TENNIS IS NOT A FAIR GAME!

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Dedicated to Michael and Evelyn Bowman, who introduced me to the delights of tennis, on the occasion of their wedding on October 19, 1996.

Abstract. Two 'equal' tennis players do not have equal chances in the tie-break. Consequently, a tennis match is not a fair game in the mathematical sense. Several remedies are proposed.

1. Introduction

Let us suppose that a tennis match takes place between two players, A and B say. A simple mathematical model for the match is as follows: we assume that A wins his service points against B with probability α and that player B wins his with probability β, and we assume that the points are independent events (see e.g. [1, chap. 2, esp. p. 28]). We should then say that A and B are 'equal' (at least when playing against each other) if $\alpha = \beta$. It turns out, as you would expect, that the two equal players have the same probability, namely $1/2$, of winning a tie-break or a set or a match. In other words there is no mathematical advantage in being the first to serve.

A more realistic model of the match would take into account the fact that the service alternates between the 'deuce' and the 'ad' courts. Most players have had the experience of facing an opponent who serves more effectively from one side than from the other: the left-hander's wide slice serve from the ad court is a good example. We can adapt our model to take this difference into account by assuming that A and B win their serves from the deuce court with probabilities p_1 and p_2 respectively, and from the ad court with probabilities q_1 and q_2 respectively. It is now no longer an easy matter to decide when two players are 'equal', because A may serve better than B from one side and B may serve better than A from the other! Most would surely agree, however, that when both $p_1 = p_2$ and $q_1 = q_2$, in which case the two players could be said to be 'identical', then the scoring system should give both players equal chances. As we shall see, this ideal is *not* met by the scoring system of the modern game, which tends to confer a modest mathematical advantage to one or other of the two identical players.

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2. Tie-break!

In fact there is only one place in the scoring system at which an asymmetry between the deuce and the ad court service occurs – the tie-break. Let us recall the rules of the tie-break. The first player serves from the deuce court for the first point. For the second and third points the second player serves from the ad and the deuce courts in that order. For the fourth point the first player serves from the ad court, and then this patttern repeats itself every four points. The tie-break is won by the first player to reach seven points by a margin of two or more points or by the first player to go ahead by a margin of two points when more than twelve points have been played. (The players also change ends after every six points, though this rule does not affect our model.)

Now let us see how the asymmetry comes about. Suppose that A serves first in the tie break and that the playing strengths of the two players are described by probabilities p_1, q_1, p_2, q_2 as above. For the moment we shall not assume that the players are 'identical'. Let P denote the *conditional probability* that A wins the tie-break given that the score reaches 6-6, and let Q denote the conditional probability that B wins given that the score reaches 7-7. When the score is 6-6, player A serves next from the deuce court. He can now win the tie-break outright by winning the next two points (with probability $p_1(1-q_2)$). If he wins only one of the next two points (with probability $p_1q_2 + (1-p_1)(1-q_2)$) then the score becomes 7-7 and his conditional probability of winning the tie-break is then $1-Q$. Hence we obtain the following equation:

$$P = p_1(1-q_2) + (p_1q_2 + (1-p_1)(1-q_2))(1-Q).$$

If the score reaches 7-7 then it is B's turn to serve from the deuce court. He wins the next two points with probability $p_2(1-q_1)$ and he wins just one of the next two points with probability $p_2q_1 + (1-p_2)(1-q_1)$. Since the serving pattern repeats itself every four points, it follows that the conditional probability of B winning the tie-break given that the score is 8-8 is the same as the conditional probability of winning given that the score is 6-6, namely $1-P$. Hence we obtain a second equation:

$$Q = p_2(1-q_1) + (p_2q_1 + (1-p_2)(1-q_1))(1-P).$$

Solving these equations, we get

$$P = \frac{(1-q_2+p_1q_2) - (1-q_1+p_1q_1)(1-p_1-q_2+2p_1q_2)}{1-(1-p_1-q_2+2p_1q_2)(1-p_2-q_1+2p_2q_1)}, \tag{1}$$

and a similar formula for Q.

When A and B are identical, let us denote by p and q the common values of p_1 and p_2 and of q_1 and q_2 respectively. Let us also denote by $P_{W66}(p,q)$ the conditional probability that the first player wins the tie-break given that the score reaches 6-6. From (1) we obtain

$$P_{W66}(p,q) = \frac{1-q+pq}{2-p-q+2pq},$$

and hence the player serving at 6-6 has an advantage over his opponent of

$$2P_{W66}(p,q) - 1 = \frac{p-q}{2-p-q+2pq},$$

which may be quite large: for example, if $p = 0.85$ and $q = 0.7$, then the first player's advantage at 6-6 is 9.146%.

