2-Clean Rings *

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Abstract. A ring R is said to be n-clean if every element can be written as a sum of an idempotent and n units. The class of these rings contains clean ring and n-good rings in which each element is a sum of n units. In this paper, we show that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-clean and that the ring $B(R)$ of all $\omega \times \omega$ row and column-finite matrices over any ring R is 2-clean. Finally, the group ring RC_n is considered where R is a local ring.

Key words: 2-clean rings, 2-good rings, free modules, row and column-finite matrix rings, group rings.

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1. Introduction

The question of when the automorphism group of a module additively generates its endomorphism ring has been of interest for many years. A ring is called n-good [12] if every element is a sum of n units. In 1953 Wolfson [14] and in 1954 Zelinsky [17] showed, independently, that every element of the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is 2-good. In 1985 Goldsmith [4] proved that the endomorphism ring of a complete module over a complete discrete valuation ring is 2-good. In [13] Wans considered free R-modules where R is a PID, and showed that if the rank of M is finite and greater than 1, then $End_R(M)$ is 2-good. Meehan [8] further showed that the endomorphism ring of a free R-module of rank at least 2 is 2-good where R is a PID. Moreover, the above question is considered by many authors on abelian groups (see [2],[7],[8]) and on general ring with an identity (see [3],[6],[11]).

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In 1977 Nicholson [10] introduced the concept of a clean ring (1-clean) which contains unit-regular rings and semiperfect rings, and showed that every clean ring must be exchange. Camillo and Yu [1] further proved that a clean ring with 2 invertible is 2-good. Recently, Xiao and Tong [16] called a ring R n-clean if every element of R is the sum of an idempotent and n units. The class of these rings contains clean rings and n-good rings. In 1974 Henriksen [6] found that for any ring R and $n > 1$, the matrix ring $M_n(R)$ is 3-good. Moreover, Vámos [12] proved that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 3-good. Motivated by the result of Henriksen and Vámos, we conjecture that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-clean.

In this paper, we answer the question in the positive. In fact, we proved that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-clean. It is also proved that the ring $B(R)$ of all $\omega \times \omega$ row and column-finite matrices over any ring R is 2-clean. Finally, the group ring RC_n is considered where R is a local ring.

Throughout this paper, rings are associative with identity and modules are unitary. $J(R)$ and $U(R)$ denote the Jacobson radical and the group of units of R, respectively.

2. BASIC PROPERTIES OF n-CLEAN RINGS

An element of a ring is called n-clean if it can be written as the sum of an idempotent and n units. A ring is called n-clean if each of its elements is n-clean. In this section, some properties of n-clean rings are given.

Proposition 1. *Let R be a ring and let $a \in R$. Then the following statements hold:*

(1) if a is n-clean then it is also l-clean for all $n \leq l$.

(2) every n-good ring is n-clean; if R is n-clean with $2 \in U(R)$ then it is $(n+1)$-good.

Proof. (1) We only need to prove that a is $n+1$-clean. Let $a \in R$ be n-clean: $a = e + u_1 + u_2 + \cdots + u_n$ where $e^2 = e \in R$ and $u_1, u_2, \cdots, u_n \in U(R)$. Note that $e = (1-e) + (2e-1)$, thus we have $a = (1-e) + (2e-1) + u_1 + \cdots + u_n$ where $2e - 1 \in U(R)$.

(2) It is clear that every n-good ring is n-clean. The second statement is due to Xiao and Tong (see [16]). □

Let $S(R)$ be the nonempty set of all proper ideal of R generated by central idempotents. An ideal $P \in S(R)$ is called a Pierce ideal of R if P is a maximal (with respect to inclusion) element of the set $S(R)$. If P is a Pierce ideal of R, then the factor ring R/P is called a Pierce stalk of R. The next result shows that the n-clean property needs to be checked only by for indecomposable rings or Pierce stalks.

Proposition 2. *Let R be a ring. Then the following are equivalent:*

(1) R is n-clean.

(2) every factor ring of R is n-clean.

(3) every indecomposable factor ring of R is n-clean.

(4) every Pierce stalk of R is n-clean.

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are directly verified.

