Social Capital and Social Quilts: Network Patterns of Favor Exchange*

Matthew O. Jackson, Tomas Rodriguez-Barraquer, and Xu Tan†

February 8, 2010

Abstract

We examine the informal exchange of favors among the members of a society. We characterize the network patterns of interaction that sustain repeated favor exchange. Networks sustainable in renegotiation-proof equilibria have certain inductive critical structures. The networks also satisfying a robustness condition, so that removing a link only results in the loss of favors in a local neighborhood, are *social quilts*: tree-like unions of completely connected subnetworks of a critical size. Allowing for general heterogeneity in the costs, benefits, and arrival probabilities of favors across individuals, robust networks are such that all links are *supported*: any pair of individuals exchanging favors must have a common friend. This provides a new measure of the local structure of social networks that contrasts with standard clustering measures.

Applying the theory, we examine favor exchange networks in 75 rural villages in southern India. We find that the support of favor links is more than twenty times higher than would arise at random, and significantly higher (beyond the 99.9 percent level) when accounting for the geographic distribution of links. We also find that the villages exhibit significantly higher fractions of supported links in their favor networks than in their purely social networks.

Keywords: Social Networks, social capital, favor exchange, social quilts, support, renegotiation-proofness

JEL Classification Codes: D85, C72, L14, Z13

*We gratefully acknowledge financial support from the NSF under grants SES–0647867 and SES-0752735. The data were gathered as part of a project with Abhijit Banerjee, Arun Chandrasekhar, and Esther Duflo, whom we thank for their work and resources, which helped make the data analysis here possible.

†All three authors are at the Department of Economics, Stanford University, Stanford, California 94305-6072 USA. Jackson is also an external faculty member at the Santa Fe Institute. Emails: jacksonm@stanford.edu, trodrig@stanford.edu, and xutan@stanford.edu.

Shorter abstract.

We examine the informal exchange of favors among the members of a society, characterizing the network patterns of exchange sustainable as renegotiation-proof equilibria. Robustness, so that deleted relationships only result in a local loss of favors, necessitates "social quilts": tree-like unions of completely connected subnetworks of a critical size. Under payoff heterogeneity robust networks are such that all links are "supported": any pair of individuals exchanging favors must have a common friend. Examining favor exchange networks in 75 villages in India, we find levels of support consistent with the theory, and significant contrasts between favor networks and purely social networks.

1 Introduction

Human beings rely on cooperation with others for their survival and growth. Although some forms of cooperation and behavior are enforced by social, religious, legal, and political institutions that have emerged over history, much of development, growth, and basic day-to-day functioning relies on a society's ability to "informally" encourage cooperative behavior. This sort of informal enforcement of cooperation ranges from basic forms of quid-pro-quo (or tit-for-tat in the game theory parlance) to more elaborate forms of social norms and culture, both of which must function without enforceable contracts or laws. Contracting costs are prohibitive for the many day-to-day sorts of favors that people exchange, ranging from offering advice to a colleague, a small loan to a friend, or emergency help to an acquaintance. Such informal favor exchange and cooperative behavior, in one sort or another, underly much of the literature on social capital.

In this paper we provide a game theoretic foundation for social enforcement of informal favor exchange, and also analyze data from 75 rural villages to examine some of the predictions of the theory. In particular, our focus is on situations where simple bilateral quid-pro-quo enforcement will not suffice. Some bilateral interactions may be infrequent enough that they fail to allow natural self-enforcement of cooperation or favor exchange. However, when such interactions are embedded in a network of interactions whose functioning can be tied to each other, then individuals can find it in their interest to cooperate given (credible) threats of ostracism or loss of multiple relationships for failure to behave well in any given relationship. We provide complete characterizations of the network patterns of favor exchange that are sustainable by a form of equilibrium satisfying two robustness criteria.

The setting that we examine is such that opportunities for one agent to do a favor for another agent arrive randomly over time. Providing a favor is costly, but the benefit outweighs the cost, so that it is efficient for agents to provide favors over time. However, it could be that the cost of providing a favor today is sufficiently high that it is not in an agent's selfish interest to provide the favor even if that means that he or she will not receive favors from that person again. Thus, networks of relationships are needed to provide sufficient incentives for favor exchange, and it may be that an agent risks losing several relationships by failing to provide a favor. We characterize the network structures that correspond to robust equilibria of favor exchanges. The robustness criteria that we examine are twofold: first, the threats of which relationships will be terminated in response to an agent's failure to deliver a favor must be credible. Credibility is captured by the game theoretic concept of "renegotiation-proofness".[1] After an agent has failed to deliver a favor, that relationship

[1]Although there are several definitions in the literature for infinitely repeated games, our games have a structure such that there is a natural definition which has an inductive structure reminiscent of that of Benoit and Krishna (1993).

is lost, but which additional relationships are lost in the continuation equilibrium, must be such that there is not another equilibrium continuation that all agents prefer to the given continuation. This sort of renegotiation-proofness rules out unreasonable equilibria such as the "grim-trigger" sort of equilibrium where once anyone fails to provide a single favor the whole society grinds to a halt and nobody provides any favors in the future. At that point, it would be in the society's interest to return to some equilibrium where at least some favors are provided. Renegotiation-proof equilibria can be complex, but have some nice intuitions underlying their structure as we explain in detail in the paper. The second sort of robustness that we examine is a robustness against social contagion. It is clear that to sustain favor exchange, an agent must expect to lose some relationships if the agent fails to deliver a favor. Those lost relationships can in turn cause other agents to lose some of their relationships since the incentives to provide favors change with the network structure. The robustness against social contagion requires that the ripple effects of some agent's failure to behave properly be confined to that agent's neighbors and not propagate throughout the network.

The robustness conditions tie down a unique type of network configuration of favor exchanges that are possible. We call those configurations "social quilts." A social quilt is a union of small cliques (completely connected subnetworks), where each clique is just large enough to sustain cooperation by all of its members and where the cliques are laced together in a tree-like pattern. One of our main theoretical results shows that configurations of favor exchange that are sustained in robust equilibria are precisely the social quilts.

With the theoretical underpinnings in hand, we then examine social networks in 75 villages in southern rural India. In particular, in these data we have information about who borrows rice and kerosene from whom, who borrows small sums of money from whom, who gets advice from whom, who seeks emergency medical aid from whom, and a variety of other sorts of relationships, as well as gps data. Using these data we can examine the networks of various forms of social interaction including specific sorts of favor exchange. In order to test the model, we extend it to allow heterogeneity in the cost and value of favors to various individuals. Under that extension, we prove that all robust equilibrium networks must exhibit a specific trait: each of its links must be supported. That is, if some agent i is linked to an agent j, then there must be some agent k linked to both of them. This is related to, but turns out to be quite different from, various clustering measures that are common in the social network literature. In line with the theoretical predictions, we find that the number of favor links that have this sort of social support is in the range of eighty to ninety percent in these villages. Moreover, the level of support is significantly higher than what would arise if links were formed at random (even with some geographic bias to formation). We provide a detailed statistical analysis of the levels of support and also find that it is significantly higher for favor relationships than other sorts of relationships.

Our research contributes to the understanding of informal favor exchange as well as social networks in several ways:

- We provide an analysis of repeated interactions where individual's decisions are influenced by the network pattern of behavior in the community, and this provides new insights into repeated games on networks.

- Our model includes dynamic choices of both favor provision and relationship choices and provides new insights into the co-evolution of networks and behavior, and in particular into the phenomenon of ostracism.

- Our analysis suggests a new source of inefficiency in informal risk and favor sharing.

- We provide an operational definition of social capital that is more specific and tighter than many existing definitions, and show how it relates to how relationships in a society are organized.

- We provide a new property of networks in terms of "link support" and show how this is distinguished from other clustering measures, both theoretically and empirically.

- We test the predictions of the model using data that include many sorts of interactions and cover 75 different villages, and find strong support for the theoretical predictions.

1.1 Related Literature

As mentioned above, there is a large literature on social capital that studies the ability of a society to foster trust and cooperation among its members.[2] Although that literature is extensive and contains important empirical studies and many intuitive ideas, it has struggled in providing firm theoretical foundations and the term "social capital" has at times been used very loosely and as a result has lost some of its bite.[3] Part of the contribution of our paper is to provide an explicit modeling of how societies can enforce cooperative favor exchange and how this is linked to the social network structure within a society. In this way, our paper provides a very concrete definition of social capital that is embedded in three components: a notion of equilibrium that embodies notions of ostracism and social expectations of individual behaviors, implications of this for resulting social network structure, and individual payoffs from the resulting behaviors.

[2]For example, see Homans (1958), Loury (1977), Bourdieu (1986), Coleman (1988, 1990), Woolcock (1998), Dasgupta (2000), Putnam (1993, 1995, 2000), Glaeser, Laibson, and Sacerdote (2002) Guiso, Sapienza, and Zingales (2004), Tabellini (2009), among others.

[3]See Sobel (2002) for an illuminating overview and critique of the literature.

Coleman (1988) discusses closure in social networks, emphasizing the ability of small groups to monitor and pressure each other to behave. Here we provide a new argument for, and a very specific variety of, closure. Here a specific form of minimal clique structures emerge because of a combination of renegotiation-proofness and a local robustness condition, rather than for informational, monitoring, or pressuring reasons. Minimal sized cliques offer both credible threats of dissolving in the face of bad behavior, and in terms of minimal contagion for a society.

The most closely related previous literature in terms of the theoretical analysis of a repeated game on a network is a series of papers that study prisoners' dilemmas in network settings, including Raub and Weesie(1990), Ali and Miller (2009), Lippert and Spagnolo(2009), and Mihm, Toth, Lang (2009).[4] In particular, Raub and Weesie(1990) and Ali and Miller (2009) show how completely connected networks shorten the travel time of information which can quicken punishment for deviations of behavior. Although cliques also play a prominent role in some of those papers, it is for very different reasons. In those settings, individuals do not have information about others' behaviors except through what they observe in terms of their own interactions. Thus, punishments only travel through the network, and the main hurdle to enforce individual cooperation is how long it takes for someone's bad behavior to come to reach their neighbors through chains of contagion. This builds on earlier work by Greif (1989), Kandori(1992), Ellison (1994), Okuno-Fujiwara and Postlewaite(1995) among others, who studied the ability of a society to sustain cooperation via threats of contagions of bad behavior. Our analysis is in a very different setting, where individuals have complete information. The quilts in our setting emerge because they *do not* lead to large contagions but instead compartmentalize the damage from an individual's defection. Moreover, the quilts consist of minimal sized cliques because only those sorts of implicit punishments are immune to renegotiation.