In order to calculate the probability that the first player wins the tie-break it is also necessary to calculate the probability that a score of 6-6 is reached. This calculation is straightforward but the formula is somewhat messy. In the first 12 points of the tie-break, player A serves three times from the deuce and the ad courts and receives three times in the deuce and the ad courts. He wins each of the four kinds of points with probabilities p_1, q_1, $\bar{p}_2 = 1 - p_2$, and $\bar{q}_2 = 1 - q_2$ respectively. At this point it is helpful to introduce some notation: given a polynomial $P = P(p_1, q_1, \bar{p}_2, \bar{q}_2)$ in the variables p_1, $\bar{p}_2$, q_1 and $\bar{q}_2$, let $S(P)$ denote the polynomial obtained by summing the *distinct* polynomials which arise from permuting the four variables. For example, if $P(p_1, q_1, \bar{p}_2, \bar{q}_2) = p_1^3 q_1 \bar{p}_2 \bar{q}_2$, then

$$S(P)(p_1, q_1, \bar{p}_2, \bar{q}_2) = p_1^3 q_1 \bar{p}_2 \bar{q}_2 + q_1^3 p_1 \bar{p}_2 \bar{q}_2 + \bar{p}_2^3 p_1 q_1 \bar{q}_2 + \bar{q}_2^3 p_1 \bar{p}_2 q_1.$$

By enumerating the possible outcomes of the first twelve points, one obtains the following formula for the probability that the tie-break reaches a score of 6-6:

$$\begin{aligned} Q(p_1, q_1, \bar{p}_2, \bar{q}_2) = {} & S(p_1^3 q_1^3 (1-p_2)^3 (1-q_2)^3) + 9S(p_1^3 q_1^2 (1-q_1)\bar{p}_2 (1-\bar{p}_2)^2 (1-\bar{q}_2)^3) \\ & + 27S(p_1^2 (1-p_1) q_1^2 (1-q_1) \bar{p}_2^2 (1-\bar{p}_2)(1-\bar{q}_2)^3) \\ & + 27S(p_1^3 q_1 (1-q_1)^2 \bar{p}_2 (1-\bar{p}_2)^2 \bar{q}_2 (1-\bar{q}_2)^2) \\ & + 81S(p_1^2 (1-p_1) q_1^2 (1-q_1) \bar{p}_2 (1-\bar{p}_2)^2 \bar{q}_2 (1-\bar{q}_2)^2). \end{aligned}$$

When the two players are identical the probability of reaching a score of 6-6 in the tie-break, denoted $P_{66}(p,q)$, is given by

$$P_{66}(p,q) = Q(p, q, 1-p, 1-q),$$

a somewhat unwieldy polynomial in p and q which we shall not write out in full. Typically $P_{66}(p,q)$ is somewhere between 0.25 and 0.3, e.g. $P_{66}(0.85, 0.7) = 0.2798$

But what happens if the tie-break is decided in 12 or fewer points? In this case the two identical players will have the same conditional probabilities of winning. This is not completely obvious, but it can be seen without any lengthy calculation as follows. If one player wins the tie-break in twelve or fewer points then the same player would be in the lead if all of the first 12 points were (hypothetically) to be played out. Conversely, if the first 12 points are played out regardless of the result of the tie-break, and one player is in the lead after the twelfth point, then that player would also be the winner of the tie-break. So the probability that player A (respectively B) wins the tie-break in 12 or fewer points is equal to the probability that player A (respectively B) is in the lead after 12 points have beeen played out. As remarked above, each of the four different kinds of service is played three times in the first 12 points, and so by symmetry the probability of being in the lead after 12 points is the same for both players, namely $(1/2)(1-P_{66}(p,q))$. Finally, then, the probability that the first player to serve wins the tie-break, denoted $P_{WTB}(p,q)$ is given by

$$\begin{aligned} P_{WTB}(p,q) &= \frac{1-P_{66}(p,q)}{2} + P_{66}(p,q)\cdot P_{W66}(p,q) \\ &= \frac{1-P_{66}(p,q)}{2} + P_{66}(p,q)\cdot \frac{1-q+pq}{2-p-q+2pq}. \end{aligned}$$

For example, if $p = 0.85$ and $q = 0.7$ then A has an advantage of 2.559% over B in the tie-break.