(3) $\Rightarrow$ (1). Suppose that (3) holds and R is not n-clean, then there is an element $a \in R$ which is not n-clean. Now let $\mathcal{S}$ be the set of all proper ideals I of R such that $\overline{a}$ is not n-clean in R/I. Clearly, $0 \in \mathcal{S}$ and the set $\mathcal{S}$ is not empty. Define a partial ordering on $\mathcal{S}$ by $'' \subseteq''$. If $\{I_\alpha : \alpha \in \Lambda\}$ is a chain in $\mathcal{S}$, let $I = \cup_{\alpha\in\Lambda} I_\alpha$. We will show that $\overline{a}$ is not n-clean in R/I. Suppose that $\overline{a}$ is n-clean in R/I. Then there exist $\overline{u}_1, \overline{u}_2, \cdots, \overline{u}_n \in U(R/I)$ (with inverses $\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_n$, respectively) and $\overline{e}^2 = \overline{e} \in R/I$ such that $\overline{a} = \overline{e} + \overline{u}_1 + \overline{u}_2 + \cdots + \overline{u}_n$. Note that $e^2 - e \in \cup_{\alpha\in\Lambda} I_\alpha$ and $u_i v_i - 1,\ v_i u_i - 1 \in \cup_{\alpha\in\Lambda} I_\alpha$, so $e^2 - e \in I_{\alpha_0},\ u_i v_i - 1 \in I_{\alpha_i}$ and $v_i u_i - 1 \in I_{\alpha_i'}$ for $\alpha_0, \alpha_i, \alpha_i' \in \Lambda$. Because $\{I_\alpha : \alpha \in \Lambda\}$ is a chain in $\mathcal{S}$, there is a maximal I_s in the set $\{I_{\alpha_0}, I_{\alpha_1}, \cdots, I_{\alpha_n}, I_{\alpha_1'}, I_{\alpha_1'}, \cdots, I_{\alpha_n'}\}$ such that $I_{\alpha_0}, I_{\alpha_i}, I_{\alpha_i'} \subseteq I_s$. That is , $\overline{a}$ is n-clean in R/I_s, a contradiction. This implies that $I \in \mathcal{S}$ is a upper bound of the chain. Because $\mathcal{S}$ is an inductive set and, by Zorn's Lemma, $\mathcal{S}$ has a maximal element I_0. By (3) R/I_0 is decomposable as a ring. Write $R/I_0 \cong R/I_1 \oplus R/I_2$ where both the ideals $I_1,\ I_2$ strictly contain I_0 and so by the choice of I_0, $\overline{a}$ is n-clean in R/I_1 and R/I_2. But then $\overline{a}$ is n-clean in R/I_0, a contradiction.

(4) $\Rightarrow$ (1). Let $\mathcal{S}$ be the set of all proper ideals I of R such that I is generated by central idempotents and the ring R/I is not n-clean. Assume that R is not n-clean. Then $0 \in \mathcal{S}$ and the set $\mathcal{S}$ is not empty. It is directly verified as above that the union of every ascending chain of ideals from $\mathcal{S}$ belongs to $\mathcal{S}$. By Zorn's Lemma, the set $\mathcal{S}$ contains a maximal element P. By condition (4), it is sufficient to prove that P is a Pierce ideal. Assume that

contrary. By the definition of the Pierce ideal, there is a central idempotent e of R such that $P+eR$ and $P+(1-e)R$ are proper ideals of R which properly contain the ideal P. Since ideals $P+eR$ and $P+(1-e)R$ do not belong to $\mathcal{S}$ and are generated by central idempotents, $R/(P+eR)$ and $R/(P+(1-e)R)$ are n-clean. Note that $R/P \cong (R/(P+eR))\times(R/(P+(1-e)R))$, it can be verified that R is n-clean. □

3. MATRIX RINGS AND ENDOMORPHISM RINGS OF FREE MODULES

In this section, we will consider the 2-cleaness of the endomorphism ring of a free R-module of rank at least 2. First we give the following simple and interesting decomposition.

Lemma 3. *Over any ring, the 2×2 and 3×3 matrices are 2-clean.*

Proof. Let R be a ring and let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R)$. Put $E = \begin{pmatrix} a_{11}-1 & 2-a_{11} \\ a_{11}-1 & 2-a_{11} \end{pmatrix}$. It is checked easily that then $E^2 = E$. Thus we have

$$A - E = \begin{pmatrix} 1 & a_{12}+a_{11}-2 \\ a_{21}-a_{11}+1 & a_{22}+a_{11}-2 \end{pmatrix}.$$

Observing the above matrix, and then there exist invertible matrices

$$P = \begin{pmatrix} 1 & 0 \\ a_{11}-a_{21}-1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 2-a_{11}-a_{12} \\ 0 & 1 \end{pmatrix}$$

such that

$$P(A-E)Q = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix},$$

where $c = a_{11}^2 + a_{11}a_{12} - a_{21}a_{12} - a_{21}a_{11} - 2a_{11} + 2a_{21} - a_{12} + a_{22}$. This shows that $A = P^{-1}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} Q^{-1} + P^{-1}\begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix} Q^{-1} + E$ is 2-clean.

