A CATEGORICAL PRIMER

CHRIS HILLMAN

Contents

1. Introduction

> The language and elementary results of category theory have now pervaded a substantial part of mathematics. Besides the everyday use of these concepts and results, we should note that categorical notions are fundamental in some of the most striking new developments in mathematics.
>
> *Nathan Jacobson* [15]
>
> Category theory is bunk.
>
> *Marshall Cohen*

Mathematics developed with great rapidity in the nineteenth century and has continued to develop with ever greater speed in this century. By the late 1930's the need for simple principles organizing the alarming proliferation of different mathematical structures was painfully apparent. This paper attempts to explain perhaps the most succesful such organizing principle which has yet emerged, the theory of categories.

The central concepts of category theory are *arrow* (or *morphism*), *functor*, *naturality*, and *adjoint*. (In a moment, I shall try to give the reader some idea of the meaning of these words.) The first three of these notions were introduced [6] by

Date: October 17, 1997.
Key words and phrases. tutorial paper, category theory.

Samuel Eilenberg and Saunders Mac Lane in 1945. Since then, category theory has developed as still another field where one studies a particular mathematical structure— albeit, the structure of a "category" is very general and abstract, unlike most important mathematical structures, which are comparatively specific and concrete. Today, some mathematicians even call themselves "category theorists"; these mathematicians prove theorems of great generality, so great that considerable effort may be required to "translate them" in order to apply them in the sort of situation likely to interest the "average mathematician". In contrast, this paper promotes the viewpoint that the four elementary concepts listed above are really very simple, and together form an organizing principle very useful for the "average mathematician".

Indeed, I would go further. One of the most striking trends in science and engineering in the second half of this century, in my view, has been the infiltration of quite sophisticated mathematics into these fields; indeed, some of the most imaginative mathematical ideas of recent years have been introduced by physicists, computer scientists and engineers, not "mathematicians". Consequently, I feel that "non-mathematicians" will also benefit from some acquaintance with category theory. Therefore, this paper is aimed at the broadest possible audience, although inevitable limits of time and energy have forced me to assume some degree of mathematical sophistication. and many of my examples will not be useful to readers unacquainted with at least undergraduate level real analysis and modern algebra. In the interest of space, most proofs are left as (valuable and by no means impossibly demanding) excercises for the reader; however, I list below a number of references which contain complete proofs of many of the most important results.

In the remainder of this section I wish to give the reader the flavor of categorical thinking, and in particular, some intuition for the four central concepts: arrow, functor, naturality, and adjoint.

Much of modern mathematics is devoted to the study of "mappings with structure" and collections of such mappings. For instance, every science and engineering student studies linear mappings between vector spaces and differentiable real valued functions, i.e. differentiable maps from $\mathbb{R}$ to $\mathbb{R}$. Math majors are almost certain to be introduced to continuous maps between topological spaces, and perhaps measure-preserving mappings between measure spaces. Typically, the composition of such "mappings with structure" is another mapping of the same type. Moreover, the *identity map* taking each element to itself is trivially a mapping with the desired structure. A *category* is essentially a collection of such "mappings with structure", called *arrows*, between certains "sets with structure", called *objects*, which is closed under composition and which contains an *identity arrow* for every object.

A huge number of such categories have been studied in modern mathematics. However, many important properties of arrows do not depend on the particular structures defining them, but only on how they combine under composition. Therefore, we may expect that considerable simplifications will acrue from *studying those properties of arrows which are expressible entirely in terms of composition.* This is the program of category theory. Hopefully it will become clear in the course of reading this paper that this program has been quite successful.

The fact that in category theory we study only those properties of our "mappings with structure" which are expressible entirely in terms of composition explains our preference for the term *arrow* rather than *morphism*, for we can diagrammatically represent the composition $\psi \circ \varphi$ where $\varphi : X \to Y$ and $\psi : Y \to Z$ as

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

Such diagrams are a very useful mental tool for grasping more complicated situations involving multiple compositions, as we shall see.

Category theory involves at least two major conceptual shifts from the way the reader is likely to have previously thought about mathematics. First, *arrows* (our "mappings with structure") rather than *objects* (the "sets with structure" our mappings map between) are now to be regarded as primary. For example, topology is, according to our point of view, not the study of topological spaces so much as the study of continous mappings. As Bell [2] puts it,

> category theory is like a language in which the "verbs" are on an equal footing with the "nouns".

Second, "structural-context-dependent properties" are to be disregarded in favor of "structural-context-free properties". This transition is best explained by example, in the setting of the simplest category of all, namely the category consisting of all ordinary mappings between sets. Two familiar properties a mapping $\varphi : X \to Y$ may or may not have are:

1. φ may be *one-one*,
2. φ may be *onto*.

The categorical expression of these properties are (respectively):

1. *postcancellation property:* for all $\alpha, \beta : E \to X$, if $\varphi \circ \alpha = \varphi \circ \beta$ then $\alpha = \beta$,
2. *precancellation property:* for all $\sigma, \tau : Y \to F$, if $\sigma \circ \varphi = \tau \circ \varphi$ then $\sigma = \tau$.

Notice that these categorical properties do not refer to "elements" or "sets" at all, only to the arrows of our category and to the composition of such arrows.

To see that the postcancellation property is equivalent to "one-one", observe that on the one hand, if φ is one-one and $\alpha \neq \beta$, i.e. for some $e \in E$, $\alpha(e) \neq \beta(e)$, then $\varphi \circ \alpha(e) \neq \varphi \circ \beta(e)$, so the contrapositive[1] of the postcancellation property holds whenever φ is one-one. On the other hand, suppose $x_1 \neq x_2$ but $\varphi(x_1) = \varphi(x_2)$. Define $\alpha, \beta : \mathbb{B} \to X$, where $\mathbb{B} = \{0, 1\}$, by

$$\alpha(0) = x_1,\ \alpha(1) = x_2$$
$$\beta(0) = x_2,\ \beta(1) = x_1$$

Then $\varphi \circ \alpha = \varphi \circ \beta$ but $\alpha \neq \beta$, showing that the postcancellation property fails unless φ is one-one.

To see that the precancellation property is equivalent to "onto", observe that on the one hand, if φ is onto and $\sigma \circ \varphi = \tau \circ \varphi$, then for any $y \in Y$ there is some $x \in X$ such that $\varphi(x) = y$ and thus $\sigma(y) = \sigma \circ \varphi(x) = \tau \circ \varphi(x) = \tau(y)$, so $\sigma = \tau$. Thus, the precancellation property holds whenever φ is onto. On the other hand, suppose y_0 is not in the image of φ. Define $\sigma, \tau : Y \to \mathbb{B}$ by setting $\sigma(y_0) = 1, \tau(y_0) = 0$ and setting $\sigma(y) = \tau(y) = 0$ for all $y \neq y_0$. Then $\sigma \circ \varphi = \tau \circ \varphi$ but $\sigma \neq \tau$, showing that the precancellation property fails unless φ is onto.

Many of the most important problems of mathematics have the form: *classify the objects of a category $\mathfrak{A}$ up to isomorphism.* (For the moment we can think of an *isomorphism* as a bijective arrow whose inverse mapping is also a arrow.) To take just one example, two of the most impressive achievements of twentieth century

[1] The contrapositive of the logical statement $p \Rightarrow q$ is the statement $\neg q \Rightarrow \neg p$; these two statements are logically equivalent. See [23].

mathematics have been the complete classication of finite dimensional complex Lie algebras (due to Cartan) and the complete classification of finite groups (due to many people; this has been called "the enormous theorem" because a complete proof would fill several volumes). Such classification problems are often simply intractable[2], but in such cases it may happen that we can introduce a simpler category $\mathfrak{B}$ and devise a *functor* which takes a arrow α of $\mathfrak{A}$ to a arrow $\mathcal{F}\beta$ of $\mathfrak{B}$, and moreover, does so in a way which preserves compositions (and thus, any notion in category theory, since as we said these are always defined entirely in terms of compositions). Note that if the object A of $\mathfrak{A}$ is the "domain" of α, this entails mapping A to the object $\mathcal{F}A$ of $\mathfrak{B}$, the "domain" of $\mathcal{F}\alpha$; likewise for codomains. The point is that if $\mathcal{F}A$ is non-isomorphic to $\mathcal{F}\mathcal{B}$, then $\mathcal{A}$ cannot be isomorphic to B; the former statement may be considerably easier to prove than the latter. Thus, functors out of $\mathfrak{A}$ provide (at the very least) a tool for "partial classification" of the objects of $\mathfrak{A}$.

In many categories we can construct the product $X \times E$ of two objects X, E. It is then natural to demand that $X \times E$ be isomorphic to $E \times X$, perhaps by the isomorphism $\omega_X : X \times E \to E \times X$ (the reason for the subscript will become apparent in a moment). However, further thought reveals that we not only desire that some such isomorphism exist, we want it to be *natural* in the sense that if we "perturb" X by replacing it by Y, where we have a "perturbing" arrow $\varphi : X \to Y$, then not only should $Y \times E$ be isomorphic to $E \times Y$, say by $\omega_Y : Y \times E \to E \times Y$, but we should have

$$(1_E \times \varphi) \circ \omega_X = \omega_Y \circ (\varphi \times 1_E)$$

where $1_E : E \to E$ is the *identity arrow* (analogous to the ordinary identity map) on E. This rather cumbersome requirement is what we mean by saying that the isomorphism $X \times E \simeq E \times X$ is "natural in X". It is easier to understand this property in the form, "the following diagram must commute:"

$$\begin{array}{ccc} X \times E & \xrightarrow{\varphi \times 1_E} & Y \times E \\ {\scriptstyle \omega_X}\downarrow & & \downarrow{\scriptstyle \omega_Y} \\ E \times X & \xrightarrow{1_E \times \varphi} & E \times Y \end{array}$$

This diagram is said to be *commutative* because we demand that that two paths from $X \times E$ to $E \times Y$ (first right then down or first down and then right) must have exactly the same effect (describe the same arrow). In any case, the point is that natural arrows belong to a collection of arrows which "behaves nicely under perturbation by other arrows".

Many useful functors "come in pairs". Specifically, we may have a natural bijection between the collection of all arrows $\mathcal{F}A \to B$ and the collection of all arrows $A \to \mathcal{G}B$. ("Natural" in the category of ordinary mappings between sets, that is.) In this case, $\mathcal{F}$ and $\mathcal{G}$ are said to be *adjoint functors*[3]. Numerous examples will be given in a later section, when (hopefully) the reader will be ready to appreciate them.

Despite this author's promise not to pass beyond the boundaries of what might be called "elementary category theory", i.e. the basic theory of arrows, functors,

---

[2]Shaharon Shelah has developed a theory of classification which in certain cases gives useful criteria for when this happens; see [13]

[3]The notion of adjoint functors was introduced [16] by Kan in 1958.

naturality, and adjoints, I must confess that this paper does contain a bit more: a brief introduction to *topos theory*. Briefly put, a topos is a category whose structure is so rich that it is capable of modelling any situation which can even be discussed in mathematical terms.

It turns out that each topos provides a model of first order logic; specifically, a formal language consisting of the operations of a Heyting algebra on terms (Heyting algebras are slight generalizations of Boolean algebras) and also existential and universal quantifiers ($\exists$ and $\forall$, respectively). In the simplest case, the category of sets, the associated logic is the usual first order logic (Boolean algebra plus existential and universal quantifiers) which suffices to do "standard mathematics". This suggests, correctly, that topoi other than the category of sets may serve as the foundation of all of mathematics. Interestingly enough, passing from Boolean algebras to more general Heyting algebras involves adopting an "intuitionistic" logic in which the law of the excluded middle may fail. To a considerable extent, topos theory succeeded in unifying some of the most important advances in logic, topology, and algebraic geometry made in this century. In particular, the *forcing construction* introduced by Paul Cohen and others to "force" certain statements to hold true turns out to be the same as the *sheafification construction* introduced by Grothendieck in algebraic geometry. This same construction also turns up (in still another disguise) in the nonstandard analysis of Abraham Robinson.

Under the optimistic assumption that by the time the reader has finished this paper, he or she will be hungry for more information about categories, I will list here some general references for further reading. The very recent textbook by McLarty [17] is an excellent and quite readable introduction to category theory and topos theory. This paper might perhaps be best regarded as an invitation to read McLarty's book. The older book by Goldblatt [9] provides a leisurely introduction to category theory and topos theory, but is less readable in places. The interesting mathematical physics textbook of Geroch [8] is based entirely on categorical notions and is quite readable. The graduate algebra text of Jacobson [15] (Volume II) contains a chapter discussing category theory. At a more advanced level, the recent book by Moerdjik and Mac Lane [20] provides a concise overview of category theory and, for those already familiar with modern algebraic geometry, a compelling introduction to topos theory. At the research level, the book [19] is a standard reference for most basic notions of category theory.

Finally, a word about the notation. This author has struggled to produce a notational system which helps the reader to keep clear the many different levels of structure in category theory. This is particularly important because, as we shall see, these levels are rather "flexible" in that by a mere change of perspective we can easily find ourselves working at a higher level. In this paper, elements of sets are denoted by lower case roman letters, while sets[4] and, more generally, objects are denoted by upper case roman letters. Arrows are denoted by lower case Greek letters, categories are (almost) denoted by upper case fraktur letters, whereas functors are denoted by upper case calligraphic letters.

[4]Except for the "usual suspects" $\mathbb{B} = \{0,1\}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Certain other exceptions to our notational rules will be noted as they occur.

SYMBOL	MEANING
x, y, z	elements
x^y	exponential element (in a Heyting algebra)
$\lceil \alpha \rceil$	the name of the predicate α (in a topos)
X, Y, Z	objects
$X \times Y$	product object
$X + Y$	sum (coproduct) object
X^Y	exponential object
$\hat{X}$	pullback (in pullback square)
$\check{X}$	pushout (in pushout square)
$*$	a unique object (e.g. in one-object category)
0	initial object
1	final object
$\overset{\lim}{\rightarrow} X_j$	direct limit of the X_j
$\overset{\lim}{\leftarrow} X_j$	inverse limit of the X_j
Ω	classifying object
$\operatorname{Qnt} X$	set of quotient objects of X
$\operatorname{Sub} X$	set of subobjects of X
Ω^X	power object of X
A, B, C	subobjects
$A \sqcap B$	meet of A, B
$A \sqcup B$	join of A, B
$\neg A$	psuedo-complement of A
$A \Rightarrow B$	psuedo-complement of A relative to B
$A \sqsubset B$	A is a subobject of B
α, β, γ	arrows (morphisms)
dom	domain operator
cod	codomain operator
id	operator assigning identity arrow to each object
$\beta \circ \alpha$	composite arrow (read right to left)
ker	kernel
coker	cokernel
im	image
1_X	identity arrow for object X
$=$	identity arrow (in a commutative diagram)
UMP	Universal Mapping Property
Υ	uniquely defined arrow (in a diagram for a UMP)
!	a unique arrow (e.g. into a final object)
π_X	canonical arrow $X \times Y \rightarrow X$
η_X	canonical arrow $X + Y \leftarrow X$
ε	equalizing arrow
κ	coequalizing arrow
$\hat{\alpha}$	pullback (in pullback square)
$\check{\alpha}$	pushout (in pushout square)
ω_X	component over X of a natural transformation

SYMBOL	MEANING
$\mathrm{Hom}(A, B)$	arrows (morphisms) from A to B
$\mathrm{Aut}\, X$	automorphisms of X
$\mathrm{End}\, X$	endomorphisms of X
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$	categories
$\mathfrak{S}, \mathfrak{T}$	topoi
$\mathfrak{C}^{\mathrm{op}}$	the opposite category for $\mathfrak{C}$ (reverse all arrows)
$\mathfrak{C}^{\rightarrow}$	the arrow category for $\mathfrak{C}$
$\mathfrak{C}/X$	slice category over X
$X/\mathfrak{C}$	coslice category under X
$\mathfrak{C}^{\downarrow\downarrow}$	the category of diagrams $X \rightrightarrows Y$ in $\mathfrak{C}$
$\mathfrak{C}^{\searrow\swarrow}$	the category of diagrams $X \to E \leftarrow Y$ in $\mathfrak{C}$
$\mathfrak{C}^{\nwarrow\nearrow}$	the category of diagrams $X \leftarrow E \to Y$ in $\mathfrak{C}$
$\mathfrak{C}^{J}$	the category of "J-shaped diagrams" in $\mathfrak{C}$, $(J, \leq)$ a preorder
$\mathfrak{A} \times \mathfrak{B}$	product category
$\mathfrak{A}^{\mathfrak{B}}$	category of "$\mathfrak{B}$-shaped diagrams" in $\mathfrak{A}$
$\mathcal{F}, \mathcal{G}, \mathcal{H}$	functors
$\mathcal{G} \circ \mathcal{F}$	composite functor (read right to left)
$\triangleright$	right adjunction operator
$\triangleleft$	left adjunction operator
$\mathcal{F} \dashv \mathcal{G}$	$\mathcal{F}$ is the left adjoint of $\mathcal{G}$ (and $\mathcal{G}$ is the right adjoint of $\mathcal{F}$)
$E_{\natural}$	hom functor induced by E
$E^{\natural}$	hom cofunctor induced by E
$(\cdot)_{\#}$	Yoneda functor
$(\cdot)^{\#}$	Yoneda cofunctor
ψ^{*}	slice change functor (pullback functor) induced by the arrow ψ
E^{*}	slice functor (pullback functor) induced by the object E
$(\cdot)^{\star}$	preimage cofunctor
Σ_{ψ}	left adjoint to ψ^{*} (naturally isomorphic to ψ_{*})
Π_{ψ}	right adjoint to ψ^{*}
$\exists_{\psi}$	existential quantifying functor induced by the arrow ψ
$\forall_{\psi}$	universal quantifying functor induced by the arrow ψ
$\exists_{E}$	existential quantifying functor induced by the object E
$\forall_{E}$	universal quantifying functor induced by the object E
$\mathfrak{S}et$	sets and mappings
ι	inclusion map
$\mathcal{P}X$	powerset of set X
$A \cap B$	intersection of subsets A, B
$A \cup B$	union of subsets A, B
A^{c}	complement of subset A
$A \subset B$	A is included in B
$X \setminus Y$	set difference
$\alpha^{-1}(\cdot)$	preimage under mapping α
$\alpha \vert E$	restriction of map α to subset E of $\mathrm{dom}\, \alpha$

SYMBOL	MEANING
$\mathfrak{P}os$	posets and order preserving mappings
$\mathfrak{L}at$	lattices and lattice homomorphisms
$\mathfrak{M}at(R)$	matrices with entries in R, R a ring
$K\mathfrak{L}in$	K-linear spaces and K-linear mappings
$G\mathfrak{S}et$	G-sets and G-equivariant maps, G a group
$R\mathfrak{M}od$	R-modules and R-homs, R a ring
$\mathfrak{G}rp$	groups and group homomorphisms
$\mathfrak{A}bg$	abelian groups
$\mathfrak{R}ng$	rings and ring homomorphisms
$\mathfrak{C}rg$	commutative rings
$\mathfrak{C}ru$	commutative rings with unit element
$\mathfrak{M}sr$	measure spaces and measure-preserving maps
$\mathfrak{T}op$	topological spaces and continuous maps
$\mathfrak{M}et$	metric spaces and contractive mappings
$\mathfrak{M}an$	smooth manifolds and smooth maps
$\mathfrak{P}air$	the category with two non-identity arrows $U \rightrightarrows V$
$\mathfrak{P}ull$	the category with two non-identity arrows $U \to W \leftarrow V$
$\mathfrak{P}ush$	the category with two non-identity arrows $U \leftarrow W \to V$
$\mathfrak{B}\mathrm{n}\,X$	bundles over X and bundle homomorphisms
E_x	stalk $\pi^{-1}(x)$ for bundle $E \xrightarrow{\pi} X$
$\mathfrak{E}\mathrm{t}\,X$	etales over X and etale homomorphisms
$\mathfrak{P}\mathrm{s}\,X$	presheaves over X and presheaf homs
$\mathfrak{S}\mathrm{h}\,X$	sheaves over X and sheaf homomorphisms
$\mathcal{S}ec$	sheaf of sections cofunctor from $\mathfrak{B}\mathrm{n}\,X$ to $\mathfrak{S}\mathrm{h}\,X$
$\mathcal{G}rm$	sheaf of germs cofunctor from $\mathfrak{P}\mathrm{s}\,X$ to $\mathfrak{E}\mathrm{t}\,X$
ev	evaluation arrow (for an exponential object)
α^E	exponential arrow $\alpha^E(\cdot) = \alpha \circ (\cdot)$
χ_A	characteristic arrow for subobject A (predicate)
$\top$	truth arrow
$-$	falsehood arrow
$\alpha \wedge \beta$	conjuction of predicates α, β
$\alpha \vee \beta$	disjunction of predicates α, β
$\neg\alpha$	negation of predicate α
$\alpha \Rightarrow \beta$	α implies β
$\kappa : 1 \to \Omega$	proposition with no free variables
$\alpha : X \to \Omega$	proposition with one free variable
$\theta : X \times Y \to \Omega$	proposition with two free variables x, y
$\langle (x,y) : \theta \rangle$	truth object of proposition θ (subobject of $X \times Y$)

2. Categories

The universe raises its head and stares at itself through me.
Russell Edson

Definition 2.1. *Suppose we have two collections*

1. *a collection of* **objects** X*,*
2. *a collection of* **arrows** *(or* **morphisms***)* φ*,*

and also four operators

1. *an operator* cod *assigning to each arrow* φ *an object* $\operatorname{cod}\varphi$*, the* **codomain** *of* φ*,*
2. *an operator* dom *assigning to each arrow* φ *an object* $\operatorname{dom}\varphi$*, the* **domain** *of* φ*,*
3. *an operator* id *assigning to each object* X *an arrow* 1_X*, the* **identity arrow**[5] *of* X*, which has* $\operatorname{dom} 1_X = \operatorname{cod} 1_X = X$*,*
4. *a binary operator, called* **composition***, assigning to* **composable pair** (α, β)*; that is, to every pair of arrows* (α, β) *with* $\operatorname{dom}\beta = \operatorname{cod}\alpha$*, an arrow* $\beta \circ \alpha$ *with*

$$\begin{aligned} \operatorname{dom}\beta\circ\alpha &= \operatorname{dom}\alpha \\ \operatorname{cod}\beta\circ\alpha &= \operatorname{cod}\beta \end{aligned}$$

These ingredients form a **category** $\mathfrak{C}$ *if*

1. *the composition operator* $\circ$ *is associative,*
2. *for each object* X*, the identity arrow* 1_X *can be cancelled from any composition, in the sense that*
   (a) *for every arrow* φ *with* $\operatorname{dom}\varphi = X$*, we have* $\varphi \circ 1_X = \varphi$*,*
   (b) *for every arrow* ψ *with* $\operatorname{cod}\psi = X$*, we have* $1_X \circ \psi = \psi$*,*

The collection of all arrows φ with $\operatorname{dom}\varphi = X$ and $\operatorname{cod}\varphi = Y$ is denoted $\operatorname{Hom}(X, Y)$. Note that "object" and "arrow" remain *undefined terms*. We shall often write an arrow as $X \xrightarrow{\varphi} Y$ or $Y \xleftarrow{\varphi} X$; either notation denotes the same arrow, with $\operatorname{dom}\varphi = X$ and $\operatorname{cod}\varphi = Y$.

Sets and ordinary mappings between them define the premier example of a category, denoted $\mathfrak{Set}$. Here, for any mapping $\varphi : X \to Y$, $\operatorname{dom}\varphi$ is the ordinary domain X and $\operatorname{cod}\varphi$ is the ordinary domain Y, the identity arrow 1_X for a set X is just the usual identity map $x \mapsto x$, and the composition operator is the ordinary composition of mappings (so if (α, β) are composable, $\beta \circ \alpha$ is defined by mapping first by α and then by β).

The collection of all sets is "too big" to itself be a set; it is a *proper class* (see [10]). Categories whose objects form a set (rather than a proper class) are called **small** categories.

In many categories, the objects are "sets with structure" and the arrows are "structure preserving mappings between objects". This structure is often algebraic in nature. Such categories are called **concrete categories**. A more formal definition will be given in Section 6; here we illustrate the idea by giving a number of examples of concrete categories and non-concrete categories.

Exercise: verify that the following form categories:

[5] We honor the special role of identity arrows by breaking our convention of always denoting arrows by lower case Greek letters.

1. $\mathfrak{Pos}$:
   (a) objects are posets $(X, \leq)$;
   (b) arrows are **order preserving maps** $\varphi : X \to Y$ where $x \leq x'$ implies $\varphi(x) \leq \varphi(x')$ for all $x, x' \in X$.
2. $\mathfrak{Lat}$:
   (a) objects are lattices $(X, \vee, \wedge)$;
   (b) arrows are **lattice homomorphisms** $\varphi : X \to Y$ where $\varphi(x \wedge x') = \varphi(x) \wedge \varphi(x')$ and $\varphi(x \vee x') = \varphi(x) \vee \varphi(x')$ for all $x, x' \in X$.
3. $K\mathfrak{Lin}$:
   (a) objects are K-linear spaces $(X, +, 0, \cdot)$;
   (b) arrows are K-**linear mappings** $\varphi : X \to Y$ where $\varphi(k \cdot x + k' \cdot x') = k \cdot \varphi(x) + k' \cdot \varphi(x')$ for all $k, k' \in K$ and all $x, x' \in X$.
4. $R\mathfrak{Mod}$:
   (a) objects are R-modules, where R is a ring;
   (b) arrows are R-homomorphisms.
   (Note that a special case is $\mathfrak{Abg}$, the category of abelian groups, which is obtained when $R = \mathbb{Z}$.)
5. $G\mathfrak{Set}$:
   (a) objects are G-sets (X, θ), where G is a group and $\theta : G \times X \to X$ is a **left action** by G on X;
   (b) arrows are G-**homomorpisms** $\varphi : X \to Y$, where $X = (X, \theta)$, $Y = (Y, \psi)$, and $\varphi \circ \theta(g) = \psi(g) \circ \varphi$ for all $g \in G$.
6. $\mathfrak{Grp}$:
   (a) objects are groups;
   (b) arrows are group homomorphisms.
7. $\mathfrak{Rng}$:
   (a) objects are rings;
   (b) arrows are ring homomorphisms.
8. $\mathfrak{Cru}$:
   (a) objects are commutative rings R with unity $1 \in R$;
   (b) arrows are ring homomorphsisms (respecting the unities).

For a very readable and elementary introduction to $\mathfrak{Pos}$ and $\mathfrak{Lat}$, see [5]. For a concise introduction to $G\mathfrak{Set}$, see [11]; for details see [22]. For an excellent introduction to $K\mathfrak{Lin}, R\mathfrak{Mod}, \mathfrak{Grp}, \mathfrak{Rng}, \mathfrak{Cru}$, see [3].