Haag and Lagunoff (2004) provide another reason favoring small cliques: heterogeneity. In their analysis large differences in preferences can preclude cooperative behavior, and so partitioning a group into more homogeneous subgroups can enable cooperative behavior which might not be feasible otherwise. Although our reasoning behind cliques comes from very different reasons, when we examine heterogeneous societies we do find assortativity in who exchanges favors with whom. Here, the reasoning is not because of direct reciprocity considerations, but because robustness requires balanced cliques and so agents need to have similar valuations of favors in order for their cliques to be critical. In this way, we provide

[4]Other studies of network structure and cooperative or various forms of risk-sharing behavior and the relationship to social network structures include Fafchamps and Lund (2003), De Weerdt and Dercon (2006), Bramoullé and Kranton (2007), Bloch, Genicot, and Ray (2007, 2008), and Karlan, Mobius, Rosenblat and Szeidl (2009).

new insights into homophily, where relationships of agents are biased towards others who have similar characteristics in terms of their values and arrival rates of favors.

Beyond the differences in the settings and the types of networks that emerge in equilibrium, our analysis of supported links provides a new look at the configurations of relationships that should emerge in a society. The importance of social pressures on fostering cooperation has deep roots in the sociology literature including seminal work by Simmel (1950), Coleman (1988) and more recently by Krackhardt (1996), among others. Measures of clustering and transitivity have grown in part out of this work. Although it is somewhat subtle, the measure of support that we define and find important in fostering favor exchange is quite different from standard clustering and transitivity measures. Those generally measure the extent to which two friends of a given agent are friends of each other. Support, in contrast, examines the number of pairs of friends that have some other friend in common. It is possible (and in fact seen in our data) to have most pairs of friends have some friend in common, without having many pairs of friends of a given agent be friends of each other. Just to make this point clear: consider a given individual who has somewhat separate groups of friends and acquaintances: those from different spheres of their lives, such as those from their work, those from their education, those associated with a hobby, and so forth. Each of those friends could have friends in common, and yet many of a given individual's friends might not be friends with each other.

Finally, our analysis of the data not only provides support for the support measure, but also uncovers significant differences between different sorts of networks. Differences between the network structure of various sorts of relationships is something that might be expected based on the different ways in which links might form across application (e.g., see Jackson (2008)) and here we add a new angle to this understanding, finding statistically distinct patterns of support in various sorts of favor and social networks. These suggest some interesting questions for future research.

2 A Model of Favor Exchange

2.1 Networks, Favors, and Payoffs

A finite set $N = \{1, \dots, n\}$ of agents are connected in a social network described by an undirected[5] graph. Given that the set of agents or nodes N is fixed throughout the analysis, we represent a network, generically denoted g, simply by the set of its links or edges. Let g^N be the set of all links (so the set of all subsets of N of size 2), and let $G = \{g \mid g \subset g^N\}$ be the set of all possible networks. For simplicity, we write ij to represent the link $\{i, j\}$, and

[5]This is not necessary for the analysis, and we comment later on possible extensions to directed networks.

so $ij \in g$ indicates that i and j are linked under the network g. We write $g - ij$ to denote the network obtained from g by deleting a link ij. For an integer k, $0 \leq k \leq n(n-1)/2$, let G_k be the set of all networks that have exactly k links, so that $G_k = \{ g \in G : |g| = k\}$.

The neighbors of agent i are denoted

$$N_i(g) = \{j \mid ij \in g\}.$$

We follow a convention that rules out self-links, and so all agents in $N_i(g)$ are distinct from i. The degree of agent i in the network g is the number of his or her neighbors denoted by $d_i(g) = |N_i(g)|$.

Time proceeds in discrete periods indexed by $t \in \{0, 1, \ldots\}$ and in any given period, there is a chance that an agent will need a favor from a friend or will be called upon to grant a favor to a friend. In particular, an agent i who is connected to an agent j (so that $ij \in g$) anticipates a probability $p > 0$ that j will need a favor from i in period t and a probability p that i will need a favor from j. It is assumed that at most one favor will be needed across all agents in any given period, and so we require that

$$n(n-1)p \leq 1.$$

we allow the sum to be less than one to admit the possibility that no favor is needed in a given period.

This is a proxy for a Poisson arrival process, where the chance that two favors are needed precisely at the same moment is 0. By letting the time between periods be small, the chance of more than one favor being called upon in the same period goes to 0. Thus, when applying the model it is important to keep in mind that periods are relatively small compared to the arrival rate of favors.

A restriction of this formulation is that p does not depend on the network structure. More generally, the chance that i needs a favor from j will depend on many things including how many other friends i has. We characterize the equilibrium networks for the general case in Section 5. We begin with the current case since it more clearly provides the basic intuitions, but the results have very intuitive analogs that are easy to describe once we have presented the simpler case.

Doing a favor costs an agent an amount $c > 0$ and the value of the favor to an agent is an amount $v > c$. "Favors" can embody many things including asking for advice, to borrow some good, to borrow money, or to perform some service. The important aspect is that the value of a favor to one agent exceeds the cost, so that it is ex ante Pareto efficient for agents to exchange favors over time. However, we examine settings where it is impossible (or too costly) for agents to write binding contracts to perform favors whenever called upon to do so. This applies in many developing countries, and also in developed countries where it is prohibitively costly and complex to write complete contracts covering the everyday sort of favors that one might need from friends. Thus, we examine self-enforcing favor exchange.

Agents discount over time according to a factor $0 < \delta < 1$. Thus, if there were just two agents who always performed favors for each other, then they would each expect a discounted stream of utility of

$$\frac{p\,(v-c)}{1-\delta}.$$

The more interesting case from a network perspective is the one that we examine, where

$$c > \frac{p\,(v-c)}{1-\delta}.$$

In this case, favor exchange between two agents in isolation is not sustainable. When called upon to perform a favor, the agent sees a cost that exceeds the future value of potential favor exchange (in isolation) and so favor exchange cannot be sustained between two people alone, but must be embedded in a larger context in order to be sustained. Sustaining favor exchange between two individuals requires a high enough frequency of arrival coupled with a high enough marginal benefit from a favor and sufficient patience. In a marriage, there are generally sufficiently many opportunities for each spouse to help the other out with some task or need that bilateral favor exchange can be sustained. However, in other contexts, where such needs are rarer - such as a need to borrow cash due to an emergency, or a need for medical advice, etc., one might need a multilateral setting to sustain favor exchange.

A society is described by the profile (N, p, v, c, δ).

2.2 The Game

The favor exchange game is described as follows.

- The game begins with some initial network in place, denoted g_0.
- Period t begins with a network g_{t-1} in place.
- Agents (simultaneously)[6] announce the links that they are willing to retain: $L_i \subset N_i(g_{t-1})$. The resulting network is $g_t' = \{ij \mid j \in L_i \text{ and } i \in L_j\}$.
- Let k_t be the number of links in g_t'. With probability $2pk_t$ need for a (single) favor arises and with probability $1 - 2pk_t$ there is no need for a favor in the period. If a favor is needed, then it could apply to any link in g_t' with equal likelihood and then go either direction. If a favor is needed, then let i_t denote the agent called upon to do the favor and j_t the agent who needs the favor, where $i_t j_t \in g_t'$.

[6]Given the equilibrium refinements that we use, whether or not the link choices are simultaneous is effectively irrelevant.

- Agent i_t chooses whether or not to perform the favor. If the favor is performed then i_t incurs the cost c and agent j_t enjoys the benefit v. Otherwise no cost or benefit is incurred.

- The ending network, denoted g_t, is $g_t' - i_t j_t$ if the need for a favor arose and it was not performed, and is g_t' otherwise.

People make two sorts of choices: they can choose with whom they associate and they can choose to do favors or not to do favors. Opportunities for favor exchange arise randomly, as in a Poisson game, and people must choose whether to act on favors as the need arises. Choices of which relationships to maintain, however, can be made essentially at any time. In the model this is captured by subdividing the period into link choices and favor choices, so that agents have a chance to adjust the network after any favor choice, as well as before any potential favor arises.

This structure embodies several things. First, favor relationships can either be sustained or not. Once a favor is denied, that relationship cannot be resuscitated. Thus, at any point in time an agent's decision is which relationships to maintain. This simplifies the analysis in that it eliminates complicated forms of punishments where various agents withhold favors from an agent over time, in order to punish an agent, but then eventually revert to providing favors. There is a way in which this is an intuitive and natural behavioral assumption, but it is effectively a bounded rationality assumption. This simplification allows us to gain a handle on sustainable network structures as the problem is already very complex (as will become clear shortly), and it appears that much of the intuition carries over to the more flexible case, but that is a subject for further research. As will be clear, this approach generates quite a rich, natural, and interesting set of conclusions. Second, we do not consider the formation of new links, but only the dissolution of links. This embodies the idea that the formation of new relationships is a longer-term process and that decisions to provide favors and/or ostracize an agent can be taken more quickly and are shorter term actions. It is important to note that we cover the case where society starts with the complete network, so we do not a priori restrict the links that might be formed.

For now, we consider a complete information version of the game, in which all agents observe all moves in the game at every node. We discuss more limited information variations in section 7.1.

An agent i's expected utility from being in a network g that he or she expects to exist forever[7] is

$$u_i(g) = \frac{d_i(g)p\,(v-c)}{1-\delta}$$

[7]This applies at any point within the period other than at the very instant at which the agent is called to receive or produce a favor.

If an agent i is called to do a favor to j and chooses to perform the favor and expects a network g to be played in perpetuity thereafter[8] then he or she expects a utility of

$$-c + \delta u_i(g).$$

Similarly if agent i is called to receive a favor from agent j and expects to receive the favor and then anticipates a network g to be played in perpetuity thereafter then his or her expected discounted stream of utility is given by

$$v + \delta u_i(g).$$

For ease of expression, we assume that the discounting parameter δ enters the agents' calculations between the announcement stage and the favor stage in any given period.[9]

2.3 Equilibrium

In this setting, any network of favor exchange g can be sustained in perpetuity as a pure strategy subgame perfect equilibrium as long as

$$c < d_i(g) \frac{\delta p \,(v - c)}{1 - \delta}$$

for every agent i. One way in which this is sustained is by a sort of "grim-trigger" strategy where all relationships are sustained and favors are provided as long as no agent refuses a favor, and once any favor is denied then all agents delete all their links and never expect to receive any favors again in the future. Thus, if each agent has enough relationships that he or she might lose, then favor exchange can be sustained. So, for instance, if

$$c < (n - 1) \frac{\delta p \,(v - c)}{1 - \delta}$$

then the complete network with the most efficient favor exchange could be sustained.