Now let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ be a 3×3 matrix over R. We first construct an idempotent in order to show 2-cleaness of B. Set

$$F = \begin{pmatrix} b_{11}-1 & b_{22}-1 & 3-b_{11}-b_{22} \\ b_{11}-1 & b_{22}-1 & 3-b_{11}-b_{22} \\ b_{11}-1 & b_{22}-1 & 3-b_{11}-b_{22} \end{pmatrix}.$$

It is directly verified that $F^2 = F$. Thus

$$B - F = \begin{pmatrix} 1 & b_{12} - b_{22} + 1 & b_{13} + b_{11} + b_{22} - 3 \\ b_{21} - b_{11} + 1 & 1 & b_{23} + b_{11} + b_{22} - 3 \\ b_{31} - b_{11} + 1 & b_{32} - b_{22} + 1 & b_{33} + b_{11} + b_{22} - 3 \end{pmatrix}.$$

We only need to show that $B - F$ is 2-good. Observing the above matrix, and then there exist invertible matrices

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{11} - b_{31} - 1 & 0 & 1 \end{pmatrix}, \ V = \begin{pmatrix} 1 & b_{22} - b_{12} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 - b_{23} - b_{11} - b_{22} \\ 0 & 0 & 1 \end{pmatrix}$ such that

$$VT(B - F)W = \begin{pmatrix} * & 0 & * \\ * & 1 & 0 \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 1 & * & * \end{pmatrix} + \begin{pmatrix} * & -1 & 0 \\ * & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Consider the two matrices U_1, U_2 occurring in the decomposition above of $VT(B - F)W$. It is straightforward to verify that the two matrices are invertible in $M_3(R)$. Thus we obtain immediately a 2-clean expression of B, i.e.,

$$B = T^{-1}V^{-1}U_1W^{-1} + T^{-1}V^{-1}U_2W^{-1} + F.$$

This completes the proof. □

Remark 4. (1). For the matrix ring $M_n(R)$, it is customary to write $GL_n(R)$ for $U(M_n(R))$. An elementary matrix is the result of an elementary row operation performed on the identity matrix. We denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by the elementary matrices, permutation matrices and -1. Observing the decompositions of the 2×2 and 3×3 matrices above, we see that, these matrices can be written as the sum of an idempotent matrix and two elements of $E_n(R)$.

(2). For any ring R, R can be embedded in the 2×2 matrix ring $M_2(R)$. That is, all rings can be embedded in a 2-clean ring by Lemma 3.

(3). We know that 2-clean rings contain clean rings and 2-good rings. However, the converse is not true. For example, the matrix ring $M_2(\mathbb{Z})$ is not clean since $\mathbb{Z}$ is not a exchange ring, and the matrix ring $M_2(\mathbb{Z}[x])$ is not 2-good (see [12, Proposition 8]).

(4). It is well known that for a clean ring R, idempotents can be lifted modulo $J(R)$. However, a 2-clean ring has not this property in general. Let

$R=\mathbb{Z}_{(2)}\cap\mathbb{Z}_{(3)}=\{m/n\in\mathbb{Q}\ :m,\ n\in\mathbb{Z},\ 2\nmid n \text{ and } 3\nmid n\}$ and set $S=M_2(R)$. Then $J(S)=J(M_2(R))=M_2(J(R))=M_2(6R)$. Let $F=\begin{pmatrix}3&0\\6&3\end{pmatrix}$. Then $F^2-F\in J(S)$, but there is no idempotent E of S such that $F-E\in J(S)$ since non-trivial idempotents of S are only of form $\begin{pmatrix}a&b\\c&1-a\end{pmatrix}$ where $bc=a-a^2$ for $a,b,c\in R$. Thus S is 2-clean by Lemma 3 but there exists an idempotent which can not be lifted modulo $J(S)$.

