)\dagger\Theta',\Gamma\dagger(\Theta,\Theta')\,]\vdash m \qquad \Delta'[m]\vdash m'}{\Delta'[\,\Delta[(\Gamma',\Gamma\dagger\Theta)\dagger\Theta',\Gamma\dagger(\Theta,\Theta')\,]\,]\vdash m'}\;PrCut}{\Delta'[\,\Delta[\,(\Gamma',\Gamma\dagger\Theta)\dagger\Theta'\,]\,]\vdash m'}\;Split.$$

(5) Otherwise, if it is principal in the first premiss and therefore non-principal in the second premiss, the cut is permuted into the second premiss; for example,

$$\dfrac{\Gamma\vdash m \qquad \dfrac{\Delta[m][\,(\Gamma'',\Gamma'\dagger\Theta)^A,(\Gamma'^A)\dagger\Theta^A\,]\vdash m'}{\Delta[m][\,(\Gamma'',\Gamma'\dagger\Theta)^A\,]\vdash m'}\;DyDist}{\Delta[\Gamma][\,(\Gamma'',\Gamma'\dagger\Theta)^A\,]\vdash m'}\;PrCut$$

transforms to

$$\dfrac{\dfrac{\Gamma\vdash m \qquad \Delta[m][\,(\Gamma'',\Gamma'\dagger\Theta)^A,(\Gamma'^A)\dagger\Theta^A\,]\vdash m'}{\Delta[\Gamma][\,(\Gamma'',\Gamma'\dagger\Theta)^A,(\Gamma'^A)\dagger\Theta^A\,]\vdash m'}\;PrCut}{\Delta[\Gamma][\,(\Gamma'',\Gamma'\dagger\Theta)^A]\,]\vdash m'}\;DyDist$$

□

6. SEMANTICS

6.1. Action Logic

THEOREM 6.1 (SOUNDNESS). *Any derivable sequent of the action logic is valid.*

PROOF. We show that the initial sequents of the sequent calculus of actions are valid and that the rules are truth-preserving.

— Axioms are routine. For example, validity of $\Sigma[\bot]\vdash q$ follows from (0) and (10) of Proposition 2.2.
— The left rules are done by induction on the structure of Σ:
  — $1L$, $\wedge L$, $\vee L$, and $\bullet L$ are routine.
  — $\blacklozenge_A L$, by Lemma 3.3 it is enough to show

$$[\![\blacklozenge_A q]\!]\leq[\![q^A]\!]$$

  which follows by definition of $[\![-]\!]$, since we have $[\![\blacklozenge_A q]\!]=[\![q^A]\!]$.
  — $\square_A L$, by Lemma 3.3 it is enough to show

$$[\![(\square_A q)^A]\!]\leq[\![q]\!].$$

  By definition of $[\![-]\!]$, we have $[\![(\square_A q)^A]\!]=[\![\blacklozenge_A\square_A q]\!]$, from this and (14) in Proposition 2.2 it follows that $[\![(\square_A q)^A]\!]\leq[\![q]\!]$.
  — $Dist$, by Lemma 3.3 it is enough to show

$$[\![(\Theta,\Theta')^A]\!]\leq[\![\Theta^A,\Theta'^A]\!].$$

  By definition of $[\![-]\!]$ and (16) in Definition 2.3 we have

$$[\![(\Theta,\Theta')^A]\!]=\blacklozenge_A[\![\Theta,\Theta']\!]=\blacklozenge_A([\![\Theta]\!]\bullet[\![\Theta']\!])\leq\blacklozenge_A[\![\Theta]\!]\bullet\blacklozenge_A[\![\Theta']\!].$$

By definition of $[\![-]\!]$, for the right hand side we have

$$\blacklozenge_A[\![\Theta]\!] \bullet \blacklozenge_A[\![\Theta']\!] = [\![\Theta^A]\!] \bullet [\![\Theta'^A]\!] = [\![\Theta^A, \Theta'^A]\!].$$

Hence, we obtain $[\![(\Theta, \Theta')^A]\!] \leq [\![\Theta^A, \Theta'^A]\!]$.

— $Unit$, by Lemma 3.3 it is enough to show

$$[\![\langle\ \rangle^A]\!] \leq [\![\ ]\!].$$

By definition of $[\![-]\!]$ and (17) in Definition 2.3 we have

$$[\![\langle\ \rangle^A]\!] = \blacklozenge_A 1 \leq 1 = [\![\langle\ \rangle]\!].$$

— The right rules.

— $\wedge R$, $\vee R$ and $\bullet R$ are routine.

— $\blacklozenge_A R$, we have to show

$$[\![\Theta]\!] \leq [\![q]\!] \quad \text{implies} \quad [\![\Theta^A]\!] \leq [\![\blacklozenge_A q]\!].$$

Assume $[\![\Theta]\!] \leq [\![q]\!]$, by property (3) of Definition 2.1 it follows that $\blacklozenge_A[\![\Theta]\!] \leq \blacklozenge_A[\![q]\!]$, by definition of $[\![-]\!]$ this is equivalent to $[\![\Theta^A]\!] \leq [\![\blacklozenge_A q]\!]$.

— $\Box_A R$, we have to show

$$[\![\Theta^A]\!] \leq [\![q]\!] \quad \text{implies} \quad [\![\Theta]\!] \leq [\![\Box_A q]\!].$$

Assume $[\![\Theta^A]\!] \leq [\![q]\!]$, by definition of $[\![-]\!]$ this is equivalent to $\blacklozenge_A[\![\Theta]\!] \leq [\![q]\!]$, by property (5) of Definition 2.1 this is equivalent to $[\![\Theta]\!] \leq \Box_A[\![q]\!]$, equivalent to $[\![\Theta]\!] \leq [\![\Box_A q]\!]$ by definition of $[\![-]\!]$.

□

THEOREM 6.2 (COMPLETENESS). *Any valid sequent of the action logic is derivable.*

PROOF. We follow the Lindenbaum-Tarski proof method of showing completeness and show the following.

(1) Let $\Theta \vdash q$ be an arbitrary valid sequent and let q' be the formula equivalent of Θ. Trivially, $q' \vdash q$ is valid. Moreover, by Proposition 3.7, $\Theta \vdash q'$ is derivable, and, by admissibility of *Cut*, if $q' \vdash q$ is derivable then so is $\Theta \vdash q$. So it suffices to consider only sequents of the form $q' \vdash q$.

(2) The relation $\cong$ on the set of all actions of the logic, defined by $q' \cong q$ iff $q \vdash q'$ and $q' \vdash q$, is an equivalence relation, i.e. it is reflexive, transitive (by the admissibility of *Cut*) and symmetric; the proof of this step is routine. Let $[q]$ denote the equivalence class of q.

(3) The order relation $\leq$ defined on these equivalence classes as $[q'] \leq [q]$ iff $q' \vdash q$ is a partial order, i.e. reflexive, transitive and anti-symmetric; the proof of this step is also routine.

(4) The operations $\wedge, \vee, \bullet, \blacklozenge_A$, and $\Box_A$ on these equivalence classes (defined in a routine fashion) are well-defined, i.e. the relation $\cong$ is really a congruence relation and we have constructed the *Lindenbaum-Tarski algebra*. The proof is as follows:
   (a) $\wedge, \vee, \bullet$ are routine.
   (b) For $\blacklozenge_A[q] := [\blacklozenge_A q]$ we need to show

$$q \vdash q' \implies \blacklozenge_A q \vdash \blacklozenge_A q'$$

which is proved by the following proof tree

$$\dfrac{\dfrac{q \vdash q'}{q^A \vdash \blacklozenge_A q'}\ \blacklozenge_A R}{\blacklozenge_A q \vdash \blacklozenge_A q'}\ \blacklozenge_A L$$

(c) For $\Box_A[q] := [\Box_A q]$ we need to show

$$q \vdash q' \implies \Box_A q \vdash \Box_A q'$$

which is proved by the following proof tree

$$\dfrac{\dfrac{q \vdash q'}{(\Box_A q)^A \vdash q'}\;\Box_A L}{\Box_A q \vdash \Box_A q'}\;\Box_A R$$

(5) The equivalence classes and their operations form a multiplicative LMAM. We prove this by providing proof trees for axioms of Definitions 2.1 (as presented in Proposition 2.2) and 2.3.

(a) The proof trees for the lattice axioms are routine, as are those for (0) of Proposition 2.2.

(b) The proof tree for one direction of the first half of axiom (1) is as follows (instances of Id refer to lemma 3.4). The proof trees for the other direction and the other half are similar.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\;Id \quad \dfrac{}{q' \vdash q'}\;Id}{q, q' \vdash q \bullet q'}\;\bullet R}{q, q' \vdash (q \bullet q') \vee (q \bullet q'')}\;\vee R \quad \dfrac{\dfrac{\dfrac{}{q \vdash q}\;Id \quad \dfrac{}{q'' \vdash q''}\;Id}{q, q'' \vdash q \bullet q''}\;\bullet R}{q, q'' \vdash (q \bullet q') \vee (q \bullet q'')]}\;\vee R}{q, (q' \vee q'') \vdash (q \bullet q') \vee (q \bullet q'')}\;\vee L}{q \bullet (q' \vee q'') \vdash (q \bullet q') \vee (q \bullet q'')}\;\bullet L$$

(c) The proof trees for the first half of axiom (2) are as follows; the proof trees for the second half are similar.

$$\dfrac{\dfrac{\dfrac{}{q \vdash q}\;Id}{q, 1 \vdash q}\;1L}{q \bullet 1 \vdash q}\;\bullet L \qquad\qquad \dfrac{\dfrac{}{q \vdash q}\;Id \quad \dfrac{}{\vdash 1}\;1R}{q \vdash q \bullet 1}\;\bullet R$$

(d) The proof trees for axioms (3) and (4) are given in parts (4)(b) and (4)(c) above.