Exercise: verify that the following form categories:

1. $\mathfrak{Msr}$:
   (a) objects are measure spaces $(X, \mathbf{m}, \mu)$, where $\mathbf{m}$ is a sigma-algebra of subsets of X and μ is a measure on $\mathbf{m}$;
   (b) arrows are **measure preserving mappings** $\varphi : X \to Y$, where $X = (X, \mathbf{m}, \mu)$, $Y = (Y, \mathbf{n}, \nu)$, and $\mu(\varphi^{-1}(F)) = \nu(F)$ for all $F \in \mathbf{n}$.
2. $\mathfrak{Top}$:
   (a) objects are topological spaces $(X, \mathbf{s})$, where $\mathbf{s}$ is the collection of open sets in X;
   (b) arrows are **continuous mappings** $\varphi : X \to Y$, where $X = (X, \mathbf{s})$, $Y = (Y, \mathbf{t})$, and $\varphi^{-1}(B) \in \mathbf{s}$ for all $B \in \mathbf{t}$.
3. $\mathfrak{Met}$:
   (a) objects are metric spaces (X, d_X), where d_X is a metric on X;

   (b) arrows are **contractive mappings** $\varphi : X \to Y$, where $d_Y(\varphi(x), \varphi(y)) \leq d_X(x, y)$.
4. $\mathfrak{B}\mathrm{n}\, X$, where X is a topological space:
   (a) objects are continuous maps $E \overset{\mu}{\to} X$;
   (b) arrows are continuous maps $\varphi : E \to F$ such that $\nu \circ \varphi = \mu$.
5. $\mathfrak{E}\mathrm{t}\, X$, where X is a topological space:
   (a) objects are **etales** or **local homeomorphisms** $E \overset{\mu}{\to} X$; that is, for every $e \in E$ there is some neighborhood V of e such that $\mu|V$ is a homeomorphism onto some open set of X;
   (b) arrows are the same as in $\mathfrak{B}\mathrm{n}\, X$.
6. $\mathfrak{M}an$:
   (a) objects are smooth manifolds M;
   (b) arrows are smooth mappings $\varphi : M \to N$.

For a concise introduction to $\mathfrak{M}sr, \mathfrak{T}op$ see the second and fourth chapters (respectively) of [7].

If the objects and arrows of $\mathfrak{A}$ are also objects and arrows of $\mathfrak{B}$, we say that $\mathfrak{A}$ is a **subcategory** of $\mathfrak{B}$. For example, $\mathfrak{M}et$ is a subcategory of $\mathfrak{T}op$. If for every pair X, Y of objects in $\mathfrak{C}$, $\mathrm{Hom}(X, Y)$ contains at most one arrow, $\mathfrak{C}$ is said to be a **skeletal** category.

Exercise: let J be a set and let $\leq$ be a **preorder** on J. That is, $\leq$ is a transitive and reflexive relation, but $j \leq j'$ and $j' \leq j$ need *not* imply that $j = j'$. Consider the elements $j \in J$ to be *objects* and consider each pair (j, k) with $j \leq k$ to be an *arrow*. Verify that this defines a *nonconcrete* category. Verify that every partial order is a preorder, and that the corresponding category is skeletal.

Exercise: consider the positive integers to be *objects*, and consider all matrices over a ring R to be *arrows*; specifically, an arrow $m \overset{A}{\leftarrow} n$ is a matrix with m rows and n columns. If we have a second matrix $\ell \overset{B}{\leftarrow} m$, define the composition $B \circ A$ to be the matrix $\ell \overset{AB}{\longleftarrow} n$, where AB is the ordinary matrix product. Verify that this defines a *nonconcrete* category, denoted $\mathfrak{M}at(R)$. What is $\mathrm{Hom}(m, n)$?

Exercise: Let $\mathfrak{C}$ be some category. Show that we can form a new (nonconcrete) category, the **arrow category** $\mathfrak{C}^{\to}$, whose objects are the *arrows* of $\mathfrak{C}$, and whose arrows are *pairs* of arrows from $\mathfrak{C}$. Specifically, if $X \overset{\varphi}{\to} Y$ and $E \overset{\psi}{\to} F$ are two arrows from $\mathfrak{C}$, then an arrow in the arrow category is a pair (α, β), where the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ E & \xrightarrow{\psi} & F \end{array}$$

commutes.

Exercise: consider any set with one element, $*$. If we declare the unique function $* \to *$ to be the identity arrow of $*$, verify that we obtain a category, the **trivial category**.

Definition 2.2. *Let $\mathfrak{C}$ be a category. A* **diagram** *in $\mathfrak{C}$ is simply a collection of objects and arrows, where given any pair of objects X, Y in the diagram, there may be many arrows, or none, from X to Y (we allow here the case $Y = X$).*

Diagrams are often defined by pictures like this:

$$
\begin{array}{ccccccccc}
 & & & & & & K & & \\
 & & & & & & \rho\uparrow & & \\
A & \xleftarrow{\alpha} & Z & \xrightarrow{\zeta} & W & \xleftarrow{\omega} & K & \xleftarrow{\kappa} & B \\
\chi\downarrow & & \| & & \| & & & & \| \\
X & \xrightarrow{\varphi} & Z & \xrightarrow{\pi} & W & \xrightarrow{\sigma} & S & & B \\
 & & \| & & \tau\downarrow & & \mu\downarrow & & \beta\uparrow \\
Y & \xrightarrow{\psi} & Z & & T & \xrightarrow{\nu} & L & = & L
\end{array} \tag{1}
$$

Here, vertical and horizontal equals signs are abbreviations for identity arrows and are used to compensate for the limitations of text processors such as latex; on a blackboard, this diagram might be drawn as in Figure 1. Exercise: label the vertices and edges of Figure 1 with the correct names given in (1). In the sequel, the reader is enjoined to redraw each diagram in this paper in the "blackboard style" of Figure 1.

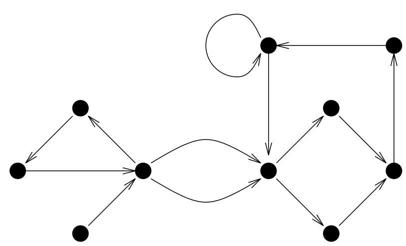

FIGURE 1. A blackboard version of (1).

A diagram is said to **commute** if any two directed paths (i.e. paths following arrows) between a given pair of objects in the diagram defines the same arrow under composition. For instance, (1) commutes iff

$$
\begin{aligned}
\chi \circ \alpha &= \varphi \\
\zeta \circ \psi &= \pi \circ \psi \\
\zeta \circ \varphi &= \pi \circ \varphi \\
\mu \circ \sigma &= \tau \circ \nu \\
\kappa \circ \beta \circ \mu \circ \sigma \circ \omega &= \rho
\end{aligned}
$$

and so forth. Here, recall that identity arrows can (by definition) be dropped without affecting any composition in which they appear. Of course, (1) is used here just for illustration; usually there are far fewer alternative paths to check. In fact, to show that a (not neccessarily planar) diagram commutes, it suffices to break it up into polygonal "faces" and show each "face" commutes.

Exercise: verify that Figure 1 pictorially defines a preorder on a set of ten elements (the vertices).

Note that diagrams can be infinite; in fact, it is often useful to picture a category $\mathfrak{C}$ as some humungous digraph whose vertices are the objects of $\mathfrak{C}$ and whose edges are the arrows of $\mathfrak{C}$. Then $\mathfrak{C}$ is a skeletal category if there is at most one edge between any ordered pair of vertices. Indeed, the notion of a small skeletal category is equivalent to the notion of a preorder.

3. Distinguished Objects and Arrows

> There are in every large chicken-yard a number of old and indignant hens who resemble Mrs. Bogart, and when they are served at Sunday noon dinner, as fricassed chicken with thick dumplings, they keep up the resemblance.
>
> Sinclair Lewis

Definition 3.1. *Let $\mathfrak{C}$ be some category. A arrow $X \xrightarrow{\eta} Y$ is* **monic** *if whenever we have two arrows $E \xrightarrow{\varphi,\psi} X$, if $\eta \circ \varphi = \eta \circ \psi$ then $\varphi = \psi$. This characteristic property of monics is called* **postcancellation**. *A arrow $X \xleftarrow{\pi} Y$ is* **epic** *if whenever we have two arrows $E \xleftarrow{\mu,\nu} X$, if $\mu \circ \pi = \nu \circ \pi$ then $\mu = \nu$. This characteristic property of epics is called* **precancellation**.

Note that the precancellation property is obtained by simply *reversing the direction of all arrows and reversing the order of all compositions* in the postcancellation property. We say that the notion of an epic arrow is **dual** to the notion of a monic arrow (and vice versa). Since all definitions in category theory are given in terms of arrows, every categorical concept has a dual concept obtained by reversing all arrows in the definition.

Exercise: show that in $K\mathfrak{Lin}, G\mathfrak{Set}, \mathfrak{Grp}$, respectively, the monic arrows are precisely the *one-one* arrows and the epic arrows are precisely the *onto* arrows. (The case of $\mathfrak{Grp}$ is quite hard!— see [8] for a solution.)

Proposition 3.2. *Let $\mathfrak{C}$ be some category. If $X \xrightarrow{\mu} Y$ and $Y \xrightarrow{\nu} Z$ are monic so is $X \xrightarrow{\nu\circ\mu} Z$, and conversely if $X \xrightarrow{\nu\circ\mu} Z$ is monic then ν must be monic. Dually, if $X \xleftarrow{\sigma} Y$ and $Y \xleftarrow{\tau} Z$ are epic so is $X \xleftarrow{\sigma\circ\tau} Z$, and conversely if $X \xleftarrow{\sigma\circ\tau} Z$ is epic then σ must be epic.*

The proof is left as an exercise.

Definition 3.3. *Let $\mathfrak{C}$ be some category. A arrow $X \xrightarrow{\sigma} Y$ is a* **section** *if there exists a arrow $X \xleftarrow{\tau} Y$ such that $\tau \circ \sigma = 1_X$. The characteristic property of a section is called* **left invertibility**. *A arrow $X \xleftarrow{\rho} Y$ is a* **retraction** *if there exists a arrow $X \xrightarrow{\iota} Y$ such that $\rho \circ \iota = 1_Y$. The characteristic property of a retraction is called* **right invertibility**. *A arrow $X \xrightarrow{\zeta} Y$ is an* **isomorphism** *if it is both left and right invertible.*

Notice that the notion of a section is dual to the notion of a retraction, and vice versa.

Exercise: verify that in the category $\mathfrak{Rng}$, the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is both monic and epic; this shows that in general it is not true that an epic and monic arrow is an isomorphism. On the other hand, show that every section is monic and every retraction is epic. For this reason, sections are sometimes called **split monic** arrows and retractions are sometimes called **split epic** arrows.

Exercise: show that every monic retraction is an isomorphism. Dually, show that every epic section is an isomorphism.

Exercise: are there any sections and/or retractions in $K\mathfrak{Lin}$?

Definition 3.4. *Let $\mathfrak{C}$ be some category. Two monic arrows α, β with codomain X are said to be equivalent, written $\alpha \sim \beta$, if there exists an isomorphism $\xi : \operatorname{dom} \alpha \to \operatorname{dom} B$ such that $\alpha = \beta \circ \xi$. The equivalence classes are the* **subobjects** *of X. Two epic arrows σ, τ with domain X are said to equivalent, written $\sigma \sim \tau$, if there is an isomorphism $\xi : \operatorname{cod} \sigma \to \operatorname{cod} \tau$ such that $\tau = \xi \circ \sigma$. The equivalence classes are* **quotient objects** *of X.*

Notice that the notion of a subobject is dual to the notion of a quotient object. The collection of subobjects of X is written $\operatorname{Sub}(X)$; dually, the collection of quotients is written $\operatorname{Qnt}(X)$. We will denote the subobject $[\alpha]$ of X which is represented by the monic arrow α with codomain X by A, and $[\beta]$ by B, etc.

Exercise: show that the obvious ordering on arrows induces a partial order on $\operatorname{Sub}(X)$ and $\operatorname{Qnt}(X)$.

We will write $A \sqsubset B$ to denote the fact that A is less than B in the partial ordering on $\operatorname{Sub}(X)$.

Exercise: show that in $\mathfrak{Set}, K\mathfrak{Lin}, G\mathfrak{Set}, \mathfrak{Grp}, \mathfrak{Rng}$, respectively, the subobjects are precisely the subsets, K-linear subspaces, stable subsets, subgroups, and subrings (respectively). How about quotient objects? (Hint: for $\mathfrak{Set}$, consider equivalence relations.)

Definition 3.5. *Let $\mathfrak{C}$ be some category. An object 1 in $\mathfrak{C}$ is said to be* **final** *if for every object X in $\mathfrak{C}$, there is a* unique *arrow $X \to 1$. Put another way, $\operatorname{Hom}(X, 1)$ contains exactly one arrow. Dually, an object 0 in $\mathfrak{C}$ is said to be* **initial** *if for every object X in $\mathfrak{C}$, there is a* unique *arrow $0 \to X$; that is, $\operatorname{Hom}(0, X)$ contains exactly one arrow.*[6]

Exercise: verify that the emptyset is initial and any one point set is final in $\mathfrak{Set}$. (Similarly for $G\mathfrak{Set}$.) Verify that the trivial K-linear space $\{0\}$ is both initial and final in $K\mathfrak{Lin}$. Similarly for $R\mathfrak{Mod}, \mathfrak{Abg}$. Verify that the trivial group $\{1\}$ is both initial and final in $\mathfrak{Grp}$. If X is a topological space, verify that the empty map $\emptyset \to X$ is an initial object in $\mathfrak{Bn}\, X$ and $X \stackrel{1_X}{\to} X$ is a final object in $\mathfrak{Bn}\, X$.

Exercise: Suppose 1 is final in some category $\mathfrak{C}$. Show that every arrow $1 \stackrel{\varphi}{\to} X$ is monic. (Dualizing shows that if 0 is initial, then every arrow $X \stackrel{\varphi}{\to} 0$ is epic.)

Neither final nor initial objects need exist in any given category. (Consider the integers $\mathbb{Z}$ with the obvious preordering $\leq$ as a category.) However, if they do exist, they are unique up to isomorphism. To see that final objects are unique, suppose that both $1, 1'$ are final objects for $\mathfrak{C}$. Then, there exists a unique arrow $1 \stackrel{\mu}{\to} 1'$ (since $1'$ is final) and also, there exists a unique arrow $1' \stackrel{\nu}{\leftarrow} 1$ (since 1 is final). Now consider the arrow $1 \stackrel{\nu \circ \mu}{\longrightarrow} 1$; it can only be 1_1 (since 1 is final, so there is a *unique* arrow $1 \to 1$); similarly $\nu \circ \mu = 1_{1'}$. We conclude that $1, 1'$ are isomorphic. A "dual" argument works for initial objects.

Definition 3.6. *Let $\mathfrak{C}$ be any category with a final object 1. Then any arrow $1 \to X$ defines an* **element** *of X.*

In other words, in category theory an "element" is a *trivial subobject*.

[6]We break our convention of denoting objects by capital roman letters in order to honor the almost universal convention in category theory of referring to initial objects as 0 and final objects as 1; this notation also makes it easier to remember certain "natural isomorphisms" which will be noted later on.

Excercise: verify that any arrow $* \to X$ in the category $\mathfrak{Set}$ (where $*$ is any one-point set) does indeed define a unique element in the usual sense. Verify that the elements in $G\mathfrak{Set}$ of the G-set X are precisely the **fixed points** of the given action by G on X. Similarly for $R\mathfrak{Mod}$, $K\mathfrak{Lin}$. What about $\mathfrak{Grp}$, $\mathfrak{Rng}$, $\mathfrak{Cru}$? Verify that an element of a bundle $E \xrightarrow{\mu} X$ is a *global section*; that is, a continuous mapping $\sigma : X \to E$ such that $\mu \circ \sigma = 1_X$.

4. Finite Operations within a Category

A comathematician is a system for turning theorems into coffee.

Tim Poston

Definition 4.1. *Suppose X, Y are objects in some category $\mathfrak{C}$. A* **product** *P is an object together with arrows $P \xrightarrow{\pi_X} X$ and $P \xrightarrow{\pi_Y} Y$, such that for any object Z equipped with arrows $Z \xrightarrow{\mu} X$ and $Z \xrightarrow{\nu} Y$, there is a* unique *arrow $Z \xrightarrow{\Upsilon} P$ such that $\Upsilon \circ \pi_X = \mu$ and $\Upsilon \circ \pi_Y = \nu$; that is, such that the diagram*

$$
\begin{array}{ccccc}
X & \xleftarrow{\mu} & Z & \xrightarrow{\nu} & Y \\
\| & & \Upsilon\big\downarrow & & \| \\
X & \xleftarrow{\pi_X} & P & \xrightarrow{\pi_Y} & Y
\end{array}
\tag{2}
$$

commutes.[7]

The product of a given pair of objects in a given category may or may not exist.

Exercise: if X, Y are two sets, show that the Cartesian product $X \times Y$ is a product of X, Y in $\mathfrak{Set}$. Similarly for $\mathfrak{Pos}$, $\mathfrak{Lat}$, $G\mathfrak{Set}$, $K\mathfrak{Lin}$, $\mathfrak{Grp}$, $\mathfrak{Abg}$, $\mathfrak{Top}$, $\mathfrak{Man}$, where $X \times Y$ is given the appropriate order (lexicographic!), meet and join, action by G, addition and scalar multiplication, group multiplication law, topology, and smooth structure, respectively. Thus, products always exist in these categories.

Exercise: if $(J, \leq)$ is a preorder, show that the product of j, k exists in $\mathfrak{J}$ iff $j, k \in J$ have a **greatest lower bound** ℓ, (and then ℓ is the desired product). Thus, for most $(J, \leq)$, a product may or may not exist for any given pair of elements.

The definition of a product involves our our second example of a definition in terms of a **universal mapping property** (UMP), in which a certain object is stipulated to posess certain properties and also to be related by a *unique* arrow to any other object satisfying the same properties. Our first example of a UMP, and the simplest, was the definition of initial and final objects. The UMP for a product says, informally speaking, that P is "the last" object with arrows to both X, Y.

In category theory, constructions of new objects from old ones are most often defined by UMP's.[8] One reason for this is that *any object defined by a UMP is unique up to isomorphism;* the argument is always the same as that given for final objects above. In the case of small categories, UMP's are often easiest to remember in terms of a certain preorder defined on the objects of the category. Namely, define $X \leq Y$ if there is an arrow $X \xrightarrow{\varphi} Y$. Then the UMP for a product says, informally speaking, that $X \times Y$ is "the last" object which has arrows from P to both X, Y, in

[7] We here break our notational convention of denoting arrows by lower case Greek letters, in order to emphasize that what matters here is that the arrow Υ is the *unique* arrow satisfying the stated property.

[8] The concept of a UMP was introduced by Mac Lane in [18].

the sense that *any object Z which also has arrows from Z to both X, Y must have an arrow into P.*

We pause here to give a few examples of how UMP's may be used to generalize familiar results concerning set theoretic constructions to more general categories.

Example 1: Recall that given any mappings $\alpha : A \to E$ and $\beta : B \to F$, we can define a **product map** $\alpha \times \beta : A \times B \to E \times F$ by setting $(\alpha \times \beta)(a, b) = (\alpha(a), \beta(b))$. Now suppose $\mathfrak{C}$ is some category in which products always exist. Given any arrows $A \xrightarrow{\alpha} E$ and $B \xrightarrow{\beta} F$ in $\mathfrak{C}$, can we can define a **product arrow** $A \times B \xrightarrow{\alpha\times\beta} E \times F$?

Since the objects of $\mathfrak{C}$ may not be "sets with structure", we can no longer simply define the desired arrow "elementwise". However, observe that we have arrows

$$E \xleftarrow{\alpha\circ\pi_A} A \times B \xrightarrow{\beta\circ\pi_B} F$$

By the UMP for the product $E \times F$, we immediately obtain a *unique* arrow Υ such that the diagram

$$\begin{array}{ccccc} E & \xleftarrow{\alpha\circ\pi_A} & A \times B & \xrightarrow{\beta\circ\pi_B} & F \\ \| & & \Upsilon\big\downarrow & & \| \\ E & \xleftarrow[\pi_E]{} & E \times F & \xrightarrow[\pi_F]{} & F \end{array}$$

commutes. Now we simply set $\alpha \times \beta = \Upsilon$.

Example 2: Recall that in $\mathfrak{Grp}$ the product of $X \times 1$, where 1 is the trivial group, is isomorphic to X. Now suppose $\mathfrak{C}$ is a category in which products always exist, and in which there is a final object 1. Can we show that $1 \times X$ is always isomorphic to X? Yes, because the diagram $X \overset{1_X}{\leftarrow} X \overset{!}{\to} 1$ satisfies the UMP of a product; that is, given any arrow $Y \overset{\varphi}{\to} X$, there is a unique arrow Υ (namely φ) making $\varphi = 1_X \circ \Upsilon$ and $! = ! \circ \Upsilon$.

Example 3: Suppose $\mathfrak{C}$ is a category in which products always exist. Then $X \times Y$ is isomorphic to $Y \times X$ for every pair of objects X, Y in $\mathfrak{C}$. To see this, first let $X \overset{\pi_X}{\leftarrow} P \overset{\pi_Y}{\to} Y$ satisfy the UMP and $Y \overset{\pi_Y}{\leftarrow} P' \overset{\pi_X}{\to} X$ satisfy the UMP of a product. It follows that there exist unique arrows Υ and Υ' such that

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & P & \xrightarrow{\pi_Y} & Y \\ \| & & \Upsilon\big\downarrow & & \| \\ Y & \xleftarrow[\pi'_Y]{} & P' & \xrightarrow[\pi'_X]{} & X \end{array}$$

and

$$\begin{array}{ccccc} Y & \xleftarrow{\pi'_Y} & P' & \xrightarrow{\pi'_X} & X \\ \| & & \Upsilon'\big\downarrow & & \| \\ X & \xleftarrow[\pi_X]{} & P & \xrightarrow[\pi_Y]{} & Y \end{array}$$

But this means that Υ, Υ' are mutually inverse arrows, and thus isomorphisms.

Since UMP's generally come in dual pairs, it follows that many important constructions in category theory will come in dual pairs. In particular, by "dualizing" (2), we obtain the following notion.

Definition 4.2. *Suppose X, Y are objects in some category $\mathfrak{C}$. A* **coproduct** *or* **sum** *S is an object together with arrows $S \overset{\eta_X}{\leftarrow} X$ and $S \overset{\eta_Y}{\leftarrow} Y$, such that for any* *object Z equipped with arrows $Z \stackrel{\mu}{\leftarrow} X$ and $Z \stackrel{\nu}{\leftarrow} Y$, there is a* unique *arrow $Z \stackrel{\Upsilon}{\leftarrow} S$ such that $\eta_X \circ \Upsilon = \mu$ and $\eta_Y \circ \Upsilon = \nu$; that is, such that the diagram*

(3)
$$\begin{array}{ccccc} X & \xrightarrow{\mu} & Z & \xleftarrow{\nu} & Y \\ \| & & \uparrow{\scriptstyle \Upsilon} & & \| \\ X & \xrightarrow{\eta_X} & S & \xleftarrow{\eta_Y} & Y \end{array}$$

commutes.

Informally speaking, this UMP says that S is "the first" object which has arrows from both X and Y. Once again, sums may or may not exist for a given pair of objects in a given category. If a sum does exist, it is unique up to isomorphism.

Exercise: if X, Y are two sets, show that the **disjoint union** $X \amalg Y$ is a sum of X, Y in $\mathfrak{Set}$. (Similarly for $G\mathfrak{Set}, \mathfrak{Top}, \mathfrak{Man}$.) If G, H are two groups, show that the **free product** $G * H$ is a sum of G, H in $\mathfrak{Grp}$. If A, B are abelian groups, show that the **direct sum** $A \oplus B$ is a sum of A, B in $\mathfrak{Abg}$. How about the direct product $A \times B$? (Similarly for $R\mathfrak{Mod}, K\mathfrak{Lin}$.) If R, S are two commutative rings, show that the **tensor product** $R \otimes S$ is a sum of R, S in $\mathfrak{Cru}$ (where $R \stackrel{\eta_R}{\to} R \otimes S$ is the map $r \mapsto r \otimes 1$, and $S \stackrel{\eta_S}{\to} R \otimes S$ is the map $s \mapsto 1 \otimes s$).

Exercise: if $(J, \leq)$ is a preorder and $j, k \in J$ have a **least upper bound** u, show that u is a sum of j, k in $\mathfrak{J}$.

Exercise: show that the sum of m, n, considered as objects in $\mathfrak{Mat}(R)$, is $m + n$. (Hint: if $k \geq m + n$, the unique matrix defined by the arrows $m \stackrel{A}{\to} k \stackrel{B}{\leftarrow} n$ is obtained by placing A, B side by side.) On the other hand, show that *products* in general do not exist in $\mathfrak{Mat}(R)$.

Exercise: dualize the construction of a product arrow to obtain the sum arrow $\alpha + \beta$, which is defined in any category in which *sums* of objects always exist.

Definition 4.3. *Let $\mathfrak{C}$ be some category, and let $X \xrightarrow{\varphi,\psi} Y$ be two arrows with the same domain and codomain. Then $E \xrightarrow{\varepsilon} X$ is an* **equalizer** *for φ, ψ if*

1. $\varphi \circ \varepsilon = \psi \circ \varepsilon$,
2. *given any $Z \stackrel{\zeta}{\to} X$ such that $\varphi \circ \zeta = \psi \circ \zeta$, there is a* unique *arrow $Z \stackrel{\Upsilon}{\to} E$ such that $\zeta = \varepsilon \circ \Upsilon$. In other words, there is a unique arrow $Z \stackrel{\Upsilon}{\to} E$ making the diagram*

(4)
$$\begin{array}{ccccc} E & \xrightarrow{\varepsilon} & X & \xrightarrow{\varphi} & Y \\ {\scriptstyle \Upsilon}\uparrow & & \| & & \| \\ Z & \xrightarrow{\zeta} & X & \xrightarrow{\psi} & Y \end{array}$$

commute.

Informally speaking, E is "the first" object with an arrow $E \stackrel{\varepsilon}{\to} X$ which "equalizes" φ, ψ in the sense that $\varphi \circ \varepsilon = \psi \circ \varepsilon$. Of course, for a given pair of arrows in a given category, such an E may or may not exist.

Exercise: if $X \stackrel{\varphi,\psi}{\to} Y$ are set mappings, show that $E \stackrel{\varepsilon}{\to} X$, where

$$E = \{x \in X : \varphi(x) = \psi(x)\}$$

and ε is the inclusion map, is an equalizer for $X \stackrel{\varphi,\psi}{\to} Y$ in $\mathfrak{Set}$. Similarly for $G\mathfrak{Set}$. What about $\mathfrak{Grp}$?