[8]after the following period's link choices,

[9]This is purely for expository ease as it slightly simplifies the expressions for the critical utility levels for different behaviors, but does not alter the basic structure of the arguments or conclusions. Effectively, it ensures that whenever an agent is either making a decision of which links to announce or whether to follow through on a favor, any potential future favors that might be influenced by the decision are discounted. If discounting happens after the favor period, then when making link choices players would not discount one round of future favors, but when making favor decisions they would. This simply makes sure that all future favors are discounted in the same way. It is also convenient to begin with an announcement phase, but again this is not essential to the conclusions.

2.3.1 Renegotiation-Proofness

While the above conclusion offers some optimism regarding a society's ability to efficiently sustain favor exchange, it rests upon drastic threats that are not always credible. If for some reason a favor was not performed and some link disappears, a society might wish to reconsider its complete dissolution. Indeed, the idea that if some person in a society acts selfishly and fails to provide a favor, the whole society collapses and no favors are ever exchanged again is drastic and unrealistic. This sort of observation is not unique to this setting, but has been an issue in repeated games for some time (e.g., see the discussion in Bernheim, Peleg and Whinston (1987)). The basic problem is that if agents have some chance to communicate with each other (and perhaps even if they cannot), then when beginning some phase of equilibrium which involves low payoffs, if there is some other *equilibrium* continuation, in which all agents are better off, then they have a strong incentive to change to the play that gives them all better payoffs. Even though this sort of "renegotiation" problem with many sorts of equilibria is well known, it is rare for researchers to do more than to acknowledge it and forge ahead. The reason for this is that properly accounting for renegotiation becomes quite complicated, especially in infinite settings where it is not even clear how to define equilibrium in the face of renegotiation (e.g., see Farrell and Maskin (1989), Bernheim and Ray (1989), and Abreu, Pearce and Stacchetti (1993)). Thus, there are few analyses of renegotiation-proofness outside of some of the original papers working out the definitions.

Our setting has a nice structure that makes it relatively easy to provide a natural definition of renegotiation-proofness and to characterize such equilibria. Before moving to the formal definitions, we present an example that illustrates the ideas.

Example **1** *The Logic of Renegotiation-Proofness*

Let there be 4 nodes and consider a case such that

$$2\frac{\delta p(v-c)}{1-\delta} > c > \frac{\delta p(v-c)}{1-\delta}.$$

Here, no link is sustainable in isolation, since the value of providing a favor c is greater than the value of the future expected stream of giving and receiving favors: $\frac{p\delta(v-c)}{1-\delta}$. However, if an agent risks losing two links by not performing a favor, then links could be sustainable depending on the configuration of the network, since $c < 2\frac{p\delta(v-c)}{1-\delta}$.

In this case, note that networks where each agent has exactly two links, for example, $g = \{12, 23, 34, 41\}$, can be sustained as a subgame perfect equilibrium. If any agent ever fails to perform a favor, then a link will be lost. For example, suppose that 1 fails to deliver a favor to 2, and so the link 12 is lost. At this point, agent 1 only has one relationship left: 14. It is now clear that agent 1 will never perform future favors for 4 and so the link

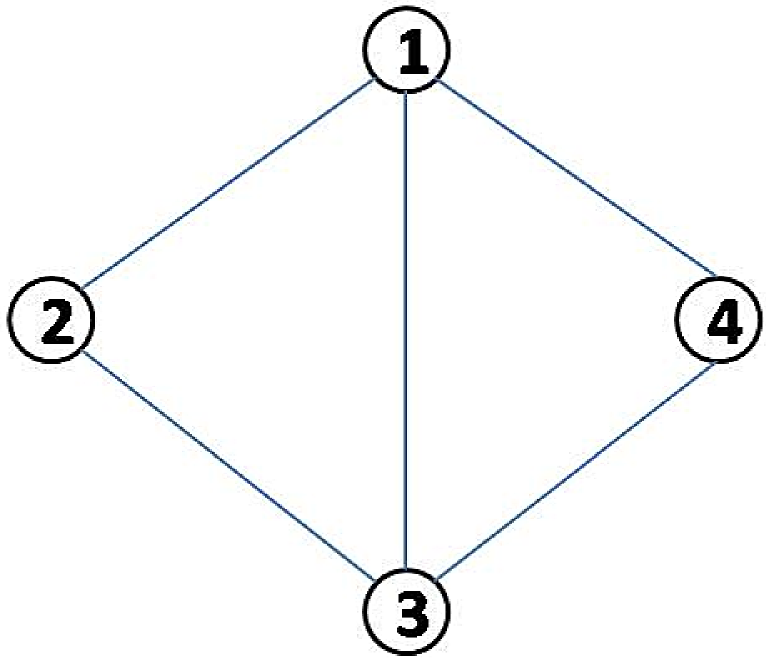


Figure 1: A five link network that is not sustained as a renegotiation-proof equilibrium

14 is effectively useless as well. The same is true of 23. Iterating on this logic, there is no subnetwork that could be sustained as a subgame perfect equilibrium. As such, an agent realizes that if he or she fails to provide a favor, then that will lead to a collapse of all favors and so the network of favor trading is sustained in equilibrium, as failing to provide one favor induces a loss of two relationships. So, starting with such a minimal network there is no difficulty with renegotiation, as following any deviation from the prescribed favor exchange the equilibrium continuation is unique. So, favor exchange sustaining this network is renegotiation-proof as an equilibrium (to be defined shortly).

The problematic subgame perfect equilibria come with $k = 5$ or more links. Consider the network $g' = \{12, 23, 34, 41, 13\}$ as pictured in Figure 1. Agents 1 and 3 each have three links and agents 2 and 4 have two links. There is a subgame perfect equilibrium sustaining this network: if any link is ever cut, then all agents cut every link in the future. However, there is no renegotiation-proof equilibrium sustaining this network. To see this, suppose that agent 1 is called upon to do a favor for agent 3. If agent 1 does not do the favor, then the resulting network is $g' = \{12, 23, 34, 41\}$. Note that g is sustainable as a subgame perfect equilibrium as argued above (and in fact is sustainable as part of a renegotiation-proof equilibrium). The logic is now that if g is reached, then it will be sustained rather than having agents delete all links, since it is not credible for agents to destroy these links as they are all strictly better off sustaining g than going to autarchy. Thus, when reaching g, in the absence of some exogenous commitment device, the agents' previous threat to delete all links lacks credibility. As a result of this, agent 1 can cut the link 13 and still expect the network g to endure, and so this is the unique best response for agent 1 and so g' is not sustainable as an equilibrium if we require that continuations not be Pareto dominated by another (renegotiation-proof) equilibrium continuation.$\diamond$

We define *renegotiation-proof networks* to be networks that are sustainable via pure strategy subgame perfect equilibria. It is easiest to define the networks directly, in a way that simultaneously implicitly defines renegotiation-proof equilibrium and explicitly tracks the networks that are sustainable via those equilibria. A full definition of renegotiation-proof equilibrium is more notation intensive, and for those interested in the details appears in the appendix, but it is not necessary for any of our results or analysis.

We define the set of pure strategy renegotiation-proof equilibria inductively.[10] The induction operates via the number of links in the starting network. In terms of notation, it will be useful to keep track of the set of all networks that have exactly k links and can be sustained in perpetuity as part of a pure strategy renegotiation-proof equilibrium if we start at that network.

We let RPN_k denote the set of *renegotiation-proof networks* with k links.

- Let $RPN_0 = \{\emptyset\}$
- Let RPN_k denote the subset of G_k such that $g \in RPN_k$ if and only if beginning with $g_0 = g$ implies there exists a pure strategy subgame perfect equilibrium[11] such that
  - on the equilibrium path g is always sustained (all favors are performed and all links are maintained), and
  - in any subgame[12] starting with some network $g' \in G_{k'}$ with $k' < k$ if g'' is played in perpetuity with some probability[13] in the continuation then $g'' \in RPN_{k''}$ for some k'' and there does not exist any $g''' \subset g'$ such that $g''' \in RPN_{k'''}$ and $u_i(g''') \geq u_i(g'')$ for all i with strict inequality for some i.[14]

The definition is inductive, since the logic of renegotiation-proofness requires that a network sustained in some continuation not be Pareto dominated by any other continuation

[10]Our analysis concentrates on pure strategy equilibria. As will become clear, considering mixed strategy equilibria would not add much to the insights regarding sustainable network structures.

[11]Pure strategy requires that agents use pure strategies at all nodes on and off the equilibrium path.

[12]This includes subgames starting at any node, not just beginning of period nodes.

[13]Even though players use pure strategies, nature randomly recognizes favors, and so there can be some randomness in a continuation path.

[14]Note that this condition implies that in any subgame starting with a network $g' \in RPN_{k'}$, g' is played in the continuation.

that itself is renegotiation-proof. The logic is inherently self-referential, and this is what generally provides difficulties in identifying an unambiguously "correct" definition in an infinite setting. Here, despite our infinite horizon, we can define renegotiation-proofness cleanly since relationships can be severed but not resuscitated and so there is a natural induction on the number of links.

We say that a network g is *renegotiation-proof* or a *renegotiation-proof network* if there exists some k such that $g \in RPN_k$.

As a further illustration of the definition, let us return to Example 1 and characterize all of the renegotiation-proof networks.

EXAMPLE **2** *Renegotiation-Proof Networks*

Let there be 4 nodes and consider a case as in Example 1 such that

$$2\frac{\delta p(v-c)}{1-\delta} > c > \frac{\delta p(v-c)}{1-\delta}.$$

Here, $RPN_1 = \emptyset$ since no isolated links are sustainable.

Similarly, $RPN_2 = \emptyset$ since any agent who only has one link will never perform a favor.

$RPN_3 = \{g = \{ij, jh, ih\} : \text{ for some distinct } h, i, j\}$. Thus, triads are sustainable, since if any agent severs a link, then that will lead to a two-link network which is not sustainable, and so becomes an empty network. Thus, not performing a favor leads to an empty network, and so it is a best response to perform a favor, anticipating favors by other agents.

$RPN_4 = \{g = \{ij, jh, h\ell, \ell i\} : \text{ for distinct } h, i, j, \ell\}$. This is an obvious extension of the logic from three-link networks.