(e) Proof trees for axiom (5) are as follows (the $\blacklozenge_A L\,Inv$ and $\Box_A R\,Inv$ rules are the inverses of the $\blacklozenge_A L$ and $\Box_A R$, proven admissible in Lemma 3.5).

$$\dfrac{\dfrac{\blacklozenge_A q \vdash q'}{q^A \vdash q'}\;\blacklozenge_A L\,Inv}{q \vdash \Box_A q'}\;\Box_A R \qquad\qquad \dfrac{\dfrac{q \vdash \Box_A q'}{q^A \vdash q'}\;\Box_A R\,Inv}{\blacklozenge_A q \vdash q'}\;\blacklozenge_A L$$

(f) The proof trees for axioms (16) and (17) are as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\;Id}{q^A \vdash \blacklozenge_A q}\;\blacklozenge_A R \quad \dfrac{\dfrac{}{q' \vdash q'}\;Id}{q'^A \vdash \blacklozenge_A q'}\;\blacklozenge_A R}{q^A, q'^A \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\;\bullet R}{(q, q')^A \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\;Dist}{(q \bullet q')^A \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\;\bullet L}{\blacklozenge_A (q \bullet q') \vdash \blacklozenge_A q \bullet \blacklozenge_A q'}\;\blacklozenge_A L \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\;1R}{\langle\,\rangle^A \vdash 1}\;Unit}{1^A \vdash 1}\;1L}{\blacklozenge_A 1 \vdash 1}\;\blacklozenge_A L$$

It follows that, for any two actions q, q', if $q' \leq q$ is true in all models (or even just in the Lindenbaum-Tarski algebra), then $q' \vdash q$ is derivable. The theorem now follows. $\Box$

6.2. Logic of Propositions

THEOREM 6.3 (SOUNDNESS). *Any derivable sequent of the logic of propositions is valid.*

PROOF. The technique is the same as that used in the previous subsection; full proofs of propositional-only rules (apart from implication, which is straightforward) have been presented in previous work [Sadrzadeh and Dyckhoff 2010], and the proofs of soundness of the dynamic rules are summarised as follows:

— $\cdot L$. By Lemma 5.3 it is enough to show

$$[\![m \cdot q]\!] \leq [\![m \dagger q]\!]$$

which follows by definition of $[\![-]\!]$.

— $\cdot R$. We have to show (in the case where Θ is non-empty)

$$[\![\Gamma]\!] \leq [\![m]\!] \quad \text{and} \quad [\![\Theta]\!] \leq [\![q]\!] \quad \text{implies} \quad [\![\Gamma', \Gamma \dagger \Theta]\!] \leq [\![m \cdot q]\!].$$

Assume $[\![\Gamma]\!] \leq [\![m]\!]$ and $[\![\Theta]\!] \leq [\![q]\!]$; by properties (23) and (24) of Definition 4.2 we obtain $[\![\Gamma]\!] \cdot [\![\Theta]\!] \leq [\![m]\!] \cdot [\![q]\!]$; by definition of $[\![-]\!]$ this is equivalent to $[\![\Gamma \dagger \Theta]\!] \leq [\![m \cdot q]\!]$. The case where the action context is empty is similar but simpler.

— DyL. Suppose that $[\![\Theta]\!] \leq [\![q]\!]$. Then (by definition of $[\![-]\!]$ and properties (24), (23) and (32) in Definition 4.2 and Proposition 4.3)

$$[\![([q]m, \Gamma) \dagger \Theta]\!] = ([\![[q]m]\!] \wedge [\![\Gamma]\!]) \cdot [\![\Theta]\!] \leq ([[\![q]\!]][\![m]\!]) \cdot [\![\Theta]\!] \leq ([[\![q]\!]][\![m]\!]) \cdot [\![q]\!] \leq [\![m]\!]$$

from which the result follows by Lemma 5.3.

— DyR. We have to show

$$[\![\Gamma \dagger q]\!] \leq [\![m]\!] \quad \text{implies} \quad [\![\Gamma]\!] \leq [\![[q]m]\!].$$

Assume $[\![\Gamma \dagger q]\!] \leq [\![m]\!]$; by definition of $[\![-]\!]$ this is equivalent to $[\![\Gamma]\!] \cdot [\![q]\!] \leq [\![m]\!]$, by property (25) of Definition 4.2 this implies $[\![\Gamma]\!] \leq [[\![q]\!]][\![m]\!] = [\![[q]m]\!]$.

— $DyDist$. By Lemma 5.3 it is enough to show

$$[\![(\Gamma', \Gamma \dagger \Theta)^A]\!] \leq [\![(\Gamma^A) \dagger (\Theta^A)]\!].$$

By definition of $[\![-]\!]$, we have $[\![(\Gamma', \Gamma \dagger \Theta)^A]\!] = \blacklozenge_A([\![\Gamma']\!] \wedge [\![\Gamma \dagger \Theta]\!])$; by property (6) of Proposition 3.12 [Sadrzadeh and Dyckhoff 2010], definition of $\wedge$, and property (22) of Definition 4.2 we have

$$\blacklozenge_A([\![\Gamma']\!] \wedge [\![\Gamma \dagger \Theta]\!]) \leq \blacklozenge_A[\![\Gamma']\!] \wedge \blacklozenge_A([\![\Gamma \dagger \Theta]\!]) \leq \blacklozenge_A([\![\Gamma \dagger \Theta]\!]) \leq \blacklozenge_A([\![\Gamma]\!] \cdot [\![\Theta]\!]) = [\![(\Gamma^A) \dagger (\Theta^A)]\!]$$

Hence, we have obtained $[\![(\Gamma', \Gamma \dagger \Theta)^A]\!] \leq [\![(\Gamma^A) \dagger (\Theta^A)]\!]$.

— On. By Lemma 5.3 it is enough to show

$$[\![\Gamma \dagger \langle\rangle]\!] \leq [\![\Gamma]\!]$$

which is trivial, by (21) of Definition 4.2 and the definition of $[\![-]\!]$ in Section 5.2.

— $Split$. By Lemma 5.3 (twice) it is enough to show

$$[\![(\Gamma \dagger \Theta) \dagger \Theta']\!] \leq [\![\Gamma \dagger (\Theta, \Theta')]\!]$$

which follows by definition of $[\![-]\!]$, using Lemma 3.2.

— $Merge$. By Lemma 5.3 it is enough to show

$$[\![\Gamma \dagger (\Theta, \Theta')]\!] \leq [\![(\Gamma \dagger \Theta) \dagger \Theta']\!]$$

which follows similarly.

— Propositional variants of action rules: these are almost identical to the proofs of soundness of the corresponding action rules, presented in Theorem 6.2. The only difference is that here we use Lemma 5.4 rather than Lemma 3.3.

□

THEOREM 6.4 (COMPLETENESS). *Any valid sequent of the logic of propositions is derivable.*

PROOF. We follow the Lindenbaum-Tarski proof method. We have to show that the LT algebra of the propositional-only part of the logic forms a HAAM on which the LT algebra of the action logic acts. The former has been shown in [Sadrzadeh and Dyckhoff 2010], apart from implication, which is straightforward; for the latter we show the following.

(1) Let $\Gamma \vdash m$ be an arbitrary valid sequent and let m' be the formula equivalent of Γ. Trivially, $m' \vdash m$ is valid. Moreover, by Proposition 5.11, $\Gamma \vdash m'$ is derivable, and, by admissibility of *Cut*, if $m' \vdash m$ is derivable then so is $\Gamma \vdash m$. So it suffices to consider only sequents of the form $m' \vdash m$.
(2) The relation $\cong$ defined on the set of propositions of the logic by $m' \cong m$ iff $m' \vdash m$ and $m \vdash m'$ is an equivalence relation and the order relation $\leq$ defined as $\vdash$ on these equivalence classes is a partial order. These are routine.
(3) The operations $\cdot$ and $[\ ]$ defined on the above equivalence classes are well-defined. The proof is as follows ($[m]$ denotes the equivalence class of m, $[q]$ denotes the equivalence class of q (as defined in the completeness proof for the action logic), but $[q]m$ and $[q][m]$ denote the dynamic modality of q on m and of $[q]$ on $[m]$, respectively).
    (a) For $[m] \cdot [q] := [m \cdot q]$ we need to show

$$m \vdash m' \quad \text{and} \quad q \vdash q' \quad \text{implies} \quad m \cdot q \vdash m' \cdot q'$$

    which is proved by the following proof tree

$$\dfrac{\dfrac{m \vdash m' \quad q \vdash q'}{m \dagger q \vdash m' \cdot q'}\ \cdot R}{m \cdot q \vdash m' \cdot q'}\ \cdot L$$

    (b) For $[q][m] := [q]m$ we need to show

$$m \vdash m' \quad \text{and} \quad q' \vdash q \quad \text{implies} \quad [q]m \vdash [q']m'$$

    which is proved by the following proof tree

$$\dfrac{\dfrac{q' \vdash q \quad \dfrac{m \vdash m'}{([q]m) \dagger q', m \vdash m'}\ Wk}{([q]m) \dagger q' \vdash m'}\ DyL}{[q]m \vdash [q']m'}\ DyR$$

(4) The above operations satisfy axioms of definition 4.2.
    — The proof trees for Heyting algebra axioms are routine.