Lemma 5. *Let R be a ring, $m,\ n\geq 1$ and $k\geq 2$. If the matrix rings $M_n(R)$ and $M_m(R)$ are both k-clean, then so is the matrix ring $M_{n+m}(R)$.*

Proof. Let $A\in M_{n+m}(R)$ be a typical $(n+m)\times(n+m)$ matrix which we will write in the block decomposition form

$$A=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix},$$

where $A_{11}\in M_n(R),\ A_{22}\in M_m(R)$ and $A_{12},\ A_{22}$ are appropriately sized rectangular matrices. By hypothesis, there exist invertible $n\times n,\ m\times m$ matrices $U_1,U_2,\cdots,U_k$ and $V_1,V_2,\cdots,V_k$, and idempotent matrices E_1,E_2 such that $A_{11}=U_1+U_2+\cdots+U_k+E_1$ and $A_{22}=V_1+V_2+\cdots+V_k+E_2$. Thus the decomposition

$$\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}=\begin{pmatrix}U_1&A_{12}\\O&V_1\end{pmatrix}+\begin{pmatrix}U_2&O\\A_{21}&V_2\end{pmatrix}+\cdots+\begin{pmatrix}U_k&O\\O&V_k\end{pmatrix}+\begin{pmatrix}E_1&O\\O&E_2\end{pmatrix}$$

shows that A is $k-$clean. □

Corollary 6. *Let $k\geq 1$. If R is a k-clean ring, then so the matrix ring $M_n(R)$ for any positive integer n.*

Proof. For $k=1$, it follows from [5, Corollary 1]. Assume that $k\geq 2$, it is clear by induction and by Lemma 5. □

Theorem 7. *Let R be a ring and let the free R-module F be (isomorphic to) the direct sum of $\alpha\geq 2$ copies of R where α is a cardinal number. Then the ring of endomorphisms E of F is 2-clean.*

Proof. Assume first that $\alpha\geq 2$ is finite so $E\cong M_\alpha(R)$. Then E is 2-clean for $\alpha=2,\ 3$ by Lemma 3 and the values of $\alpha<\omega$ for which E is 2-clean are closed under addition by Lemma 5. So E is 2-clean for all finite α.

Assume now that α is infinite. Then $E\cong M_2(E)$ follows from $F\cong F\oplus F$, and so E is 2-clean by Lemma 3. □

4. ROW AND COLUMN-FINITE MATRIX RINGS

Let $B(R)$ be the ring of all $\omega \times \omega$ row and column-finite matrices over a ring R. Fix a free R-module $F = \bigoplus_{i=1}^{\infty} f_i R$ on countably many generators, and for each $k \in \mathbb{N}$ let $F_k = \bigoplus_{i=k}^{\infty} f_i R$. A moment's reflection, using the standard correspondence between R-endomorphisms of F_R and $\omega \times \omega$ column-finite matrices over R relative to the basis $\{f_i\}_{i=1}^{\infty}$, confirms that

$$B(R) \cong \{\phi \in End_R(F) : \text{for each } k \in \mathbb{N},\ \exists\ m \in \mathbb{N} \text{ with } \phi(F_m) \subseteq F_k\}.$$

Hence we identify $B(R)$ with this ring of transformations. Next we will consider the 2-cleanness of $B(R)$. The proof of the following result is a modification of that in [8, Theorem 3.5].

Theorem 8. *Let R is ring. Then the row and column-finite matrix ring $B(R)$ is 2-clean.*

Proof. Note that $B(R) \cong B(M_2(R))$, so we may assume that R is 2-clean by Lemma 3. Let $\phi \in B(R)$. Recall that φ is defined by

(*a*) α-endomorphism if $\varphi(f_i R) \subseteq \bigoplus_{k>i} f_i R$ for all $i < \omega$;
(*b*) β-endomorphism if $\varphi(f_i R) \subseteq \bigoplus_{k=1}^{i-1} f_i R$ for all $i < \omega$;
(*c*) d-endomorphism if $\varphi(f_i R) \subseteq f_i R$ for all $i < \omega$.