Exercise: if A, B are abelian groups and $A \overset{\varphi,\psi}{\longrightarrow} B$ are group homomorphisms, verify that $\varphi - \psi$ is a new homomorphism from A to B and that $\ker(\varphi - \psi) \overset{\iota}{\to} A$, where ι is the inclusion map, is an equalizer for $A \overset{\varphi,\psi}{\longrightarrow} B$ in $\mathfrak{Abg}$. Observe that for any A, B there is a special arrow $A \overset{0}{\to} B$ whose image is $0 \in B$. Verify that the equalizer of $A \overset{\varphi}{\to} B \overset{0}{\leftarrow} A$ is $E \overset{\varepsilon}{\to} A$, where E is **kernel** $\ker \varphi$ and ε is the inclusion map. Similarly for $K\mathfrak{Lin}$, $R\mathfrak{Mod}$. What about $\mathfrak{Grp}$, $\mathfrak{Cru}$, $\mathfrak{Rng}$?

As the reader will have already guessed, there is a dual to the notion of an equalizer.

Definition 4.4. *Let $\mathfrak{C}$ be some category, and let $X \overset{\varphi,\psi}{\longrightarrow} Y$ be two arrows with the same domain and codomain. Then $Y \overset{\kappa}{\longrightarrow} K$ is a* **coequalizer** *for φ, ψ if*

1. *$\kappa \circ \varphi = \kappa \circ \psi$,*
2. *given any $Y \overset{\zeta}{\to} Z$ such that $\zeta \circ \varphi = \zeta \circ \psi$, there is a* unique *arrow $K \overset{\Upsilon}{\to} Z$ such that $\zeta = \Upsilon \circ \kappa$. In other words, there is a unique arrow $K \overset{\Upsilon}{\to} Z$ making the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\kappa} & K \\ \| & & \| & & \Upsilon\downarrow \\ X & \xrightarrow{\psi} & Y & \xrightarrow{\zeta} & Z \end{array} \tag{5}$$

commute.

Informally speaking, K is the "first" object with an arrow $Y \overset{\kappa}{\to} K$ which "co-equalizes" φ, ψ in the sense that $\kappa \circ \varphi = \kappa \circ \psi$. Of course, for a given pair of objects φ, ψ in a given category, such a K may or may not exist.

Exercise: show that if $X \overset{\varphi,\psi}{\longrightarrow} Y$ are set mappings, then $Y \overset{\kappa}{\to} K$, where K is obtained by "moding out" $Y \amalg Y$ according to the smallest equivalence relation such that $\varphi(x) \sim \psi(x)$ for all $x \in X$, and where κ is the natural projection, is the coequalizer of $X \overset{\varphi,\psi}{\longrightarrow} Y$ in $\mathfrak{Set}$. (See [5] for the notion of the smallest equivalence relation swallowing a given relation on Y and for natural projections.)

Exercise: show that if A, B are abelian groups and $A \overset{\varphi,\psi}{\to} B$ are group homomorphisms, then $B \to B/\operatorname{im}(\varphi - \psi)$ is the coequalizer of $A \overset{\varphi,\psi}{\to} B$ in $\mathfrak{Abg}$. Verify that the coequalizer of $A \overset{\varphi,0}{\to} B$, where $A \overset{0}{\to} B$ takes every $x \in A$ to $0 \in B$, is $B \overset{\pi}{\to} K$, where K is the **cokernel** $\operatorname{coker} B = B/(\operatorname{im}\varphi)$ and κ is the obvious projection onto the quotient. Similarly for $K\mathfrak{Lin}$, $R\mathfrak{Mod}$.

Exercise: show that if G, H are groups and $G \overset{\varphi,\psi}{\longrightarrow} H$ are group homomorphisms, then $G \to H/N$, where N is the smallest normal subgroup containing

$$\left\{\varphi(g)\psi(g)^{-1} : g \in G\right\}$$

is the coequalizer of $G \overset{\varphi,\psi}{\longrightarrow} H$ in $\mathfrak{Grp}$. Similarly, show that the equalizer of $G \overset{\varphi,1}{\longrightarrow} H$ in $\mathfrak{Grp}$ is the **cokernel** H/N, where N is the smallest normal subgroup swallowing $\varphi(G)$.

Proposition 4.5. *In any category, every equalizer is monic. Dually, every coequalizer is epic. Furthermore, every epic equalizer, and every monic coequalizer, is an isomorphism.*

Definition 4.6. *Suppose we have the following arrows in some category* $\mathfrak{C}$*:*

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow{\mu} & Z \end{array}$$

A **pullback square** *for these arrows is a commuting diagram*

$$\begin{array}{ccc} \hat{Z} & \xrightarrow{\hat{\mu}} & Y \\ {\scriptstyle \hat{\nu}}\downarrow & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow{\mu} & Z \end{array} \tag{6}$$

where whenever

$$\begin{array}{ccc} W & \xrightarrow{\beta} & Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow{\mu} & Z \end{array}$$

is another commuting diagram, there is a unique *arrow* $W \xrightarrow{\Upsilon} \hat{Z}$ *such that* $\alpha \circ \Upsilon = \hat{\mu}$ *and* $\beta \circ \Upsilon = \hat{\nu}$*.*

We say that $\hat{\mu}$ is obtained by pulling back μ along ν and that $\hat{\nu}$ is obtained by pulling back ν along μ. Informally speaking, $\hat{Z}$ is "the last" object such that (6) commutes. Of course, such an object may or may not exist for a given pair of arrows in a given category.

Exercise: if $X \xleftarrow{\mu} Z \xrightarrow{\nu} Y$ are set mappings, then

$$\hat{Z} = \{(x, y) \in X \times Y : \mu(x) = \nu(y)\}$$

and $\hat{\mu}(x, y) = y$, $\hat{\nu}(x, y) = x$ define a pullback square in $\mathfrak{Set}$. (Similarly for $G\mathfrak{Set}$.)

Exercise: there is a principle saying that every construction in $\mathfrak{Top}$ can be extended to $\mathfrak{Bn}\,X$ by "bundling" the $\mathfrak{Top}$ construction applied to each stalk. Here, if $E \xrightarrow{\mu} X$ is a bundle, and $x \in X$, then $E_x = \mu^{-1}(x)$ is the **stalk over** x. In particular, we can construct a pullback square for any arrows $E \xrightarrow{\varphi} S \xleftarrow{\psi} F$ in $\mathfrak{Bn}\,X$, where $F \xrightarrow{\nu} X$ and $S \xrightarrow{\sigma} X$ are bundles. Define

$$\hat{S} = \{(e, f) \in E \times F : \mu(e) = \nu(f)\}$$

and $\hat{\psi}(e, f) = e, \hat{\varphi}(e, f) = f$ and $\hat{\sigma}(e, f) = \mu(e) = \nu(f)$. Then $\hat{S} \xrightarrow{\hat{\sigma}} X$ is a bundle over X (note that $E \times F$ is given the product topology and that $\hat{S}$ is a closed subset of $E \times F$). Verify that this construction is a pullback in $\mathfrak{Bn}\,X$ and also that for each x, the stalk $\hat{S}_x$ is obtained by taking the pullback of the set mappings

$$E_x \xrightarrow{\mu|E_x} S_x \xleftarrow{\nu|F_x} F_x$$

Proposition 4.7. *Let* $\mathfrak{C}$ *be a category.*

1. *The pullback of a monic arrow (along any arrow) is monic.*
2. *The pullback of* $X \xrightarrow{\varphi} Y \xleftarrow{\psi} X$*, if it exists, is the equalizer* $X \xleftarrow{\varepsilon} E \xrightarrow{\varepsilon} X$*.*
3. *If* 1 *is final, the pullback of* $X \to 1 \leftarrow Y$*, if it exists, is the product* $X \leftarrow X \times Y \to X$*.*

The proof is left as an exercise.

Exercise: verify the following claims.

1. In $\mathfrak{Set}$, the pullback of $X \overset{\varphi}{\to} Y \overset{\iota}{\leftarrow} B$ (where B is a subset of Y and ι is the inclusion map) is $X \leftarrow A \to B$, where A is the **preimage**
$$\varphi^{-1}(B) = \{x \in X : \varphi(x) \in B\}$$
Similarly for $\mathfrak{Pos}$, $\mathfrak{Lat}$, $K\mathfrak{Lin}$, $R\mathfrak{Mod}$, $G\mathfrak{Set}$, $\mathfrak{Grp}$, $\mathfrak{Rng}$, where $\varphi^{-1}(B)$ is a subposet, sublattice, linear subspace, submodule, stable subset, subgroup, and subring respectively.
2. In $\mathfrak{Set}$, the pullback of $X \overset{\varphi}{\to} Y \overset{\varphi}{\leftarrow} X$ is the **kernel congruence**
$$\ker\varphi = \{(x, x') \in X \times X : \varphi(x) = \varphi(x')\}$$
in the sense of universal algebra [15]. This is an equivalence relation (considered as a subset of $X \times X$) which partitions X into the preimages of elements of Y. Similarly for $\mathfrak{Pos}$, $\mathfrak{Lat}$, $G\mathfrak{Set}$, where now $\ker\varphi$ is a **congruence relation** [5][11], so that the quotient $X/\ker\varphi$ is defined. Must this be modified for $\mathfrak{Grp}, R\mathfrak{Mod}, \mathfrak{Rng}$? The **fundamental theorem** of posets, lattices, G-sets, groups, etc., states that $X/\ker\varphi$ is isomorphic to $\operatorname{im}\varphi$. (See, for instance, [11].)

Lemma 4.8 (Pullback Lemma)**.** *Suppose that the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow \\ A & \xrightarrow{\sigma} & B & \xrightarrow{\tau} & C \end{array} \tag{7}$$

commutes. Then

1. *If the two small squares are pullback squares, so is the outer "rectangle".*
2. *If the right hand square and the outer "rectangle" are pullback squares, so is the left hand square.*

The proof is left as an exercise (but see p. 67 of [9]).

Definition 4.9. *Suppose we have the following arrows in some category $\mathfrak{C}$:*

$$\begin{array}{ccc} & & Y \\ & & \uparrow\nu \\ X & \xleftarrow{\mu} & Z \end{array}$$

A **pushout square** *for these arrows is a commuting diagram*

$$\begin{array}{ccc} \check{Z} & \xleftarrow{\check{\mu}} & Y \\ \check{\nu}\uparrow & & \uparrow\nu \\ X & \xleftarrow{\mu} & Z \end{array} \tag{8}$$

where whenever

$$\begin{array}{ccc} W & \xleftarrow{\beta} & Y \\ \alpha\uparrow & & \uparrow\nu \\ X & \xleftarrow{\mu} & Z \end{array}$$

is another commuting diagram, there is a unique *arrow $W \overset{\Upsilon}{\leftarrow} \hat{Z}$ such that $\Upsilon \circ \alpha = \check{m}u$ and $\Upsilon \circ \beta = \check{n}u$.*

We say that $\check{\mu}$ is obtained by pushing out μ along ν and that $\check{\nu}$ is obtained by pushing out ν along μ. Informally speaking, $\check{Z}$ is "the first" object such that (8) commutes. Of course, such an object may or may not exist for a given pair of arrows in a given category.

Exercise: if $Y \stackrel{\mu}{\leftarrow} Z \stackrel{\nu}{\rightarrow} Y$ are set mappings, then $\check{Z} = (X \amalg Y)/\sim$ where $\sim$ is the smallest equivalence relation such that $x \sim y$ whenever there exists some z with $\mu(z) = x, \nu(z) = y$, and $\check{\mu}(y) = [y]$, $\check{\nu}(x) = [x]$ defines a pushout square in $\mathfrak{Set}$. Similarly for $G\mathfrak{Set}$.

Exercise: show that if $A \stackrel{\alpha}{\leftarrow} H \stackrel{\beta}{\rightarrow} B$ are *monic* arrows in $\mathfrak{Grp}$, then $A *_H B$ (the free product of A times B with amalgamated subgroup H) is a pushout. What about $R\mathfrak{Mod}$, $\mathfrak{Cru}$, $\mathfrak{Rng}$? (See [24].)

Exercise: construct a pushout square in $\mathfrak{Bn}\,X$ for the arrows $E \stackrel{\varphi}{\leftarrow} S \stackrel{\psi}{\rightarrow} F$. (Hint: quotient topology.) Verify that each stalk of the pushout $\check{S}$ arises as the pushout of the corresponding stalks of E, S, F.

Proposition 4.10. *Let $\mathfrak{C}$ be a category.*

1. *The pushout of an epic arrow (along any arrow) is epic.*
2. *The pushout of $X \stackrel{\varphi}{\leftarrow} Y \stackrel{\psi}{\rightarrow} X$, if it exists, is the coequalizer $X \stackrel{\kappa}{\rightarrow} K \stackrel{\kappa}{\leftarrow} X$.*
3. *If I is initial, the pushout of $X \leftarrow I \rightarrow Y$, if it exists, is the sum $X \rightarrow X \times S \leftarrow X$.*

Exercise: verify that the pushout in $\mathfrak{Set}$ of $Y \stackrel{\varphi}{\rightarrow} X \stackrel{\varphi}{\rightarrow} Y$ is the **cokernel** $(Y \uplus Y)/\sim$ where $y \sim y'$ if for some $x \in X$, $\varphi(x) = y = y'$.

Lemma 4.11 (Pushout Lemma)**.** *Suppose that the diagram*

(9)
$$\begin{array}{ccccc} X & \stackrel{\varphi}{\longleftarrow} & Y & \stackrel{\psi}{\longleftarrow} & Z \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow \\ A & \stackrel{\sigma}{\longleftarrow} & B & \stackrel{\tau}{\longleftarrow} & C \end{array}$$

commutes. Then

1. *If the two small squares are pushout squares, so is the outer "rectangle".*
2. *If the left hand square and the outer "rectangle" are pushout squares, so is the right hand square.*

We postpone until Section 9 the important operation of exponentiation in a category, because this construction is more natural in the context of adjoint functors.

5. Limit Operations within a Category

Definition 5.1. *Suppose we have a diagram consisting of various arrows $X \stackrel{\varphi}{\leftarrow} X'$, where X, X' are objects in the diagram. (There might be many arrows, or none, between any particular pair of objects.) Suppose there is an object L of $\mathfrak{C}$ together with arrows $X \stackrel{\lambda}{\leftarrow} L$ for each object X in the diagram, such that for all arrows $X \stackrel{\varphi}{\leftarrow} X'$ in the diagram,*

$$\begin{array}{ccc} L & = & L \\ {\scriptstyle\lambda}\downarrow & & {\scriptstyle\lambda'}\downarrow \\ X & \stackrel{\varphi}{\longleftarrow} & X' \end{array}$$

commutes. Then the diagram consisting of the arrows $X \xleftarrow{\varphi} X'$ *of the given diagram* together with *the new arrows* $X \xleftarrow{\lambda} L$ *is called a* **cone** *and* L *is its* **vertex***.*

This cone is an **inverse limit** *(or* **limit***) of the original diagram if given any other cone* $X_j \xleftarrow{\mu_j} M$*, there is a unique arrow* $L \xleftarrow{\Upsilon} M$*.*

Informally speaking, L (if it exists) is "the last" object in $\mathfrak{C}$ which can be taken as the vertex of a cone over the diagram in question. If an inverse limit does exist, if is unique up to isomorphism. The term "inverse limit" is used because, informally speaking, the limit is taken "against the arrows".

Exercise:

1. Consider the diagram consisting of two objects X, Y (and no arrows) in some category $\mathfrak{C}$. Verify that every inverse limit of this diagram is a *product* of X, Y in $\mathfrak{C}$. Now consider the diagram consisting of any collection of objects $\{X_j : j \in J\}$ (and no arrows). Observe that we can define $\prod_{j \in J} X_j$ as "the" inverse limit (if it exists) of this diagram.
2. Consider the diagram $Y \xleftarrow{\varphi, \psi} X$ (two objects with two arrows). Verify that any inverse limit must be an *equalizer* of $Y \xleftarrow{\varphi, \psi} X$ in $\mathfrak{C}$. Similarly for infinite collections of maps in $\mathrm{Hom}(X, Y)$.
3. Consider the diagram $M \xrightarrow{\mu} X \xleftarrow{\nu} Y$ (three objects with two arrows). Verify that any inverse limit must be a *pullback* $M \xleftarrow{\hat{\nu}} \hat{X} \xrightarrow{\hat{\mu}} Y$. Similarly for infinite collections of arrows into X.

In practice, we are often particularly interested in taking inverse limits for a special type of diagram.

Definition 5.2. *Let* $(J, \leq)$ *be a preorder. Suppose that for each* $j \leq k$ *where* $j, k \in J$ *we have an arrow* $X_j \xleftarrow{\varphi_{jk}} X_k$ *in some category* $\mathfrak{C}$*. Suppose further that whenever* $i \leq j \leq k$ *where* $i, j, k \in J$*, we have* $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$*; that is, the diagram*

$$\begin{array}{ccc} X_j & \xleftarrow{\varphi_{jk}} & X_k \\ \varphi_{ij} \downarrow & & \varphi_{ik} \downarrow \\ X_i & = & X_i \end{array}$$

commutes. Then the collection of arrows $\{\varphi_{jk} : j \leq k\}$ *forms an* **inverse system** *(or* **system***) in* $\mathfrak{C}$*.*

Note that to check that the arrows $X_j \xleftarrow{\mu_j} M$ define a cone over the inverse system, it suffices to check that

$$\begin{array}{ccc} M & = & M \\ \mu_j \downarrow & & \mu_k \downarrow \\ X_j & \xleftarrow{\varphi_{jk}} & X_k \end{array}$$

commutes whenever $j \leq k$ in the preorder.

Exercise: consider an inverse system $X_j \xleftarrow{\varphi_{jk}} X_k$ in $\mathfrak{Set}$. Show that the inverse limit is $X_j \xleftarrow{\lambda_j} L$ where

$$L = \left\{ (x_j) \in \prod_{j \in J} X_j : x_j = \varphi_{jk}(x_k) \ \forall j \leq k \right\}$$

(note that L is a subset of the Cartesian product $\prod X_j$) and where λ_j is the projection onto X_k. Similarly for $\mathfrak{Pos}$, $\mathfrak{Lat}$, $G\mathfrak{Set}$, $R\mathfrak{Mod}$, $K\mathfrak{Lin}$.

Exercise: consider an inverse system $X_j \overset{\varphi_{jk}}{\leftarrow} X_k$ in $\mathfrak{Top}$. Show that the vertex M of any cone must be given the *largest* topology on M making all the functions $X_j \overset{\mu_j}{\leftarrow} M$ continuous. This is often called the **pullback topology**. What about $\mathfrak{Msr}$, $\mathfrak{Man}$?

Definition 5.3. *Suppose we have a diagram consisting of various arrows $X \overset{\varphi}{\to} X'$, where X, X' are objects in the diagram (there might be many arrows, or none, between any given pair of objects in the diagram). Suppose there is an object K of $\mathfrak{C}$ together with arrows $X \overset{\kappa}{\to} K$ for each X in the diagram, such that for any $X \overset{\varphi}{\to} X'$ in the diagram,*

$$\begin{array}{ccc} L & = & L \\ {\scriptstyle\kappa}\uparrow & & {\scriptstyle\kappa'}\uparrow \\ X & \xleftarrow{\;\varphi\;} & X' \end{array}$$

commutes. Then the diagram consisting of the original arrows $X \overset{\varphi}{\to} X'$ together with the new arrows $X \overset{\kappa}{\to} K$ is called a **cocone** *and K is its* **vertex**.

This cocone is a **direct limit** *(or* **colimit***) of the original diagram if given any other cocone $X \overset{\nu}{\to} N$, there is a unique arrow $K \overset{\Upsilon}{\to} N$.*

Informally speaking, K must be "the first" object which can be the vertex of a cocone for the original diagram. The term *direct* limit is used because the limit is taken "with the arrows".

Exercise:

1. Consider the diagram consisting of two objects X, Y and no arrows in $\mathfrak{C}$. Verify that any direct limit must be a *sum* of X, Y in $\mathfrak{C}$. Similarly for arbitrary collections $\{X_j : j \in J\}$.
2. Consider the diagram $X \overset{\varphi,\psi}{\longrightarrow} Y$ (two objects with two arrows). Verify that any direct limit must be a *coequalizer* of φ, ψ in $\mathfrak{C}$. Similarly for an arbitrary collection of arrows in $\mathrm{Hom}(X, Y)$.
3. Consider the diagram $A \to X \leftarrow B$ (three objects with two arrows). Verify that any direct limit must be a *pushout* $A \leftarrow \check{X} \to B$. Similarly for an arbitrary collection of arrows $X_j \to X$.

In practice, we are often particularly interested in taking the direct limit of a special type of diagram (dual to the inverse system).

Definition 5.4. *Let $(J, \leq)$ be a preorder. Suppose that for each $j \leq k$ where $j, k \in J$ we have an arrow $X_j \overset{\varphi_{jk}}{\to} X_k$ in some category $\mathfrak{C}$. Suppose further that whenever $i \leq j \leq k$ where $i, j, k \in J$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$; that is, the diagram*

$$\begin{array}{ccc} X_j & \xrightarrow{\;\varphi_{jk}\;} & X_k \\ {\scriptstyle\varphi_{ij}}\uparrow & & {\scriptstyle\varphi_{ik}}\uparrow \\ X_i & = & X_i \end{array}$$

commutes. Then the collection of arrows $\{\varphi_{jk} : j \leq k\}$ forms a **direct system** *(or* **cosystem***) in $\mathfrak{C}$.*

Note that to check that the arrows $X_j \overset{\nu_j}{\to} N$ define a cone over the direct system, it suffices to check that

$$\begin{array}{ccc} N & = & N \\ {\scriptstyle \nu_j}\uparrow & & {\scriptstyle \nu_k}\uparrow \\ X_j & \xrightarrow{\varphi_{jk}} & X_k \end{array}$$

commutes whenever $j \leq k$ in the preorder.

Exercise: consider a direct system $X_j \overset{\varphi_{jk}}{\to} X_k$ in $\mathfrak{Set}$. Verify that the direct limit is given by certain functions $X_j \overset{\kappa_j}{\to} K$. Here

$$K = \left(\biguplus_{j \in J} X_j \right) / \sim$$

and $\sim$ is the equivalence relation defined by saying that $x_j \sim x_k$ whenever there is some $i \leq j, k$ with $\varphi_{ij}(x_i) = x_j$ and $\varphi_{ik}(x_i) = x_k$. Note that $[x_j]$ consists of certain elements $x_k \in X_k$ for all $j \leq k$, so the classes $[x_j]$ correspond to points $x_j \in X_j$. This means we can take the κ_j to the obvious embedding $x_j \mapsto [x_j]$). Similarly for $G\mathfrak{Set}$, where K has the quotient action.

Incidently, the reason the *inverse* limit is also called the *limit* in category theory while the *direct* limit is also called the *colimit* is that, as the reader may already have observed, objects defined by UMP's saying they are "the first" object satisfying certain conditions are called "cothings" whereas objects defined by the dual UMP saying they are "the last" object satisfying the dual conditions (obtained by reversing all arrows) are just called "things". The reader should contemplate Figure 2, where the UMP's for various constructions we have considered are indicated by schematic diagrams. The organization of these diagrams into dual pairs should make it clear why the equalizer, pullback, and inverse limit all tend to be *subobjects* of a *product*, whereas the coequalizer, pushout, and direct limit all tend to be *quotient objects* of a *sum*.

Exercise: consider a direct system $X_j \overset{\varphi_{jk}}{\to} X_k$ in $\mathfrak{Top}$. Verify that the vertex N of any cocone $X_j \overset{\nu_j}{\to} N$ must be given the *smallest* topology making all the ν_j continuous. This topology is often called the **pushout topology**.

6. Functors

> He scratched his head and wondered whose fault this was; wondering if this not after all another of those meaningless cruelties happening far from the Bureaucracy of Heaven?
>
> Russell Edson

Definition 6.1. *Let $\mathfrak{C}, \mathfrak{D}$ be two categories. Suppose we have a rule which*

1. *assigns to every object X in $\mathfrak{C}$ an object $\mathcal{F}X$ in $\mathfrak{D}$,*
2. *assigns to every arrow φ in $\mathfrak{C}$ an arrow $\mathcal{F}\varphi$ in $\mathfrak{D}$.*

Then $\mathcal{F}$ is a **functor** *(or* **covariant functor***) from $\mathfrak{C}$ to $\mathfrak{D}$ if*

1. *for all arrows φ in $\mathfrak{C}$, we have*

$$\begin{aligned} \operatorname{dom} \mathcal{F}\varphi &= \mathcal{F}(\operatorname{dom} \varphi) \\ \operatorname{cod} \mathcal{F}\varphi &= \mathcal{F}(\operatorname{cod} \varphi) \end{aligned}$$

2. *for all objects X in $\mathfrak{C}$, we have $\mathcal{F}1_X = 1_{\mathcal{F}X}$,*

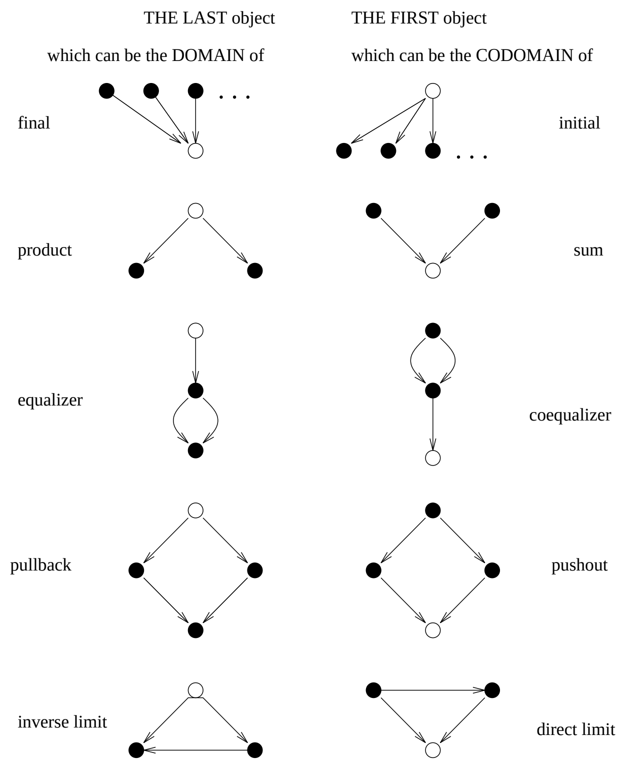

FIGURE 2. Schematic *commuting* diagrams illustrating various dual pairs of UMP's, ("co-things" on the right), where in each case every black vertex represents some given object and the white vertex represents the object to be defined by the UMP, in terms of the given object(s).