— The proof trees for axiom (20) are as follows :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{}{q \vdash q}\,Id}{m \dagger q \vdash m \cdot q}\,\cdot R \quad \dfrac{}{q' \vdash q'}\,Id}{(m \dagger q) \dagger q' \vdash (m \cdot q) \cdot q'}\,\cdot R}{m \dagger (q, q') \vdash (m \cdot q) \cdot q'}\,Merge}{m \dagger (q \bullet q') \vdash (m \cdot q) \cdot q'}\,\bullet L}{m \cdot (q \bullet q') \vdash (m \cdot q) \cdot q'}\,\cdot L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad \dfrac{\dfrac{}{q \vdash q}\,Id \quad \dfrac{}{q' \vdash q'}\,Id}{q, q' \vdash q \bullet q'}\,\bullet R}{m \dagger (q, q') \vdash m \cdot (q \bullet q')}\,\cdot R}{(m \dagger q) \dagger q' \vdash m \cdot (q \bullet q')}\,Split}{(m \cdot q) \dagger q' \vdash m \cdot (q \bullet q')}\,\cdot L}{(m \cdot q) \cdot q' \vdash m \cdot (q \bullet q')}\,\cdot L$$

— The proof tree for axiom (21) is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{m \dagger \langle\rangle, m \vdash m}\,Id}{m \dagger \langle\rangle \vdash m}\,On}{m \dagger 1 \vdash m}\,1L}{m \cdot 1 \vdash m}\,\cdot L$$

— The proof tree for axiom (22) is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id}{m^A \vdash \blacklozenge_A m}\,\blacklozenge_A R \quad \dfrac{\dfrac{}{q \vdash q}\,Id}{q^A \vdash \blacklozenge_A q}\,\blacklozenge_A R}{(m \dagger q)^A, (m^A) \dagger q^A \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,\cdot R}{(m \dagger q)^A \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,DyDist}{(m \cdot q)^A \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,\cdot L}{\blacklozenge_A (m \cdot q) \vdash \blacklozenge_A m \cdot \blacklozenge_A q}\,\blacklozenge_A L$$

— The proof trees for axioms (23) and (24) are as follows (instances of Id refer to Lemma 5.5)

$$\dfrac{\dfrac{\dfrac{}{m \vdash m}\,Id \quad q \vdash q'}{m \dagger q \vdash m \cdot q'}\,\cdot R}{m \cdot q \vdash m \cdot q'}\,\cdot L \qquad \dfrac{\dfrac{m \vdash m' \quad \dfrac{}{q \vdash q}\,Id}{m \dagger q \vdash m' \cdot q}\,\cdot R}{m \cdot q \vdash m' \cdot q}\,\cdot L$$

$$\dfrac{\dfrac{q \vdash q' \quad \dfrac{}{m \vdash m}\,Id}{([q']m) \dagger q \vdash m}\,DyL}{[q']m \vdash [q]m}\,DyR \qquad \dfrac{\dfrac{\dfrac{}{q \vdash q}\,Id \quad m \vdash m'}{([q]m) \dagger q \vdash m'}\,DyL}{[q]m \vdash [q]m'}\,DyR$$

— The proof trees for axiom (25) are as follows (the $DyR\,Inv$ and $\cdot L\,Inv$ rules are the inverses of the DyR and $\cdot L$ rules, proven admissible in Lemma 5.7)

$$\dfrac{\dfrac{m \vdash [q]m'}{m \dagger q \vdash m'}\,DyR\,Inv}{m \cdot q \vdash m'}\,\cdot L \qquad \dfrac{\dfrac{m \cdot q \vdash m'}{m \dagger q \vdash m'}\,\cdot L\,Inv}{m \vdash [q]m'}\,DyR$$

As before, the argument now is that, if $m' \vdash m$ is a non-derivable primitive sequent, then $m' \leq m$ is not true in the algebra thus constructed (the *Lindenbaum-Tarski algebra*). The theorem now follows. □

7. MODELLING AND REASONING ABOUT SCENARIOS

A *scenario* is based on a set of agents, a set of basic propositions and a set of basic actions, from which compound propositions and actions are generated using the rules given above. A scenario may also have some specific or general features, representing features of the real world and agents' beliefs about these features.

The literature on Epistemic and Dynamic Epistemic Logic describes various different ways of representing these features and beliefs: for instance $\sigma \leq \Box_A \sigma$ (for all A in $\mathcal{A}$) represents that the announcement σ is public, and $\sigma \leq \Box_B 1$ represents that the announcement σ is private to a subset of $\mathcal{A}$ not including B, since 1 represents the "skip" action, as formalized in [Baltag and Moss 2004].

There are situations where some basic actions are inapplicable. For instance, consider a coin toss, where σ represents an honest announcement that the coin has come down tails (denoted by basic proposition T), then σ cannot apply to the basic proposition H that the coin has come down heads; we represent this by $H \leq [\sigma]\bot$. On the other hand, if σ' was a dishonest announcement of tails, made when the coin had actually come down heads, we would have $T \leq [\sigma']\bot$. Likewise, if σ is the action of moving a robot R south, but is inapplicable (e.g. because of an obstacle) when R is in a certain position, then $P \leq [\sigma]\bot$, where P is the proposition that R is in that position.

More generally, we say that a proposition k is in the *kernel* of a basic action σ if, when k is true, the action σ cannot honestly be performed and, when k is false, the action σ cannot dishonestly be performed; we represent this by the inequality $k \cdot \sigma \leq \bot$ or, equivalently, $k \leq [\sigma]\bot$. Note that honesty or dishonesty is a property of the performance of the action rather than of the action itelf or of an agent who performs the action; but in loose terms we often talk of a "dishonest agent" or a "dishonest action", and infer dishonesty of the action (or even its performance) from that of the agent. Note that an announcement may be honest if performed at one moment and dishonest if performed at a different moment. (Reasoning about such matters is not treated here.)

After a basic action, the agents' beliefs are changed by the discarding of the propositions whose update by the action would lead to failure. For instance, if an agent believes that the coin has come down either heads or tails, when in reality it has come down tails, we have that $T \leq \Box_A(H \vee T)$. But after the announcement, σ, that it has come down tails, A will update his belief and we will have $T \leq [\sigma]\Box_A T$. The updated beliefs may be true or false, based on the initial beliefs of the agents. For instance, if agent A believes that the dishonest announcement of tails (made when the coin has actually come down heads) is an honest announcement of tails, that is $\sigma' \leq \Box_A \sigma$, then he would falsely believe in T, that is $H \leq [\sigma']\Box_A T$ whereas in reality we have that H.

We will allow, in a scenario, certain relationships between basic propositions, limited to those of the form that say that two basic propositions are inconsistent or are complementary, as in $p \wedge p' = \bot$ and $\top = p \vee p'$.

Definition 7.1. Given a set $\mathcal{A}$ of "agents", a set B of "basic actions" and a set C of "basic propositions", a *scenario* over $(\mathcal{A}, \mathsf{B}, \mathsf{C})$ is a finite set of inequalities, which we call *assumption inequalities*, of the form:

$$k \leq [\sigma]\bot \quad \text{with } k \in M, \sigma \in \mathsf{B} \tag{1}$$
$$p \leq [\sigma]n \quad \text{with } p \in \mathsf{C}, \sigma \in \mathsf{B}, n \in M \tag{2}$$
$$p \leq \Box_A n \quad \text{with } p \in \mathsf{C},\ A \in \mathcal{A}, n \in M \tag{3}$$
$$\sigma \leq \Box_A w \quad \text{with } \sigma \in \mathsf{B},\ A \in \mathcal{A}, w \in Q \tag{4}$$
$$p \wedge p' \leq \bot \quad \text{with } p, p' \in \mathsf{C} \tag{5}$$
$$\top \leq p \vee p' \quad \text{with } p, p' \in \mathsf{C} \tag{6}$$

Q and M here refer to the sets of actions and propositions generated from the triple $(\mathcal{A}, \mathsf{B}, \mathsf{C})$ as described in earlier sections. Note that the inequalities are particular rather than general, e.g. an inequality (1) does not say that $k \leq [\sigma]\bot$ holds for all k in M and all σ in B but for some particular pair (k, σ).

Inequality (1) says that a basic action σ is inapplicable when a proposition k holds; inequality (2) says that, when basic proposition p is true, then, after the basic action σ, the proposition n is true; inequalities (3) and (4) say (respectively) that, when basic proposition p is true, agent A believes that proposition n is true, and that, when basic action σ happens, agent A believes that action w happens.

A special case of (2) is when $n \equiv p$; this is referred to in [Dyckhoff et al. 2012] as the *stability* of p under σ.

An algebra models a scenario iff these inequalities hold in the algebra. The problem now arises of adding the inequalities to our proof system. Addition of inequalities as initial sequents (e.g. (1) as $k \vdash [\sigma]\bot$) might be an option, but cut-admissibility will then fail. We might add some propositional implications to the antecedent of the problem to be solved; but this doesn't deal with inequalities of type (4), which concern actions rather than propositions, and (in examples) it adds to the sequents clumsy detail that we prefer to hide by incorporation into rules, at the expense of complicating the metatheory slightly.