3. Game and Set!

First we find the probability of winning a game. Recall that a game is won by the player who first wins four or more points by a margin of two. Suppose that the game reaches deuce (3-3). By considering the outcome of the next two points one sees that the conditional probability that the server wins the game given that the game has a deuce is given by

$$P_{WD}(p,q) = \frac{pq}{pq+(1-p)(1-q)}.$$

The probability that the game has a deuce is given by

$$P_D(p,q) = p^3(1-q)^3 + 9p^2q(1-p)(1-q)^2 + 9pq^2(1-p)^2(1-q) + q^3(1-p)^3,$$

and that the server wins the game with a score of 4-0, 4-1 or 4-2 by

$$P_{ND}(p,q) = p^2q^2 + 2p^2q(p(1-q)+q(1-p)) + pq(p^2(1-q)^2 + 6pq(1-p)(1-q) + 3q^2(1-p)^2).$$

Finally, for the probability that the server will win the game, we have

$$P_{WG}(p,q) = P_{ND}(p,q) + P_D(p,q)\cdot P_{WD}(p,q).$$

Before considering the probability of winning a set, let us observe some interesting consequences of the above formula. First, $P_{WG}(p, q) = P_{WG}(q, p)$, which means that neither the ad nor the deuce court is favored over the other, which is as it should be of course! For a strong server (i.e. when $p+q > 1$), on the other hand, there is some advantage to having two *unequal* serves as the following figures indicate:

$$P_{WG}(0.8, 0.5) = 0.84800, \quad P_{WG}(0.65, 0.65) = 0.82964,$$

and

$$P_{WG}(0.8, 0.7) = 0.95091, \qquad P_{WG}(0.75, 0.75) = 0.94921.$$

In the first set of figures, a player for whom $p = 0.8$ and $q = 0.5$ will win the same *expected* number of points on his serve in the long run as an 'equal' player for whom $p = q = 0.65$. But the former player has a higher probability (approx. 1.8355% higher) of winning a game than the latter. The second set of figures illustrates the same curious phenomenon (here the advantage is only 0.169%). For a weak server (i.e. $p + q < 1$) the situation is the opposite, and it is now advantageous to have 'equal' serves!

Now we can calculate the probability of winning a set that is played between our two identical players A and B. Recall that the set is won by the first player to win six or seven games by a margin of two or by the winner of the tie-break should a score of 6-6 be reached. Each player wins his own service games with probability $P_{WG} = P_{WG}(p, q)$. The probability that the set ends in a tie-break is then given by

$$\begin{aligned} P_{TB}(p,q) = &(P_{WG}^2 + Q_{WG}^2)(P_{WG}^{10} + 25P_{WG}^8 Q_{WG}^2 + 100P_{WG}^6 Q_{WG}^4 + 100P_{WG}^4 Q_{WG}^6 \\ &+ 25P_{WG}^2 Q_{WG}^8 + Q_{WG}^8), \end{aligned}$$

where Q_{WG} denotes $1 - P_{WG}$. If there is no tie-break then both players have the same conditional probability of winning the set. In the event of a tie-break the first player to serve in the match will also be the first to serve in the tie-break. Finally, then, the probability that the set is won by the first player to serve is given by

$$P_{WS}(p, q) = \frac{1 - P_{TB}}{2} + P_{TB} \cdot P_{WTB}.$$

For example, if $p = 0.85$ and $q = 0.7$ then the first player has an advantage of 1.806% over his opponent for the set.

4. Game, Set and Match!

We have now seen that when $p > q$ the first player to serve has a better than evens chance of winning the set. Now let us consider what effect this has on the match as a whole, and for simplicity we shall consider only the case of a three set match. Suppose that A serves first in the first set. One's first guess might be that the advantage to A in the first set would be neutralized in the second set by the change of service. This would indeed be correct if B were to serve first in the second set, but of course this might not be the case! According to the rules of tennis the service alternates throughout the match with tie-breaks counting as ordinary games for this purpose. So if the first set ends in a tie-break or in a score of 6-1 or 6-3 then B will indeed serve first in the second set. But if the first set ends with a score of 6-0, 6-2, 6-4 or 7-5 then A will serve first in the second set. This has the consequence that A continues to retain a very small advantage over the whole match. For example, if $p = 0.85$ and $q = 0.70$, then A has an advantage of 0.692% over B in a three set match, assuming that each set is to be decided by a tie-break if necessary. If the third set is to be played without a tie-break, on the other hand, then the first player's advantage is a mere 0.087%.

The precise expression for the probability of winning the match turns out to be very complicated: as explained above, we must take into account not only the result of each set but also the total number of games in each set as the evenness or oddness of this number determines who serves first in the subsequent set. Instead of boring you with some gruesome formulas we are content to tabulate some sample figures in the appendix!