Following the argument in Example 1 it is easy to check that there are no five-link renegotiation-proof equilibria. Thus $RPN_5 = \emptyset$.

Next, note that $RPN_6 = \emptyset$ as well. To see this, note that if some agent i deletes a link ij, then a continuation must result in a pure strategy renegotiation-proof equilibrium, which would be either a triad, four link network (with a cycle), or the empty network. The remaining four link network that has a cycle Pareto dominates any other continuation. Thus, if an agent i severs a link ij, then the agent is sure that he or she will still have two links in the continuation and so only loses one link.$\diamond$

3 Characterizing Renegotiation-Proof Networks

In this section we provide a complete characterization of renegotiation-proof networks. Before providing the complete characterization, however, we provide some intuitive sufficient conditions that give insight into the structure of renegotiation-proof networks.

Let m be the whole number defined by

$$m\frac{\delta p(v-c)}{1-\delta} > c > (m-1)\frac{\delta p(v-c)}{1-\delta}. \tag{1}$$

It is clear that there is at most one such m and that $m \geq 1$.

We work with the generic case, ignoring exact equality on either side above. The parameter m captures how many relationships, each with a future value of $\frac{\delta p(v-c)}{1-\delta}$, an agent has to risk losing in the future in order to have incentives to perform a favor today at the cost of c.

Throughout the remainder of the paper, the definitions will all be relative to m, and so we take it to be fixed and defined by (1) and omit explicit mention of it in some of the definitions.

We begin with a formal statement of the proposition on subgame perfect equilibria that motivates our analysis of renegotiation-proof equilibria.

PROPOSITION **1** *A network is sustainable in perpetuity on the equilibrium path of a subgame perfect equilibrium of the favor exchange game if and only if each agent has either 0 links or at least m links.*

The proof of Proposition 1 is obvious and thus omitted.

Now we examine renegotiation-proof networks.

3.1 Critical Networks and Renegotiation-Proofness

Before proceeding to the complete characterization of renegotiation-proof networks, we examine some natural classes of renegotiation-proof equilibria. These sufficient conditions provide an intuitive look at equilibrium structure and help motivate the complete characterization.

Let

$$G(m) = \{g \mid \forall i, d_i(g) \geq m \text{ or } d_i(g) = 0\}$$

denote the set of networks in which each node has either at least m links or 0 links. So $G(m)$ is the set of networks sustainable as subgame perfect equilibria. Note that any renegotiation-proof network must be in $G(m)$ and since any network sustained in any off-equilibrium path continuation must also be a renegotiation-proof network it must also be contained in $G(m)$.

One way to build a sustainable network is to ensure that agents have just enough links to sustain favors, but such that if any links are deleted then the whole network must collapse. That is if any link is deleted, then one of the agents involved in that link could no longer be counted upon to deliver favors on other links, and so then the other links of that agent would have to be deleted, leading to further contagions of link deletion. With this in mind we provide the following definition.

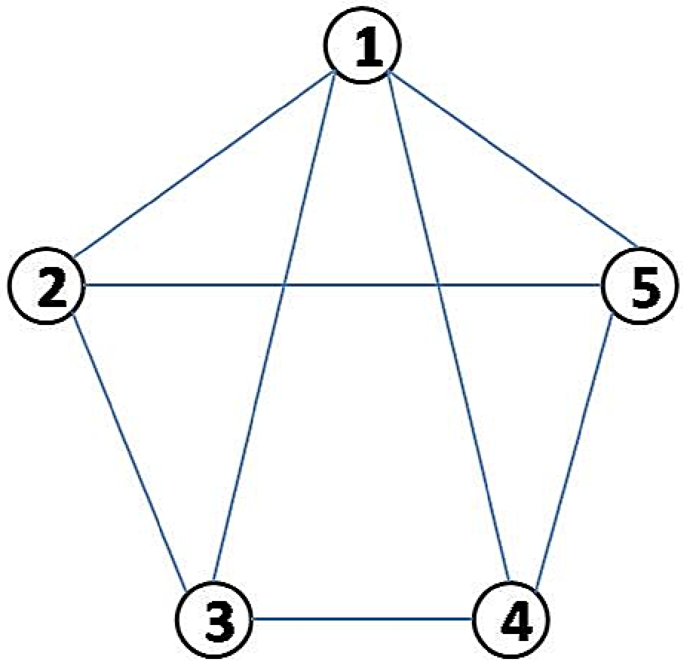


Figure 2: A minimally critical network for $m = 3$, but where node 1 has 4 links.

3.1.1 Minimally Critical Networks

A network g is *minimally critical* if $g \in G(m)$ and there doesn't exist a nonempty $g' \neq g$ such that $g' \subset g$ and $g' \in G(m)$.

An easy way to build a minimally critical network is to have each agent have exactly m links and to have a path from each agent to every other agent. However, we remark that minimal criticality does not require all nodes to have have exactly m links (nor does all nodes having exactly m links imply minimal criticality). Figure 2 pictures a minimally critical network for a case with $m = 3$ where node 1 has four links. There is no proper nonempty subnetwork in which all nodes that still have links also have at least 3 links each. Thus, if node 1 (or any other node) severs any link, the entire network will collapse and node 1 loses all four links.

As the following example shows, it is also possible to have more than one node have more than m links, as long as those two nodes are not adjacent.

Example **3** *A minimally critical network such that two nodes have more than m links.*

Consider the following network, which is pictured in Figure 3:

$\{12, 13, 14, 15, 23, 45, 26, 36, 46, 56\}$.

In this network nodes 1 and 6 have degree four. This is minimally critical and is a renegotiation-proof network when (1) holds for $m = 3$. If any node, including 1 or 6, drop a

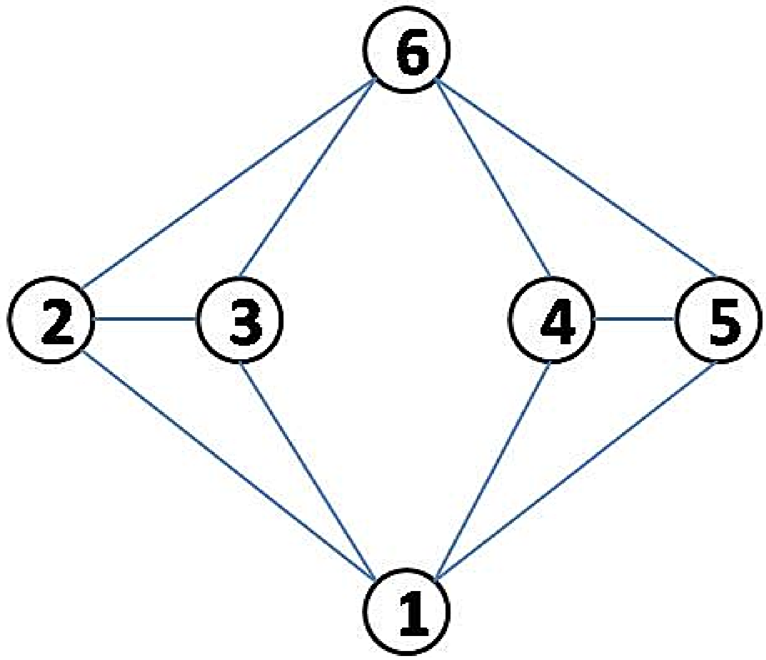


Figure 3: A minimally critical network where two agents have an excess number of links

link, then some node's degree drops below 3 and there is no subnetwork that is sustainable.◇

Proposition 2 *Any nonempty network $g \in G(m)$ contains a nonempty minimally critical network, and any minimally critical network is renegotiation-proof.*

Minimally critical networks provide an important and interesting class of networks that are renegotiation-proof. In the sense of Proposition 2, they are a foundational class of networks.

3.1.2 Unions of Minimally Critical Networks

Let us explore the extent to which one can build richer classes of networks that are renegotiation-proof by agglomerating minimally critical networks.

A first question is, "Are unions of minimally critical networks renegotiation proof networks?"

The first point is that one has to be careful as to how one builds a union of networks. To see this, suppose that $m = 2$ and we consider a union of two minimally critical networks which are two triads. If the "union" is two disjoint networks with no intersecting nodes, then it is clear that the resulting network will be renegotiation-proof. However, if the union involves duplication of a link, then the resulting network might not be. For example, consider

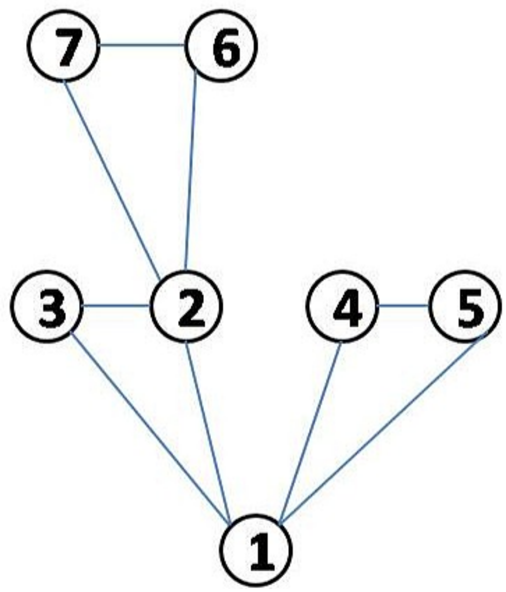


Figure 4: A tree-union of minimally critical networks that is not a minimally critical network, but is still renegotiation-proof.

the network in Figure 1. This can be seen as the union of two triads where the link 13 is shared by the triads. We already know that this is not renegotiation-proof. Thus, we need to be careful that if the networks intersect, then they do not share links.

With this in mind, we define "tree unions" of networks.

A union of several networks $g_1, ..., g_K$ is called a *tree union* if the networks can be ordered in a way $g_1, ..., g_K$ such that successive unions

$$U_1 = g_1, \ldots, U_k = U_{k-1} \cup g_k, \ldots, U_K = \bigcup_{k=1\ldots K} g_k$$

are such that each additional network has at most one node in common with the preceding union: $|N(U_{k-1}) \cap N(g_k)| \leq 1$.

One thing to note is that a tree union of minimally critical networks is not necessarily minimally critical, as illustrated in the following example.