We solve this problem in a manner similar to that in [Negri and von Plato 1998]: rather than adding initial sequents, we incorporate the inequalities as *assumption rules*, as follows. We give, with numbering as before, the assumption inequality being incorporated, the rule, the rule name and the constraints. There are two rules numbered (4): the second is just a propositional variant of the first. Note that these rules are for arbitrary Γ, Δ, Λ, Σ, q and m but just for the particular A, σ, w, p, k and n mentioned by the assumption inequalities of the scenario, and made explicit in the suffix of the name of the rule.

1	$k \leq [\sigma]\bot$	$\dfrac{\Gamma \vdash k}{\Delta[\Gamma \dagger \sigma] \vdash m}$	$Ker_{(\sigma,k)}$	$k \in M, \sigma \in \mathsf{B}$
2	$p \leq [\sigma]n$	$\dfrac{\Delta[(\Gamma, p) \dagger \sigma, n] \vdash m}{\Delta[(\Gamma, p) \dagger \sigma] \vdash m}$	$PrApp_{(\sigma,p,n)}$	$p \in \mathsf{C}, \sigma \in \mathsf{B}, n \in M$
3	$p \leq \Box_A n$	$\dfrac{\Delta[(\Gamma, p)^A, n] \vdash m}{\Delta[(\Gamma, p)^A] \vdash m}$	$PrApp_{(A,p,n)}$	$p \in \mathsf{C},\ A \in \mathcal{A}, n \in M$
4	$\sigma \leq \Box_A w$	$\dfrac{\Sigma[w] \vdash q}{\Sigma[\sigma^A] \vdash q}$	$AcApp_{(A,\sigma,w)}$	$\sigma \in \mathsf{B},\ A \in \mathcal{A}, w \in Q$
4	$\sigma \leq \Box_A w$	$\dfrac{\Lambda[w] \vdash m}{\Lambda[\sigma^A] \vdash m}$	$AcApp_{(A,\sigma,w)}$	$\sigma \in \mathsf{B},\ A \in \mathcal{A}, w \in Q$
5	$p \wedge p' \leq \bot$	$\overline{\Delta[p, p'] \vdash m}$	$Incon_{(p,p')}$	$p, p' \in \mathsf{C}$
6	$\top \leq p \vee p'$	$\dfrac{\Delta[p, \Gamma] \vdash m \quad \Delta[p', \Gamma] \vdash m}{\Delta[\Gamma] \vdash m}$	$Comp_{(p,p')}$	$p, p' \in \mathsf{C}$

We abbreviate the special case $PrApp_{(\sigma,p,p)}$ by the name $Stab_{(p,\sigma)}$.

LEMMA 7.2. *The calculus with the assumption rules admits* Wk *and* Contr.

THEOREM 7.3. *The calculus with the assumption rules admits* DyCut *and* PrCut.

PROOF. The *Ker* rules have σ basic; the *PrApp* and *AcApp* rules have p [and σ] (resp. σ) basic; the *Incon* rule has p, p' basic; so, no new principal cuts can be formed. The admissibility of the various forms of *Cut* is then just an adaptation of the previous proof, with more permutations but no new principal cuts. A similar method is used in [Negri and von Plato 1998]. □

THEOREM 7.4. *The calculus with the assumption rules for a scenario is sound and (for primitive sequents) complete with respect to the algebras that model the scenario.*

PROOF.

— For soundness, consider any interpretation $[\![-]\!]$ in any algebra that models the scenario; we have to show that the rules are truth-preserving.
  (1) $Ker_{(\sigma,k)}$. Suppose $[\![\Gamma]\!] \leq [\![k]\!]$, then by assumption inequality (1), we have that $[\![\Gamma]\!] \leq [\![[\sigma]\bot]\!]$, which by definition of $[\![-]\!]$ and adjunction is equivalent to $[\![\Gamma \dagger \sigma]\!] \leq [\![\bot]\!]$. By Lemma 5.3 this implies that $[\![\Delta[\Gamma \dagger \sigma]]\!] \leq [\![\Delta[\bot]]\!]$, and since $[\![\Delta[\bot]]\!] \leq [\![m]\!]$, we obtain $[\![\Delta[\Gamma \dagger \sigma]]\!] \leq [\![m]\!]$.
  (2) $PrApp_{(\sigma,p,n)}$. By assumption inequality (2), we have that $[\![p]\!] \leq [\![[\sigma]n]\!]$, which by definition of $[\![-]\!]$ and adjunction is equivalent to $[\![p]\!] \cdot [\![\sigma]\!] \leq [\![n]\!]$. By definition of $\wedge$, from this we obtain $[\![\Gamma]\!] \cdot [\![\sigma]\!] \wedge [\![p]\!] \cdot [\![\sigma]\!] \leq [\![n]\!]$. From this, by transitivity and Property (27), Proposition 4.3, we obtain that $[\![\bigwedge \Gamma \wedge p]\!] \cdot [\![\sigma]\!] \leq [\![n]\!]$, equivalent to $[\![(\Gamma, p) \dagger \sigma]\!] \leq [\![n]\!]$, hence also that $[\![(\Gamma, p) \dagger \sigma]\!] \leq [\![(\Gamma, p) \dagger \sigma, n]\!]$. From this by Lemma 5.3, we obtain $[\![\Delta[(\Gamma, p) \dagger \sigma]]\!] \leq [\![\Delta[(\Gamma, p) \dagger \sigma, n]]\!]$, and hence that $[\![\Delta[(\Gamma, p) \dagger \sigma, n]]\!] \leq [\![m]\!]$ implies that $[\![\Delta[(\Gamma, p) \dagger \sigma]]\!] \leq [\![m]\!]$.
  (3) $PrApp_{(A,p,n)}$ and $AcApp$. Similar to the above.
  (4) $Incon_{(p,p')}$. Follows from assumption inequality (5), Lemma 5.3 and rule $\bot L$.
  (5) $Comp_{(p,p')}$. Follows from assumption inequality (6), rule $\vee L$, and admissible rule $\top L^-$ of Lemma 5.10.

— For completeness, we have to show that the Lindenbaum-Tarski algebra of the calculus with the assumption rules forms an algebra which satisfies the assumption inequalities.
  (1) The proof tree for the assumption inequality (1) is as follows:

$$\dfrac{\dfrac{\dfrac{}{k \vdash k}\ Id}{k \dagger \sigma \vdash \bot}\ Ker_{(k,\sigma)}}{k \vdash [\sigma]\bot}\ DyR$$

  (2) The proof tree for assumption inequality (2) is as follows:

$$\dfrac{\dfrac{\dfrac{}{p \dagger \sigma, n \vdash n}\ Id}{p \dagger \sigma \vdash n}\ PrApp_{(\sigma,p,n)}}{p \vdash [\sigma]n}\ DyR$$

  (3) The proof tree for assumption inequality (3) is as follows; that for (4) is similar.

$$\dfrac{\dfrac{\dfrac{}{p^A, n \vdash n}\ Id}{p^A \vdash n}\ PrApp_{(A,p,n)}}{p \vdash \Box_A n}\ \Box_A R$$

  (4) The proof tree for assumption inequality (5) is just an application of $\wedge L$ and $Incon_{(p,p')}$ rules.

(5) The proof tree for assumption inequality (6) is as follows:

$$\dfrac{\dfrac{\overline{p, \top \vdash p}\ Id}{p, \top \vdash p \vee p'}\ \vee R_1 \qquad \dfrac{\overline{p', \top \vdash p'}\ Id}{p', \top \vdash p \vee p'}\ \vee R_2}{\top \vdash p \vee p'}\ Comp_{(p,p')}$$

□

8. EXAMPLE SCENARIOS

8.1. Drinking logicians

An illustrative scenario is that of a bar in which three logicians are asked by a barman "Do you all want a beer?". The first logician answers "I don't know", as does the second. The third answers "Yes". We formalise the reasoning that, if the third logician wants a beer, then, after the first two answers, she knows that everyone wants a beer and so can answer "Yes".

We use B_i for the basic proposition that logician i wants a beer, and $\overline{B}_i$ for the basic proposition that she doesn't.

The basic action σ_1 is used for the answer by logician 1 that she doesn't know; similarly for σ_2. The action σ_1 is inapplicable if logician 1 doesn't want a beer, because in that case the honest answer would be "No". Similarly for σ_2; so we have two inequalities of form (1): $\overline{B}_i \leq [\sigma_i]\bot$ for $i = 1, 2$. No announcement changes anyone's wish to have a beer or not: so, for $i = 1, 2$ and $j = 1, 2, 3$ we have inequalities of type (2): $B_i \leq [\sigma_j]B_i$ and $\overline{B}_i \leq [\sigma_j]\overline{B}_i$. Each logician knows her own preference about having a beer; so, for $i = 1, 2, 3$ we have inequalities of type (3): $B_i \leq \Box_i B_i$ and $\overline{B}_i \leq \Box_i \overline{B}_i$. Each logician hears the other logicians' answers and trusts them to be honest; so, for each $i = 1, 2$ and $j = 1, 2, 3$ we have an inequality of type (4): $\sigma_i \leq \Box_j \sigma_i$. We include inequalities of the form (5), which we won't use, to say that B_i and $\overline{B}_i$ are inconsistent; and those of the form (6), to say, for each i, that B_i and $\overline{B}_i$ are complementary: $\top \leq B_i \vee \overline{B}_i$.