3. *for any arrows* φ, ψ *in* $\mathfrak{C}$ *with* $\operatorname{cod}\varphi = \operatorname{dom}\psi$*, we have* $\mathcal{F}(\psi\circ\varphi) = (\mathcal{F}\psi)\circ(\mathcal{F}\varphi)$.

In other words, a functor is a map between two categories which respects the categorical structure, namely the operators $\operatorname{dom}, \operatorname{cod}, \operatorname{id}, \circ$.

One way to think about the meaning of this is to observe that any functor from $\mathfrak{C}$ to $\mathfrak{D}$ takes commuting diagrams in $\mathfrak{C}$ to commuting diagrams in $\mathfrak{D}$; for instance, if

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ S & \xrightarrow[\nu]{} & T \end{array}$$

commutes, so does

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}\mu} & \mathcal{F}Y \\ {\scriptstyle\mathcal{F}\sigma}\downarrow & & \downarrow{\scriptstyle\mathcal{F}\tau} \\ \mathcal{F}S & \xrightarrow[\mathcal{F}\nu]{} & \mathcal{F}T \end{array}$$

In this paper, functors will always be denoted by calligraphic capital letters.

Let $\mathcal{F}$ be a functor from $\mathfrak{C}$ to $\mathfrak{D}$. Given any two objects X, Y of $\mathfrak{C}$, we can restrict the mapping on arrows of $\mathfrak{C}$ which is defined by $\mathcal{F}$ to the set $\mathrm{Hom}(X, Y)$ If this restriction is always a one-one mapping, $\mathcal{F}$ is called **faithful**. If it is always onto, $\mathcal{F}$ is called **full**.

Because functors are built out of maps, they can be composed in an obvious way. Moreover, the obvious identity map on objects and arrows defines a functor, the **identity functor**, from $\mathfrak{C}$ to itself. This suggests considering categories whose objects are some collection of categories and whose arrows are some collection (closed under composition) of functors between these categories (clearly this collection must also contain the appropriate identity functors).

Exercise: given a group G, let $\mathcal{F}G$ be the underlying set. Similarly, given a group homomophism φ let $\mathcal{F}\varphi$ be the underlying map. Verify this defines a (faithful but non-full) functor, called the **forgetful functor**, from $\mathfrak{G}rp$ to $\mathfrak{S}et$. Similarly for $\mathfrak{P}os$, $\mathfrak{L}at$, $G\mathfrak{S}et$, $R\mathfrak{M}od$, $\mathfrak{R}ng$, $\mathfrak{C}ru$.

Exercise: suppose $\mathfrak{A}$ is a subcategory of $\mathfrak{B}$. Verify that the mapping taking $\varphi : X \to Y$ (considered as an arrow of $\mathfrak{A}$) to itself (considered as an arrow of $\mathfrak{B}$), defines a functor, called the **inclusion functor** $\mathcal{I}$.

Exercise: define a map taking any arrow $X \stackrel{\varphi}{\to} Y$ of $\mathfrak{C}$ to the identity map on $*$, the unique object of the trivial category, $\mathfrak{Z}$. Verify that this defines a functor, called the **trivial functor**.

Exercise: Let $\mathfrak{C}$ be some category and fix an object E of $\mathfrak{C}$.

1. Suppose that products always exist in $\mathfrak{C}$. Show that we obtain a functor by taking the object X to $X \times E$ and taking the arrow $X \stackrel{\varphi}{\to} Y$ to the **product arrow** $X \times E \stackrel{\varphi \times 1_E}{\longrightarrow} Y \times E$.
2. Suppose that sums always exist in $\mathfrak{C}$. Show that we obtain a functor by taking the object X to $X + E$ and taking the arrow $X \stackrel{\varphi}{\longrightarrow} Y$ to the **sum arrow** $X + E \stackrel{\varphi + 1_E}{\to} Y + E$.

Definition 6.2. *Let $\mathfrak{C}, \mathfrak{D}$ be two categories. Suppose we have a rule which*

1. *assigns to every object X of $\mathfrak{C}$ an object $\mathcal{G}X$ of $\mathfrak{D}$,*
2. *assigns to every arrow φ of $\mathfrak{C}$ an arrow $\mathcal{G}\varphi$ of $\mathfrak{D}$.*

Then $\mathcal{G}$ is a **cofunctor** *(or* **contravariant functor***) from $\mathfrak{C}$ to $\mathfrak{D}$ if*

1. *for all arrows φ of $\mathfrak{C}$, we have*
$$\begin{aligned} \mathrm{dom}(\mathcal{G}\varphi) &= \mathcal{G}(\mathrm{cod}\,\varphi) \\ \mathrm{cod}(\mathcal{G}\varphi) &= \mathcal{G}(\mathrm{dom}\,\varphi) \end{aligned}$$
2. *for all objects X of $\mathfrak{C}$, we have $\mathcal{G}1_X = 1_{\mathcal{G}X}$,*
3. *for any arrows φ, ψ of $\mathfrak{C}$ with $\mathrm{dom}\,\psi = \mathrm{cod}\,\varphi$, we have $\mathcal{G}(\psi \circ \varphi) = (\mathcal{G}\varphi) \circ (\mathcal{G}\psi)$.*

In other words, a cofunctor respects identity arrows but interchanges domain and codomain of arrows and also reverses the order of all compositions. The difference between functors and cofunctors is probably most easily understood in terms of diagrams: if $\mathcal{G}$ is a cofunctor from $\mathfrak{C}$ to $\mathfrak{D}$, then whenever

$$\begin{array}{ccc} X & \stackrel{\mu}{\longrightarrow} & Y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ S & \underset{\nu}{\longrightarrow} & T \end{array}$$

commutes, so does

$$\begin{array}{ccc} \mathcal{G}X & \xleftarrow{\mathcal{G}\mu} & \mathcal{G}Y \\ {\scriptstyle \mathcal{G}\sigma}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathcal{G}\tau} \\ \mathcal{G}S & \xleftarrow[\mathcal{G}\nu]{} & \mathcal{G}T \end{array}$$

The point is that *all arrows are reversed when we pass from from one category to another via a cofunctor.* This "reversal of arrows" is the reason for the prefix *co* in the term *cofunctor.*

Exercise: given a category $\mathfrak{C}$, show that reversing the direction of all arrows gives a new category, the **opposite** category $\mathfrak{C}^{\mathrm{op}}$. Show that the cofunctors from $\mathfrak{C}$ are exactly the functors from $\mathfrak{C}^{\mathrm{op}}$.

Exercise: given a set X, let $\mathcal{P}X$ be the **powerset** $\mathcal{P}X = \{A \subset X\}$, and given a mapping $\varphi : X \to Y$, let $\mathcal{P}\varphi$ be the mapping taking $B \in \mathcal{P}Y$ to $\varphi^{-1}(B) \in \mathcal{P}X$. Verify that this defines a cofunctor, the **preimage functor**, from $\mathfrak{Set}$ to itself.

Exercise: given an abelian group A, its **character group** is $\mathcal{C}A$, the group of all homomorphisms $\chi : A \to \mathcal{C}$ (with pointwise multiplication). Given a homomorphism $\varphi : A \to B$, where A, B are abelian groups, define $\mathcal{C}\varphi : \mathcal{C}A \leftarrow \mathcal{C}B$ by $(\mathcal{C}\varphi)(\chi) = \varphi \circ \chi$ for each $\chi \in \mathcal{C}B$. Verify that this defines a cofunctor from $\mathfrak{Abg}$ to itself.

Exercise:

1. Suppose $\mathfrak{C}$ is some category such that pullbacks always exist in $\mathfrak{C}$. Then we can define a cofunctor, the **poset of subobjects functor**, from $\mathfrak{C}$ to $\mathfrak{Pos}$ by taking each object X of $\mathfrak{C}$ to the poset $\mathrm{Sub}(X)$, and taking each arrow $X \stackrel{\varphi}{\to} Y$ to the mapping $\mathrm{Sub}(X) \leftarrow \mathrm{Sub}\, Y$, obtained by taking the arrow $B \stackrel{\beta}{\to} Y$ (representing some subobject of Y) to the pullback of β along φ (which represents some subobject of X).
2. Dualize this to obtain the **poset of quotients functor** from $\mathfrak{C}$ to $\mathfrak{Pos}$, in the case when pushouts always exist in $\mathfrak{C}$.

Exercise: let $\mathfrak{C}$ be any category and fix an object E of $\mathfrak{C}$.

1. Define a rule taking every object X of $\mathfrak{C}$ to the set $E_\natural X = \mathrm{Hom}(E, X)$ and taking the arrow $\varphi : X \to Y$ to the mapping $E_\natural\varphi = \varphi \circ (\cdot) : E_\natural X \to E_\natural Y$ defined by taking the arrow $\sigma : E \to X$ to the mapping $(E_\natural\varphi)(\sigma) = \varphi \circ \sigma : E \to Y$. Show this defines a functor from $\mathfrak{C}$ to $\mathfrak{Set}$, the **hom functor** induced by E, denoted by $E_\natural$.
2. Define a rule taking every object X of $\mathfrak{C}$ to the set $E^\natural X = \mathrm{Hom}(X, E)$ and taking every arrow $\varphi : X \to Y$ to the mapping $E^\natural\varphi = (\cdot) \circ \varphi : E^\natural X \leftarrow E^\natural Y$ defined by taking each arrow $\tau : Y \to E$ to the arrow $(E^\natural\varphi)(\tau) = \tau \circ \varphi : X \to E$. Show this defines a cofunctor from $\mathfrak{C}$ to $\mathfrak{Set}$, the **hom cofunctor** induced by E, denoted by $E^\natural$.

In the last exercise, we have broken our convention that a functor or cofunctor is denoted by a calligraphic capital letter, in order to emphasize the dependence of the definitions on E. In situations like this where one has a functor and a closely related cofunctor, it is standard practice to denote the functor by some distinctive subscript, e.g. $X \stackrel{\varphi}{\to} Y$ goes to $E_\natural X \stackrel{E_\natural\varphi}{\longrightarrow} E_\natural Y$, and the cofunctor by the same symbol used as a superscript, e.g. $E^\natural X \stackrel{E^\natural\varphi}{\longleftarrow} E^\natural Y$.

7. Naturality

> Father said, well, I hardly expected this.
>
> And mother said, well, this was really not quite expected, but past the initial shock one learns to expect what has already happened.
>
> Russell Edson

Definition 7.1. *Let $\mathfrak{A}, \mathfrak{B}$ be two categories and let $\mathcal{F}, \mathcal{G}$ be two functors from $\mathfrak{A}$ to $\mathfrak{B}$. A* **natural transformation** *from $\mathcal{F}$ to $\mathcal{G}$ is a rule assigning to each object X of $\mathfrak{A}$ an arrow $\mathcal{F}X \xrightarrow{\omega_X} \mathcal{G}X$ of $\mathfrak{B}$ such that for each arrow $X \xrightarrow{\varphi} Y$ in $\mathfrak{A}$, the diagram*

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}\varphi} & \mathcal{F}Y \\ {\scriptstyle \omega_X}\downarrow & & \downarrow{\scriptstyle \omega_Y} \\ \mathcal{G}X & \xrightarrow[\mathcal{G}\varphi]{} & \mathcal{G}Y \end{array}$$

commutes.

Here, the arrows ω_X are called the **components** of the natural transformation. If they are in fact isomorphisms, we have a **natural isomorphism** between $\mathcal{F}$ and $\mathcal{G}$.

Natural transformations between pairs of cofunctors are defined similarly.

Exercise: given a functor $\mathcal{F}$ from $\mathfrak{C}$ to $\mathfrak{D}$, verify that we can define a new functor from $\mathfrak{C}$ to $\mathfrak{Set}$ by taking the arrow $\varphi : X \to Y$ to the mapping $(\mathcal{F}\varphi) \circ (\cdot) : \mathrm{Hom}(\mathcal{F}X, \mathcal{F}E) \to \mathrm{Hom}(\mathcal{F}Y, \mathcal{F}E)$. Verify that $\mathcal{F}$ defines a natural transformation from the postcomposition functor induced by E to this functor. (The component $\omega_X : \mathrm{Hom}(E, X) \to \mathrm{Hom}(\mathcal{F}E, \mathcal{F}X)$ is defined in the obvious way.)

Exercise: suppose $\omega_{\{\cdot\}}$ is a natural transformation from $\mathcal{F}$ to $\mathcal{F}'$ and suppose $\omega'_{\{\cdot\}}$ is a natural transformation from $\mathcal{F}'$ to $\mathcal{F}''$. Show that $\omega'_{\{\cdot\}} \circ \omega_{\{\cdot\}}$ defines a natural transformation from $\mathcal{F}$ to $\mathcal{F}''$.

Exercise: suppose $\mathcal{E}$ is a functor from $\mathfrak{A}$ to $\mathfrak{B}$, that $\mathcal{F}, \mathcal{G}$ are functors from $\mathfrak{B}$ to $\mathfrak{C}$, and that $\mathcal{H}$ is a functor from $\mathfrak{C}$ to $\mathfrak{D}$, and suppose that $\omega_{\{\cdot\}}$ is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$. Show that we obtain a natural transformation from $\mathcal{H} \circ \mathcal{F} \circ \mathcal{E}$ to $\mathcal{H} \circ \mathcal{G} \circ \mathcal{E}$.

Exercise: consider the category of finite dimensional real vector spaces and linear mappings (a subcategory of $K\mathfrak{Lin}$, where $K = \mathbb{R}$). Let $\mathcal{D}$ be the dualizing functor taking V to V^* and $U \xrightarrow{\varphi} W$ to $U^* \xleftarrow{\varphi^*} W^*$ where $\varphi^*(g) = g \circ \varphi$, and let $\mathcal{I}$ be the identity functor. Show that for each V, there is an isomorphism $V \xrightarrow{\xi_V} V^*$. However, there is no way of choosing the ξ_V so as to construct a *natural* isomorphism $V \simeq V^*$.

Natural isomorphisms are the *natural* way to express formal properties of the categorical constructions (product, sum, etc.) we have discussed. To demonstrate this, we must introduce a simple operation on categories.

Definition 7.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two categories. The objects of the* **product category** *$\mathfrak{A} \times \mathfrak{B}$ are pairs (A, B) where B is an object of $\mathfrak{A}$ and B is an object of $\mathfrak{B}$, and the arrows of $\mathfrak{A} \times \mathfrak{B}$ are pairs (α, β) where $A \xleftarrow{\alpha} A'$ is an arrow of $\mathfrak{C}$ and $B \xleftarrow{\beta} B'$ is an arrow of $\mathfrak{D}$. Here,*

$$\begin{aligned} \mathrm{dom}(\alpha, \beta) &= (\mathrm{dom}\,\alpha, \mathrm{dom}\,\beta) \\ \mathrm{cod}(\alpha, \beta) &= (\mathrm{cod}\,\alpha, \mathrm{cod}\,\beta) \end{aligned}$$

and $1_(A, B) = (1_A, 1_B)$. *Also, given* (α', β'), *where* $A' \stackrel{\alpha'}{\leftarrow} A''$ *and* $B' \stackrel{\beta'}{\leftarrow} B''$, *the composition in* $\mathfrak{A} \times \mathfrak{B}$ *is*

$$(\alpha', \beta') \circ (\alpha, \beta) = (\alpha' \circ \alpha, \beta' \circ \beta)$$

Exercise: suppose $\mathfrak{C}$ is some category in which products always exist, and fix an object E of $\mathfrak{C}$. Consider the map taking each arrow $X \stackrel{\varphi}{\to} Y$ of $\mathfrak{C}$ to the arrow $X \times E \stackrel{\varphi \times 1_E}{\longrightarrow} Y \times E$ of $\mathfrak{C}$, where $\varphi \times 1_E$ is the product arrow defined in Section 4. Verify this defines a functor $\mathcal{F}$ from $\mathfrak{C}$ to itself. Similarly, consider the map which takes each arrow $X \stackrel{\varphi}{\to} Y$ of $\mathfrak{C}$ to the arrow $E \times X \stackrel{1_E \times \varphi}{\longrightarrow} E \times Y$ of $\mathfrak{C}$. Verify this defines a functor $\mathcal{G}$ from $\mathfrak{C}$ to itself. Show that there is a natural isomorphism from $\mathcal{F}$ to $\mathcal{G}$. This statement can be abbreviated by saying that for all X in $\mathfrak{C}$ there is an isomorphism $X \times E \simeq E \times X$ which is "natural in X". Now observe that, by symmetry, we have an isomorphism $X \times Y \simeq Y \times X$ which is "natural in both variables individually".

Exercise: Define two functors from $\mathfrak{C} \times \mathfrak{C}$ to $\mathfrak{C}$, as follows. Let $\mathcal{F}$ take the object (X, Y) of $\mathfrak{C} \times \mathfrak{C}$ to the object $X \times Y$ of $\mathfrak{C}$, and let it act in the obvious way on arrows. Let $\mathcal{G}$ take (X, Y) to $Y \times X$ and let it act in the obvious way on arrows. Show that we have an isomorphism $X \times Y \simeq Y \times X$ which is "natural in both variables jointly".

Exercise: state and prove lemmas to the effect that there are isomorphisms

$$(X \times Y) \times Z \simeq X \times (Y \times Z)$$
$$(X + Y) + Z \simeq X + (Y + Z)$$
$$X \times (Y + Z) \simeq (X \times Y) + (X \times Z)$$
$$1 \times X \simeq X$$
$$0 + X \simeq X$$

(natural in all three variables).

Exercise: suppose products always exist in $\mathfrak{A}$, and suppose that $\mathcal{F}, \mathcal{G}$ are two functors from $\mathfrak{A}$ to $\mathfrak{B}$. Show that the arrows $X \stackrel{\pi_X}{\leftarrow} X \times Y \stackrel{\pi_Y}{\to} Y$ induce a unique functor $\mathcal{F} \times \mathcal{G}$ possessing natural transformations to both $\mathcal{F}$ and $\mathcal{G}$.

Exercise: verify that the natural projections $G \stackrel{\omega_G}{\to} G/[G, G]$ give a natural transformation from the identity functor on $\mathfrak{G}rp$ to the functor on $\mathfrak{G}rp$ defined by taking G to its **abelianization** $G/[G, G]$ and taking the group homomorphism $\varphi : G \to H$ to the induced homomorphism $\varphi_* : G/[G, G] \to H/[H, H]$. (Here $[G, G]$ is the **commutator subgroup** of G; see [15].)

Exercise: recall that the character functor $\mathcal{C}$ is a cofunctor from $\mathfrak{A}bg$ to itself, so that the composed functor $\mathcal{C}\mathcal{C}$ is a functor from $\mathfrak{A}bg$ to itself. Given an abelian group A, define a homomorphism $\omega_A : A \to \mathcal{C}\mathcal{C}A$ by taking $a \in A$ to the character of $\mathcal{C}A$ defined by $\chi \mapsto \chi(a)$. Verify that this defines a natural transformation from the identity functor on $\mathfrak{A}bg$ to $\mathcal{C}\mathcal{C}$.

8. Operations on Categories

> One morning a man awakened to find strings coming through the window attached to his hands and feet.
>
> ... I'm not a marionette, he says, his voice rising with the question, am I? Am I a marionette?
>
> Russell Edson

Definition 8.1. *Fix an object X of some category $\mathfrak{C}$. Consider the arrows in*

$$\mathrm{Hom}(\cdot, X) = \{\sigma : \mathrm{cod}\,\sigma = X\}$$

to be objects, and consider any arrow $E \stackrel{\eta}{\to} F$ of $\mathfrak{C}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\eta} & F \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ X & = & X \end{array}$$

commutes to be an arrow from $E \stackrel{\alpha}{\to} X$ to $F \stackrel{\beta}{\to} X$ This defines a category called the **slice category** *over X, denoted $\mathfrak{C}/X$.*

If X is a topological space, the category $\mathfrak{Top}/X$ is just $\mathfrak{Bn}\,X$.

Exercise: show that there is a final object in $\mathfrak{C}/X$. (Hint: consider the identity arrow for X in $\mathfrak{C}$.) Observe how your argument blurs the distinction between objects and arrows of $\mathfrak{C}/X$.

Exercise: suppose products always exist in $\mathfrak{C}$. Fix an object E of $\mathfrak{C}$. Show that we can define a functor E^*, the **slice functor**, from $\mathfrak{C}$ to $\mathfrak{C}/E$ as follows. Take X to the canonical arrow $E \times X \to E$ and take $\varphi : X \to Y$ to $1_E \times \varphi$.

Exercise: suppose $\mathfrak{C}$ is a category in which sums always exist.

1. Show that we can "add" the objects of the slice category $\mathfrak{C}/X$. More precisely, given $S \stackrel{\sigma}{\to} X$ and $T \stackrel{\tau}{\to} X$, define an arrow $S + T \stackrel{\sigma+\tau}{\to} X$, where $S + T$ is the sum of the objects S, T of $\mathfrak{C}$. (Hint: use the UMP of a sum.)
2. Prove the identity $1_{S+T} = 1_S + 1_T$.
3. Show that $\sigma + \tau$ is isomorphic to $\tau + \sigma$ in $\mathfrak{C}/X$.
4. Show that $\mathrm{Hom}(S + T, X)$ is in bijection with $\mathrm{Hom}(S, X) \uplus \mathrm{Hom}(T, X)$.
5. Now suppose that there is an initial object I in $\mathfrak{C}$. Show that S must be isomorphic to $S + I$. (Hint: show that $1_S + \Upsilon$, where $I \stackrel{\Upsilon}{\to} S$, is a left inverse of $S \stackrel{\eta_X}{\to} S + I$. Now use $1_{S+T} = 1_S + 1_T$ to obtain a right inverse.)
6. Conclude that the objects $\sigma + 1_X, \sigma$ are isomorphic in $\mathfrak{C}/X$.
7. Show that isomorphism classes of objects in $\mathfrak{C}/X$ form a commutative monoid.

Exercise: define a category $S/\mathfrak{C}$, called the **coslice category**, which is "dual" to the slice category $\mathfrak{C}/S$.

The categories $S/\mathfrak{C}$ and $\mathfrak{C}/S$ are often called **comma categories**.

Exercise: suppose $\mathfrak{C}$ is a category in which products always exist.

1. Show that we can "multiply" the objects of the coslice category $X/\mathfrak{C}$. More precisely, given $X \stackrel{\mu}{\to} M$ and $X \stackrel{\nu}{\to} N$ define an arrow $X \stackrel{\mu\times\nu}{\to} M \times N$.
2. Show that $\mu \times \nu$ is isomorphic to $\nu \times \mu$ in $X/\mathfrak{C}$.
3. Prove the identity $1_M \times 1_N = 1_{M\times N}$.
4. Show that $\mathrm{Hom}(X, M \times N)$ is in bijection with $\mathrm{Hom}(X, M) \times \mathrm{Hom}(X, N)$.
5. Now suppose there is a final object F in $\mathfrak{C}$, and show that $M \times F$ is always isomorphic to M.
6. Conclude that the objects $1_M, 1_M \times 1_F$ are isomorphic in $X/\mathfrak{C}$.
7. Show that the isomorphism classes of objects in $X/\mathfrak{C}$ form a commutative monoid.

Exercise: Fix an arrow $E \stackrel{\psi}{\to} F$ of $\mathfrak{C}$.

1. Suppose that pushouts always exist in $\mathfrak{C}$. Show that we obtain a functor ψ_* from $E/\mathfrak{C}$ to $F/\mathfrak{C}$, called the **coslice change functor**, as follows. Given an object $\sigma : E \to X$ of $E/\mathfrak{C}$, we have the situation

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ {\scriptstyle\sigma}\big\downarrow & & \\ X & & \end{array}$$

so we can push out σ along ψ to obtain the object $\psi_*\sigma : F \to \check{X}$ of $F/\mathfrak{C}$. Similarly, given an arrow of $E/\mathfrak{C}$; that is, an arrow $\xi : X \to Y$ such that

$$\begin{array}{ccc} E & = & E \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ X & \xrightarrow{\xi} & Y \end{array} \tag{10}$$

commutes, we can pushout out (10) to obtain an arrow of $F/\mathfrak{C}$.

2. Suppose that pullbacks always exist in $\mathfrak{C}$. Show that we obtain a cofunctor ψ^* from $\mathfrak{C}/F$ to $\mathfrak{C}/E$, called the **slice change cofunctor**, as follows. Given an object $\sigma : X \to E$ of $E/\mathfrak{C}$, we have the situation

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle\sigma} \\ E & \xrightarrow{\psi} & F \end{array}$$

so we can pull back σ along ψ to obtain the object $\psi^*\sigma : \hat{F} \to E$ of $E/\mathfrak{C}$. Similarly, given an arrow of $F/\mathfrak{C}$; that is, an arrow $\xi : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\xi} & Y \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ F & = & F \end{array} \tag{11}$$

commutes, we can pull back (11) along ψ to obtain an arrow $\psi^*\xi$ of $\mathfrak{C}/S$.

The two functors defined in the preceeding exercise are also called the **change of base functors**.

Exercise: try to adapt the preceeding exercise to obtain a functor taking an arrow $X \to Y$ to a poset hom $\mathrm{Qnt}(X) \to \mathrm{Qnt}\,Y$ and a cofunctor taking an arrow $X \to Y$ to a poset hom $\mathrm{Sub}(X) \leftarrow \mathrm{Sub}\,Y$.

Definition 8.2. *Let $\mathfrak{A}, \mathfrak{B}$ be two categories. The objects of the* **functor category** $\mathfrak{B}^{\mathfrak{A}}$ *are the functors from $\mathfrak{A}$ to $\mathfrak{B}$. The arrows are natural transformations from one such functor to another.*

Exercise: what is the identity arrow of a functor from $\mathfrak{C}$ to $\mathfrak{D}$? What is the definition of composition? (See [9] for a detailed solution.) Show that the isomorphisms of $\mathfrak{D}^{\mathfrak{C}}$ are precisely the natural isomorphisms between functors from $\mathfrak{C}$ to $\mathfrak{D}$.

Exercise: let $\mathfrak{C}$ be a small category and fix an object E of $\mathfrak{C}$. Define a map taking a cofunctor $\mathcal{F}$ from $\mathfrak{C}$ to $\mathfrak{Set}$ to the set $\mathcal{F}X$ and taking a natural transformation ω from $\mathcal{F}$ to $\mathcal{G}$ to the mapping $\omega_X : \mathcal{F}X \to \mathcal{G}X$. Verify that this defines a cofunctor from $\mathfrak{Set}^{\mathfrak{C}^{\mathrm{op}}}$ to $\mathfrak{Set}$.