5. CHANGE THE RULES!

Those players who habitually lose the 'big points' do not need to be told that tennis is not a fair game! But the fact that tennis may not be a 'fair game' in the mathematical sense, as we have seen, is somewhat disturbing and may cause one to look for a change in the rules which will remedy the problem. In fact, there are many ways of changing the rules to eliminate the asymmetry between the ad and the deuce courts described above. Here we list three such possibilities:

(1) Change the tie-break rules so that the tie-break is won either by the first player to win seven points or, if a score of 6-6 is reached in the tie-break, by the first player to go ahead by a margin of two *when the total number of points played is divisible by four*. With this change the tie-break could no longer end with a score of 8-6, but it could end with a score of either 9-6 or 9-7.

(2) Change the rules so that if player A serves first in the first set then player B *always serves first* in the second set. This would give both players equal chances of winning the match provided the final set is not decided by a tie-break.

(3) Change the rules so that if player A serves first in the first set then player B *serves first in a second set tie-break if there is one*. Provided again that the final set is not decided by a tie-break, this would give both players equal chances of winning the match.

All three possible changes to the rules have their pros and cons. My own preference is for (1) because it completely eliminates the asymmetry between the ad and deuce courts in the tie-break. Unfortunately, it would be quite possible for a player to 'win' the tie-break under the current rules (by a score of 9-7, say), but then to go on to lose the tie-break under the rules of (1), a difficult proposition to swallow when memory of the old rules is still fresh!

The second proposal has the advantage that it leaves the tie-break rules unchanged, but it has the disadvantage that it violates the hallowed principle that the service should always alternate throughout the match.

The third proposal would not violate this principle because the second set tie-break would still 'count' as a service game for the player who *would have* served first in the tie-break under the old rules! In itself this would be a fairly innocuous change as the tie-break is not in any *real* sense a service game for either player. However, the third proposal has the distinct disadvantage that it is necessary to remember who served first in the first set to decide who serves first in the second set tie-break, a case of 'action at a distance'!

Finally, while (2) and (3) do succeed in evening the chances over the whole match, they do not address the imbalance in the tie-break, and consequently the final set would have to be played without a tie-break to ensure equal chances for both players. (1), on the other hand, gives equal chances in a tie-break, and so the tie-break could also be used in the final set.

APPENDIX

Several of the probabilities (expressed as percentages) discussed above are tabulated here for some typical values of p and q at intervals of 5%. The calculations were done with *Mathcad 4.0*. Let us recall the meaning of these quantities:

p – probability of winning a serve from the deuce court;

q – probability of winning a serve from the ad court;

P_{W66} – conditional probability that the tie-break is won by the first player given that the score of 6-6 is reached;

P_{WTB} – probability that the tie-break is won by the first player;

P_{WS} – probability that the set is won by the first player;

P_{3TB} – probability that the match is won by the first player when each set is to be decided by a tie-break if necessary;

P_{2TB} – probability that the match is won by the first player when only the first two sets are to be decided by a tie-break if necessary.

p	q	P_{W66}	P_{WTB}	P_{WS}	P_{3TB}	P_{2TB}
70%	65%	51.602%	50.389%	50.119%	50.053%	50.015%
75%	65%	53.174%	50.793%	50.306%	50.127%	50.032%
80%	65%	54.716%	51.222%	50.598%	50.233%	50.051%
85%	65%	56.230%	51.689%	51.040%	50.393%	50.065%
90%	65%	57.716%	52.211%	51.669%	50.658%	50.067%
95%	65%	59.174%	52.809%	52.500%	51.101%	50.045%
75%	70%	51.562%	50.400%	50.189%	50.074%	50.016%
80%	70%	53.086%	50.822%	50.481%	50.181%	50.032%
85%	70%	54.573%	51.279%	50.903%	50.346%	50.043%
90%	70%	56.024%	51.790%	51.475%	50.609%	50.042%
95%	70%	57.440%	52.378%	52.204%	51.009%	50.026%
80%	75%	51.515%	50.419%	50.286%	50.108%	50.014%
85%	75%	52.985%	50.872%	50.688%	50.275%	50.023%
90%	75%	54.411%	51.380%	51.217%	50.530%	50.023%
95%	75%	55.797%	51.971%	51.882%	50.889%	50.014%
85%	80%	51.461%	50.452%	50.390%	50.165%	50.008%
90%	80%	52.873%	50.964%	50.895%	50.409%	50.010%
95%	80%	54.237%	51.572%	51.531%	50.741%	50.006%
90%	85%	51.404%	50.518%	50.498%	50.236%	50.003%
95%	85%	52.754%	51.151%	51.136%	50.559%	50.002%
95%	90%	51.344%	50.656%	50.653%	50.324%	50.000%

References

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