The problem is then formalised as the sequent $B_3 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)$.

Using inequalities of form (6), we can expand this to the three sequents

$$B_1, \overline{B}_2, B_3 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)$$

$$\overline{B}_1, B_3 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)$$

$$B_1, B_2, B_3 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)$$

By weakening, we can replace the first by

$$\overline{B}_2 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)$$

for which a proof is as follows:

$$\dfrac{\dfrac{\dfrac{\overline{\overline{B}_2 \vdash \overline{B}_2}\ Id}{(\overline{B}_2) \dagger \sigma_2 \vdash \Box_3(B_1 \wedge B_2 \wedge B_3)}\ Ker_{(\sigma_2, \overline{B}_2)}}{(\overline{B}_2) \dagger \sigma_1 \wedge \sigma_2 \vdash \Box_3(B_1 \wedge B_2 \wedge B_3)}\ \wedge L}{\overline{B}_2 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)}\ DyR$$

Similarly, the second sequent can be proved, and we are left with the third, done as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{ThreePremisses}{(B_1,B_2,B_3)^3 \dagger (\sigma_1 \wedge \sigma_2)^3 \vdash B_1 \wedge B_2 \wedge B_3}\ \wedge R}{((B_1,B_2,B_3) \dagger (\sigma_1 \wedge \sigma_2))^3 \vdash B_1 \wedge B_2 \wedge B_3}\ DyDist}{(B_1,B_2,B_3) \dagger (\sigma_1 \wedge \sigma_2) \vdash \Box_3(B_1 \wedge B_2 \wedge B_3)}\ \Box_3 R}{B_1,B_2,B_3 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(B_1 \wedge B_2 \wedge B_3)}\ DyR$$

The proof of the first premiss is :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((B_1)^3,B_1) \dagger \sigma_1, B_1 \vdash B_1}\ Id}{((B_1)^3,B_1) \dagger \sigma_1 \vdash B_1}\ Stab_{(B_1,\sigma_1)} \quad \dfrac{\dfrac{}{(B_1)^3,\overline{B}_1 \vdash \overline{B}_1}\ Id}{((B_1)^3,\overline{B}_1) \dagger \sigma_1 \vdash B_1}\ Ker_{(\sigma_1,\overline{B}_1)}}{((B_1)^3) \dagger \sigma_1 \vdash B_1}\ Comp_{(B_1,\overline{B}_1)}}{((B_1)^3) \dagger (\sigma_1)^3 \vdash B_1}\ AcApp_{(3,\sigma_1,\sigma_1)}}{((B_1)^3) \dagger (\sigma_1 \wedge \sigma_2)^3 \vdash B_1}\ \wedge L}{((B_1,B_2,B_3)^3) \dagger (\sigma_1 \wedge \sigma_2)^3 \vdash B_1}\ Wk$$

and the second is done similarly, discarding σ_2 rather than σ_1. As for the third one

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(B_3) \dagger \sigma_1, B_3 \vdash B_3}\ Id}{(B_3) \dagger \sigma_1 \vdash B_3}\ Stab_{(B_3,\sigma_1)}}{(B_3) \dagger (\sigma_1)^3 \vdash B_3}\ AcApp_{(3,\sigma_1,\sigma_1)}}{((B_3)^3) \dagger (\sigma_1)^3 \vdash B_3}\ PrApp_{(3,B_3,B_3)}}{((B_3)^3) \dagger (\sigma_1 \wedge \sigma_2)^3 \vdash B_3}\ \wedge L}{((B_1,B_2,B_3)^3) \dagger (\sigma_1 \wedge \sigma_2)^3 \vdash B_3}\ Wk$$

Using assumption inequality (4), we can also model noisy variants of this scenario, where, for instance, logician 3 mishears logician 2 and believes that she answers "No". As a result, logician 3 will believe that logician 2 doesn't want to drink beer, hence will also answer "No", leading to no-one drinking, although everyone wants to. In this case, the basic action ν_2 is used for the "No" answer of logician 2; we have an inequality of type (1): $B_2 \le [\nu_2]\bot$, some inequalities of type (2), similar to the ones specified above, and an inequality of type (4): $\sigma_2 \le \Box_3\nu_2$. The problem is then formalised as the sequent $B_3 \vdash [\sigma_1 \wedge \sigma_2]\Box_3(\overline{B}_1 \wedge \overline{B}_2 \wedge B_3)$. The proof is similar to the above, with the application of $AcApp_{(3,\sigma_2,\nu_2)}$ in the last branch.

8.2. Honest announcements by dirty children

Another illustrative scenario is that of the dirty (aka "muddy") children. It goes as follows: n children are playing in the mud and k of them have dirty foreheads. Each child can see the other children's foreheads, but cannot see his own. Their father announces to them "At least one of you has a dirty forehead." and then asks them "Do you know that it is you who has a dirty forehead?". After $k-1$ rounds of honest "No" answers by all the children, the dirty ones learn that they are dirty (but the clean ones are still unsure). We can also reason about dishonest variants of the puzzle. For instance, if the father is a liar and announces that at least one child is dirty even if there is none, then all the children will wrongly believe that they are dirty. Or it may be that the dirty children are liars and after the first $k-1$ honest answers they lie that they still do not know that they are dirty; as a result of this the clean children wrongly believe that they are dirty.

For a formalization of the honest version, assume the children are enumerated and the first k are dirty. Consider the basic propositions D_i for child i is dirty and C_i for child i is clean. We denote father's initial announcement by basic action σ and children's "No" answers by basic action σ'.

Father's initial announcement σ cannot happen when there are no dirty children; hence we have the inequality of the form (1): $\wedge_{i=1}^{n} C_i \leq [\sigma]\bot$. Children's "No" answers cannot happen if any of them knows that they are dirty; hence we also have the inequalities of form (1): $\Box_i D_i \leq [\sigma']\bot$.

The basic propositions are stable under all the basic actions, i.e. we have inequalities of form (2): $p \leq [\sigma]p$ and $p \leq [\sigma']p$ for p in $\{D_i, C_i\}$.

For $i \neq j$, we have that child i can see the state of child j's forehead; so we have inequalities of form (3): $D_j \leq \Box_i D_j$ and $C_j \leq \Box_i C_j$.

The basic actions are honest public announcements; so we have inequalities of form (4): $\sigma \leq \Box_i \sigma$ and $\sigma' \leq \Box_i \sigma'$.

We have simple relationships between them, exactly those of the form (5): $D_i \wedge C_i \leq \bot$. and (6): $\top \leq D_i \vee C_i$.

Translating these inequalities into rules, we have the following (with, in the $PrApp$ rules, $i \neq j$):

$$\frac{\Gamma \vdash \wedge_{i=1}^{n} C_i}{\Delta[\Gamma \dagger \sigma] \vdash m}\ Ker_{(\sigma, \wedge_{i=1}^{n} C_i)} \qquad \frac{\Gamma \vdash \vee_{i=1}^{n} \Box_i D_i}{\Delta[\Gamma \dagger \sigma'] \vdash m}\ Ker_{(\sigma', \vee_{i=1}^{n} \Box_i D_i)}$$

$$\frac{\Delta[(\Gamma, D_i) \dagger \sigma, D_i] \vdash m}{\Delta[(\Gamma, D_i) \dagger \sigma] \vdash m}\ Stab_{(D_i, \sigma)} \qquad \frac{\Delta[(\Gamma, C_i) \dagger \sigma, C_i] \vdash m}{\Delta[(\Gamma, C_i) \dagger \sigma] \vdash m}\ Stab_{(C_i, \sigma)}$$

$$\frac{\Delta[(\Gamma, D_i) \dagger \sigma', D_i] \vdash m}{\Delta[(\Gamma, D_i) \dagger \sigma'] \vdash m}\ Stab_{(D_i, \sigma')} \qquad \frac{\Delta[(\Gamma, C_i) \dagger \sigma', C_i] \vdash m}{\Delta[(\Gamma, C_i) \dagger \sigma'] \vdash m}\ Stab_{(C_i, \sigma')}$$

$$\frac{\Delta[(\Gamma, D_j)^i, D_j] \vdash m}{\Delta[(\Gamma, D_j)^i] \vdash m}\ PrApp_{(i, D_j, D_j)} \qquad \frac{\Delta[(\Gamma, C_j)^i, C_j] \vdash m}{\Delta[(\Gamma, C_j)^i] \vdash m}\ PrApp_{(i, C_j, C_j)}$$

$$\frac{\Delta[\Gamma \dagger \sigma] \vdash m}{\Delta[\Gamma \dagger \sigma^i] \vdash m}\ AcApp_{(i, \sigma, \sigma)} \qquad \frac{\Delta[\Gamma \dagger \sigma'] \vdash m}{\Delta[\Gamma \dagger \sigma'^i] \vdash m}\ AcApp_{(i, \sigma', \sigma')}$$

$$\frac{}{\Delta[D_i, C_i] \vdash m}\ Incon_{(D_i, C_i)} \qquad \frac{\Delta[\Gamma, D_i] \vdash m \quad \Delta[\Gamma, C_i] \vdash m}{\Delta[\Gamma] \vdash m}\ Comp_{(D_i, C_i)}$$