Example 4 *A Tree Union of Minimally Critical Networks*

Let $m = 2$ and consider the network of three linked triads $g = \{12, 23, 13, 14, 15, 45, 26, 27, 67\}$ as pictured in Figure 4. This is not minimally critical since if 1 cuts the link 12, then all nodes in the sub-network still have at least 2 links. Nonetheless, (as we will verify below) this network is renegotiation-proof.$\diamond$

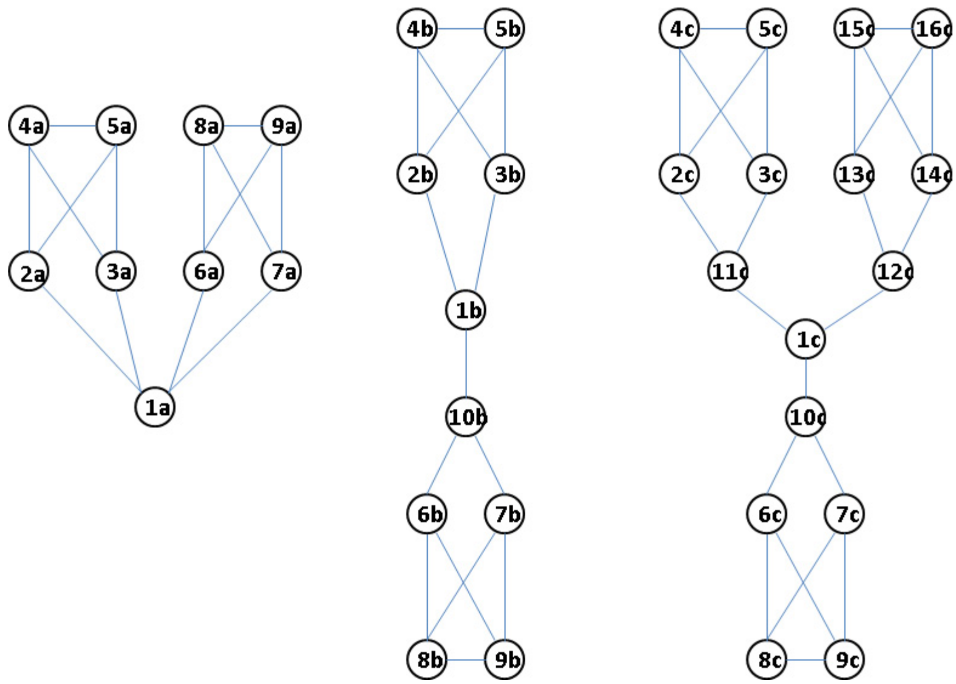


Figure 5: A tree union of minimally critical networks that is not renegotiation-proof.

Although Example 4 shows that it is possible to have a network that is a tree union of minimally critical networks not be minimally critical, and yet still renegotiation-proof, that is not true of all tree unions, as the following example shows.

Example **5** *A tree union of minimally critical networks that is not renegotiation-proof.*

Let $m = 3$ and consider the three minimally critical networks g_a, g_b and g_c shown in Figure 5. Let g'_a be a network with the same structure as g_a but with a different group of agents $\{1a', ..., 9a'\}$, and define g'_b, g''_b and g'''_b similarly. Consider two tree unions of these minimally critical networks:

- $U_1 = g_a \cup g'_a \cup g_c$. intersecting at the node $1a = 1a' = 1c$;
- $U_2 = g_b \cup g'_b \cup g''_b \cup g'''_b$, intersecting at the node $1b = 1b' = 1b'' = 1b'''$.

Structurally, $U_2 - \{6b, 7b, 8b, 9b, 10b\}$ is the same as U_1. The claim is U_1 and U_2 cannot both be both renegotiation proof networks. Otherwise, starting from U_2 if agent $1b$ refuses a favor to agent $10b$, the network played in the continuation has to be $U_2 - \{6b, 7b, 8b, 9b, 10b\}$ since it is renegotiation-proof (having the same structure as U_1) and noting that the nodes $\{6b, 7b, 8b, 9b, 10b\}$ must lose their links in any continuation. Thus, agent $1b$ only loses one link and would prefer not to do a favor for $10b$, contradicting the supposition that U_2 is a renegotiation-proof network.$\diamond$

One special character of networks in this example that potentially prevents the unions to be renegotiation-proof is there are some "critical" nodes in the networks such as $1a$, $1b$ and $1c$.

A node is called *critical* if deleting this node increases the number of components in the network.

In other words, a node is critical if it plays an essential role in connecting different agents who will be in different components without the critical node. For example, without $1a$, agents $2a$ and $6a$ won't be connected in g_a. Another way to present this character is by noting that any path between $2a$ and $6a$ has to contain $1a$ such that there is no simple cycle containing agents $2a$ and $6a$. The lemma below implies the equivalence of these two presentations of the special character of networks in Example 5.

LEMMA **1** *Consider a path-connected network involving links among at least three nodes. For each pair of path-connected nodes in the network there is a simple cycle containing them if and only if there is no critical node in the network.*

So in the following, we look at a nice subclass of minimally critical networks that don't have these critical nodes. And it turns out that tree unions of networks in this subclass are renegotiation-proof.

3.1.3 Simply Minimally Critical Networks

A useful subclass of minimally critical networks is the class in which any two nodes are connected via a simple cycle. Such networks can be agglomerated quite nicely.

A network g is called *simply minimally critical* if $d_i(g)$ equals m or 0 for every i, and for any pair of nodes i and j there is a simple cycle containing them.

Clearly a simply minimally critical network is minimally critical. An obvious example of a simply minimally critical network is a clique of $m+1$ nodes.[15] To get a deeper feeling for what simplicity implies, see the network pictured in Figure 6 which is minimally critical but not simply minimally critical.

EXAMPLE **6** *A Minimally Critical Network with a Bridge.*

Consider the network pictured in Figure 6:
$g = \{12, 13, 24, 27, 35, 36, 45, 46, 67, 75, 18, 89, 8-10, 9-11, 9-14, 10-12, 10-13, 11-12, 11-13, 13-14, 14-12\}$

In this network every node has exactly 3 links. There is a bridge: the link 18. This network is renegotiation-proof when (1) holds for $m=3$. If any node drops a link, then all

[15] A clique is a completely connected (sub-)network

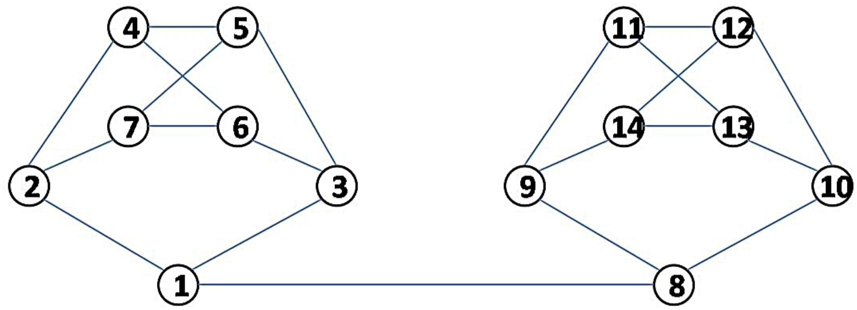


Figure 6: A minimally critical network with a bridge

links are dropped since there is no proper subnetwork where each node in the subnetwork has at least 3 links. However, this network is not simply minimally critical$\diamond$

Thus, the idea of simply minimally critical networks is that each each agent has exactly m links and is cyclicly tied to every other agent. A nice feature of simply minimally critical networks is that they make nice "building blocks" in that they can be agglomerated via tree unions to create renegotiation-proof networks.

Proposition **3** *A tree union of simply minimally critical networks is renegotiation-proof.*

Before moving on, we note one useful observation for identifying simply minimally critical networks.

A *Hamiltonian network*[16] is a network with a simple cycle visiting all nodes. So a Hamiltonian network is sufficient, but not necessary, for there to be a simple cycle containing any given pair of nodes in a network. Thus, any minimally critical network that is a Hamiltonian where each node has m links is simply minimally critical, but not vice versa.[17]

[16] See Jackson (2008) for more background.

[17] Consider the network $g = \{13, 34, 35, 45, 46, 56, 62, 17, 78, 79, 89, 8-10, 9-10, 10-2, 1-11, 11-12, 11-13, 12-13, 12-14, 13-14, 14-2\}$ and $m = 3$. g is minimally critical since every node has exactly m links and any pair of nodes has a cycle containing them. However, g is not a Hamiltonian network since there are three "highways" connecting 1 and 2 such that there is no way a simple cycle can contain all nodes.

3.1.4 Critical Networks

Minimally critical networks are a useful and intuitive class of renegotiation-proof networks, but we can extend the insight behind them to identify a much richer class of renegotiation-proof networks. The basic idea is that to offer proper incentives for sustaining favors, if an agent deletes a link in a network or fails to provide a favor, it is not necessary that the whole network collapse, merely that the agent expect to lose at least m links in the sequel. That is captured by the following definition.

We say that a network g is *critical*, if

- $g \in G(m)$
- for any i and $ij \in g$, there is no subnetwork $g' \subset g - ij$ such that $d_i(g') > d_i(g) - m$ and $g' \in G(m)$.

The following proposition follows easily.

PROPOSITION **4** *A critical network is renegotiation-proof.*

Proposition 4 follows from Theorem 1, but we mention the idea behind how this can be proven directly. In order to prove this, we need to show that there exists a pure strategy renegotiation-proof equilibrium such that for any i and $ij \in g$, if i is called to grant a favor to j and refuses the favor, i must lose at least m links in any continuation network g'. Renegotiation-proofness requires that the continuation g' be a subnetwork $g' \subset g - ij$ and be sustainable as a pure strategy renegotiation-proof equilibrium and so it must be in $G(m)$. By the definition of criticality, it then follows that any such $g' \in G(m)$ be such that $d_i(g') \leq d_i(g) - m$ (and there always exists at least one such g' since the empty network is in $G(m)$), and so provides the necessary incentives.

To see that criticality is not necessary for renegotiation-proofness we revisit Example 4: let $m = 2$ and $g = \{12, 23, 13, 14, 15, 45, 26, 27, 67\}$ is a tree union of three triads. This is not critical since if 1 cuts the link 12, then all nodes in the sub-network still have at least 2 links. We shall see that it is renegotiation-proof below.

3.2 A Complete Characterization of Renegotiation-Proof Networks: Transitively Critical Networks

We now turn to the complete characterization of renegotiation-proof networks.

Let $D(g)$ denote the profile of degrees associated with a network g:

$$D(g) \equiv (d_1(g), \ldots, d_n(g)).$$

Write $D(g) > D(g')$ if $D(g) \geq D(g')$ and $D(g) \neq D(g')$.