Then ϕ can obviously be expressed as

$$\phi = \eta + \rho + \delta,$$

where η is an α-endomorphism, ρ is a β-endomorphism and δ is a d-endomorphism. Since $\phi \in B(R)$, for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\phi(F_m) \subseteq F_k$. By the definitions of $\eta,\ \rho$ and δ, we check easily that $\eta(F_m) \subseteq F_k$, $\rho(F_m) \subseteq F_k$ and $\delta(F_m) \subseteq F_k$. For the α-endomorphism η, by [8, Proposition 3.2], there exists a strictly ascending sequence of integers $0 < r_0 < r_1 < r_2 < \cdots$ such that $\eta(f_i R) \subseteq \bigoplus_{k=i+1}^{s+2-1} f_k R$ for all $r_s \leq i < r_{s+1}$. Using this sequence we define endomorphisms $\eta_1,\ \eta_2$ of F as follows

$$\eta_1 f_i = \begin{cases} \eta f_i & \text{for } r_{2t} \leq i < r_{2t+1}; \\ 0 & \text{for } r_{2t+1} \leq i < r_{2t+2}, \end{cases}$$

and

$$\eta_2 f_i = \begin{cases} 0 & \text{for } r_{2t} \leq i < r_{2t+1}; \\ \eta f_i & \text{for } r_{2t+1} \leq i < r_{2t+2}. \end{cases}$$

Clearly, η_1 and η_2 are α-endomorphisms of F with $\eta = \eta_1 + \eta_2$, and for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\eta_1(F_m) \subseteq F_k$ and $\eta_2(F_m) \subseteq F_k$. By [8, Lemma 3.4], we have that $\eta_1,\ \eta_2$ are both locally nilpotent. Next we decompose the β-endomorphism ρ. For each $i<\omega$, we have

$$\rho f_i = \sum_{k<i} f_k r_{ik} = \sum_{\substack{k<i \\ k\in I_1}} f_k r_{ik} + \sum_{\substack{k<i \\ k\in I_2}} f_k r_{ik},$$

where $I_1 = \bigcup_{t<\omega}\{k \mid r_{2t} \le k < r_{2t+1}\}$ and $I_2 = \bigcup_{t<\omega}\{k \mid r_{2t+1} \le k < r_{2t+2}\}$. We define $\rho_1,\ \rho_2$ correspondingly, i.e.,

$$\rho_1 f_i = \sum_{\substack{k<i \\ k\in I_1}} f_k r_{ik} \quad \text{and} \quad \rho_2 f_i = \sum_{\substack{k<i \\ k\in I_2}} f_k r_{ik}.$$

Clearly, $\rho = \rho_1 + \rho_2$ and $\rho_1,\ \rho_2$ are both locally nilpotent. We check easily that for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\rho_1(F_m) \subseteq F_k$ and $\rho_2(F_m) \subseteq F_k$. Note that $\rho_1\eta_2 = 0 = \rho_2\eta_1$ by definitions of $\eta_1,\ \eta_2,\ \rho_1,\ \rho_2$, so $\eta_1+\rho_2$ and $\eta_2+\rho_1$ are also locally nilpotent. Now we consider the d-endomorphism δ. For each $i<\omega$, there exists an element r_i of R such that $\delta f_i = f_i r_i$. Since R is 2-clean, there are $e_i^2 = e_i \in R$ and units $u_{i1},\ u_{i2}$ of R such that

$$\delta f_i = f_i u_{i1} + f_i u_{i2} + f_i e_i.$$

defining $\delta_e f_i = f_i e_i$ and $\delta_j f_i = f_i u_{ij}\ \ (i<\omega,\ j=1,\ 2)$. So $\delta = \delta_1 + \delta_2 + \delta_e$ and $\delta_1,\ \delta_2,\ \delta_e$ are d-endomorphisms of F. Note that for each $k \in \mathbb{N}$, set $m = k$, we get $\delta_1(F_m) \subseteq F_k$, $\delta_2(F_m) \subseteq F_k$ and $\delta_e(F_m) \subseteq F_k$. Thus we consider the decomposition of ϕ

$$\begin{aligned}
\phi &= \eta + \rho + \delta \\
&= \eta_1 + \eta_2 + \rho_1 + \rho_2 + \delta_1 + \delta_2 + \delta_e \\
&= (\eta_1 + \rho_2 + \delta_1) + (\eta_2 + \rho_1 + \delta_2) + \delta_e \\
&= \delta_1(\delta_1^{-1}(\eta_1 + \rho_2) + 1) + \delta_2(\delta_2^{-1}(\eta_2 + \rho_1) + 1) + \delta_e.
\end{aligned}$$