For reasons of space, we consider just the case when $n = 3$ and $k = 2$. Below is the proof tree for the property that, after one round of "No" answers, the dirty child $j = 1$

knows that she is dirty.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D_1, D_1 \dagger \sigma' \vdash D_1}\,Id}{D_1 \dagger \sigma' \vdash D_1}\,Stab(D_1,\sigma')}{(D_1,(D_1,(D_1,D_2,C_3)^1)\dagger\sigma)\dagger\sigma'\vdash D_1}\,Wk}{((D_1,(D_1,D_2,C_3)^1)\dagger\sigma)\dagger\sigma'\vdash D_1}\,Stab(D_1,\sigma)}{(D_1,(D_1,D_2,C_3)^1))\dagger(\sigma,\sigma')\vdash D_1}\,Merge \qquad \dfrac{\dots}{(C_1,(D_1,D_2,C_3)^1)\dagger(\sigma,\sigma')\vdash D_1}}{((D_1,D_2,C_3)^1)\dagger(\sigma,\sigma')\vdash D_1}\,Comp_{(D_1,C_1)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((D_1,D_2,C_3)^1)\dagger(\sigma,\sigma')\vdash D_1}{((D_1,D_2,C_3)^1)\dagger(\sigma,\sigma'^1)\vdash D_1}\,AcApp(1,\sigma',\sigma')}{((D_1,D_2,C_3)^1)\dagger(\sigma^1,\sigma'^1)\vdash D_1}\,AcApp(1,\sigma,\sigma)}{((D_1,D_2,C_3)^1)\dagger(\sigma,\sigma')^1\vdash D_1}\,Dist}{((D_1,D_2,C_3)\dagger(\sigma,\sigma'))^1\vdash D_1}\,DyDist}{(D_1,D_2,C_3)\dagger(\sigma,\sigma')\vdash \Box_1 D_1}\,\Box_1 R}{(D_1,D_2,C_3)\dagger(\sigma\bullet\sigma')\vdash \Box_1 D_1}\,\bullet L}{D_1,D_2,C_3\vdash[\sigma\bullet\sigma']\Box_1 D_1}\,DyR}{D_1\wedge D_2\wedge C_3\vdash[\sigma\bullet\sigma']\Box_1 D_1}\,\wedge L$$

where the second premiss is derived thus:

$$\dfrac{\dfrac{\dfrac{}{D_2,(D_2,(C_1,(D_1,D_2,C_3)^1)^2)\dagger\sigma\vdash D_2}\,Id}{(D_2,(C_1,(D_1,D_2,C_3)^1)^2)\dagger\sigma\vdash D_2}\,Stab(D_2,\sigma) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C_1,\dots\vdash C_1}\,Id \quad \dfrac{\dfrac{}{C_2,\dots\vdash C_2}\,Id \quad \dfrac{}{\dots,C_3\vdash C_3}\,Id}{C_2,C_1,C_3,\dots\vdash C_2\wedge C_3}\,\wedge R}{C_2,C_1,C_3,\dots\vdash \wedge_{i=1}^3 C_i}\,\wedge R}{C_2,C_1,(C_1,C_3,(D_1,D_2,C_3)^1)^2\vdash\wedge_{i=1}^3 C_i}\,PrApp(2,C_3,C_3)}{C_2,(C_1,C_3,(D_1,D_2,C_3)^1)^2\vdash\wedge_{i=1}^3 C_i}\,PrApp(2,C_1,C_1)}{C_2,(C_1,(D_1,D_2,C_3)^1)^2\vdash\wedge_{i=1}^3 C_i}\,PrApp(1,C_3,C_3)}{(C_2,(C_1,(D_1,D_2,C_3)^1)^2)\dagger\sigma\vdash D_2}\,Ker(\sigma,\wedge_{i=1}^3 C_i)}{((C_1,(D_1,D_2,C_3)^1)^2)\dagger\sigma\vdash D_2}\,Comp_{(D_2,C_2)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{((C_1,(D_1,D_2,C_3)^1)^2)\dagger\sigma\vdash D_2}{((C_1,(D_1,D_2,C_3)^1)^2)\dagger(\sigma^2)\vdash D_2}\,AcApp(2,\sigma,\sigma)}{((C_1,(D_1,D_2,C_3)^1)\dagger\sigma)^2\vdash D_2}\,DyDist}{(C_1,(D_1,D_2,C_3)^1)\dagger\sigma\vdash\Box_2 D_2}\,\Box_2 R}{(C_1,(D_1,D_2,C_3)^1)\dagger\sigma\vdash\Box_1 D_1\vee\Box_2 D_2\vee\Box_3 D_3}\,\vee R2}{((C_1,(D_1,D_2,C_3)^1)\dagger\sigma)\dagger\sigma'\vdash D_1}\,Ker(\sigma',\vee_{i=1}^3\Box_i D_i))}{(C_1,(D_1,D_2,C_3)^1)\dagger(\sigma,\sigma')\vdash D_1}\,Merge$$

8.3. Dishonest announcement by father of dirty children

Consider a variant of the scenario, where no child is dirty but the father dishonestly announces that at least one child is dirty and the children believe him to be honest; we denote this dishonest announcement by the basic action $\overline{\sigma}$, in contrast to the honest announcement σ (not actually made, but with the same content). We have two inequalities of type (1): $\wedge_{i=1}^3 C_i \leq [\sigma]\bot$ and $\vee_{i=1}^3 D_i \leq [\overline{\sigma}]\bot$, two pairs of families of inequalities of type (2): $D_i \leq [\sigma]D_i$, $C_i \leq [\sigma]C_i$, $D_i \leq [\overline{\sigma}]D_i$ and $C_i \leq [\overline{\sigma}]C_i$, one family of inequalities of type (3): $C_j \leq \Box_i C_j$, and one family of inequalities of type (4): $\overline{\sigma} \leq \Box_i \sigma$; these lead respectively to the rules $Ker_{(\sigma,\wedge_{i=1}^3 C_i)}$, $Ker_{(\overline{\sigma},\vee_{i=1}^3 D_i)}$, $Stab_{(D_i,\sigma)}$ and $Stab_{(C_i,\sigma)}$ for

$i = 1, \ldots, 3$ (and the same for $\overline{\sigma}$), $PrApp_{(i,C_j,C_j)}$ for $i, j = 1, \ldots, 3$ and $AcApp_{(i,\overline{\sigma},\sigma)}$ for $i = 1, \ldots, 3$.

Here is a proof that, if none of the children are dirty, then, after father's lie that at least one child is dirty, the clean child $i = 2$ believes that she is dirty.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((C_1,C_2,C_3)^2,D_2)\dagger\sigma, D_2 \vdash D_2}\,Id}{((C_1,C_2,C_3)^2,D_2)\dagger\sigma \vdash D_2}\,Stab(D_2,\sigma) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{C_1,\ldots \vdash C_1}\,Id \quad \dfrac{}{C_2,\ldots \vdash C_2}\,Id \quad \dfrac{}{\ldots,C_3 \vdash C_3}\,Id}{C_2,C_1,C_3 \vdash C_1 \wedge C_2 \wedge C_3}\,\wedge R}{(C_1,C_2,C_3)^2, C_2, C_1, C_3 \vdash C_1 \wedge C_2 \wedge C_3}\,Wk}{(C_1,C_2,C_3)^2, C_2, C_1 \vdash C_1 \wedge C_2 \wedge C_3}\,PrApp(2,C_3,C_3)}{(C_1,C_2,C_3)^2, C_2 \vdash C_1 \wedge C_2 \wedge C_3}\,PrApp(2,C_1,C_1)}{((C_1,C_2,C_3)^2, C_2)\dagger\sigma \vdash D_2}\,Ker(\sigma,\wedge C_i)}{((C_1,C_2,C_3)^2)\dagger\sigma \vdash D_2}\,Comp_{(D_2,C_2)}}{((C_1,C_2,C_3)^2)\dagger\overline{\sigma}^2 \vdash D_2}\,AcApp(2,\overline{\sigma},\sigma)}{((C_1,C_2,C_3)\dagger\overline{\sigma})^2 \vdash D_2}\,DyDist}{(C_1,C_2,C_3)\dagger\overline{\sigma} \vdash \Box_2 D_2}\,\Box_2 R}{C_1,C_2,C_3 \vdash [\overline{\sigma}]\Box_2 D_2}\,DyR}{C_1 \wedge C_2 \wedge C_3 \vdash [\overline{\sigma}]\Box_2 D_2}\,\wedge L$$