We define *transitively critical* networks as follows.

Given a whole number m satisfying (1), let $TC_k(m) \subset G_k$ denote the set of transitively critical networks with k links.

- Let $TC_0(m) = \{\emptyset\}$.
- Inductively on k, $TC_k(m) \subset G_k$ is such that $g \in TC_k(m)$ if and only if for any i and $ij \in g$, there exists $g' \subseteq g - ij$ such that $g' \in TC_{k'}(m)$, $d_i(g') \leq d_i(g) - m$, and there is no $g'' \in TC_{k''}(m)$ such that $g'' \subset g - ij$ and $D(g'') > D(g')$.

Even though this is also an inductive definition (not surprisingly, given that renegotiation-proof equilibria are so defined), it does not involve any incentive descriptions and is effectively an algorithm that can identify equilibria directly from m. In fact, we use this below to develop a computer program that calculates renegotiation-proof networks.

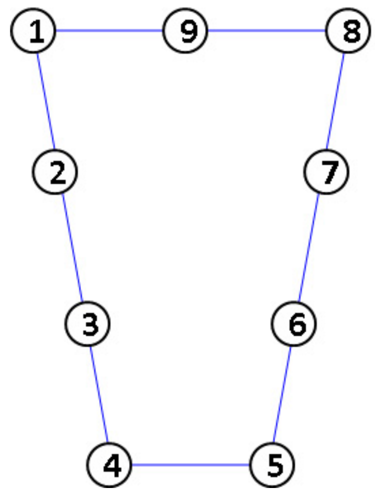
Theorem 1 *A network is renegotiation-proof if and only if it is transitively critical.*

Although one might expect the proof to be straightforward given that both definitions are inductive, there are some subtleties and challenges in proving Theorem 1. The main one is that there are many strategy profiles that may in principle sustain a collection of networks in a subgame perfect manner, in a way such that the collection satisfies the self-consistency property demanded by renegotiation-proofness. The issue then is that to show that some network is not renegotiation-proof we must in principle be sure that none among the large number of the different strategy profiles that could sustain it, actually work. Moreover, showing that some network is renegotiation-proof involves first showing that some other networks are not in the set. The way in which we tackle this difficulty is by arguing that we can avoid the vast set potential equilibria that could be used to sustain networks and can focus on a nicely behaved set of strategy profiles. The details appear in Appendix A.

4 Robustness

We now turn to our further notion of robustness. The idea is that a network is robust if it relies only on local "damage" due to a failure to provide a favor, rather than more global sorts of damage. That is, failure to provide a favor will require some lost links and there is a question of how far that loss of links propagates. We begin with a simple observation.

Observation 1 *If (1) holds for $m \geq 2$, g is a renegotiation-proof network, and $ij \in g$, then $g - ij$ is not a renegotiation proof network.*



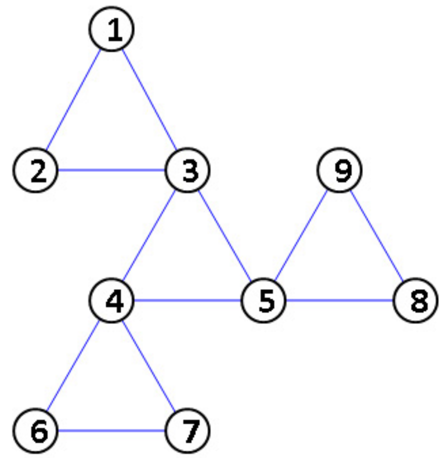


Figure 7: Two renegotiation-proof networks when $m = 2$ and $n = 9$: a non robust one and a robust one.

The observation follows since otherwise the continuation from i or j failing to do each other a favor would only result in the loss of one link, and so they would not do each other favors and g would not be sustainable.

The important implication of the observation is that beginning from a network that is renegotiation-proof, if a link is deleted then the network will necessarily further degrade in terms of what is sustainable. There may be different ways in which things could degrade. Here is where the idea of robustness comes into play. Robustness against social contagion seeks to minimize the extent to which the loss of links propagates beyond the original deviator(s) in the network.

Example **7** *Robustness*

Suppose that (1) holds for $m = 2$, and there are $n = 9$ nodes. Figure 7 lists two renegotiation-proof networks: one network is a single cycle containing all nodes, $\{12, 23, 34, 45, \ldots, 91\}$; the other network is a 2-"quilt", $\{12, 23, 13, 34, 45, 35, 46, 47, 74, 58, 59, 89\}$.

Note that if any link is deleted from the first network, then it completely collapses. If any link is deleted from the second network, only two other links are deleted and they are limited to a local neighborhood of the original link that is deleted.$\diamond$

The second network in Example 7 is more "robust" than the first one in the sense that the damage by the deletion of a link is more "local" in a sense that we now discuss.

4.1 Robustness Against Social Contagion

We say that a network g is *robust against social contagion* if it is renegotiation-proof and sustained as part of a pure strategy subgame perfect equilibrium with $g_0 = g$ such that in any subgame continuation from any renegotiation proof $g' \subset g$, and for any i and $ij \in g'$, if i does not perform the favor for j when called upon, then the continuation leads to g'' such that if $h\ell \notin g''$ then $h \in N_i(g') \cup \{i\}$ and $\ell \in N_i(g') \cup \{i\}$.

Robustness requires that a network be sustainable as part of an equilibrium such that in any continuation starting from some renegotiation proof (sub-)network, if some link is deleted then the only other links that are deleted in response must only involve the agent deleting the link and his or her neighbors. In a well-defined sense this localizes the damage to society.

We now characterize the networks that are robust against social contagion, or robust for short.

4.1.1 Social Quilts

The networks that are robust against social contagion have a particular structure to them that is described as follows.

An *m-clique* is a complete network with $m + 1$ nodes so that every node has exactly m links. m-cliques are an important class of simply minimally critical networks.

Note that a clique g with $m + 2$ nodes (each having $m + 1$ links) is not renegotiation-proof. To see this, suppose the contrary and have some i delete a link ij. In order for this not to be a valid deviation, it must be that i loses all links in the continuation, so that the continuation is such that there are at most $m + 1$ nodes and each has just m links. This is Pareto dominated by a network g' with all $m + 2$ nodes such that all nodes have m links except for possibly one node. There is such a network that is a subset of $g - ij$ (which is such that all but i and j have $m + 1$ links and i and j each have m links). It also follows that g' is critical and thus renegotiation-proof. Thus, we have a contradiction.

Let us say that a network g is an *m-quilt* if g can be written as a tree-union of m-cliques. Figure 4 shows a 2-quilt. The following example shows a non-tree union of cliques that is not a tree-union and is not robust against social contagion.

EXAMPLE **8** *A union of m-cliques that is not an m-quilt and is not robust.*

Let $m = 2$ and consider the network $g = \{12, 23, 13, 14, 15, 45, 26, 27, 67, 46, 68, 84\}$ as in Figure 8. It is a union of four linked 2-cliques (triads) and any two of these cliques intersect in at most one node. However, it is not a 2-quilt since there is a simple cycle $C = \{12, 26, 64, 41\}$ involving 4 nodes which is more than $m + 1$. The presence of this cycle makes it not robust against social contagion: $g' = \{12, 26, 64, 41\} \subset g$ and $g' \in RPN$, however g' is not robust.$\diamond$

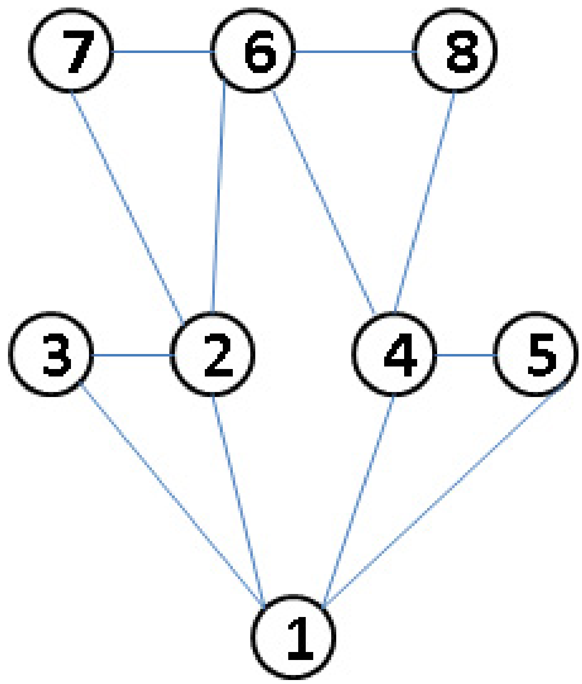


Figure 8: A union of m-cliques that is not an m-quilt.

Here are some useful properties of m-quilts, where $m > 1$:

- There are no bridges.
- The removal of a link does not change the distance between any two nodes except the two losing the link, and that distance increases just by 1.
- The removal of any link increases the diameter by at most 1, so there are no "long-distance" links.

THEOREM **2** *A network is robust against social contagion if and only if it is a social quilt.*

This is a special case of the robustness with asymmetric values, which is discussed in the next section, and so we defer the description of the proof until then.

4.1.2 Weak Robustness

It turns out that one can weaken the definition of robustness and still end up with exactly the class of social quilts. A weaker notion of renegotiation-proofness suffices in the definition.

Let us say that a network g is *weakly renegotiation-proof* if it is sustainable by a pure strategy subgame perfect equilibrium and at any subgame that starts with $g' \subset g$ that is minimally critical, g' is sustained in the equilibrium continuation.

Let $WPRN_k$ denote the set of all networks that have exactly k links and can be sustained in perpetuity as part of a pure strategy weakly renegotiation-proof equilibrium.

Lemma **2** *All renegotiation-proof networks are weakly renegotiation-proof; that is, $RPN_k \subset WRPN_k$ for all k.*

Weakly renegotiation-proof networks are a richer set than renegotiation-proof networks. This is not completely obvious, since the latter definition is inductive; however, it follows since weakly renegotiation-proof networks don't have inductive restrictions on the degrading path. Both definitions require that in any subgame starting with a network g' that cannot degrade further without collapsing (thus being minimally critical), g' be played in perpetuity. Otherwise, weak renegotiation-proofness puts no additional restrictions on the networks in continuation whereas renegotiation-proofness does. Weak renegotiation-proofness allows richer punishments than renegotiation-proofness, while it still rules out things like grim trigger. While it may be on less solid ground as a solution concept, it is useful in proving some results, which then also hold a fortiori for the stronger concept of renegotiation-proofness.