Exercise: observe that for the trival category $\mathfrak{J}$, $\mathfrak{T}op^{\mathfrak{J}}$ is the category of **pointed topological spaces** (X, x), where an arrow from (X, x) to (Y, y) is a continous map taking x to y. Given an object (X, x) of $\mathfrak{T}op^{\mathfrak{J}}$, let $\mathcal{H}(X, x) = \pi_1(X, x)$ be the **homotopy group** consisting of the homotopy classes of paths in X which begin and end at x. Similarly, given an arrow $(X, x) \stackrel{\varphi}{\to} (Y, y)$ of $\mathfrak{T}op^{\mathfrak{J}}$ let $\mathcal{H}\varphi : \mathbf{h}X \to \mathbf{h}Y$ be the group homomorphism defined by $[\alpha] \mapsto [\varphi \circ \alpha]$ (note this is well defined.) Verify that this defines a functor from $\mathfrak{T}op^{\mathfrak{J}}$ to $\mathfrak{G}rp$. (See, for instance, [8] for more about the homotopy functor).

Exercise: Let X be a topological space. If $\mathcal{T}$ is the collection of open sets of X, considered as a poset under inclusion, let $\mathfrak{O}X$ be the corresponding preorder category. Consider cofunctors from $\mathfrak{O}X$ to $\mathfrak{S}et$ to be objects, and consider natural transformations among such objects to be arrows. Verify that this defines a category. It is called the category **presheaves over** X and is denoted $\mathfrak{P}\mathrm{s}\, X$. If $U \subset V$, so that we obtain an arrow $\mathcal{F}U \leftarrow \mathcal{F}V$, given $t \in \mathcal{F}V$, let $t|U$ denote the image of t under this arrow. (We are thinking of $\mathcal{F}V$ as a set of functions defined on open sets "sitting over" V.)

Exercise: Define a cofunctor from $\mathfrak{P}\mathrm{s}\, X$ to $\mathfrak{E}\mathrm{t}\, X$ as follows. Start with a presheaf $\mathcal{P}$ over X. Given U, V open neighborhoods of $x \in X$, if $s \in \mathcal{F}U$ and $t \in \mathcal{F}V$, define $s \sim_x t$ if there is some neighborhood W of x such that $s|U \cap W = t|V \cap W$. Verify that this is an equivalence relation. For each $x \in X$, define the set of **germs at** x to be

$$\mathcal{P}_x = \{[s] : s \in \mathcal{P}U : U \text{ open nghbd of } x\}$$

Now let $\mathcal{P}_x$ be the inverse limit over the inverse system defined by the neighborhoods of x (ordered by inclusion). Now let $\mathcal{G}rm\mathcal{P} = \uplus_{x \in X} \mathcal{P}_x$ be given the pullback topology, via the map $\mathcal{G}rm\mathcal{P} \stackrel{\mu}{\to} X$ defined by $\mu([s]) = x$ where $[s] \in \mathcal{P}_x$. Verify that μ is not only continuous but a local homeomorphism, and that $\mathcal{G}rm$ defines a cofunctor. It is called the **sheaf of germs cofunctor** from $\mathfrak{P}\mathrm{s}\, X$ to $\mathfrak{E}\mathrm{t}\, X$.

Exercise:

1. Define a category with two objects U, V and four arrows, $1_U, 1_V$ and $U \stackrel{\mu,\nu}{\longrightarrow} V$. Verify that this is a preorder category. It is denoted $\mathfrak{P}air$ or $\downarrow\downarrow$. Verify that the objects of the functor category $\mathfrak{C}^{\downarrow\downarrow}$, the **category of pairs**, are diagrams in $\mathfrak{C}$ with the "shape" $X \stackrel{\varphi,\psi}{\longrightarrow} Y$. Verify that an arrow in $\mathfrak{C}^{\downarrow\downarrow}$ from $X \stackrel{\varphi,\psi}{\longrightarrow} Y$ to $A \stackrel{\sigma,\tau}{\longrightarrow} B$ is a pair of commuting diagrams

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[\sigma]{} & B \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[\tau]{} & B \end{array}$$

Verify that here α, β are the components of a natural transformation. (The first functor takes the nonidentity arrows of the category $\mathfrak{P}air$ to the arrows $X \stackrel{\varphi,\psi}{\to} Y$ of $\mathfrak{C}$, while the second takes the nonidentity arrows of $\mathfrak{P}air$ to the arrows $A \stackrel{\sigma,\tau}{\to} B$ of $\mathfrak{C}$.) Verify that the map taking an arrow $X \stackrel{\varphi}{\to} Y$ to the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X,1_X} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow[1_Y,1_Y]{} & Y \end{array}$$

defines a functor, called the **diagonal functor**, from $\mathfrak{C}$ to $\mathfrak{C}^{\downarrow\downarrow}$.

2. Define a category with three objects U, V, W and five arrows $1_U, 1_V, 1_W$, and $U \overset{\mu}{\to} W \overset{\nu}{\leftarrow} V$. Verify that this is in fact a category. It is denoted $\mathfrak{P}ull$ or $\searrow\swarrow$. Verify that the objects of $\mathfrak{C}^{\searrow\swarrow}$ are diagrams in $\mathfrak{C}$ with the "shape" $X \overset{\varphi}{\to} Z \overset{\psi}{\leftarrow} Y$. Verify that an arrow in $\mathfrak{C}^{\searrow\swarrow}$ from $X \overset{\varphi}{\to} Z \overset{\psi}{\leftarrow} Y$ to $A \overset{\sigma}{\to} C \overset{\tau}{\leftarrow} B$ has the form

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Z & \xleftarrow{\psi} & Y \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\beta}\downarrow \\ A & \xrightarrow{\sigma} & C & \xleftarrow{\tau} & B \end{array}$$

Verify that here α, β, γ are the components of a natural transformation. Verify that the map taking an arrow $X \overset{\varphi}{\to} Y$ to the commuting diagram

$$\begin{array}{ccccc} X & \xrightarrow{1_X} & X & \xleftarrow{1_X} & X \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi}\downarrow \\ Y & \xrightarrow{1_Y} & Y & \xleftarrow{1_Y} & Y \end{array}$$

defines a functor, called the **diagonal functor**, from $\mathfrak{C}$ to $\mathfrak{C}^{\searrow\swarrow}$.

3. Define a category with three objects U, V, W and five arrows $1_U, 1_V, 1_W$, and $U \overset{\mu}{\leftarrow} W \overset{\nu}{\to} V$. It is denoted $\mathfrak{P}ush$ or $\nwarrow\nearrow$. Verify that the objects of $\mathfrak{C}^{\nwarrow\nearrow}$ are diagrams in $\mathfrak{C}$ with "shape" $X \overset{\varphi}{\leftarrow} Z \overset{\psi}{\to} Y$. Verify that an arrow in $\mathfrak{C}^{\nwarrow\nearrow}$ from $X \overset{\varphi}{\leftarrow} Z \overset{\psi}{\to} Y$ to $A \overset{\sigma}{\leftarrow} C \overset{\tau}{\to} B$ has the form

$$\begin{array}{ccccc} X & \xleftarrow{\varphi} & Z & \xrightarrow{\psi} & Y \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\beta}\downarrow \\ A & \xleftarrow{\sigma} & C & \xrightarrow{\tau} & B \end{array}$$

Verify that here α, β, γ are the components of a natural transformation. Verify that the map taking the arrow $X \overset{\varphi}{\to} Y$ to the commuting diagram

$$\begin{array}{ccccc} X & \xleftarrow{1_X} & X & \xrightarrow{1_X} & X \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi}\downarrow \\ Y & \xleftarrow{1_Y} & Y & \xrightarrow{1_Y} & Y \end{array}$$

defines a functor, called the **diagonal functor**, from $\mathfrak{C}$ to $\mathfrak{C}^{\nwarrow\nearrow}$.

This suggests thinking of $\mathfrak{C}^{\mathfrak{D}}$ as diagrams in $\mathfrak{C}$ of "shape" $\mathfrak{D}$, where $\mathfrak{D}$ is considered as a humungous digraph. We can think of the objects of $\mathfrak{C}^{\mathfrak{D}}$ as **labeled digraphs** where the edges are labled by arrows of $\mathfrak{C}$ and the vertices are labled by objects of $\mathfrak{C}$.

Exercise:

1. Show that we can construct a cofunctor from $\mathfrak{C}$ to $\mathfrak{Set}^{\mathfrak{C}}$, called the **Yoneda cofunctor**, as follows. Given $\psi : E \to F$, we obtain a natural transformation $\psi^{\#}$ from $F_\natural$ to $E_\natural$ (hom functors induced by F, E respectively) via diagrams like

$$\begin{array}{ccc} E_\natural X = \mathrm{Hom}(E,X) & \xrightarrow{\varphi\circ(\cdot)=E_\natural\varphi} & E_\natural Y = \mathrm{Hom}(E,Y) \\ {\scriptstyle \psi^{\#}_X=(\cdot)\circ\psi}\Big\uparrow & & \Big\uparrow{\scriptstyle \psi^{\#}_Y=(\cdot)\circ\psi} \\ F_\natural X = \mathrm{Hom}(F,X) & \xrightarrow{\varphi\circ(\cdot)=F_\natural\varphi} & F_\natural Y = \mathrm{Hom}(F,Y) \end{array}$$

2. Show that we can construct a functor from $\mathfrak{C}$ to $\mathfrak{Set}^{\mathfrak{C}^{op}}$, called the **Yoneda functor**, as follows. Given $\psi : E \to F$, we obtain a natural transformation $\psi_{\#}$ from $E^\natural$ to $F^\natural$ (hom cofunctors induced by E, F respectively) via diagrams like

$$\begin{array}{ccc} E^\natural X = \mathrm{Hom}(X,E) & \xrightarrow{\varphi\circ(\cdot)=E^\natural\varphi} & E^\natural Y = \mathrm{Hom}(Y,E) \\ {\scriptstyle (\psi_{\#})_X=\psi\circ(\cdot)}\Big\downarrow & & \Big\downarrow{\scriptstyle (\psi_{\#})_Y=\psi\circ(\cdot)} \\ F^\natural X = \mathrm{Hom}(X,F) & \xrightarrow{\varphi\circ(\cdot)=F^\natural\varphi} & F^\natural Y = \mathrm{Hom}(Y,F) \end{array}$$

One of the fundamental theorems of category theory, the **Yoneda Embedding Theorem**, concerns these functors.

Theorem 8.3 (Yoneda). *The Yoneda functor is both full and faithfull; likewise for the Yoneda cofunctor.*

That is, we have *bijections* (in fact, natural bijections)

$$\mathrm{Hom}(E^\natural, F^\natural) \longleftrightarrow \mathrm{Hom}(E,F) \longleftrightarrow \mathrm{Hom}(F_\natural, E_\natural)$$

Sketch of proof: the principle idea is that given a natural transformation ω from $E^\natural$ to $F^\natural$, taking the arrow $\varphi : X \to Y$ to the commuting diagram

$$\begin{array}{ccc} E^\natural X & \xleftarrow{E^\natural\varphi} & E^\natural Y \\ {\scriptstyle \omega_X}\Big\downarrow & & \Big\downarrow{\scriptstyle \omega_Y} \\ F^\natural X & \xleftarrow{F^\natural\varphi} & F^\natural Y \end{array}$$

this must be induced by the *unique* arrow $\psi : E \to F$ defined by observing that 1_E is an element of $E^\natural E = \mathrm{Hom}(E,E)$, and therefore, $\omega_E(1_E)$ is an element of $F^\natural E = \mathrm{Hom}(E,F)$; setting $\psi = \omega_E(1_E)$ one can now check that this does indeed induce ω and that furthermore ψ is the unique arrow inducing ω. The other half of the Yoneda theorem now follows by duality.

This theorem is much more useful than it might at first appear. In Section 11 we shall a typical application, in which we are able to transfer algebraic structure from $E^\natural$ to E by invoking the fact that every a natural transformation from $E^\natural$ to $F^\natural$. induces a unique arrow $E \to F$.

9. Adjoints

> The isolation and explication of the notion of *adjointness* is perhaps the most profound contribution that category theory has made to the history of general mathematical ideas.
>
> *Robert Goldblatt* [9]

Adjoint functors are bunk.

Walt Pohl

Consider the UMP for a product $A \times B$. It says that given object X and arrows $A \overset{\alpha}{\leftarrow} X \overset{\beta}{\rightarrow} B$, there is a *unique* arrow

$$X \xrightarrow{\Upsilon = \triangleright(\alpha,\beta)} A \times B$$

such that the diagram

$$\begin{array}{ccccc} A & \xleftarrow{\alpha} & X & \xrightarrow{\beta} & B \\ \| & & \Upsilon \downarrow & & \| \\ A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \end{array}$$

commutes. But the pair of arrows α, β in $\mathfrak{C}$ can be regarded as a single arrow in the product category $\mathfrak{C} \times \mathfrak{C}$, so this says that given $(\alpha, \beta) \in \mathrm{Hom}_{\mathfrak{C}\times\mathfrak{C}}((X, X), (A, B))$, there is a unique $\Upsilon \in \mathrm{Hom}_{\mathfrak{C}}(X, A \times B)$ such that the diagram commutes. Conversely, given Υ, we can recover

$$(\alpha, \beta) = (\pi_A \circ \Upsilon, \beta = \pi_B \circ \Upsilon) = \triangleleft \Upsilon$$

Thus, the UMP for a product guarantees that the hom sets

$$\underset{\mathfrak{C}\times\mathfrak{C}}{\mathrm{Hom}}((X, X), (A, B)) \simeq \underset{\mathfrak{C}}{\mathrm{Hom}}(X, A \times B)$$

are isomorphic in $\mathfrak{Set}$; that is, they are in bijection via the mutually inverse maps

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}\times\mathfrak{C}}((X, X), (A, B)) & \xrightarrow{\triangleright} & \mathrm{Hom}_{\mathfrak{C}}(X, A \times B) \\ \| & & \| \\ \mathrm{Hom}_{\mathfrak{C}\times\mathfrak{C}}((X, X), (A, B)) & \xleftarrow[\triangleleft]{} & \mathrm{Hom}_{\mathfrak{C}}(X, A \times B) \end{array}$$

Moreover, *this bijection is "natural"* in the sense that it respects "perturbations" of X and (A, B). More precisely, given a **preperturbation** $X \overset{\varphi}{\leftarrow} X'$ in $\mathfrak{C}$ and a **postperturbation** $(A, B) \overset{(\mu,\nu)}{\longleftarrow} (A', B')$ in $\mathfrak{C}$, we have the new arrows

(12)
$$A' \xleftarrow{\mu\circ\alpha\circ\varphi} X' \xrightarrow{\nu\circ\beta\circ\varphi} B'$$

in $\mathfrak{C}$; applying the UMP for $A' \times B'$ guarantees that there is a unique arrow $A' \times B' \leftarrow X'$ through which the arrows (12) factor. Naturality means that this arrow is precisely the composition $A' \times B' \xleftarrow{\Upsilon' \circ \Upsilon \circ \varphi} X$ we would "naturally" expect, where $A' \times B' \overset{\Upsilon'}{\leftarrow} A \times B$ is the unique arrow induced by the arrows

$$A' \xleftarrow{\mu\circ\pi_A} A \times B \xrightarrow{\nu\circ\pi_B} B'$$

(The commuting diagram

$$\begin{array}{ccccc}
 & & X' & & \\
 & & \varphi \big\downarrow & & \\
A & \xleftarrow{\alpha} & X & \xrightarrow{\beta} & B \\
\big\| & & \Upsilon \big\downarrow & & \big\| \\
A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\
\mu \big\downarrow & & \Upsilon' \big\downarrow & & \nu \big\downarrow \\
A' & \xleftarrow{\pi_{A'}} & A' \times B' & \xrightarrow{\pi_{B'}} & B'
\end{array}$$

may help in following this discussion.)

We have established a **natural bijection**

$$\underset{\mathfrak{C}\times\mathfrak{C}}{\mathrm{Hom}}(\mathcal{D}X, (A, B)) \simeq \underset{\mathfrak{C}}{\mathrm{Hom}}(X, \mathcal{P}(A, B))$$

where $\mathcal{D}$ is the **diagonal functor** from $\mathfrak{C}$ to $\mathfrak{C} \times \mathfrak{C}$ (taking the object X of $\mathfrak{C}$ to the object (X, X) of $\mathfrak{C} \times \mathfrak{C}$) and $\mathcal{P}$ is the **product functor** from $\mathfrak{C} \times \mathfrak{C}$ back to $\mathfrak{C}$ (taking the object (A, B) of $\mathfrak{C} \times \mathfrak{C}$ to the object $A \times B$ of $\mathfrak{C}$). Such natural bijections are quite important and they occur throughout mathematics.

Definition 9.1. *Suppose $\mathcal{F}$ is a functor from $\mathfrak{A}$ to $\mathfrak{B}$ and $\mathcal{G}$ is a functor from $\mathfrak{B}$ back to $\mathfrak{A}$, such that there is a natural bijection*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{B}}(\mathcal{F}A, B) & \xrightarrow{\triangleright} & \mathrm{Hom}_{\mathfrak{A}}(A, \mathcal{G}B) \\
\big\| & & \big\| \\
\mathrm{Hom}_{\mathfrak{B}}(\mathcal{F}A, B) & \xleftarrow[\triangleleft]{} & \mathrm{Hom}_{\mathfrak{A}}(A, \mathcal{G}B)
\end{array}$$

Then $\mathcal{F}$ is a **left adjoint** *of $\mathcal{G}$ and $\mathcal{G}$ is a* **right adjoint** *of $\mathcal{F}$.*

In this situation, the map $\triangleright$ is sometimes called the **right adjunction operator** and $\triangleleft$ is called the **left adjunction operator**. When $\mathcal{F}, \mathcal{G}$ are adjoint, we schematically indicate the natural bijection $\mathfrak{B} \overset{\mathcal{F}}{\underset{\mathcal{G}}{\leftrightarrows}} \mathfrak{A}$ like this:

(13)

$$\begin{array}{r|l}
\mathfrak{B} & \mathfrak{A} \\
\mathcal{F}A & A \\
{\triangleleft\,\psi = \varphi}\big\downarrow & \big\downarrow{\psi = \triangleright\,\varphi} \\
B & \mathcal{G}B \\
\mathcal{F} & \mathcal{G}
\end{array}$$

Here, "natural" means that given any preperturbation $A' \xrightarrow{\alpha} A$ and any postperturbation $B \xrightarrow{\beta} B'$, we have

$$\triangleright(\beta \circ \varphi \circ \mathcal{F}\alpha) = (\mathcal{G}\beta) \circ (\triangleright\,\varphi) \circ \alpha$$

The perturbations have an effect indicated by the diagram

$$
\begin{array}{r|l}
\mathfrak{B} & \mathfrak{A} \\
\mathcal{F}A' & A' \\
{\scriptstyle \mathcal{F}\alpha}\downarrow & {\scriptstyle \alpha}\downarrow \\
\mathcal{F}A & A \\
{\scriptstyle \varphi}\downarrow & {\scriptstyle \rhd\varphi}\downarrow \\
B & \mathcal{G}B \\
{\scriptstyle \beta}\downarrow & {\scriptstyle \mathcal{G}\beta}\downarrow \\
B' & \mathcal{G}B' \\
\mathcal{F} & \mathcal{G}
\end{array}
$$

Likewise

$$\lhd(\mathcal{G}\beta \circ \psi \circ \alpha) = \beta \circ (\lhd \psi) \circ (\mathcal{F}\alpha)$$

Proposition 9.2. *Let $\mathcal{F}$ be a functor from $\mathfrak{A}$ to $\mathfrak{B}$. If it has a left adjoint functor* **l**, *this is unique up to natural isomorphism. Similarly for right adjoint functors.*

Exercise: suppose $\mathcal{F}$ is left adjoint to $\mathcal{F}'$, and that $\mathcal{G}$ is left adjoint to $\mathcal{G}'$, where $\mathcal{F}$ is a functor from $\mathfrak{A}$ to $\mathfrak{B}$ and $\mathcal{F}'$ is a functor from $\mathfrak{B}$ to $\mathfrak{C}$. Show that the adjunctions compose; conclude that $\mathcal{G} \circ \mathcal{F}$ is left adjoint to $\mathcal{G}' \circ \mathcal{F}'$.

Proposition 9.3. *Let $\mathcal{F}$ be a functor from $\mathfrak{A}$ to $\mathfrak{B}$ and let $\mathcal{G}$ be a functor from $\mathfrak{B}$ back to $\mathfrak{A}$, so that we can compose $\mathcal{F}$, $\mathcal{G}$ in either order. Then $\mathcal{F}, \mathcal{G}$ are adjoint iff*

1. *$\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to the identity functor on $\mathfrak{A}$,*
2. *$\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to the identity functor on $\mathfrak{B}$.*

In this case,

1. *for every object B of $\mathfrak{B}$, $(\mathcal{G} \circ \mathcal{F})(B) = \omega_B$ is the unique arrow such that $\lhd\omega_B = 1_{\mathcal{F}B}$,*
2. *for every object A of $\mathfrak{A}$, $(\mathcal{F} \circ \mathcal{G})(A) = \omega_A$ is the unique arrow such that $\lhd\omega_A = 1_{\mathcal{G}A}$.*

The composite functor $\mathcal{G} \circ \mathcal{F}$ from $\mathfrak{A}$ to itself is called the **unit of adjunction**, *while the functor $\mathcal{F} \circ \mathcal{G}$ from $\mathfrak{B}$ to itself is called the* **counit of adjunction**.

This shows that adjoint functors are "mutually inverse" up to natural isomorphism. The proof is left as an excercise (see [19] for a full solution).

Proposition 9.4. *Let $\mathcal{G}$ be a functor from $\mathfrak{B}$ to $\mathfrak{A}$ and let $\mathcal{F}$ be a functor from $\mathfrak{A}$ back to $\mathfrak{B}$. Suppose $\mathcal{F} \dashv \mathcal{G}$ are adjoints with the natural bijection indicated by (13). Let $A \stackrel{\omega_A}{\to} \mathcal{G} \circ \mathcal{F}A$ be the components of the natural isomorphism of $\mathcal{G} \circ \mathcal{F}$ with the identity functor on $\mathfrak{A}$. Then for all objects A of $\mathfrak{A}$, the object $\mathcal{F}A$ of $\mathfrak{B}$ is a* **free $\mathfrak{B}$ object over** *A, in the sense that given any object B in $\mathfrak{B}$ and any arrow $A \stackrel{\alpha}{\to} \mathcal{G}B$*

of $\mathfrak{A}$*, there is a unique arrow* $\mathcal{F}A \xrightarrow{\Upsilon} B$ *such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\omega_A} & \mathcal{G} \circ \mathcal{F}A \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathcal{G}\Upsilon} \\ \mathcal{G}B & = & \mathcal{G}B \end{array}$$

commutes.

This provides the categorical way of understanding the hierarchy of structure in mathematics. In general, whenever we have a forgetful functor $\mathcal{G}$ from $\mathfrak{B}$ to $\mathfrak{A}$, we have a left adjoint $\mathcal{F}$ which augments (if neccessary) and places just the right structure on an object X of $\mathfrak{A}$ to make it into an object of $\mathfrak{B}$. Such left adjoints are called **free constructions**.

Exercise: let $\mathcal{G}$ be the forgetful functor from $R\mathfrak{Mod}$ to $\mathfrak{Set}$. Show that it has a left adjoint $\mathcal{F}$, where $\mathcal{F}X$ is the free R-module over the set X. Similarly for $G\mathfrak{Set}$ and $\mathfrak{Grp}$. What about $\mathfrak{Pos}$, $\mathfrak{Lat}$? What are the units and counits of adjunction?

Exercise: consider the forgetful functors from (respectively) real associative linear algebras to real Lie algebras to real linear spaces to abelian groups to sets. Verify that the first of these is the right adjoint of the functor which constructs the "universal enveloping algebra" of a Lie algebra (the free-est associative algebra over the Lie algebra); similarly verify that appropriate free constructions give the left adjoints of the remaining forgetful functors. What are the units and counits of adjunction? (See [8][19] for details.)

Adjoint functors also provide a useful way to understand various notions of "completion" in mathematics. Such constructions arise whenever the inclusion functor taking a subcategory into its parent has a left adjoint, in this case we call the subcategory **reflective**. The left adjoint functor is called a **completion construction**.

Exercise: verify that lattice homomorphisms between *complete* lattices form a category $\mathfrak{L}$. (A lattice is complete if every set of elements has a least upper bound and a greatest lower bound; in general this is true only if the set is finite). Verify that $\mathfrak{L}$ is a subcategory of $\mathfrak{Pos}$. Show that the inclusion functor has a left adjoint which is the functor taking a poset X to it's Dedekind-MacNeille completion (see [5] or [12] for this construction).

Exercise: verify that continous mappings between compact Hausdorff spaces form a category $\mathfrak{K}$ which is a subcategory of $\mathfrak{Top}$. Verify that the inclusion functor has a left adjoint which is the functor taking a topological space X to its Stone-Cech compactification.

Exercise: verify that uniformly continuous mappings between metric spaces, and complete metric spaces, respectively, form categories. The latter is a subcategory of the former; verify that the left adjoint of the inclusion functor is the functor taking a metric space X to its completion (as a metric space).

Exercise: suppose $\mathcal{F}$ is left adjoint to $\mathcal{G}$, where $\mathcal{F}$ is a functor from $\mathfrak{A}$ to $\mathfrak{B}$. Show the following

1. If $A \xrightarrow{\pi} \mathcal{G}B$ is epic then $\mathcal{F}A \xrightarrow{\lhd\pi} B$ is epic. Dually, if $\mathcal{F}A \xrightarrow{\mu} B$ is monic then $A \xrightarrow{\rhd\mu} \mathcal{G}B$ is monic.
2. If 0 is initial in $\mathfrak{A}$, $\mathcal{F}0$ is initial in $\mathfrak{B}$. Dually, if 1 is final in $\mathfrak{B}$, $\mathcal{G}1$ is final in $\mathfrak{A}$.

3. $\mathcal{F}$ preserves all coproducts; in fact, all coequalizers; indeed, all colimits. Dually, $\mathcal{G}$ preserves all limits.

Here is another example of a pair of adjoint functors, which is of considerable independent interest. Given an arrow $\varphi : X \times E \to Y$ in $\mathfrak{Set}$, for each $x \in X$ we can define $\Upsilon(x)$ to be the map taking $e \mapsto \varphi(x, e)$. This gives an arrow $\Upsilon : X \to Y^E$, where we define $Y^E = \{\nu : E \to Y\}$. Conversely, given Υ we can recover φ by observing that

$$\varphi(x, e) = \Upsilon(x)(e) = \mathrm{ev}(\Upsilon(x), e)$$

where setting $\mathrm{ev}(\nu, e) = \nu(e)$ for all $\nu : E \to Y$ in Y^E defines the **evaluation map** $\mathrm{ev} : Y^E \times E \to Y$. This means that we have a bijection

$$\mathrm{Hom}(X \times E, Y) \simeq \mathrm{Hom}(X, Y^E)$$

Moreover, this bijection is natural in the sense that it respects preperturbations $X' \to X$ and postpertubations $Y \to Y'$.