8.4. Dishonest announcements by dirty children

Consider a second dishonest variant with a new basic action $\overline{\sigma'}$. In this action, the k dirty children dishonestly say "No" and the clean children honestly say "No". The dirty children are believed by the clean children to be honest. We extend the details of the scenario from Subsection 8.2. We have two new inequalities of type (1): $\vee_{j=1}^{k} \neg\Box_j D_j \leq [\overline{\sigma'}]\bot$, representing that the action $\overline{\sigma'}$ is inapplicable if any dirty child does not believe she is dirty, and $\vee_{l=k+1}^{n} \Box_l D_l \leq [\overline{\sigma'}]\bot$, representing that the action $\overline{\sigma'}$ is inapplicable if any clean child does believe she is dirty. We also have two families of inequalities of type (2): $D_i \leq [\overline{\sigma'}]D_i$ and $C_i \leq [\overline{\sigma'}]C_i$ for $i = 1, \ldots, n$, representing that the announcement $\overline{\sigma'}$ is (like σ and σ') epistemic, i.e. has no effect on the cleanliness of children, and two families of inequalities of type (4): $\overline{\sigma'} \leq \Box_j \overline{\sigma'}$ (for $j \leq k$) and $\overline{\sigma'} \leq \Box_l \sigma'$ (for $l > k$), representing that dirty children take account of their own dishonesty but the clean children believe, after the dishonest announcement $\overline{\sigma'}$, just what they would have believed if the honest announcement σ' had been made. We thus get, *inter alia*, the following new assumption rules (for $i = 1, \ldots, n$, for $j = 1, \ldots, k$ and for $l = k+1, \ldots, n$) :

$$\dfrac{\Gamma \vdash \vee_{j=1}^{k} \neg\Box_j D_j}{\Delta[\Gamma \dagger \overline{\sigma'}] \vdash m}\ Ker(\overline{\sigma'}, \vee_{j=1}^{k} \neg\Box_j D_j) \qquad \dfrac{\Gamma \vdash \vee_{l=k+1}^{n} \Box_l D_l}{\Delta[\Gamma \dagger \overline{\sigma'}] \vdash m}\ Ker(\overline{\sigma'}, \vee_{l=k+1}^{n} \Box_l D_l)$$

$$\dfrac{\Delta[(\Gamma, D_i) \dagger \overline{\sigma'}, D_i] \vdash m}{\Delta[(\Gamma, D_i) \dagger \overline{\sigma'}] \vdash m}\ Stab_{(D_i,\overline{\sigma'})} \qquad \dfrac{\Delta[(\Gamma, C_i) \dagger \overline{\sigma'}, C_i] \vdash m}{\Delta[(\Gamma, C_i) \dagger \overline{\sigma'}] \vdash m}\ Stab_{(C_i,\overline{\sigma'})}$$

$$\dfrac{\Delta[\Gamma \dagger \overline{\sigma'}] \vdash m}{\Delta[\Gamma \dagger \overline{\sigma'}^{j}] \vdash m}\ AcApp_{(j,\overline{\sigma'},\overline{\sigma'})} \qquad \dfrac{\Delta[\Gamma \dagger \sigma'] \vdash m}{\Delta[\Gamma \dagger \overline{\sigma'}^{l}] \vdash m}\ AcApp_{(l,\overline{\sigma'},\sigma')}$$

Here is the proof tree that (with $k = 2$), after father's announcement σ, a round σ' of honest "No" answers , and a round $\overline{\sigma'}$ of partially dishonest "No" answers, the clean child $i = 3$ believes that she is dirty. The rest of the left branch is identical to the proof tree of the honest version (excluding its first two steps).

$$
\begin{array}{ll}
\hline
D_3, (D_3, D_3 \dagger \sigma') \dagger \sigma' \vdash D_3 & Id \\
\hline
(D_3, D_3 \dagger \sigma') \dagger \sigma' \vdash D_3 & Stab(D_3, \sigma') \\
\hline
(D_3 \dagger \sigma') \dagger \sigma' \vdash D_3 & Stab(D_3, \sigma') \\
\hline
D_3 \dagger (\sigma', \sigma') \vdash D_3 & Merge \\
\hline
(((D_1, D_2, D_3)^3, D_3) \dagger \sigma, D_3) \dagger (\sigma', \sigma') \vdash D_3 & Wk \\
\hline
(((D_1, D_2, D_3)^3, D_3) \dagger \sigma) \dagger (\sigma', \sigma') \vdash D_3 & Stab(D_3, \sigma) \\
\hline
((D_1, D_2, C_3)^3, D_3) \dagger (\sigma, \sigma', \sigma') \vdash D_3 & Merge
\end{array}
\qquad
\begin{array}{ll}
\hline
((D_1, D_2, C_3)^3, C_3) \dagger (\sigma, \sigma') \vdash \Box_1 D_1 & (*) \\
\hline
(((D_1, D_2, C_3)^3, C_3) \dagger (\sigma, \sigma') \vdash \Box_1 D_1 \vee \Box_2 D_2 & \vee R_1 \\
\hline
(((D_1, D_2, C_3)^3, C_3) \dagger (\sigma, \sigma')) \dagger \sigma' \vdash D_3 & Ker_{(\sigma', \vee_{i=1}^{2} \Box_i D_i)} \\
\hline
((D_1, D_2, C_3)^3, C_3) \dagger (\sigma, \sigma', \sigma') \vdash D_3 & Merge
\end{array}
$$

$$
\begin{array}{ll}
\hline
((D_1, D_2, C_3)^3) \dagger (\sigma, \sigma', \sigma') \vdash D_3 & Comp_{(D_3, C_3)} \\
\hline
((D_1, D_2, C_3)^3) \dagger (\sigma^3, \sigma'^3, \overline{\sigma'}^3) \vdash D_3 & AcApp_{(3,\sigma,\sigma)}, AcApp_{(3,\sigma',\sigma')}, AcApp_{(3,\overline{\sigma'},\sigma')} \\
\hline
((D_1, D_2, C_3) \dagger (\sigma, \sigma', \overline{\sigma'}))^3 \vdash D_3 & DyDist, Dist, Dist \\
\hline
(D_1, D_2, C_3) \dagger (\sigma, \sigma', \overline{\sigma'}) \vdash \Box_3 D_3 & \Box_3 R \\
\hline
(D_1, D_2, C_3) \dagger (\sigma \bullet \sigma' \bullet \overline{\sigma'}) \vdash \Box_3 D_3 & \bullet L \\
\hline
D_1, D_2, C_3 \vdash [\sigma \bullet \sigma' \bullet \overline{\sigma'}] \Box_3 D_3 & DyR \\
\hline
D_1 \wedge D_2 \wedge C_3 \vdash [\sigma \bullet \sigma' \bullet \overline{\sigma'}] \Box_3 D_3 & \wedge L
\end{array}
$$

8.5. Navigation Examples

Consider a robot M in a building with four levels: basic proposition L_i means that the robot is at level i (for $i = 1, \ldots, 4$). Inconveniently, the levels are unlabelled. The robot can move up (represented by basic action u) from level i to level $i+1$, for $i = 1, 2, 3$; and can move down (represented by basic action d) from level i to level $i-1$, for $i = 2, 3, 4$. This scenario is represented by the following inequalities: those of type (1) are that $L_1 \leq [d]\bot$ and $L_4 \leq [u]\bot$ (the robot cannot go down from level 1 nor up from level 4); those of type (2) are that $L_i \leq [u]L_{i+1}$ for $i = 1, 2, 3$ and $L_i \leq [d]L_{i-1}$ for $i = 2, 3, 4$; those of type (3) are that $L_i \leq \Box_M L_i$ (for $i = 1, 4$) and that $L_i \leq \Box_M(L_2 \vee L_3)$ (for $i = 2, 3$), since at level 1 or 4 the robot knows, from the lack of options to go down (resp. up), what level it is at, but at level 2 or 3 there is less certainty; those of type (4) are that $u \leq \Box_M u$ and $d \leq \Box_M d$; those of type (5) are that $L_i \wedge L_j \leq \bot$ (for $(i \neq j)$ (the robot cannot be at two distinct levels simultaneously) and that of type (6) is that $\top \leq \bigvee_{i=1} 4L_i$ (the robot is at one or other of the levels). We omit the routine translation of these inequalities into assumption rules.

As an example, we show (except that some weakenings are omitted) that, after two moves down from level 3, the robot knows that it is at level 1:

$$
\begin{array}{ll}
\hline
L_1^M, L_1 \vdash L_1 & Id \\
\hline
L_1^M \vdash L_1 & PrApp_{(M, L_1, L_1)} \\
\hline
(L_2 \dagger d)^M \vdash L_1 & PrApp_{(d, L_2, L_1)} \\
\hline
((L_3 \dagger d) \dagger d)^M \vdash L_1 & PrApp_{(d, L_3, L_2)} \\
\hline
(L_3 \dagger (d, d))^M \vdash L_1 & Merge \\
\hline
L_3 \vdash [d \cdot d]\Box_M L_1 & Dy_R, \Box_M R, \cdot L
\end{array}
$$

We also show that, after a move down from level 3, the robot knows that it is at level 2. Informally, the argument is that initially it knows it is at level 2 or 3 (which it cannot distinguish), so after a move down it knows it is level 1 or 2; it can distinguish these, so it can infer that it is at level 2. This informal argument uses parts of the scenario (namely that one cannot go down from level 1) which can in fact be dispensed with, as follows (in an argument not so easily expressed informally):