We can also use weak renegotiation-proofness as a basis for a robustness definition. We say that a network g is *weakly robust against social contagion* if it is weakly renegotiation-proof and sustained by a pure strategy subgame perfect equilibrium with $g_0 = g$ such that in any subgame continuation from some weakly renegotiation proof $g' \subset g$, and for any i and $ij \in g'$, if i does not perform the favor for j when called upon, then the continuation leads to g'' such that if $h\ell \notin g''$ then $h \in N_i(g') \cup \{i\}$ and $\ell \in N_i(g') \cup \{i\}$.

Weak robustness turns out to be equivalent to robustness.

Proposition **5** *A network is weakly robust against social contagion if and only if it is robust against social contagion.*

Proposition 5 comes from proving that weak robustness against social contagion implies that a network must be a social quilt, which is then in turn robust against social contagion.

4.2 The Relative Number of Robust Networks Compared to Subgame Perfect Equilibria

We now present some results that contrast the set of subgame perfect networks with the set of robust networks. Whereas almost all networks are subgame perfect equilibria, a fraction going to 0 of networks are robust. Thus robustness is a very discriminating refinement of the set of equilibrium networks providing pointed predictions.

Proposition **6** *Fix m and let n grow.*

- *The fraction of networks that are sustainable as subgame perfect equilibria goes to* 1.
- *The fraction of social quilts (and thus robust networks) goes to* 0.

5 Asymmetric Payoffs

Before we examine data concerning favor exchange settings, we extend the model to allow for asymmetries in payoffs. Given the heterogeneity in characteristics of agents in the societies we examine, it is clear that they may face different costs and benefits from favor exchange, and so this extension is needed to provide predictions to take to the data.

In particular, suppose that the probabilities of favors, and their values and costs are specific to relationships. Moreover, let the probability that i needs a favor from j depend on the degree of agent i, $d_i(g)$. Suppose that doing a favor for an agent j costs an agent i an amount $c_{ji} > 0$ and the value of the favor to an agent i from an agent j is an amount v_{ij}. Let $p_{ij}(d_i(g))$ denote the probability that i needs a favor from j. Moreover, this also allows us to discuss directed networks, as $P_{ij} = 0$ and $P_{ji} > 0$ suggest that only j needs favors from i and i never needs favors from j.

Agents discount over time according to a factor $0 < \delta_i < 1$. Agents' expected utilities are similarly as that in the symmetric case. An agent i's expected future utility from being in a network g where all favors are provided in perpetuity is

$$u_i(g) = \delta_i \frac{\sum_{j \in N_i(g)} p_{ij}(d_i(g))v_{ij} - p_{ji}(d_j(g))c_{ji}}{1-\delta_i}.$$

Thus, if agent i currently provides a favor to agent j with $ij \in g$, i's current expected discounted utility stream is $-c_{ji} + u_i(g)$, whereas if agent i is called to receive a favor from agent j it is $v_{ij} + u_i(g)$.

Let us consider settings such that

$$c_{ji} > \delta_i \frac{p_{ij}(d_i)v_{ij} - p_{ji}(d_j)c_{ji}}{1-\delta_i}.$$

for each ij and d_i and d_j. Thus, no relationship is sustainable on its own.

Our definitions of renegotiation-proof networks and robustness are exactly as previously stated.

A link $ij \in g$ is *supported* if there exists agent k different from i and j such that $ik \in g$ and $jk \in g$.

So a link is supported if it is part of a triad. Support is a necessary condition for robustness.

Theorem 3 *A network g is robust against social contagion only if all links in g are supported.*

Figure 9: The contrast between support and a standard clustering or transitivity measure

Support is an important prediction, since it differs from standard clustering measures, as seen in Figure 9. For example, it is possible that a standard measure of clustering[18] of a network is close to 0 while support is close to or even equal to 1.[19] In fact, as we shall see below, the support measure in the observed networks is quite high while standard clustering measures are much lower.

5.1 A Special Heterogeneous Case

An interesting case that generalizes the homogeneous case and yet is not as fully general as the case examined above is one where agents may have specific values and costs to favors v_i and c_i that are not dependent upon whom they are linked to, and also where the favor probabilities are not agent dependent. In that case, each agent is characterized by his or her own m_i such that

$$m_i \frac{\delta_i p(v_i - c_i)}{1-\delta_i} > c > (m_i - 1)\frac{\delta_i p(v_i - c_i)}{1-\delta_i}. \qquad (2)$$

[18]See Jackson (2008) for various definitions of clustering and transitivity.

[19]For example, consider an agent i who has many friends: $N_i(g) = (j_1, k_1, j_2, k_2, \ldots, j_M, k_M)$ such that j_m and k_m are linked to each other for each m but such that there are no other relationships between the friends. The support measure here is 1 since every link is part of a triad. Yet, the clustering for i is very small: $\frac{M}{M(M-1)/2}$, which simplifies to $\frac{2}{M}$ and goes to 0 as M grows.

For this case, our previous results have direct analogs.

We define *transitively critical* networks as before, simply changing the reference number of links to be agent specific.

Given $m = (m_1, \dots, m_n)$, let $TC_k(m) \subset G_k$ denote the set of transitively critical networks with k links.

- Let $TC_0(m) = \{\emptyset\}$.
- Inductively on k, $TC_k(m) \subset G_k$ is such that $g \in TC_k(m)$ if and only if for any i and $ij \in g$, there exists $g' \subseteq g - ij$ such that $g' \in TC_{k'}(m)$, $d_i(g') \leq d_i(g) - m_i$, and there is no $g'' \in TC_{k''}(m)$ such that $g'' \subset g - ij$ and $D(g'') > D(g')$.

Next, in order to define the analog of social quilts we need to define an analog of a minimal clique. In the fully symmetric case a minimally critical clique was simply one where each i had m_i links. Now, however, different agents may have different critical numbers of favor relationships that they must fear losing in order to give them incentives to exchange favors. There cannot be too much asymmetry in these critical numbers across the members of a clique or else some subset of the clique could sever some relationships and still have it be sustainable. For example if $m_i = 2$ for two agents and $m_i = 3$ for another two agents, making a clique from these four agents will not be renegotiation-proof. The first two agents could sever the link between them and end up with a (transitively) critical network.

A *minimally critical clique* with m' nodes is a clique that has m' nodes and such that $m_i \leq m' - 1$ for each i in the clique and $m_i < m' - 1$ for at most one i.

Next, in order to define social quilts we also need to be careful about how tree-unions work. A tree union of minimally critical cliques is not always robust, as the following example illustrates.

Example **9** *A tree union of minimally critical cliques that is not robust.*

Consider two minimally critical cliques with agents $\{1, 2, 3, 4\}$ and $\{1, 5, 6, 7\}$, respectively, where $m_i = 3$ for all i except $m_4 = m_7 = 2$. The tree-union of these two cliques is denoted g as in Figure 10. Then deleting links 14 and 17 leads to a subnetwork of g that is critical and renegotiation-proof. Indeed, all agents end up with exactly their critical number of links except agent 1 who has one extra link. It then follows that g is not robust, since this subnetwork would violate a local contagion condition. $\diamond$

Thus, we have to be careful in defining tree-unions when some of the cliques have asymmetries in the agents' respective numbers of critical links. When we unite two critical networks at some agent like agent 1, we add extra links for that agent and some of them might become non-critical. This leads to the following definition.

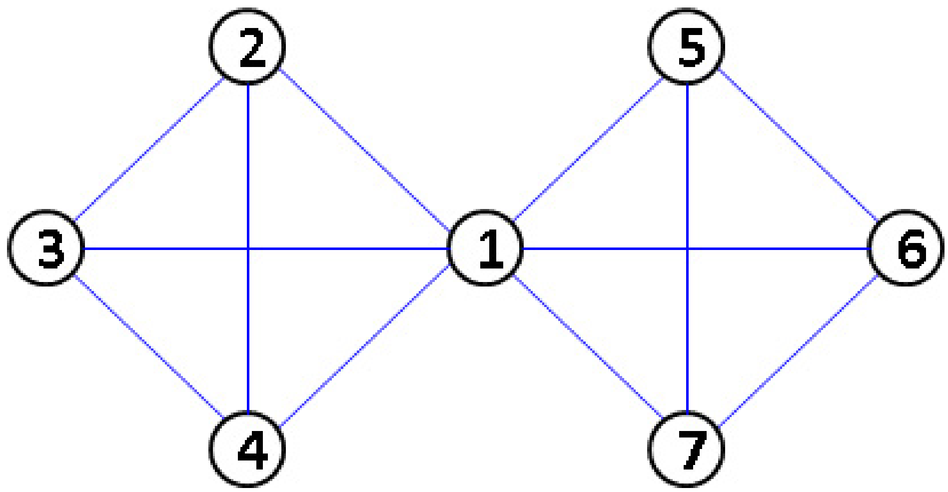


Figure 10: A tree union of minimally critical cliques that is not robust

Given a profile $m = (m_1, \ldots, m_n)$, a *strong tree union* of networks $g_1, ..., g_K$ is a tree union of those networks such that in each step of the union $U_k = U_{k-1} \cup g_{(k)}$ the node in common, if there is one, is the node with the smallest m_i in $g_{(k)}$.

Of course, a strong tree-union is the same as a tree-union in the case where all agents have the same critical number.

Now we define a *social quilt* to be a strong tree union of minimally critical cliques.

Social quilts thus defined are sufficient and necessary for robustness.

THEOREM 4 *A network is renegotiation-proof if and only if it is transitively critical, and a network is robust against social contagion if and only if it is a social quilt.*

Given our previous discussion of critical networks, it is a simple extension to see that transitive criticality is sufficient and necessary for renegotiation-proofness, and social quilts are renegotiation-proof. The minimally critical cliques limit contagion to be local in nature. The subtle and difficult part of the proof of Theorem 4 is in showing that only social quilts are robust. For example, why is a complete network not robust? This requires an involved argument, which draws upon both the renegotiation-proofness and the local aspect of punishments. Roughly, the intuition is as follows. First, any robust network must contain some cliques, as an agent who cheats must lose some number of links, which must all be local. In terms of continuation equilibria, any smallest sustainable subnetwork of a given network must be a clique. This follows since any deviation must lead to the loss of all its links since it is the smallest, and by locality the agents must all be neighbors. Moreover, it must be of minimal size by renegotiation-proofness as otherwise the society could renegotiate to keep a minimal sized clique which would contradict this being the smallest sustainable subnetwork.