Definition 9.5. *Suppose $\mathfrak{C}$ is a category in which products always exist. Fix an object E and suppose that for all objects Y we have an object Y^E and an arrow $Y^E \xrightarrow{\varepsilon_Y} Y$ such that given any arrow $X \times E \xrightarrow{\psi} Y$, we have a unique arrow $X \xrightarrow{\Upsilon} Y^E$ (called the* **transpose** *of ψ) such that the diagram*

$$\begin{array}{ccc} Y^E \times E & \xrightarrow{\varepsilon_Y} & Y \\ {\scriptstyle \Upsilon \times 1_E}\uparrow & & \| \\ X \times E & \xrightarrow[\psi]{} & Y \end{array} \tag{14}$$

commutes. Then Y^E is called an **exponential object***.*

Exponentials may not exist for a given pair of objects Y, E, but if an exponential Y^E does exist, it is unique up to isomorphism.

Exercise: suppose $\mathfrak{C}$ is a category with pullbacks, exponentials, and a final object 1. Prove that there are natural isomorphisms

$$1^X \simeq 1, \qquad X^1 \simeq X, \qquad (Y \times Z)^X \simeq Y^X \times Z^X$$

and

$$(X^Y)^Z \simeq X^{Y \times Z} \simeq X^{Z \times Y} \simeq (X^Z)^Y$$

Exercise: suppose that exponentials always exist in $\mathfrak{C}$. Given an arrow $X \xrightarrow{\varphi} Y$, show that there is a unique arrow φ^E making the diagram

$$\begin{array}{ccc} Y^E \times E & \xrightarrow{\varepsilon_Y} & Y \\ {\scriptstyle \varphi^E \times 1_E}\uparrow & & \uparrow{\scriptstyle \varphi} \\ X^E \times E & \xrightarrow[\varepsilon_X]{} & X \end{array}$$

commute. Verify that this defines a functor, called the **exponential functor**, from $\mathfrak{C}$ to itself, together with a natural transformation from the exponential functor onto the identity functor. In the special case $\mathfrak{C} = \mathfrak{Set}$, verify that $\varphi^E = \varphi \circ (\cdot)$; that is, it takes $\mu : E \to X$ to $\varphi \circ \mu : E \to Y$.

Exercise: suppose Y, E are objects of $G\mathfrak{Set}$ with actions $\psi : G \times Y \to Y$ and $\eta : G \times E \to E$. Define Y^E to be the set of mappings (*not* neccessarily G-homs) $\mu : E \to Y$ with the action defined by letting g send μ to the map

$$e \mapsto \eta(g)(\mu\{\psi(g^{-1})(e)\})$$

Informally, we can write $(g\mu)(e) = g\mu(g^{-1}e)$, which is easier to understand! Note that the G-homs in Y^E are precisely the fixed points or one-point subobjects under this action; that is, they are the (categorical) "elements" of Y^E. Verify that the evaluation map $Y^E \times E \stackrel{\mathrm{ev}}{\to} Y$ is a G-hom. Conclude that exponentials always exist in $G\mathfrak{Set}$. What is the exponential functor?

Exercise: suppose $E \stackrel{\mu}{\to} X$ and $F \stackrel{\nu}{\to} X$ are objects of $\mathfrak{Bn}\, X$. We define a new bundle $F^E \stackrel{\zeta}{\to} X$ as follows. First, given $x \in X$, the stalk $(F^E)_x$ is

$$(F^E)_x = \left\{ (\varphi | E_x, x) : E_x \to F_x, \text{ such that } \varphi \in \operatorname*{Hom}_{\mathfrak{Bn}\, X}(E, F) \right\}$$

In other words, $(F^E)_x$ consists of the restrictions to the stalk E_x of the various bundle morphisms φ, where each such restriction has been *labled by the point* x. (Note the relation to the exponential in $\mathfrak{Set}$; namely $(F_x)^{(E_x)}$.) Next define $F^E = \biguplus_{x \in X} (F^E)_x$ and set $\zeta(\varphi | E_x, x) = x$. Give F^E the pullback topology from X via ζ. Verify $F^E \stackrel{\zeta}{\to} X$ is an object of $\mathfrak{Bn}\, X$. Define $\varepsilon((\varphi | E_x, x), e) = \varphi(e)$ and verify this is an arrow of $\mathfrak{Bn}\, X$. Conclude that exponentials always exist in $\mathfrak{Bn}\, X$. How about $\mathfrak{Et}\, X$?

Returning to the bijection $\mathrm{Hom}(X \times E, Y) \simeq \mathrm{Hom}(X, Y^E)$, and recalling from a previous section the definition of the product functor from $\mathfrak{C}$ to itself which takes X to $X \times E$, we now recognize the exponential functor $(\cdot)^E$ as the right adjoint of the product functor $(\cdot) \times E$:

$$\begin{array}{c|c} \mathfrak{C} & \mathfrak{C} \\ & \\ X \times E & X \\ \downarrow & \downarrow \\ Y & Y^E \\ & \\ (\cdot) \times E & (\cdot)^E \end{array}$$

When $(J, \leq)$ is a preorder, recall that an arrow in the preorder category $\mathfrak{J}$ has the form $j \to j'$ where $j \leq j'$. Thus, adjoint functors between two preorder categories are very easy to characterize: they are order preserving maps $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F} j \leq k$ iff $j \leq \mathcal{G} k$. In the next few exercises, we explore some examples of such functors.

Exercise: let X, Y be sets and let $R \subset X \times Y$ define a relation from X to Y. Then we obtain a dual pair of complete lattices L, M as explained in [12], each ordered by inclusion. Consider $(L, \subset)$ as a category $\mathfrak{L}$ and consider the **dual** of M, namely $(M, \supset)$ (note the order reversal), as a category $\mathfrak{M}$. Then the galois connection maps $\triangleright, \triangleleft$ give adjoint functors as indicated in the diagram

$$\begin{array}{c|c} \mathfrak{L} & \mathfrak{M} \\ \\ \lhd B & B \\ \Big\downarrow & \Big\downarrow \\ A & \rhd A \\ \\ \lhd & \rhd \end{array}$$

Note that our use in this paper of $\rhd, \lhd$ to denote the mutually inverse pair of natural bijections associated with an adjunction generalizes their use in [12] to denote the maps defining a galois connection.

Exercise: suppose $X \xrightarrow{\varphi} Y$ is a map. We can consider the powerset $\mathcal{P}X$ (ordered by inclusion) as a preorder category, denoted here by $\mathfrak{P}$. Likewise, we can consider $\mathcal{P}Y$ as a preorder category, denoted here by $\mathfrak{Q}$. Recall that taking an object B of $\mathfrak{Q}$ to an object $\varphi^{-1}(B)$ of $\mathfrak{P}$ defines a functor, the **preimage functor**; similarly taking an object A of $\mathfrak{P}$ to an object $\varphi(A)$ of $\mathfrak{Q}$ defines a functor, the **image functor**. Show that the image functor is a left adjoint of the preimage functor, as indicated by the diagram

$$\begin{array}{c|c} \mathfrak{Q} & \mathfrak{P} \\ \\ \varphi(A) & A \\ \Big\downarrow & \Big\downarrow \\ B & \varphi^{-1}(B) \\ \\ \text{image} & \text{preimage} \end{array}$$

Exercise: We will show that the fundamental set-theoretic operations (complementation, intersection, and union) are all examples of adjoint functors. Fix a set X and let $\mathfrak{P}$ be the preorder category defined by the powerset $\mathcal{P}X$ (ordered by inclusion).

1. Observe that $A \cap E$ is a product in $\mathfrak{P}$. Verify that the map taking each arrow $A \subset B$ in $\mathfrak{P}$ to the arrow $A \cap E \subset B \cap E$ defines the product functor. Verify that this has a right adjoint $(\cdot)^E$ defined by taking the arrow $A \subset B$ to the arrow $A \cup E^c \subset B \cup E^c$, where $(\cdot)^c$ means complementation in X. Conclude that in $\mathfrak{P}$ exponentials always exist. Next, dualize the diagram defining an exponential. Observe that in $\mathfrak{P}$ $A \cup E$ is a sum. Show that the sum functor $(\cdot) \cup E$ has a left adjoint $(\cdot) \setminus E$ which is "dual" to exponentiation in $\mathfrak{P}$. The natural bijections are indicated by the diagrams

$$\begin{array}{c|c} \mathfrak{P} & \mathfrak{P} \\ \\ A \cap E & A \\ \Big\downarrow & \Big\downarrow \\ B & B \cup E^c \\ \\ (\cdot) \cap E & (\cdot)^E \end{array} \qquad\qquad \begin{array}{c|c} \mathfrak{P} & \mathfrak{P} \\ \\ A \setminus E & A \\ \Big\downarrow & \Big\downarrow \\ B & B \cup E \\ \\ (\cdot) \setminus E & (\cdot) \cup E \end{array}$$

2. Now consider the product category $\mathfrak{P} \times \mathfrak{P}$. Verify that the map taking the arrow $A \subset B$ of $\mathfrak{P}$ to the arrow $(A, A) \leq (B, B)$ of $\mathfrak{P} \times \mathfrak{P}$ defines a functor, called the **diagonal functor**. Show that union and intersection (both defined as functors from $\mathfrak{P} \times \mathfrak{P}$ to $\mathfrak{P}$) give left and right adjoints, respectively, of the diagonal functor, as indicated by diagrams

$$\begin{array}{c|c} \mathfrak{P} & \mathfrak{P} \times \mathfrak{P} \\ & \\ A \cup B & (A, B) \\ \downarrow & \downarrow \\ C & (C, C) \\ & \\ \text{union} & \text{diagonal} \end{array} \qquad \begin{array}{c|c} \mathfrak{P} \times \mathfrak{P} & \mathfrak{P} \\ & \\ (A, A) & A \\ \downarrow & \downarrow \\ (B, C) & B \cap C \\ & \\ \text{diagonal} & \text{intersection} \end{array}$$

Exercise: we will show that all the operations within a given category $\mathfrak{C}$ which were discussed in Sections 4 and 5 are all examples of adjoint functors. In each case, note how the universal mapping property gives the desired natural bijection between the appropriate hom sets.

1. Verify that the initial and final objects (if any) of $\mathfrak{C}$ are obtained as the left and right adjoints (respectively) of the trivial functor from $\mathfrak{C}$ to the trivial category, as indicated by the diagrams:

$$\begin{array}{c|c} \mathfrak{C} & \mathfrak{Z} \\ & \\ 0 & * \\ \Upsilon \big\downarrow & \big\downarrow ! \\ Y & * \\ & \\ \text{initial} & \text{trivial} \end{array} \qquad \begin{array}{c|c} \mathfrak{Z} & \mathfrak{C} \\ & \\ * & X \\ ! \big\downarrow & \big\downarrow \Upsilon \\ * & 1 \\ & \\ \text{trivial} & \text{final} \end{array}$$

   On the left, Υ denotes the unique arrow from 0 to Y whose existence is guaranteed by the UMP for an initial object in $\mathfrak{C}$, and on the right, Υ denotes the unique arrow from X to 1 whose existence is guaranteed by the UMP for a final object in $\mathfrak{C}$.

2. Verify that the sum and product (if they exist) of any pair of objects in $\mathfrak{C}$ are obtained as the left and right adjoints (respectively) of the diagonal functor from $\mathfrak{C}$ to $\mathfrak{C} \times \mathfrak{C}$, as indicated by the diagrams:

$$\begin{array}{c|c} \mathfrak{C} & \mathfrak{C} \times \mathfrak{C} \\ & \\ A + B & (A,\ B) \\ \Upsilon \big\downarrow & {}^{\varphi}\big\downarrow \ \ \big\downarrow^{\psi} \\ X & (X,\ X) \\ & \\ \text{sum} & \text{diagonal} \end{array} \qquad \begin{array}{c|c} \mathfrak{C} \times \mathfrak{C} & \mathfrak{C} \\ & \\ (X,\ X) & X \\ {}^{\varphi}\big\downarrow \ \ \big\downarrow^{\psi} & \big\downarrow \Upsilon \\ (A,\ B) & A \times B \\ & \\ \text{diagonal} & \text{product} \end{array}$$

   On the left, Υ denotes the unique arrow $A + B \stackrel{\Upsilon}{\to} X$ guaranteed by the UMP of a sum whenever we have arrows $A \stackrel{\varphi}{\to} X \stackrel{\psi}{\leftarrow} B$, whereas on the right,

Υ denotes the unique arrow $X \xrightarrow{\Upsilon} A \times B$ guaranteed by the UMP of a product whenever we have arrows $A \xleftarrow{\varphi} X \xrightarrow{\psi} B$.

3. Verify that the coequalizer and equalizer objects of $A \xrightarrow{\varphi,\psi} B$ (if they exist) are obtained as the left and right adjoints (respectively) of the diagonal functor from $\mathfrak{C}$ to the category of pairs $\mathfrak{C}^{\downarrow\downarrow}$, as indicated by the diagrams:

$$
\begin{array}{c|ccc|c}
\mathfrak{C} & & \mathfrak{C}^{\downarrow\downarrow} \qquad\qquad \mathfrak{C}^{\downarrow\downarrow} & & \mathfrak{C} \\
K & A \xrightarrow{(\varphi,\psi)} B & & X \xrightarrow{(1_X,1_X)} X & X \\
\Upsilon\downarrow & \alpha\downarrow \qquad \downarrow\beta & & \alpha\downarrow \qquad \downarrow\beta & \downarrow\Upsilon \\
X & X \xrightarrow[(1_X,1_X)]{} X & & A \xrightarrow[(\varphi,\psi)]{} B & E \\
\text{coequalizer} & \text{diagonal} & & \text{diagonal} & \text{equalizer}
\end{array}
$$

4. Verify that the pushout of $A \xleftarrow{\alpha} S \xrightarrow{\beta} B$ (if it exists) is obtained as the left adjoint of the diagonal functor from $\mathfrak{C}$ to $\mathfrak{C}^{\nwarrow\nearrow}$, whereas the pullback of $A \xrightarrow{\alpha} T \xleftarrow{\beta} B$ (if it exists) is obtained as the right adjoint of the diagonal functor from $\mathfrak{C}$ to $\mathfrak{C}^{\searrow\swarrow}$, as indicated by the diagrams:

$$
\begin{array}{c|cc|c}
\mathfrak{C} & \mathfrak{C}^{\nwarrow\nearrow} & \mathfrak{C}^{\searrow\swarrow} & \mathfrak{C} \\
\check{S} & A \xleftarrow{\varphi} S \xrightarrow{\psi} B & X \xrightarrow{1_X} X \xleftarrow{1_X} X & X \\
\Upsilon\downarrow & \alpha\downarrow \qquad\qquad \downarrow\beta & \alpha\downarrow \qquad\qquad \downarrow\beta & \downarrow\Upsilon \\
X & X \xleftarrow[1_X]{} X \xrightarrow[1_X]{} X & A \xrightarrow[\varphi]{} T \xleftarrow[\psi]{} B & \hat{T} \\
\text{pushout} & \text{diagonal} & \text{diagonal} & \text{pullback}
\end{array}
$$

5. Verify that the direct limit over J and the inverse limit over J (if they exist) define respecively right and left adjoints to the diagonal functor from $\mathfrak{C}$ to $\mathfrak{C}^J$, as indicated in the diagrams:

$$
\begin{array}{c|cc|c}
\mathfrak{C} & \mathfrak{C}^J & \mathfrak{C}^J & \mathfrak{C} \\
\varinjlim X_j & X_j \xrightarrow{\alpha_{jk}} X_k & M \xrightarrow{1_M} M & M \\
\Upsilon\downarrow & \varphi_j\downarrow \qquad \downarrow\varphi_k & \varphi_j\downarrow \qquad \downarrow\varphi_k & \Upsilon\downarrow \\
N & N \xrightarrow[1_N]{} N & X_j \xrightarrow[\alpha_{jk}]{} X_k & \varprojlim X_j \\
\text{direct limit} & \text{diagonal} & \text{diagonal} & \text{inverse limit}
\end{array}
$$

Exercise: let $\mathfrak{C}$ be a category.

1. Suppose that E is an object in $\mathfrak{C}$. Show that we can define a functor Σ_E from $\mathfrak{C}/E$ to $\mathfrak{C}$ as follows. Take the object $\sigma : X \to E$ of $\mathfrak{C}/E$ to $\operatorname{dom}\sigma = X$, and take the arrow $\varphi : X \to Y$ of $\mathfrak{C}/E$ to itself (considered as an arrow of $\mathfrak{C}$).

Recall that if products exist in $\mathfrak{C}$, we can define the slice functor E^* from $\mathfrak{C}$ to $\mathfrak{C}/E$. Show that Σ_E is the left adjoint of E^*.

2. Suppose that $E \xrightarrow{\psi} F$ is an arrow in $\mathfrak{C}$. Show that we can define a functor Σ_ψ from $\mathfrak{C}/E$ to $\mathfrak{C}/F$ as follows. Take the object σ of $\mathfrak{C}/E$ to the object $\psi \circ \sigma$ of $\mathfrak{C}/B$, and take the arrow $\varphi : X \to Y$ of $\mathfrak{C}/E$ to itself (considered as an arrow of $\mathfrak{C}/F$). Recall that if pullbacks exist in $\mathfrak{C}$, we can define a functor ψ^* from $\mathfrak{C}/F$ to $\mathfrak{C}/E$. Show that Σ_ψ is the left adjoint of ψ^*. (Hint: use the UMP of the pullback; the key point is to prove that σ factors through $\varphi^*\tau$ iff $\varphi \circ \sigma$ factors through τ.)

The natural bijections in question are indicated by following diagram:

$$
\begin{array}{c|c}
\mathfrak{C} & \mathfrak{C}/E \\
& \\
\operatorname{dom}\sigma & \sigma \\
\downarrow & \downarrow \\
Y & E^*Y \\
& \\
\Sigma_E & E^*
\end{array}
\qquad\qquad
\begin{array}{c|c}
\mathfrak{C}/E & \mathfrak{C}/F \\
& \\
\psi \circ \sigma & \sigma \\
\downarrow & \downarrow \\
\tau & \psi^*\tau \\
& \\
\Sigma_\psi & \psi^*
\end{array}
$$

Exercise (not for the faint of heart): let X be a topological space.

1. Define a **sheaf over** X to be a presheaf $\mathcal{F}$ such that given an open cover of U open in X, say $U = \cup_{j \in J} U_j$ (where the U_j are open sets of X), and given a family $\{t_j \in \mathcal{F}U_j\}$ which is "self" consistent on all intersections $U_j \cap U_k$, the t_j can be "pasted together" to form a *unique* $t \in \mathcal{F}U$ such that $t|U_j = t_j$ for all $j \in J$. (Here, recall that if $U \subset V$, so that we have a set mapping $\mathcal{F}U \leftarrow \mathcal{F}V$, we denote by $t|U$ the image of $t \in \mathcal{F}V$ under this "restriction" mapping.)
2. Verify that the cofunctor taking U to the continuous real valued functions on U, $C(U, \mathbb{R})$, defines a sheaf.
3. Verify that the definition of a sheaf may be reformulated as follows: a sheaf over X is a presheaf $\mathcal{F}$ over X such that we require that each open cover $U = \cup_{j \in J} U_j$ of an open set $U \subset X$ has an *equalizer*

$$
\begin{array}{ccccc}
\mathcal{F}U & \xrightarrow{\varepsilon} & \prod_j \mathcal{F}U_j & \xrightarrow{\mu} & \prod_{(j,k)} \mathcal{F}(U_j \cap U_k) \\
& & \| & & \| \\
& & \prod_j \mathcal{F}U_j & \xrightarrow{\nu} & \prod_{(j,k)} \mathcal{F}(U_j \cap U_k)
\end{array}
$$

where $\varepsilon(t) = (t|U_j)_{j \in J}$ and where μ takes $(t_j)_{j \in J}$ to $(t_j|U_j \cap U_k)_{k \in J}$, whereas ν takes $(t_j)_{j \in J}$ to $(t_k|U_j \cap U_k)_{j \in J}$.
4. Verify that sheaves over X form the objects of a category whose arrows are natural transformations between sheaves over X.
5. Given an object $E \xrightarrow{\mu} X$ of $\mathfrak{Bn}\, X$, define a presheaf by taking an open set U in X to $\mathcal{F}U$ defined by the pullback square

$$
\begin{array}{ccc}
\mathcal{F}U & \xrightarrow{\hat{\iota}} & E \\
{\scriptstyle \hat{\mu}}\downarrow & & \downarrow{\scriptstyle \mu} \\
U & \xrightarrow[\iota]{} & X
\end{array}
$$

where ι is the inclusion map. Verify that in fact this is a sheaf. Verify that this defines a cofunctor, the **sheaf of sections cofunctor**, $\mathcal{S}ec$, from $\mathfrak{Bn}\,X$ to $\mathfrak{Sh}\,X$.

6. Recall that $\mathcal{G}rm$, the sheaf of germs cofunctor, takes $\mathfrak{Ps}\,X$ to $\mathfrak{Et}\,X$. Since every object of $\mathfrak{Sh}\,X$ is in $\mathfrak{Ps}\,X$ and every object of $\mathfrak{Et}\,X$ is in $\mathfrak{Bn}\,X$, we can consider $\mathcal{S}ec$ and $\mathcal{G}rm$ to be functors between $\mathfrak{Bn}\,X$ and $\mathfrak{Ps}\,X$. Verify that $\mathcal{S}ec$ is a left adjoint of $\mathcal{G}rm$.
7. Verify that $\mathfrak{Et}\,X$ and $\mathfrak{Sh}\,X$ are isomorphic categories.

10. Topoi

> Any topos may be regarded as a mathematical domain of discourse or "world" in which mathematical concepts can be interpreted and mathematical constructions performed.
>
> J. L. Bell [2]

One of the most remarkable properties of $\mathfrak{Set}$ is that we can represent the subobjects (subsets) A of X not just as equivalence classes of monic arrows into X, but as **characteristic functions** $\chi_A : X \to \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$ and

$$\chi_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

The notion of characteristic functions can be put in categorical form, as follows.

Definition 10.1. *Suppose $\mathfrak{C}$ is a category which posesses a final object* 1. *A* **subobject classifier** *consists of an object Ω, called the* **classifying object**, *and a* monic *arrow $1 \xrightarrow{\top} \Omega$, called* **truth**, *such that given any* monic *arrow $\operatorname{dom}\alpha \xrightarrow{\alpha} X$, there is a* unique *arrow $X \xrightarrow{\chi_\alpha} \Omega$, called the* **characteristic arrow of** α, *such that the diagram*

$$\begin{array}{ccc} \operatorname{dom}\alpha & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow{\scriptstyle \chi_\alpha} \\ 1 & \xrightarrow{\top} & \Omega \end{array} \tag{15}$$

is a pullback square.[9]

Note that if we choose a different representative β of $A = [\alpha]$, the diagram

$$\begin{array}{ccc} \operatorname{dom}\beta & \xrightarrow{\beta} & X \\ \downarrow & & \downarrow{\scriptstyle \chi_\alpha} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

must still be a pullback square, so $\chi_\beta = \chi_\alpha$. In short, characteristic arrows describe *subobjects* (as they should), and we can write χ_A instead of χ_α. As this little argument tends to emphasize, the condition that (15) be a pullback square is much stronger than merely requiring that (15) commute.

As usual, a classifying object might not exist, but if it does, it is unique up to isomorphism.

---

[9] We have broken our notational convention to agree with a universally observed convention in topos theory which denotes classifing objects by Ω, and an almost universally observed convention in logic and topos theory which denotes the truth arrow by $\top$.

Exercise: recall that every equalizing arrow is a monic arrow. Conversely, if $\mathfrak{C}$ is a category with a final object and a classifier, and if we define $\top_X$ to be the composite $X \to 1 \xrightarrow{\top} \Omega$, show that every monic arrow is an equalizing arrow.

Exercise: verify that in $\mathfrak{Set}$, the mapping $* \xrightarrow{\top} \mathbb{B}$ taking the single point of $*$ to $1 \in \mathbb{B}$, gives the truth arrow for the classifying object $\mathbb{B}$. Similarly for $G\mathfrak{Set}$.

Exercise: define a bundle $\mathbb{B} \times X \xrightarrow{\pi_2} X$ and define $X \xrightarrow{\top} \mathbb{B} \times X$ by $\top(x) = (1, x)$. Verify that this defines a classifier in $\mathfrak{Bn}\, X$. What is the characteristic arrow of a typical monic arrow in $\mathfrak{Bn}\, X$?

Definition 10.2. *A* **topos** *is a category $\mathfrak{T}$ which posseses:*

1. *a final object* 1*,*
2. *an initial object* 0*,*
3. *a subobject classifier* $1 \xrightarrow{\top} \Omega$*,*

and which also posseses

1. *a pullback square for all arrows* $X \to Z \leftarrow Y$*,*
2. *a pushout square for all arrows* $X \leftarrow Z \to Y$*,*
3. *an exponential object* X^Y *for all objects* X, Y*.*

At this point we know that $\mathfrak{Set}$, $G\mathfrak{Set}$ (for any group G), and $\mathfrak{Bn}\, X$ (for any topological space X) all give examples of topoi.

Exercise: if G is a topological group and X is a topological space, define a **G-bundle over** X to be a bundle $E \xrightarrow{\mu} X$ such that G acts continuously upon E, with μ a G-hom. Define the obvious notion of morphism to make the collection of G-bundles over X into a category and verify that the operations defined for $\mathfrak{Bn}\, X$ extend to the new category, so that it is in fact a topos. Similarly for G-sheaves over X.

Exercise: if X is a topological space, show that $\mathfrak{Ps}\, X$ is a topos. Similarly for $\mathfrak{Sh}\, X$.

Exercise: if $\mathfrak{S}, \mathfrak{T}$ are topoi, show that $\mathfrak{S} \times \mathfrak{T}$ is a topos.

See [17][9][20] for many more examples of topoi.

Proposition 10.3. *Let $\mathfrak{T}$ be a topos. Then*

1. *for each pair of objects X, Y we have a natural bijection*
$$\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(1, Y^X)$$
2. *for each object X, we have natural bijections*
$$\mathrm{Sub}(X) \simeq \mathrm{Hom}(X, \Omega) \simeq \mathrm{Hom}(1, \Omega^X)$$

The proof is left as a short and easy exercise. (Outline: the first is immediate from $X \simeq 1 \times X$ and the fact that $(\cdot)^X$ is the right adjoint of $(\cdot) \times X$. Then use the UMP's for a classifier and a pullback square to obtain $\mathrm{Sub}(X) \simeq \mathrm{Hom}(X, \Omega)$; then combine results.)