Note that $\delta_1^{-1}(\eta_1 + \rho_2)$ is locally nilpotent since δ_1^{-1} is d-endomorphism and $\eta_1 + \rho_2$ is locally nilpotent, and so $\delta_1^{-1}(\eta_1 + \rho_2) + 1$ is an automorphism of F. Hence $\delta_1(\delta_1^{-1}(\eta_1 + \rho_2) + 1)$ is also an automorphism of F. Similarly, $\delta_2(\delta_2^{-1}(\eta_2 + \rho_1) + 1)$ is an automorphism of F. Clearly, by the definitions of δ_e, δ_e is idempotent endomorphism of F. It is checked easily that $\eta_1 + \rho_2 + \delta_1,\ \eta_2 + \rho_1 + \delta_2,\ \delta_e \in B(R)$ since $B(R)$ is a ring. Thus we complete the proof. □

Remark 9. From the proof of Theorem 8, we may consider row and column-finite matrix rings over a 2-good ring similarly. In fact, we obtain that if R

is 2-good then so is the row and column-finite matrix ring $B(R)$, and that for any ring R the row and column-finite matrix ring $B(R)$ is 3-good.

5. 2-CLEAN GROUP RINGS

Given a group G and a ring R, denote the group ring by RG. In this section, we consider the group ring RC_n where R is a local ring and C_n is a cyclic group of order n. Some results of Xiao and Tong [16] are extended.

Theorem 10. *Let R be a local ring with $\overline{R} = R/J(R)$ and let C_n be a cyclic group of order n. If $char\overline{R} \neq 2$, then RC_n is 2-good.*

Proof. If $char\overline{R} = 0$ or $(char\overline{R},\ n) = 1$, then $\overline{n}$ and $\overline{2}$ are invertible in $\overline{R}$. Note that $\overline{R}$ is a division ring, then $\overline{R}C_n$ is semisimple from $n \cdot \overline{1} = \overline{n} \in U(\overline{R})$, and so $\overline{R}C_n$ is clean. This implies that $\overline{R}C_n$ is 2-good by [1, Proposition 10]. We know that if G is locally finite then $J(R)G \subseteq J(RG)$ by [15]. Clearly, $J(R)C_n \subseteq J(RC_n)$, and then $\overline{R}C_n \cong RC_n/J(R)C_n \twoheadrightarrow RC_n/J(RC_n)$. So the factor ring $RC_n/J(RC_n)$ is 2-good since 2-good rings are closed under factor rings. By [12, Proposition 3], RC_n is also 2-good. If $n = mp^k$ where $char\overline{R} = p \neq 2,\ k \geq 1$, and $(m,\ p) = 1$. Then $C_n \cong C_{p^k} \times C_m$, and so $RC_n \cong (RC_{p^k})C_m$. By [9, Theorem], RC_{p^k} is also a local ring and $charRC_{p^k} = p$. The rest is proved similarly as above since $(p,\ m) = 1$. Thus we complete the proof. □

By Theorem 10, we obtain the following corollary immediately

Corollary 11. *Let R be a local ring with $\overline{R} = R/J(R)$ and let C_n be a cyclic group of order n. If $char\overline{R} \neq 2$, then RC_n is 2-clean.*

Corollary 12. ([16, Theorem 2.3]) *If C_3 is a cyclic group of order 3, then the group ring $\mathbb{Z}_{(p)}C_3$ is 2-clean for any prime number $p \neq 2$.*

Remark 13. The group ring RC_n which satisfies the conditions of Theorem 10 need not be clean. In [5], Han and Nicholson showed that the group ring $\mathbb{Z}_{(7)}C_3$ is not clean where $\mathbb{Z}_{(7)} = \{m/n \in \mathbb{Q} : 7 \nmid n\}$.

Let $C_m = \{1, g, g^2, \cdots, g^{m-1}\}$ with $g^m = 1$ where m is odd. Set $S = \{1, 2, \cdots, m-1\}$. Define $\sigma : S \longrightarrow S$ by $i \longmapsto 2i \pmod{m}$. It is checked easily that σ is a permutation of $\{1, 2, \cdots, m-1\}$. Let F be a field with $charF = 2$ and let $e = e_0 + e_1 g + \cdots + e_{m-1}g^{m-1} \in FC_m$ be an idempotent. Note that $2 = 0$ and $g^n = 1$, so $e^2 = e_0^2 + e_{\sigma(1)}g^{\sigma(1)} + \cdots + e_{\sigma(m-1)}g^{\sigma(m-1)}$. Suppose that σ

is a cyclic permutation. Then we have $e_0^2 = e_0$ and $e_1^2 = e_1 = e_2 = \cdots = e_{m-1}$, and so idempotents of FC_m are $0,\ 1,\ 1 + g + \cdots + g^{m-1},\ g + g^2 + \cdots + g^{m-1}$.