Here is a cut-free proof (with principal items boxed for clarity):

$$\dfrac{\dfrac{\dfrac{}{(L_3^M, L_2) \dagger d, \boxed{L_1}, \boxed{L_3} \vdash L_2}\ InCon_{(L_1,L_3)}}{\boxed{(L_3^M, L_2) \dagger d}, L_3 \vdash L_2}\ PrApp_{(d,L_2,L_1)} \qquad \dfrac{\dfrac{}{\ldots, \boxed{L_2}, L_3 \vdash \boxed{L_2}}\ Id}{\boxed{(L_3^M, L_3) \dagger d}, L_3 \vdash L_2}\ PrApp_{(d,L_3,L_2)}}{\boxed{(L_3^M, L_2 \vee L_3) \dagger d}, L_3 \vdash L_2}\ \vee L$$

$$\dfrac{\dfrac{\dfrac{(\text{as above})}{(\boxed{L_3^M}) \dagger d, L_3 \vdash L_2}\ PrApp_{(M,L_3,L_2 \vee L_3)}}{(L_3^M) \dagger \boxed{d^M}, L_3 \vdash L_2}\ AcApp_{(M,d,d)}}{\boxed{(L_3 \dagger d, L_2)^M}, L_3 \vdash L_2}\ DyDist$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots, \boxed{L_2} \vdash \boxed{L_2}}\ Id \qquad \boxed{(L_3 \dagger d, L_2)^M}, L_3 \vdash L_2}{(L_3 \dagger d, L_2)^M, \boxed{L_2 \vee L_3} \vdash L_2}\ \vee L}{\boxed{(L_3 \dagger d, L_2)^M} \vdash L_2}\ PrApp_{(M,L_2,L_2 \vee L_3)}}{(\boxed{L_3 \dagger d})^M \vdash L_2}\ PrApp_{(d,L_3,L_2)}}{L_3 \dagger d \vdash \boxed{\Box_M L_2}}\ \Box_M R}{L_3 \vdash \boxed{[d]\Box_M L_2}}\ DyR$$

9. SUMMARY, RELATED AND FUTURE WORK

The algebraic semantics presented here is (apart from the addition of implication and thus of intuitionistic negation) a finite version of the algebra of [Baltag et al. 2007; Sadrzadeh 2006], inspired by [Abramsky and Vickers 1993], in the sense that the completion of a LMAM which acts on a DLAM (essentially an HAAM without implication) is an *epistemic system*. As proved in these papers, the relational semantics of dynamic epistemic logics, as presented in [Baltag and Moss 2004], lifts to such an epistemic system. With the addition of implication, there is thus an applicable intuitionistic modal logic of propositions and actions, with adjoint epistemic and dynamic modalities, with a known algebraic semantics.

To provide an analytic (and thus implementable) alternative to algebraic reasoning for the logic, we developed a nested sequent calculus for it; we proved that it admits weakening, contraction and cut, and that it is sound and complete with regard to the algebraic semantics. We illustrated the applicability of the rules of this calculus to reasoning about properties of dynamic epistemic scenarios, by developing and exploiting "assumption rules" that model the scenarios but do not affect the admissibility results of the calculus.

Existing proof theoretic treatments, e.g. [Negri and Maffezioli 2011; Balbiani et al. 2010], of dynamic epistemic logics, i.e. logics with both dynamic and epistemic modalities, only consider the specific actions of public announcements. The calculus of [Baltag et al. 2007] allows for arbitrary types (public/private, honest/dishonest) of announcement, and for other kinds of action such as navigation; but it lacks cut-admissibility. Hence, our work shows that some rather general dynamic epistemic logics can be represented in terms of a cut-free sequent calculus, allowing root-first proof search, albeit with some non-determinism.

[Galatos and Jipsen 2012] show how "[residuated] frames provide a uniform treatment for semantic proofs of cut-elimination", so one might speculate that similar methods would enable at least a semantic proof in our case; but our case (where, to provide the same kind of residuation, division operators \ and / adjoint to composition could be added without difficulty) is more complex: we include adjoint modal operators and have a multi-sorted syntax and semantics. Moreover, for philosophical reasons, we value constructive cut-admissibility arguments.

[Fitting 2012] shows how, in the single-sorted case, nested sequent calculi can be translated to (and from) labelled tableau calculi (or labelled sequent calculi, as in [Negri and Maffezioli 2011]); thus, a mechanical translation to a relational semantics is in principle feasible. How this relates to the relational semantics of dynamic epistemic logics in the sense of [Baltag and Moss 2004] or of [van Ditmarsch et al. 2007] could (in further work) be clarified.

The original models of DEL (as in [Baltag and Moss 2004]) were presented in terms of an 'update' operation on Kripke structures, which change the model in a dynamic way. Later, algebraic and coalgebraic semantics were developed for DEL (e.g. [Baltag et al. 2007; Cîrstea and Sadrzadeh 2007]), where the update operation was no longer model-changing. This is also the case for the algebraic semantics of this paper. On the syntactic side, the original proof system of DEL was a Hilbert-style axiomatization, where the model-changing feature was mainly reflected by an axiom called *Action-Knowledge*. This axiom expresses how belief after an action is computed from belief before it. A generalisation of this axiom was developed in [Baltag et al. 2007], where it was shown how the equivalence between the above two ideas can be weakened to an implication; our *DyDist* rule is the proof-theoretic version of this generalised version.

Our sequent calculi have a weak form of the sub-formula property (of a kind studied in [Brünnler and Guglielmi 2004]) adequate to allow the calculi to be called "analytic"; proof search is therefore effective (even if not efficient) and decidability is routine (since there are none of the issues from first-order logic of choosing arbitrary terms). The implementation of an earlier version of this sequent calculus [Truffaut 2011] provides an automated tool for encoding situations and scenarios involving both epistemic operators and actions. Suitable refinements of the calculus, addressing issues such as termination and backtracking, have yet to be developed.

REFERENCES

ABRAMSKY, S. AND VICKERS, S. J. 1993. Quantales, observational logic and process semantics. *Mathematical Structures in Computer Science 3*, 161–227.

BALBIANI, P., VAN DITMARSCH, H., HERZIG, A., AND DE LIMA, T. 2010. Tableaux for public announcement logic. *Journal of Logic and Computation 20*, 55–76.

BALTAG, A., COECKE, B., AND SADRZADEH, M. 2007. Epistemic actions as resources. *Journal of Logic and Computation 17*, 555–585.

BALTAG, A. AND MOSS, L. 2004. Logics for epistemic programs. *Synthese 139*, 165–224.

BRÜNNLER, K. 2009. Deep sequent systems for modal logic. *Archive for Mathematical Logic 48,* 6, 551–577.

BRÜNNLER, K. AND GUGLIELMI, A. 2004. A first order system with finite choice of premises. In *First-Order Logic Revisited*. Logos Verlag, London, 59–74.

CÎRSTEA, C. AND SADRZADEH, M. 2007. Coalgebraic epistemic update without change of model. In *Algebra and Coalgebra in Computer Science, Second International Conference (CALCO)*, T. Mossakowski, U. Montanari, and M. Haveraaen, Eds. Lecture Notes in Computer Science Series, vol. 4624. Springer, Berlin Heidelberg, 158–172.

DYCKHOFF, R., SADRZADEH, M., AND TRUFFAUT, J. 2012. Algebra, proof theory and applications for a logic of propositions, actions and adjoint modal operators. *ENTCS (MFPS Proceedings) 286*, 157–172.

FAGIN, R., HALPERN, J. Y., MOSES, Y., AND VARDI, M. Y. 1995. *Reasoning about Knowledge*. MIT Press, Cambridge (Massachusetts) and London (England).

FITTING, M. 2012. Prefixed tableaus and nested sequents. *Annals of Pure and Applied Logic 163*, 291–313.

GALATOS, N. AND JIPSEN, P. 2012. Residuated frames with applications to decidability. *Transactions of the American Mathematical Society 0*, 0–0.

GORÉ, R., POSTNIECE, L., AND TIU, A. 2011. On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. *Logical Methods in Comp. Science 7*, 1–38.

HUTH, M. AND RYAN, M. 2000. *Logic in Computer Science*. Cambridge Univ. Press, Cambridge.

KASHIMA, R. 1994. Cut-free sequent calculi for some tense logics. *Studia Logica 53*, 119–135.

MOORTGAT, M. 1995. Multimodal linguistic inference. *Logic J. of the IGPL 3*, 371–401.

NEGRI, S. AND MAFFEZIOLI, P. 2011. A proof theoretical perspective on public announcement logic. *Logic and Philosophy of Science 9*, 49–59.

NEGRI, S. AND VON PLATO, J. 1998. Cut elimination in the presence of axioms. *Bulletin of Symbolic Logic 4*, 418–435.

PLAZA, J. 2007. Logics of public communications. *Synthese 158*, 165–179.

POGGIOLESI, F. 2010. *Gentzen Calculi for Modal Propositional Logic*. Springer, Dordrecht Heidelberg London New York.

RESTALL, G. 2000. *An Introduction to Substructural Logics*. Routledge, London.

SADRZADEH, M. 2006. Actions and resources in epistemic logic. Ph.D. thesis, Université du Québec à Montréal.

SADRZADEH, M. AND DYCKHOFF, R. 2010. Positive logic with adjoint modalities: Proof theory, semantics and reasoning about information. *Review of Symbolic Logic 3,* 3, 351–373.

STRASSBURGER, L. 2013. Cut elimination in nested sequents for intuitionistic modal logics. In *FoSSaCS*, F. Pfenning, Ed. Lecture Notes in Computer Science Series, vol. 7794. Springer, Berlin Heidelberg, 209–224.

TRUFFAUT, J. 2011. *Implementation and Improvements of a Cut-Free Sequent Calculus for Dynamic Epistemic Logic*. Department of Computer Science, University of Oxford, Oxford.

VAN DITMARSCH, H., VAN DER HOEK, W., AND KOOI, B. 2007. *Dynamic Epistemic Logic*. Springer, Dordrecht.