The significance of this proposition is that for any topos $\mathfrak{T}$,

1. we have a *natural internal representation* in $\mathfrak{T}$ of each (genuine) element of the set $\mathrm{Hom}(X, Y)$ as a (categorical) element of the object X^Y,
2. we have a *natural internal representation* in $\mathfrak{T}$ of each (genuine) element of $\mathrm{Sub}(X)$ as a (categorical) element of the object Ω^X.

In fact, we have three equivalent ways to think about subobjects of a given object X in a topos $\mathfrak{T}$:

1. an equivalence class of monic arrows $\operatorname{dom}\alpha \overset{\alpha}{\to} X$,
2. an arrow $X \overset{\chi}{\to} \Omega$,
3. a (categorical) element $1 \overset{a}{\to} \Omega^X$.

This is so important that we pause to give a rather detailed explanation of how to use the structure of $\mathfrak{T}$ to pass between these three representations. First, to get from a subobject A to the corresponding characteristic arrow χ_A, we need only appeal to the UMP for a classifier and to show that the result does not depend on which representative we choose. To get from a characteristic arrow χ back to the corresponding subobject A, we need only pull back χ along $\top$ to get a representative of A. Second, to get from a characteristic arrow χ to the corresponding element $1 \overset{a}{\to} \Omega^X$, where $a = \ulcorner\chi\urcorner$ is called the **name** of χ, we use the UMP of Ω^X to obtain a unique arrow making the diagram

$$\begin{array}{ccc} \Omega^X \times X & \xrightarrow{\ \varepsilon\ } & \Omega \\ {\scriptstyle \ulcorner\chi\urcorner \times 1_X}\uparrow & & \uparrow{\scriptstyle \varphi} \\ 1 \times X & \xrightarrow{\ \eta\ } & X \end{array}$$

commute, where on the bottom we have an isomorphism. To get from an element $1 \overset{a}{\to} \Omega^X$ back to the corresponding characteristic arrow $X \overset{\chi}{\to} \Omega$, we simply take the composition $\varepsilon \circ (a \times 1_X) \circ \eta^{-1}$.

Write $x \in_Z X$ whenever we have an arrow $Z \overset{x}{\to} X$. We say that x is **generalized element**, specifically a Z-element, of X. We should think of a Z-element of X is as a "point" $x \in X$ which varies as we range over Z. For example, in $\mathfrak{T}op$ every closed curve is nothing but an S^1-element, a closed surface is an S^2-element. Given an arrow $X \overset{\varphi}{\to} Y$, write $\varphi[x]$ for $\varphi \circ x$. Note that $\varphi[x] \in_Z Y$. Using these notions, we can obtain new characterizations of final objects and monic arrows (but not initial objects or epic arrows).

Proposition 10.4. *Let $X \overset{\varphi,\psi}{\to} Y$ be two arrows in a category $\mathfrak{C}$. Then $\varphi = \psi$ iff for all Z in $\mathfrak{C}$ and all $x \in_Z X$, $\varphi[x] = \psi[x]$.*

Exercise: let $\mathfrak{C}$ be a category.

1. Show that F is final in $\mathfrak{C}$ iff for all objects Z of $\mathfrak{C}$, there exists a unique generalized element $z \in_Z F$.
2. Show that $A \overset{\alpha}{\to} X$ is monic iff for all Z and all $z, z' \in_Z A$, $\alpha[z] = \alpha[z']$ implies $z = z'$.
3. Show that initial objects and epic arrows have no such characterizations.

Observe that an element $1 \overset{x}{\to} X$ of X is in fact an element of the subobject $[\alpha]$, exactly when there is an arrow $1 \overset{a}{\to} X$ such that $x = \alpha \circ a$. Examining the UMP for the classifier, we conclude this happens iff

$$\top = \chi_A \circ x$$

This gives an *internal criterion* in $\mathfrak{T}$ for when an element x of X is in fact an element of A.

Recall that in $\mathfrak{S}et$ every map (arrow) φ can be *uniquely factored* as an onto map (epic arrow) followed by an inclusion (monic arrow), with the intermediate object being the image of φ. It turns out that the same thing happens in any topos.

Lemma 10.5. *Let* $X \xrightarrow{\varphi} Y$ *be any arrow in a topos* $\mathfrak{T}$*. Then* $\varphi = \eta \circ \pi$ *where* $X \xrightarrow{\pi} E$ *is* epic *and* $E \xrightarrow{\eta} Y$ *is* monic*.*

Here $[\eta] = \operatorname{im} \varphi$ is a subobject of Y called the **image** of φ.
Sketch of proof: take the pullback of

$$X \xrightarrow{\ \varphi\ } Y \xleftarrow{\ \varphi\ } X$$

Then take the pushout of

$$X \xleftarrow{\ \hat{\varphi}\ } \hat{Y} \xrightarrow{\ \hat{\varphi}\ } X$$

The UMP of a pushout gives a unique arrow $\check{\hat{Y}} \to Y$, yielding the factorization $\varphi = \Upsilon \circ \check{\hat{\varphi}}$. Argue that Υ is monic whereas $\check{\hat{\varphi}}$ is epic.

Using this **monic-epic factorization**, we can show that $\operatorname{Sub}(X)$ is in fact a lattice, just like the powerset of a set.

Corollary 10.6. *For every object* X *of a topos* $\mathfrak{T}$*,* $\operatorname{Sub}(X)$ *is a lattice. Moreover,* $\operatorname{Sub}(X)$ *has both a maximal element (represented by* $X \xrightarrow{1_X} X$*) and a minimal element (represented by* $0 \xrightarrow{!} X$*).*

Sketch of proof: a lattice is a poset such that any two elements have a least lower bound and a greatest upper bound. In the case of $\operatorname{Sub}(X)$ the order relation is $A \sqsubset B$ iff there is a monic arrow $\operatorname{dom} \alpha \to \operatorname{dom} \beta$, where α, β represent A, B respectively. Given two monic maps

$$\operatorname{dom} \alpha \xrightarrow{\alpha} X \xleftarrow{\beta} \operatorname{dom} \beta$$

take their pullback. Since pullbacks of monics are monics, this implies that $\hat{X} \xrightarrow{\hat{\alpha}} \operatorname{dom} \alpha$ defines a subobject of A, whereas $\hat{X} \xrightarrow{\hat{\beta}} \operatorname{dom} \beta$ defines a subobject of B. Moreover, $\beta \circ \hat{\alpha} = \alpha \circ \hat{\beta}$ represent the same subobject $A \sqcap B$ of X, which is a common subobject of A, B; moreover, by the defining property of pullback squares it is the *greatest* common subobject, so we have found the desired greatest lower bound. Next, construct the sum $\operatorname{dom} \alpha + \operatorname{dom} \beta$, giving the diagram

$$\begin{array}{ccc}
X & \xleftarrow{\ \beta\ } & \operatorname{dom} \beta \\
{\scriptstyle \alpha}\uparrow & & \downarrow \\
\operatorname{dom} \alpha & \longrightarrow & \operatorname{dom} \alpha + \operatorname{dom} \beta
\end{array}$$

Use the UMP of a sum to find a unique arrow $\operatorname{dom} \alpha + \operatorname{dom} \beta \to X$ and take its *image* to be $A \sqcup B$. This is a subobject of X which is a common superobject of both A, B. Next, show that

$$\begin{array}{ccc}
A \sqcap B & \longrightarrow & B \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \sqcup B
\end{array}$$

is a pushout square (as well as a pullback square!); this means exactly that $A \sqcup B$ is the desired *least* common superobject, so we have found the desired least upper bound.

It follows from elementary lattice theory that $\sqcap, \sqcup$ obey the following algebraic laws, which give an equivalent axiomatic definition of a lattice (see [5]):

1. idempotent laws,

$$A \sqcap A = A$$
$$A \sqcup A = A$$

2. commutative laws

$$A \sqcap B = B \sqcap A$$
$$A \sqcup B = B \sqcup A$$

3. associative laws

$$A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$$
$$A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$$

4. absorption laws,

$$A \sqcap (A \sqcup B) = A$$
$$A \sqcup (A \sqcap B) = A$$

Corollary 10.7. *The classifying object* Ω *of a topos* $\mathfrak{T}$ *is always a lattice with maximal element* $1 \xrightarrow{\top} \Omega$ *and minimal element* $1 \xrightarrow{-} \Omega$.

Here, the arrow $1 \xrightarrow{-} \Omega$, called **false**, is the characteristic arrow of the unique arrow $0 \to 1$, where 0 is of course an initial object of $\mathfrak{T}$. (As usual, taking a different initial object gives the same characteristic arrow $-$.)

Sketch of proof: the basic idea is to apply the Yoneda Embedding Theorem. Begin by observing that we have a mapping

$$\mathrm{Sub}(X) \times \mathrm{Sub}(X) \xrightarrow{\sqcup} \mathrm{Sub}(X)$$

But $\mathrm{Sub}(X)$ is in bijection with $\mathrm{Hom}(X, \Omega)$, so this may be considered a mapping

$$\mathrm{Hom}(X, \Omega) \times \mathrm{Hom}(X, \Omega) \xrightarrow{\sqcup} \mathrm{Hom}(X, \Omega)$$

or, since $\mathrm{Hom}(X, \Omega) \times \mathrm{Hom}(X, \Omega)$ is in (natural) bijection with $\mathrm{Hom}(X, \Omega \times \Omega)$, a mapping

$$(\Omega \times \Omega)^{\natural}\, X \xrightarrow{\sqcup} \Omega^{\natural} X$$

(where $\mathrm{Hom}(E, X) = E^{\natural} X$ as usual).

Now, this mapping is natural in X; that is, if $\varphi : X \to Y$, then for any subobjects $A, B \in \mathrm{Sub}(X)$, we have

$$(\varphi^{\star} A) \sqcup (\varphi^{\star} B) = \varphi^{\star}(A \sqcup B)$$

The point is that we now have a natural transformation from $(\Omega \times \Omega)^{\natural}$ to $\Omega^{\natural}$, and by Yoneda this is induced by a unique arrow

$$\Omega \times \Omega \xrightarrow{\vee} \Omega$$

Moreover, this arrow induces $\sqcup$ functorially, so $\vee$ respects the same algebraic laws as $\sqcup$. Similarly for $\wedge$ (defined from $\sqcap$).

We can close this circle of ideas by observing that given two arrows $X \xrightarrow{\alpha, \beta} \Omega$, we can define arrow $\alpha \wedge \beta$ and $\alpha \vee \beta$ by the compositions

$$\begin{array}{ccc} X \times X & \xrightarrow{(\alpha,\beta)} & \Omega \times \Omega \\ {\scriptstyle \alpha \wedge \beta} \downarrow & & \downarrow {\scriptstyle \wedge} \\ \Omega & = & \Omega \end{array} \qquad \begin{array}{ccc} X \times X & \xrightarrow{(\alpha,\beta)} & \Omega \times \Omega \\ {\scriptstyle \alpha \vee \beta} \downarrow & & \downarrow {\scriptstyle \vee} \\ \Omega & = & \Omega \end{array}$$

Regarding α, β as characteristic arrows of subobjects A, B respectively, we obtain the expected identities

$$\chi_{A \sqcap B} = \chi_A \wedge \chi_B, \qquad \chi_{A \sqcup B} = \chi_A \vee \chi_B$$

No discussion of topoi would be complete without mention of the "persistence principle": *categorical operations on topoi often yield new topoi.* The most important example of this principle is the following.

Theorem 10.8. *Suppose $\mathfrak{T}$ is a topos. Then*

1. *for each object E of $\mathfrak{T}$, the slice category $\mathfrak{T}/E$ is also a topos, with classifying object $E \times 1 \stackrel{1_E \times \top}{\longrightarrow} E \times \Omega$.*
2. *for each arrow $\varphi : E \to F$ of $\mathfrak{T}$, the slice change cofunctor from $\mathfrak{T}/F$ to $\mathfrak{T}/E$*
   (a) *preserves colimits, exponentials, and classifying objects,*
   (b) *has both a left adjoint Σ_φ which preserves limits,*
   (c) *has a right adjoint Π_φ which also preserves limits.*

This theorem was so important in the development of topos theory that it is often called the **Fundamental Theorem of Topoi**. For a proof see [1][17]. The adjunctions $\Sigma_\psi \dashv \psi^* \dashv \Pi_\psi$ are indicated in the following diagrams:

$$\begin{array}{c|c|c}
\mathfrak{T}/F & \mathfrak{T}/E & \mathfrak{T}/F \\
\psi \circ \alpha & \alpha & \\
\downarrow & \downarrow & \\
\beta & \psi^* \beta & \beta \\
 & \downarrow & \downarrow \\
 & \gamma & \Pi_\psi \gamma \\
\Sigma_\psi & \psi^* & \Pi_\psi
\end{array}$$

Exercise: let $\mathfrak{T}$ be a topos. Fix an object E of $\mathfrak{T}$. Show that the slice functor E^* from $\mathfrak{T}$ to $\mathfrak{T}/E$ has a *right* adjoint Π_E, defined as follows. Take $\alpha : X \to E$ to the object $\Pi_E \alpha$ defined by pulling back $\alpha^E : X^E \to E^E$ along $\lceil 1_E \rceil : 1 \to E^E$, so that

$$\begin{array}{ccc}
\Pi_E \alpha & \longrightarrow & X^E \\
\downarrow & & \downarrow {\scriptstyle \alpha^E} \\
1 & \xrightarrow{\lceil 1_E \rceil} & E^E
\end{array}$$

is a pullback diagram. Take $\varphi : X \to Y$ (considered as an arrow of $\mathfrak{T}/E$ to the arrow $\Pi_E \varphi$ defined by pulling back the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\alpha} & E \\
{\scriptstyle \varphi} \downarrow & & \| \\
Y & \xrightarrow{\beta} & E
\end{array}$$

along $\lceil 1_E \rceil : 1 \to E^E$. (That is, pull back top and bottom and recall that there is a unique map making the "prism" commmute.) Combining with a previous exercise gives the adjunctions $\Sigma_E \dashv E^* \dashv \Pi_E$, as indicated in the following diagram:

$\mathfrak{T}$	$\mathfrak{T}/E$	$\mathfrak{T}$
$\operatorname{dom}\alpha$	α	
$\downarrow$	$\downarrow$	
Y	$E \times Y \xrightarrow{\pi} E$	Y
	$\downarrow$	$\downarrow$
	γ	$\Pi_E \gamma$
Σ_E	E^*	Π_E

Exercise: use the preceeding exercise to obtain $\exists_\psi \dashv \psi^* \dashv \forall_\psi$. (This is a bit tricky; see [17] for hints.)

Exercise: let X be an object in a topos $\mathfrak{T}$ and consider $\mathrm{Sub}(X)$ to be preorder category. Show that we obtain a functor $\mathcal{L}_X$ from $\mathfrak{T}/X$ to $\mathrm{Sub}(X)$ by taking $\sigma : E \to X$ to $\operatorname{im}\sigma \in Sub(X)$ and taking the arrow $\psi : E \to F$ from σ to $\tau : F \to X$, where $\sigma = \tau \circ \psi$, to the arrow $\operatorname{im}\sigma \sqsubset \operatorname{im}\tau$ of $\mathrm{Sub}(X)$. Conversely, given $A \in Sub(X)$, choose a representative monic $\alpha : \operatorname{dom}\alpha \to X$ and call this $\mathcal{L}_X A$; verify that any arrow $A \sqsubset B$ of $\mathrm{Sub}(X)$ induces one of $\mathfrak{T}/X$. Show that $\mathcal{L}_X \dashv \mathcal{I}_X$. Conclude that $\mathrm{Sub}(X)$ (as a preorder category) is a subcategory of $\mathfrak{T}/X$. Fix an arrow $\varphi : X \to Y$ in a topos $\mathfrak{T}$. Conclude that we have a functor $\varphi^\star$, called the **preimage functor**, from $\mathrm{Sub}(Y)$ to $\mathrm{Sub}(X)$. (Hint: $\varphi^\star = \mathcal{L}_X \circ \varphi^* \circ \mathcal{I}_Y$.) Does it have left and/or right adjoints?

11. Logic in a Topos

> Anything can follow from anything else, provided that nothing is taken as the basis.
>
> John Cage

We have shown that if X is an object of a topos $\mathfrak{T}$ then $\mathrm{Sub}(X)$ is a lattice with both minimal and maximal elements. Moreover, we were able to transfer this structure to the classifying object Ω using the Yoneda theorem. In particular, when $\mathfrak{T} = \mathfrak{Set}$ we recover the familiar fact the power set of a set is a lattice (under union and intersection), and that $\mathbb{B}$ is a lattice. However, in the latter case we know that in fact both powersets and $\mathbb{B}$ are in fact *Boolean algebras*. This means that we can model the elementary calculus of propositions [10][23] (implication relation together with the operations of disjunction, conjunction, and negation) using the calculus of subsets (inclusion relation together with the operations of union, intersection and complementation).

Let us recall very briefly how this works.

1. The elements $0, 1$ of $\mathbb{B}$ may be regarded as *truth values*. As categorical elements, these are the mappings $1 \xrightarrow{\top} \mathbb{B}$ (called *truth*) $1 \xrightarrow{-} \mathbb{B}$ (called *false*) respectively.
2. The binary logical operators $\wedge, \vee$ can be regarded as mappings $\wedge, \vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$; likewise the monic operator $\neg$ can be regarded as a mapping $\neg : \mathbb{B} \to \mathbb{B}$. That is, the logical operators work with truth values; this is the Boolean algebra structure on $\mathbb{B}$.

3. Every mathematical statement, or *proposition*, mentioning variables $x_1, x_2, \ldots x_n$ which are understood to belong to sets (or other objects) $X_1, X_2, \ldots X_n$ can be regarded as a function from $X_1 \times X_2 \times \ldots X_n$ to $\mathbb{B}$. For example, consider $\sigma, \tau : \mathbb{R} \to \mathbb{B}$ where

$$\sigma(x) = \begin{cases} 1 & x > 2 \\ 0 & \text{else} \end{cases}$$
$$\tau(x) = \begin{cases} 1 & x > 5 \\ 0 & \text{else} \end{cases}$$

and $\varphi : \mathbb{N}^4 \to \mathbb{B}$ where

$$\varphi(j, k, \ell, m) = \begin{cases} 1 & j^m + k^m = \ell^m \\ 0 & \text{else} \end{cases}$$

In short, propositions are the *characteristic functions* of $\mathfrak{Set}$.

4. Every such proposition is associated with a *truth set*, namely the subset of the domain on which the proposition holds true. In otherwords, the truth set is the preimage of $1 \in \mathbb{B}$; this is, of course, the same as saying that the truth set of the characteristic $\chi_A : X \to \mathbb{B}$ is none other than $A \in \text{Sub}(X)$. It is standard to write $\{(x, y) : \psi\}$ for the truth set of $\psi : X \times Y \to \mathbb{B}$. For example,

$$\{x : \sigma\} = (2, \infty), \qquad \{x : \tau\} = (5, \infty)$$
$$\{(j, k, l, m) : \varphi\} = \{ \text{ solutions of } j^m + k^m = \ell^m\}$$

5. Inclusions among truth sets corresponds to implications among the corresponding propositions, according to the rule

$$\{x : \alpha\} \subset \{x : \beta\} \text{ iff } \alpha \text{ implies } \beta$$

For example, $(5, \infty) \subset (2, \infty)$, so we say that τ implies σ, meaning that $\tau(x) = 1$ only if $\sigma(x) = 1$ as well. (Needless to say, this makes sense only for propositions involving the same variable sets, i.e. functions into $\mathbb{B}$ sharing a common codomain.)

6. We can compute either with truth sets (using the set operations intersection, union and complementation) or with truth values (using the logical operations conjuction, disjunction, and negation), using the basic dictionary:

$$\begin{aligned} \{x : \alpha \wedge \beta\} &= \{x : \alpha\} \cap \{x : \beta\} \\ \{x : \alpha \vee \beta\} &= \{x : \alpha\} \cup \{x : \beta\} \\ \{x : \neg\alpha\} &= \{x : \alpha\}^c \end{aligned}$$

For example,

$$\{x : \sigma \wedge \neg\tau\} = (2, \infty) \cap (5, \infty)^c = (2, 5]$$

The set operations on $\text{Sub}(X)$ give the Boolean algebra structure on $\text{Sub}(X)$.

7. A proposition is said to be *valid* if the corresponding truth set is the entire domain. For example, the statement that "τ implies σ" is valid; on the other hand, the statement "σ implies τ" is not, because the contradiction of this statement, $\sigma \wedge \neg\tau$, has the nonempty truth set just computed.

8. It is convenient to introduce a third binary logical operator, called *material implication* and written $\Rightarrow : \Omega \times \Omega \to \Omega$, defined such that $\alpha \Rightarrow \beta$ is the proposition which is false only for those values x such that $\alpha(x) = 1$ but $\beta(x) = 0$. Thus, $\alpha \Rightarrow \beta$ is valid iff α implies β. For example,
$$\{x : \sigma \Rightarrow \tau\} = \{x : \sigma\}^c \cup \{x : \tau\} = (-\infty, 2] \cup (5, \infty)$$

So far we have been working with propositions having one or more *free variables*; that is, unknown quantities which are taken to belong to some set. However, we can also have propositions such as the (incorrect!) statement "$2 > 3$" which we can regard as a mapping $\kappa : 1 \to \mathbb{B}$ (where 1 is any one-element set, i.e. a final object of $\mathfrak{Set}$) defined by
$$\kappa(x) = \begin{cases} 1 & 2 > 3 \\ 0 & \text{else} \end{cases}$$
To take a more interesting example, we can imagine a proposition $\lambda : 1 \to \mathbb{B}$ defined by
$$\lambda(x) = \begin{cases} 1 & 1867930291 \text{ is prime} \\ 0 & \text{else} \end{cases}$$
Given a proposition in one or more free variables, we can *bind* them using quantifiers. For example, in the proposition $\exists x\ \psi : Y \to \mathbb{B}$, x is bound by $\exists$, whereas in $\forall y\ \psi : X \to \mathbb{B}$, y is bound by $\forall$. Of course, the meaning of $\exists$ and $\forall$ is that $\{y : \exists x\ \psi\}$ contains all $y \in Y$ such that $\psi(x, y) = 1$ for *some* $x \in X$, while $\{y : \forall x\ \psi\}$ contains those $y \in Y$ such that $\psi(x, y) = 1$ for *every* $x \in X$. For example, if $\rho : \mathbb{N}^2 \to \mathbb{B}$ is defined by
$$\rho(m, n) = \begin{cases} 1 & m < n \\ 0 & \text{else} \end{cases}$$
then $\exists n\ \rho : \mathbb{N} \to \mathbb{B}$ is valid, and so is $\forall m\ \exists n\ \rho : 1 \to \mathbb{B}$.

Note that we have the inclusion
$$\{y : \forall x\ \psi\} \subset \{y : \exists x\ \psi\}$$
corresponding to the logical implication
$$\forall x\ \psi \text{ implies } \exists x\ \psi$$

The Boolean calculus together with the quantifiers $\exists, \forall$ form the *classical first order logic* underlying standard mathematics.[10] To sum the previous discussion, classical first order logic involves characteristic arrows and subobjects in the topos $\mathfrak{Set}$, and can thus be *modeled* in that topos. It is natural to conjecture that any topos can model first order logic in an analogous fashion.

Let $\mathfrak{T}$ be a topos. We already know that Ω and the various $\mathrm{Sub}(X)$ are lattices. To model first order logic, we must do two things. First, we must provide Ω with additional structure modeling logical implication; specifically, we must define a material implication arrow $\Rightarrow : \Omega \times \Omega \to \Omega$. Second, we must somehow capture the notion of existential and universal quantifiers using the topos structure.

The first problem is fairly easily solved. Let $\pi : \Omega \times \Omega \to \Omega$ be the projection onto the first factor, and take the equalizer of $\vee, \pi : \Omega \times \Omega \to \Omega$. This gives a

[10] Higher order logics exist (and are handily treated by topos theory), but we will not discuss them.

subobject of $\Omega \times \Omega$ whose characteristic arrow is the desired binary logical operator $\Rightarrow : \Omega \times \Omega \to \Omega$.

Exercise: if $\mathfrak{S}et$, verify that the equalizer of $\pi, \vee$ is

$$\{(x,y) : x \wedge y = x\} = \{(x,y) : x \leq y\}$$

Verify that the operator $\Rightarrow : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ agrees with our original definition.

Next, define $\neg : \Omega \to \Omega$ to be the characteristic arrow of $1 \stackrel{-}{\to} \Omega$.

Exercise: verify that plugging any combination of the maximal and minimal truth values ($\top, -$ respectively) into the operators $\neg, \vee, \wedge, \Rightarrow$, always gives the usual Boolean results; e.g. $\neg - = \top$, $- \wedge \top = -$, $- \vee \top = \top$, etc. However, note that for many topoi, the classifier has elements (truth values) other than $\top, -$.

Now for the suprise: the law of the excluded middle $\neg\neg\alpha = \alpha$ fails for general topoi $\mathfrak{T}$. Thus, Ω cannot, in general, be a Boolean algebra. In fact, it is in general a *Heyting algebra*; these are generalizations of Boolean algebras which first arose in the formal study of intuitionistic logic (where the law of the excluded middle is denied). A topos $\mathfrak{T}$ whose classifier is a Boolean algebra is called a **Boolean topos**; otherwise $\mathfrak{T}$ will have a *weaker* logic than the classical first order logic: in particular, the law of the excluded middle fails, although the De Morgan laws may still hold.

Fortunately, our previous work has prepared us well for a simple definition of a Heyting algebra. Given a lattice L, we have a natural partial order defined by $x \leq y$ iff $x \wedge y = x$ (equivalently, $x \vee y = y$), so we may consider L as a preorder category $\mathfrak{L}$.

Exercise: verify that for any $x, y \in L$, $x \wedge y$ and $x \vee y$ define a product and sum (respectively) in $\mathfrak{L}$. If L has a **top element** 1 (satisfying $1 \wedge x = x$ for all x) and **bottom element** 0 (satisfying $0 \vee x = x$ for all x), verify that these are final and initial objects of $\mathfrak{L}$ (respectively).

If L is a distributive lattice with top and bottom elements $0, 1$ (respectively), and if $\mathfrak{L}$ has an exponential, L is said to be a **Heyting algebra**; in this context y^x is written $x \Rightarrow y$ and is called the **psuedocomplement** of x **relative to** y. The adjunction

$$\begin{array}{c|c}
\mathfrak{L} & \mathfrak{L} \\
x \wedge e & x \\
\downarrow & \downarrow \\
y & e \Rightarrow y \\
(\cdot) \wedge e & e \Rightarrow (\cdot)
\end{array}$$

amounts to saying that $x \Rightarrow y$ is the unique element $z \in L$ such that

$$z \leq (x \Rightarrow y) \text{ iff } z \wedge x \leq y$$

The psueocomplement $x \Rightarrow 0$ is usually called simply the **psuedocomplement** of x and is denoted $\neg x$.