Theorem 14. *Let R be a local ring with $char\overline{R} = 2$ and let C_n be a cyclic group of order n. Write $n = m \cdot 2^k$ $(k \geq 0)$ where $(m,\ 2) = 1$. If $\overline{R}$ ia a field and σ is a cyclic permutation of $\{1, 2, \cdots, m-1\}$, then the group ring RC_n is semiperfect.*

Proof. Suppose $k \geq 1$. Then $C_n \cong C_{2^k} \times C_m$ from $(m,\ 2) = 1$, and so $RC_n \cong (RC_{2^k})C_m$. By [9, Theorem], RC_{2^k} is local. Since $\overline{R}$ is a field and $\overline{R}C_{2^k} \twoheadrightarrow \overline{RC_{2^k}}$ is a ring epimorphism, $\overline{RC_{2^k}}$ is a field and $char\overline{RC_{2^k}} = char\overline{R} = 2$. Hence we may assume $n = m$. Note that $\overline{R}C_m$ is semisimple by $(m,\ 2) = 1$ and $J(R)C_m \subseteq J(RC_m)$, so $J(R)C_m = J(RC_m)$. This shows that $\overline{RC_m} \cong \overline{R}C_m$ with $char\overline{R} = 2$. Since $\overline{R}$ is a field and σ is a cyclic permutation of $\{1, 2, \cdots, m-1\}$, $\overline{R}C_m$ has only four idempotents, and so all idempotents in $\overline{RC_m}$ are $\overline{0},\ \overline{1},\ \overline{1} + \overline{g} + \cdots + \overline{g}^{m-1},\ \overline{g} + \overline{g}^2 + \cdots + \overline{g}^{m-1}$. We find easily idempotents in RC_m, $f_1 = 0,\ f_2 = 1,\ f_3 = m^{-1}(1 + g + \cdots + g^{m-1}),\ f_4 = m^{-1}((m-1) - g - g^2 - \cdots - g^{m-1})$ such that $\overline{f}_1 = \overline{0},\ \overline{f}_2 = \overline{1},\ \overline{f}_3 = \overline{1} + \overline{g} + \cdots + \overline{g}^{m-1},\ \overline{f}_4 = \overline{g} + \overline{g}^2 + \cdots + \overline{g}^{m-1}$. This shows that RC_m is semiperfect. $\square$

The following result is immediate by Theorem 14 and by [1, Theorem 9].

Corollary 15. *Let R be a local ring with $char\overline{R} = 2$ and let C_n be a cyclic group of order n. Write $n = m \cdot 2^k$ $(k \geq 0)$ where $(m,\ 2) = 1$. If $\overline{R}$ is a field and σ is a cyclic permutation of $\{1, 2, \cdots, m-1\}$, then the group ring RC_n is clean.*

Corollary 16. *([16, Theorem 3.2]) If C_3 is a cyclic group of order 3, then the group ring $\mathbb{Z}_{(2)}C_3$ is clean.*

Remark 17. The condition which σ is cyclic in Theorem 14 can not be removed. In fact, it is determined only by m whether the permutation σ of $\{1, 2, \cdots, m-1\}$ is cyclic. We calculate that σ is cyclic in the case $m = 3, 5, 11, 13, \cdots$. However, set $m = 7$ or 9, σ is not cyclic. Here, $\mathbb{Z}_{(2)}C_7$ is not semiperfect. In fact, in $\mathbb{Z}_2[X]$, $X^7 - \overline{1} = (X + \overline{1})(X^3 + X - \overline{1})(X^3 + X^2 + \overline{1})$. But in $\mathbb{Z}_{(2)}[X]$, $X^7 - 1 = (X-1)(X^6 + X^5 + X^4 + X^3 + X^2 + X + 1)$ and $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ is irreducible. So $\mathbb{Z}_{(2)}C_7$ is not semiperfect by [15, Theorem 5.8]. Note that $\overline{\mathbb{Z}_{(2)}C_7}$ is semisimple, then idempotents cannot be lifted modulo $J(\mathbb{Z}_{(2)}C_7)$, and so $\mathbb{Z}_{(2)}C_7$ is not clean.

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