Exercise: verify that in a Boolean algebra (a lattice equipped with a complement $(\cdot)^c$ operation satisfying appropriate properties) we can take $x \Rightarrow y$ to be $x^c \vee y$, and then $\neg x = x^c$.

Exercise: consider the preorder category $\mathfrak{O}X$ constructed from the open sets of X (ordered by inclusion). Verify that $U \cap V$ and $U \cup V$ define the sum and product (respectively) while $\emptyset$ and X are initial and final objects (respectively). Verify that defining $(U \Rightarrow V) = (U^c)^o \cup V$ makes the open sets of X into a **complete Heyting algebra** (i.e. a complete lattice which is also a Heyting algebra; note that the wedge over an infinite index set is the *interior* of the corresponding intersection!) Verify that $\neg U = (U^c)^o$.

We should (and do) have the obvious identities involving characteristic arrows:

$$\begin{array}{rcl} \chi_{A \sqcap B} & = & \chi_A \wedge \chi_B \\ \chi_{A \sqcup B} & = & \chi_A \vee \chi_B \\ \chi_{A \Rightarrow B} & = & \chi_A \Rightarrow \chi_B \\ \chi_{\neg A} & = & \neg \chi_A \end{array}$$

Let us write $\langle x : \alpha \rangle$ for the "truth-object" of $\alpha : X \to \Omega$. We should (and do) have the obvious identities relating the Heyting algebra structure on $\mathrm{Sub}(X)$ to logical operations on predicates:

$$\begin{array}{rcl} \langle x : \alpha \wedge \beta \rangle & = & \langle x : \alpha \rangle \sqcap \langle x : \beta \rangle \\ \langle x : \alpha \vee \beta \rangle & = & \langle x : \alpha \rangle \sqcup \langle x : \beta \rangle \\ \langle x : \alpha \Rightarrow \beta \rangle & = & \langle x : \alpha \rangle \Rightarrow \langle x : \beta \rangle \\ \langle x : \neg \alpha \rangle & = & \neg \langle x : \alpha \rangle \end{array}$$

Exercise: explicitly construct the various Heyting operators for Ω in the cases of $G\mathfrak{S}et$ and $\mathfrak{B}\mathrm{n}\,X$.

Fix an arrow $\psi : E \to Y$. Recall that the slice change functor ψ^* has both left and right adjoints, $\Sigma_\psi \dashv \psi^* \dashv \Pi_\psi$. The preimage functor $\psi^\star$ is closely related to ψ, so it is not suprising that we obtain similar adjunctions $\exists_\psi \dashv \psi^\star \dashv \forall_\psi$, as indicated in the diagram:

$$\begin{array}{c|c|c} \mathrm{Sub}(F) & \mathrm{Sub}(E) & \mathrm{Sub}(F) \\ & & \\ \exists_\psi A_1 & A_1 & \\ \downarrow & \downarrow & \\ B & \psi^\star B & B \\ & \downarrow & \downarrow \\ & A_2 & \forall_\psi A_2 \\ & & \\ \exists_\psi & \psi^\star & \forall_\psi \end{array}$$

Notice that the content of the assertion that $\exists_\psi \dashv \psi^\star \dashv \forall_\psi$ is precisely the claim that

$$\exists_\psi A_1 \sqsubset B \sqsubset \forall_\psi A_2$$

iff

$$A_1 \sqsubset \psi^\star B \sqsubset A_2$$

Notions of first order logic	Categorical notions for $\mathfrak{T}$
A truth value	An element of Ω, i.e. $1 \to \Omega$
A monic logical operator	An arrow $\Omega \to \Omega$
A binary logical operator	An arrow $\Omega \times \Omega \to \Omega$
Proposition κ (no free variables)	$\kappa : 1 \to \Omega$
Proposition α (with free variable x)	the characteristic of some $A \in \mathrm{Sub}(X)$
Proposition φ (free variables x, y)	the characteristic of some $R \in \mathrm{Sub}(X \times Y)$
Proposition $\exists x\ \varphi$ (free variable y)	the characteristic of $\exists_\pi R \in \mathrm{Sub}(Y)$, where $\exists_\pi$ is left adjoint of $\pi^\star$, where $\pi : X \times Y \to Y$
Proposition $\forall x\ \varphi$ (free variable y)	the characteristic of $\forall_\pi R \in \mathrm{Sub}(Y)$, where $\forall_\pi$ is right adjoint of $\pi^\star$

FIGURE 3. How to model first order logic in a topos $\mathfrak{T}$.

Exercise: in the case $\mathfrak{T} = \mathfrak{S}et$, verify that given $A \in \mathrm{Sub}(X)$, $B \in \mathrm{Sub}(Y)$, and $\varphi : X \to Y$, we have

$$\begin{aligned} \exists_\psi A &= \varphi(A) \\ \psi^\star B &= \varphi^{-1}(B) \\ \forall_\psi A &= \varphi(A) \setminus \varphi(A^c) \end{aligned}$$

In particular, in the case of the canonical projection $\pi : X \times Y \to Y$, for $R \in \mathrm{Sub}(X, Y)$, the truth set of some $\psi : X \times Y \to \mathbb{B}$, verify that we have

$$\begin{aligned} \exists_\pi R &= \{y : \exists x\ \psi\} \\ \forall_\pi R &= \{y : \forall x\ \psi\} \end{aligned}$$

(Lawvere was apparently the first to realize that adjoint functors can replace quantifiers in the foundations of mathematics.)

Proposition 11.1 (Beck)**.** *Let $\mathfrak{T}$ be a topos. Then whenever*

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

is a pullback square, then in the diagram

$$\begin{array}{ccc} \mathrm{Sub}(E) & \begin{array}{c} \xrightarrow{\exists_\psi} \\ \xleftarrow{\psi^\star} \\ \xrightarrow{\forall_\psi} \end{array} & \mathrm{Sub}(F) \\ {\scriptstyle\exists_\sigma}\uparrow\downarrow\uparrow{\scriptstyle\forall_\sigma} & & {\scriptstyle\exists_\tau}\uparrow\downarrow\uparrow{\scriptstyle\forall_\tau} \\ \mathrm{Sub}(X) & \begin{array}{c} \xrightarrow{\exists_\varphi} \\ \xleftarrow{\varphi^\star} \\ \xrightarrow{\forall_\varphi} \end{array} & \mathrm{Sub}(Y) \end{array}$$

True in any topos	True only in a Boolean topos
	$(\alpha \vee \neg\alpha) = \top$
$\alpha \Rightarrow \neg\neg\alpha$	$\neg\neg\alpha \Rightarrow \alpha$
$\neg\alpha \ \wedge \ \neg\beta \Leftrightarrow \neg(\alpha \vee \beta)$	
$\neg\alpha \ \vee \ \neg\beta \Rightarrow \neg(\alpha \wedge \beta)$	$\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \ \wedge \ \neg\beta$
$\forall x \ \neg\psi \Leftrightarrow \exists x \ \psi$	
$\exists x \ \psi \Rightarrow \neg(\forall x \ \neg\psi)$	$\neg(\forall x \ \neg\psi) \Rightarrow \exists x \ \psi$
$\forall x \ \psi \Rightarrow \neg(\exists x \ \neg\psi)$	$\neg(\exists x \ \neg\psi) \Rightarrow \forall x \ \psi$

FIGURE 4. A rough guide to topos logic.

we have the following four equalities:

$$\begin{aligned} \varphi^\star \circ \exists_\tau &= \exists_\sigma \circ \psi^\star \\ \varphi^\star \circ \forall_\tau &= \forall_\sigma \circ \psi^\star \\ \tau^\star \circ \exists_\varphi &= \exists_\psi \circ \sigma^\star \\ \tau^\star \circ \forall_\varphi &= \forall_\psi \circ \sigma^\star \end{aligned}$$

The relation between the six cofunctors $\Sigma_\psi \dashv \psi^* \dashv \Pi_\psi$ and $\exists_\psi \dashv \psi^\star \dashv \forall_\psi$, and the four functors $\mathcal{L}_E \dashv \mathcal{I}_E$ and $\mathcal{L}_F \dashv \mathcal{I}_F$ is given in the following Lemma.

Proposition 11.2. *In the following diagram (called the* **doctrinal diagram***),*

$$\begin{array}{ccc} \mathrm{Sub}(E) & \begin{array}{c} \xrightarrow{\exists_\psi} \\ \xleftarrow{\psi^\star} \\ \xrightarrow{\forall_\psi} \end{array} & \mathrm{Sub}(F) \\ \mathcal{L}_E \uparrow \ \downarrow \mathcal{I}_E & & \mathcal{L}_F \uparrow \ \downarrow \mathcal{I}_F \\ \mathfrak{T}/E & \begin{array}{c} \xrightarrow{\Sigma_\psi} \\ \xleftarrow{\psi^*} \\ \xrightarrow{\Pi_\psi} \end{array} & \mathfrak{T}/F \end{array}$$

we have the following four natural isomorphisms:

$$\begin{aligned} \exists_\psi \circ \mathcal{L}_F &\simeq \mathcal{L}_E \circ \Sigma_\psi \\ \mathcal{I}_F \circ \forall_\psi &\simeq \Pi_\psi \circ \mathcal{I}_E \\ \psi^* \circ \mathcal{I}_F &\simeq \mathcal{I}_E \circ \psi^\star \\ \psi^\star \circ \mathcal{L}_F &\simeq \mathcal{L}_E \circ \psi^* \end{aligned}$$

Exercise: verify the claims made in Figure 4.

12. Models in a Topos

> A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects: on the one hand, topology and algebraic geometry, and on the other hand, logic and set theory.
>
> Saunders Mac Lane and Ieke Moerdijk [20]

In this final (rather sketchy) section, we explore briefly the idea that a (not quite arbitrary) topos can be used to model any mathematical concept whatsoever.

We begin by examining in some detail how the notion of a *group* can be recast in categorical form. Indeed, this part of our discussion will work for any category $\mathfrak{C}$ with a final object 1, in which finite products always exist.

Definition 12.1. *A* $\mathfrak{G}rp$**-object** *of* $\mathfrak{C}$ *is an object* G *together with:*

1. *a group product operator* $\mu : G \times G \to G$*,*
2. *an group inversion operator* $\rho : G \to G$*,*
3. *an group identity element* $\varepsilon : 1 \to G$*,*

such that

1. ε *behaves like a two-sided identity element, i.e.*

$$\begin{array}{ccccc} 1 & \xrightarrow{(\varepsilon, 1_G)} & G \times G & \xleftarrow{(1_G, \varepsilon)} & 1 \\ {\scriptstyle !}\big\uparrow & & {\scriptstyle \mu}\big\downarrow & & {\scriptstyle !}\big\uparrow \\ G & \xrightarrow{1_G} & G & \xleftarrow{1_G} & G \end{array}$$

   commutes,

2. ρ *acts like inversion, i.e.*

$$\begin{array}{ccccc} G & \xrightarrow{(\rho, 1_G)} & G \times G & \xleftarrow{(1_G, \rho)} & G \\ {\scriptstyle !}\big\downarrow & & {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle !} \\ 1 & \xrightarrow{\varepsilon} & G & \xleftarrow{\varepsilon} & 1 \end{array}$$

   commutes,

3. μ *is associative, i.e.*

$$\begin{array}{ccccc} (G \times G) \times G & \xrightarrow{\mu \times 1_G} & G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle \simeq}\big\downarrow & & & & \big\| \\ G \times (G \times G) & \xrightarrow{1_G \times \mu} & G \times G & \xrightarrow{\mu} & G \end{array}$$

   commutes.

A $\mathfrak{G}rp$**-arrow** *of* $\mathfrak{C}$ *is an arrow* $\varphi : G \to H$ *between two group objects such that*

1. φ *respects the identities, i.e.*

$$\begin{array}{ccc} 1 & \xrightarrow{\varepsilon_G} & G \\ \big\| & & {\scriptstyle \varphi}\big\downarrow \\ 1 & \xrightarrow{\varepsilon_H} & H \end{array}$$

   commutes,

2. φ *respects multiplication, i.e.*

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu_G} & G \\ {\scriptstyle \varphi \times \varphi}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow \\ H \times H & \xrightarrow{\mu_H} & H \end{array}$$

   commutes.

Exercise: verify that $\mathfrak{G}rp$-arrows between $\mathfrak{G}rp$-objects form a subcategory of $\mathfrak{C}$, denoted $\mathfrak{G}rp_{\mathfrak{C}}$. Show that this category has products and equalizers (and thus kernels in the usual sense of group theory). Verify that $\mathfrak{G}rp_{\mathfrak{T}op}$ is the category of topological groups, while $\mathfrak{Grp}_{\mathfrak{Man}}$ is the category of Lie groups, $\mathfrak{Grp}_{\mathfrak{Sh}\,X}$ is the category of sheaves of groups over X, and $\mathfrak{Abg}_{\mathfrak{Sh}\,X}$ is the category of sheaves of *abelian* groups over X. This suggests that if $\mathfrak{C}$ is itself a subcategory of $\mathfrak{D}$, then $\mathfrak{Grp}_{\mathfrak{C}}$ is a subcategory of $\mathfrak{Grp}_{\mathfrak{D}}$ as well as of $\mathfrak{C}$. Is it?

Exercise: Fix a group G, a ring R, and a field K, and let $\mathfrak{C}$ be as above. Construct the categories $\mathfrak{Rng}_{\mathfrak{C}}$, $G\mathfrak{Set}_{\mathfrak{C}}$, $R\mathfrak{Mod}_{\mathfrak{C}}$, and $K\mathfrak{Lin}_{\mathfrak{C}}$ following the model of $\mathfrak{Grp}_{\mathfrak{C}}$.

Since toposes always have a final object and products, these categories "know" the structure of the elementary algebraic categories, athough the theory of $\mathfrak{Grp}_{\mathfrak{T}}$ may very well be quite different from the theory of $\mathfrak{Grp}_{\mathfrak{Set}}$. The most interesting toposes from the point of view of logic can model many more mathematical structures; in particular, they contain a notion of *number*.

Definition 12.2. *Let* $\mathfrak{T}$ *be a topos. A* **natural number object** *is an object* N *equipped with*

1. *a* **zero element** $\zeta : 1 \to N$,
2. *a* **successor arrow** $\sigma : N \to N$,

such that for every object X *and arrow* $\varphi : X \to X$ *and element* $x : 1 \to X$*, there is a unique* υ *such that the diagram*

$$\begin{array}{ccccc} 1 & \xrightarrow{\zeta} & N & \xrightarrow{\sigma} & N \\ \| & & \downarrow{\scriptstyle\upsilon} & & \downarrow{\scriptstyle\upsilon} \\ 1 & \xrightarrow{\zeta} & N & \xrightarrow{\sigma} & N \end{array}$$

As usual, a natural object may not exist, but if it does it is unique up to isomorphism. The idea is that a natural number object contains the "recursive structure" needed to carry out finite induction using the logic of $\mathfrak{T}$. We can then proceed to define addition on the natural numbers (the elements of N), and so forth; see [17].

Exercise: find a natural number object for a topos other than $\mathfrak{Set}$.

Exercise: suppose $\mathfrak{C}$ is a category with a final object, products, sums, coequalizers, and a natural number object. Show that $\mathfrak{Grp}_{\mathfrak{C}}$ has sums and coequalizers (and thus cokernels).

Once we have the natural numbers, since sums are available in $\mathfrak{T}$ it is not hard to see how to construct negative integers. Because we can model an equivalence relation on (say) ordered pairs of integers as an appropriate arrow, we can now obtain rational numbers; see [17] for details. Finally, we can mimic either the Cauchy or Dedekind constructions (in $\mathfrak{Set}$) of the real numbers. But now another suprise: in general these give different results, although we can say that for any topos $\mathfrak{T}$ (possessing a natural number object) both constructions work and that every *Cauchy real number* is also a *Dedekind real number*. In particular, in the topos $\mathfrak{Sh}\,X$, the Cauchy reals are the *locally constant* real valued functions on X (if X is connected, this of course means they are in bijection with the ordinary real numbers), whereas the Dedekind reals correspond to the real valued functions on X.

This suggests, correctly, that the kind of mathematics we will create depends heavily upon which topos we choose as our foundation. An even more striking example of this principle involves the most notorious of all "dubious" assertions in the standard foundations for classical mathematics, namely the *Axiom of Choice*. It states that for any onto mapping $\varphi : X \to Y$, there is a **selection function** $\sigma : X \to Y$ which selects one point from each fiber $\varphi^{-1}(y)$, where of course $\varphi \circ \sigma =$

Nature of local set theory	Condition on the corresoponding topos $\mathfrak{T}$
Classical logic	Ω is a Boolean algebra ($\neg\neg\psi = \psi$ holds)
Weaker than classical logic	DeMorgan laws hold in Ω but $\neg\neg\psi = \psi$ fails
Still weaker logic	$\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \ \wedge \ \neg\beta$ fails
Consistent logic	$0, 1$ are non-isomorphic objects
Complete	Ω has only two elements ($\top$ and $-$)
Axiom of choice holds	Every epic is a retraction

FIGURE 5. The nature of the mathematics founded upon a given topos depends on the structure of that topos.

1_Y. (The Axiom of Choice is logically equivalent to Zorn's Lemma, the Hausforff Maximal Principle, and a number of other ideas which play an important role in modern mathematics.) But this is easily restated very succintly in categorical terms: *the Axiom of Choice holds in a topos $\mathfrak{T}$ iff every epic is a retraction.* Moreover, Diaconescu showed that this happens only if $\mathfrak{T}$ is a Boolean topos; see [1][2][17].

Definition 12.3. *Let $\mathfrak{T}$ be a topos. A* **Lawvere-Tierney operator** *on $\mathfrak{T}$ is an arrow $j : \Omega \to \Omega$ such that:*

1. $j \circ \top = \top$,
2. $j \circ j = j$,
3. $j \circ \wedge = \wedge \circ (j \times j)$.

Given $A \in \mathrm{Sub}(X)$, we write $J\ A$ for the subobject of X whose characteristic is $j \circ \chi_A$.

Exercise: show that for any $A, B \in \mathrm{Sub}(X)$,

1. $A \sqsubset J\ A$,
2. $J\ A = JJ\ A$,
3. $A \sqsubset B$ implies $J\ A \sqsubset J\ B$,
4. $J\ A \sqcap J\ B = J\ (A \sqcap B)$,
5. $J\varphi^\star(A) = \varphi^\star(J\ A)$.

The first three properties here say that J is an **algebraic closure operator** on Ω. The topological closure operator on a topological space is another example of an algebraic closure operator, but the fourth property shows that we do *not* want to think of J as topological closure. Nonetheless, there *is* a topological interpretation, and j is traditionally called a **topology** on $\mathfrak{T}$, although it certainly does not define a topology in the usual sense of that word. In modal logic, one has monic propositional modifiers (monic logical operators) which are interpreted to mean something like

1. "it is neccessarily true that" (alethic mode),
2. "it is known that" (epistemic mode),
3. "it is believed that" (doxastic mode),
4. "it ought to be true that" (deontic mode).

According to Tierney, the monic logical operator $J : \Omega \to \Omega$ should be interpreted as "it is locally true that"; in other words, we have a notion of a "local neighborhood" of the value taken on by some variable x appearing in the proposition.

Exercise: show that the following define Lawvere-Tierney operators:

1. $j = \neg\neg$ (this gives the **double negation topology**),
2. $j = \gamma \Rightarrow (\cdot)$, where $\gamma : 1 \to \Omega$ is any truth value (the **open topology**),

3. $j = \gamma \wedge (\cdot)$, where $\gamma : 1 \to \Omega$ is any truth value (the **closed toplogy**).

Proposition 12.4. *Let $\mathfrak{T}$ be a topos, and let $j = \neg\neg$. Then $\mathfrak{T}_J$ is a Boolean topos.*

See [20] for a proof.

If $\psi : X \to \Omega$ satisfies $j \circ \psi = \top_X$, ψ is said to be ***j*-true**.

In the case $\mathfrak{T} = \mathfrak{Sh}\, X$, let U be the open set of X which is the projection of the places where the section $\gamma : 1 \to \Omega$ coincides with the truth section, and observe that the three examples have the following interpretations: $\psi : X \to \Omega$ is j-true if

1. "ψ is true so far as global elements are concerned",
2. the truth-object of ψ includes U,
3. the truth-object of ψ includes the complement of U.

Definition 12.5. *Let $\mathfrak{T}$ be a topos and let j be a Lawvere-Tierney operator on Ω. A* ***j*-sheaf** *X is an object of $\mathfrak{T}$ such that for all objects F and subobjects $E \in \mathrm{Sub}(F)$ with $J\ E = F$, there is a unique arrow such that*

$$\begin{array}{ccc} \mathrm{dom}\,\psi & \xrightarrow{\ \sigma\ } & X \\ {\scriptstyle\psi}\downarrow & & \| \\ F & \xrightarrow{\ !\ } & X \end{array}$$

commutes, where of course $[\psi] = E$.

Exercise: verify that arrows between j-sheaves in $\mathfrak{T}$ form a full subcategory.

Theorem 12.6 (Lawvere-Tierney)**.** *Let $\mathfrak{T}$ be a topos. The collection of sheaf maps between j-sheaves is a* subtopos *of $\mathfrak{T}$, written $\mathfrak{T}_j$, which is a reflective subcategory of $\mathfrak{T}$; that is, the inclusion functor $\mathcal{I}_j$ from $\mathfrak{T}_j$ to $\mathfrak{T}$ has a left adjoint $\mathcal{L}_j \dashv \mathcal{I}_j$. Moreover, $\mathcal{L}_j$ preserves finite limits and all colimits and $\mathcal{I}_j$ preserves all limits.*

An adjoint pair $\mathcal{L} \dashv \mathcal{I}$ satisfying the stated conditions is called a **geometric morphism**.

Exercise: show that the collection of geometric morphisms between topoi forms a category.

Theorem 12.7. *If $\mathfrak{T}$ is a topos, then for any category $\mathfrak{C}$, the category $\mathfrak{T}^{\mathfrak{C}}$ is a topos.*

See [17][20] for a proof.

In particular, if P is a poset, let $\mathcal{P}$ be the corresponding preorder category. Then $\mathfrak{Set}^{\mathcal{P}}$ is a topos. It is hardly ever a Boolean topos; indeed, the classifying object usually has many more than two elements. Cohen's idea for forcing a certain property to be true in a "nonstandard set theory" now comes down to this. We interpret the elements of P as stages of "knowlege", where $p \leq q$ means that q is a later (and more extensive) stage of knowledge than p. Note that each element of $\mathfrak{Set}^{\mathcal{P}}$ is a sort of "net" of sets indexed by P. There is a natural notion of asymptotic agreement between two such elements of $\mathfrak{Set}^{\mathcal{P}}$; moding out by this equivalence relation we obtain $\overline{\mathfrak{Set}}$, the **Cohen extension** of $\mathfrak{Set}$. This will be a Boolean topos. Another way of describing this construction is to note that $\mathfrak{Set}^{\mathcal{P}}$ is essentially the presheaf category over $\mathcal{P}$, and $\overline{\mathfrak{Set}}$ is $\mathcal{P}_{\neg\neg}$. For an appropriate choice of P, the topos $\overline{\mathfrak{Set}}$ corresponds to a nonstandard set theory where classical logic obtains (and the Axiom of Choice holds), but the Continuum Hypothesis fails.

For other choices of P, alternative propositions can be "forced" to be true or false; hence the term **forcing techniques**.

Remarkably, essentially the same construction was arrived at by Grothendieck in the context of trying to set up a programme for proving the Weil conjectures (these were soon proven by Deligne, using the sheaf-theoretic ideas provided by Grothendieck). (See [20] for details of the ideas of both Cohen and Grothendieck.) Similar ideas also turn up (in disguise) in Abraham Robinson's nonstandard analysis. (See [17].) These startling and entirely unexpected unifications were (and remain) a major triumph of topos theory.

References

1. Michael Barr and Charles Wells, *Toposes, Theories, and Triples*, New York: Springer-Verlag, 1985.
2. J. L. Bell, *Toposes and Local Set Theories*. Oxford: Clarendon Press, 1988.
3. Garrett Birkoff and Saunders Mac Lane, *A Survey of Modern Algebra*. Fourth edition. New York: Macmillan, 1977.
4. William Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Second edition. New York: Academic Press, 1986.
5. B. A. Davey and H. A. Priestly, *Introduction to Lattices and Order*. Cambridge: Cambridge University Press, 1990.
6. S. Eilenberg and S. Mac Lane, "General Theory of Natural Equivalences", *Trans. Amer. Math. Soc.* 58 (1945): 231–244.
7. Gerald B. Folland, *Real Analysis: Modern Techniques and their Applications*. New York: Wiley, 1984.
8. Robert Geroch, *Mathematical Physics*. Chicago: University of Chicago Press, 1985.
9. Robert Goldblatt, *Topoi: the Categorial Analysis of Logic*. New York: North-Holland, 1979.
10. Paul R. Halmos, *Naive Set Theory*. New-York, Springer-Verlag, 1974.
11. Chris Hillman, "Outline of the Theory of G-Sets", eprint, 1995, available at the author's web site, URL
    `http://www.math.washington.edu/~hillman/papers.html`
12. —, "What is a Concept?", eprint, 1996, available at the author's web site.
13. Wilfrid Hodges, "What is a Structure Theory?", *Bulletin of the London Mathematical Society* 19 (1987) 209–237.
14. Thomas W. Hungerford, *Algebra*. New York: Springer-Verlag, 1976.
15. Nathan Jacobson, *Basic Algebra*. Two volumes. Second edition. New York: Freeman, 1985.
16. D. M. Kan, "Adjoint Functors", *Trans. Amer. Math. Soc.* 87 (1958) 294–329.
17. Colin McLarty, *Elementary Categories, Elementary Toposes*. Oxford: Oxford Science Publications, 1995.
18. Saunders Mac Lane, "Duality for groups", *Bulletin of the American Mathematical Society* 56 (1950): 485–516.
19. —, *Categories for the Working Mathematician*. New York: Springer-Verlag, 1971.
20. Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic: a First Introduction to Topos Theory*. New York: Springer-Verlag, 1992.
21. M. Makkai and G. Reyes, *First Order Categorical Logic*. Lecture Notes in Mathematics Vol. 611. New York: Springer-Verlag, 1977.
22. Peter M. Neumann, Gabrielle A. Stoy, and the late Edward C. Thompson, *Groups and Geometry*. New York : Oxford University Press, 1994.
23. W. V. Quine, *Methods of Logic*. Fourth edition. Cambridge: Harvard University Press, 1982.
24. Derek J. S. Robinson, *A Course in the Theory of Groups*. New York: Springer-Verlag, 1982.

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195-4350

E-mail address: `hillman@math.washington.edu`