The proof then works by using some graph theoretic reasoning to show that any network that is not a social quilt has some subnetwork that is a minimal critical network, and hence a smallest sustainable subnetwork, but is not a clique. Thus, if a network is not a social quilt, there is some way in which it could be broken down so that the eventual contagion in a last stage of destruction would necessarily be non-local.

6 An Analysis of Favor Networks in Rural India

We now analyze a large data set of a variety of networks that include various forms of favor exchange as well as other sorts of social interaction and relationships. In particular, we examine the support measures in the social networks these villages.

6.1 Description of the Data

The data are from 75 rural villages in Karnataka, an area of southern India within a few hours from Bangalore. The average population per village is 926.48. The survey was designed as part of a study of the deployment of a micro-finance program (see Banerjee et al. (2010))

Only half of the households were surveyed, which could bias our support measures (downwards) as we discuss below. The selection of households was based on a stratified random sampling technique in order to control for selection biases; with households being stratified by religion (Hindu, Muslim, Christian) and also by geographic sub-locations based on a full census of the villages that was conducted just prior to and in conjunction with the survey.

Each surveyed individual was asked to name the people he or she has various sorts of relationships with. The relationships that were queried in the survey are listed in Table 1[20].

There are several potential sources of measurement error in these data. First, the fact that not all people are surveyed means that there are missing nodes in the data, as well as links. Without any particular selection of which nodes are missing (given the random selection of households), the missing data would bias the measure of support downwards, since support looks at observed links ij and then asks whether i and j have a common neighbor. If that neighbor is missing from the data, then support can be underestimated. Second, there are the usual measurement issues with survey data: people may forget to mention some of their connections, people get fatigued by interviews, and the survey did not allow individuals to name more than five other people in some of the categories. As an illustration of the problem note that in relative relationships both parties in a pair name each other as relatives only in 10.27% percent of the pairs in which at least one member

[20]In the borrowing and lending relationships, fifty Rupees are roughly a dollar and the per capita income in India is on the order of three dollars per day.

Relationships in Survey

Codename	Question in Survey
Friends	Name the 5 non-relatives whom you speak to the most.
Visit-go	In your free time, whose house do you visit?
Visit-come	Who visits your house in his or her free time?
Borrow-kerorice	If you need to borrow kerosene or rice, to whom would you go?
Lend-kerorice	Who would come to you if he or she needed to borrow kerosene or rice?
Borrow-money	If you suddenly needed to borrow Rs. 50 for a day, whom would you ask?
Lend-money	Whom do you trust enough that if he or she needed to borrow Rs. 50 for a day you would lend it the him or her?
Advice-come	Who comes to you for advice?
Advice-go	If you had to make a difficult personal decision, whom would you ask for advice?
Medical-help	If you had a medical emergency and were alone at home, whom would you ask for help in getting to a hospital?
Relatives	Name any close relatives, aside those in this household, who also live in this village. Plus people in the same household.
Temple-company	Whom do you go to temple with?

Table 1: The Contents of the Survey

names the other as a relative. For other relationships, the probabilities are similar as for relatives.[21] Thus, in building networks, we say that agent i borrows money from agent j if and only if either agent i reports borrowing money from j or agent j reports lending money to agent i, and we do the same with respect to lend-money, borrow-kerorice and lend-kerorice, visit-come and visit-go and advice-come and advice-go relationships. Note that these are all directed relationships. In the case of friends, medical-help, relatives and temple-company we define that agent i has a relationship of the type in question with agent j if and only if at least one of them acknowledge so. One can also work with other variations on these definitions, and in an appendix we report on some of these variations, which do not

[21]Note that the questions were worded in ways to avoid basic perception issues that are associated with questions such as "who are your friends?" Based on wording that is more explicit about particular interactions (borrowing rice, asking for medical advice, etc.) the relationships are more concrete.

significantly alter the conclusions.

6.2 Measuring Support

The data are quite rich in providing a variety of different sorts of relationships. Thus, we can enrich our measures of support to account for the fact that individuals are involved in various types of relationships. In particular, we can ask whether relationships of one type are supported through relationships of another type. We can define the support of a network g' relative to another network g as $S(g', g)$,

$$S(g', g) = \frac{\sum_{ij \in g'} 1_{\{\exists k, ik \in g, kj \in g\}}}{\sum_{ij \in g'} 1}$$

which is the proportion of links in g' whose nodes have common neighbors in g. Note that $g = g'$ is allowed and then this reduces to *self-support*.

We refer to g' as the *base network* and g as the *context network*.

The reason for considering variations on the support measure is that it is quite possible that exchange of one type of favor is supported via relationships involving exchange of some other sort of favor or some other valuable interaction.

The support measure can be very high. For instance, if we examine the exchange of kerosene and rice and let $g' = max(BK, LK)$ (so there is a link if two individuals borrow or lend kerosene with each other) and $g = All$, the network of all relationships, then the support measure is $S(g', g) = 85.18\%$ when averaged across all villages.

In what follows we restrict our attention to base and context pairs (g', g) that seem a priori consistent with the assumptions of our model and contrast them with relationships of other nature that may not require support to be sustainable[22]

Thus, we group these relationships by the following steps. First, in terms of the money relationships we define B&L-money that there is one such link between individual i and j if and only if i both borrows and lends to j. Thus, pure lending relationships are excluded from our analysis. Similarly, we define Visit-c&g, B&L-kerorice, and Advice-c&g requiring mutual relationships to make the data consistent. Second, these relationships are put into three different categories, including Physical Favor (PF), Intangible Favor (IF) and Social Relationships (SR). All1 relationship is the union of these three, and All2 relationship is the union of All1, relatives and temple-company.

[22]There are other features of the data that call for care. For example, in the borrowing and lending money relationships, we need to deal with the existence of money lenders. To address this particular issue we only consider relationships in which both parties lend to and borrow money from each other.

$$All2 = \begin{cases} Relatives \\ Temple - company \\ All1 = \begin{cases} PF = & \text{B\&L-money} \bigcup \text{B\&L-kerorice;} \\ IF = & \text{Advice-c\&g} \bigcup \text{Medical-help;} \\ SR = & \text{Visit-c\&g} \bigcup \text{Friends.} \end{cases} \end{cases}$$

Using the data, we calculate the self support measures for Physical Favor, Intangible Favor, Social Relationships and All Relationships, and also the support measures of the first three relationships based on All relationship network for all 75 networks. Table 2 shows the average support measure mentioned above among 75 villages.

Support Measure	Self	All1	All2
Physical Favors	0.2807	0.6002	0.7200
Intangible Favors	0.2587	0.5721	0.7198
Social Relationships	0.3795	0.5569	0.6530
All Relationships	–	0.5556	0.6931

Table 2: The Average Support Measure

Our theory suggests that in robust favor exchange networks, the support measure should 100 percent, presuming that each relationship requires support in order to sustain favor exchange. However, in the data we see support that is less than 100 percent. Of course, this could be due to some relationships having frequent enough interaction to be self-sustaining. It could also be due to various forms of measurement error, such as missing nodes and also potentially missing links even within the observed networks. We now explore some aspects of support in more detail.

6.2.1 How Observed Support Compares to that Expected in a Random Network

One way to get a feeling for how much support we observe is to compare the observed level of support with that which would arise if the same number of links were instead distributed purely at random.

In a random network with a the probability p of a link and n as the population of the network, the expected support measure can be approximated as follows: the average degree is $D = p \cdot (n-1)$, and the chance any given link is supported is roughly $S = 1-(1-p)^{(D-1)}$, since the chance a link ij is not supported is that none of the other $D-1$ friends of agent

i are friends of agent j which is $(1-p)^{(D-1)}$.[23] From the data, we can calculate p and n for each village and estimate what the support measure S would be if the network were generated uniformly at random. Table 3 shows the average support measure in real networks are substantially larger than those expected in random networks.

	S(PF, PF)	**S(IF, IF)**	**S(SR, SR)**	**S(All1, All1)**	**S(All2, All2)**
Real network	0.2807	0.2587	0.3795	0.5556	0.6931
Random network	0.0137	0.0172	0.0400	0.0960	0.1487

Table 3: Average Support Measure for Real and Random Networks

While these numbers are suggestive, we now provide a rigorous statistical test to see if the support is higher than would be generated at random, where random also allows for geographic biases.

6.2.2 Geography and Support

Some of the relationships in these data are bound to be at least partly correlated by geographic closeness, since it is natural to expect some sorts of favor exchange among geographic neighbors , and geographic closeness is a transitive relation. Therefore networks that we observe may inherit some support from this geographic determinacy in a manner unrelated to the network structure based favor exchange that we have examined. In order to address this issue, we examine a geographically-biased-random network formation model and then see whether the support measures from that model differ in a statistically significant way from the observed support measures.

We proceed as follows.

- For each village we decomposed the observed links of each type into deciles according to the geographic proximity of the members of the pair in question, as measured by the households' GPS coordinates. Based upon this decomposition we constructed an empirical link distribution for each specific relationship and each village.

- We then carried out 50 simulations for each village and relationship. In each case we constructed a random graph based on the corresponding empirical distribution of links by geographic location. In order to guarantee that each simulated base network

[23] This approximates things since not all nodes are expected to have degree D. Slightly more accurate estimates could be obtained by working with the specific degree distribution that would be generated, or some other degree distribution. We do not pursue those, since we provide a more rigorous test with the geographic based networks in any case.

was a subset of the context network, we produced the context network by augmenting the simulated base network with the appropriate number of randomly drawn links according to the appropriate conditional distribution.

- We measured the support of each simulated network, and by comparing it to the observed support for the corresponding village and relationship, created a realization of a random variable with value 1 if the simulated support measure exceeded the observed support and 0 otherwise.

- Pooling all the random variables generated according to this method for a given relationship across all villages, we performed a one sided test of the null hypothesis that the random variable was binomially distributed with equal probability of being 1 or 0.[24] As shown in Table 4, for every one of relationships, the observed support is significantly greater (with p-values smaller than 0.0001 in all cases) than the one generated by the geographically biased random graph models.

Base-Context	**p-value**
PF-PF	0.000
PF-SR	0.000
PF-All	0.000
IF-IF	0.000
IF-SR	0.000
IF-All	0.000
SR-SR	0.000

Table 4: Binomial one sided test

6.2.3 Bounding Measurement error

Another thing that we do is examine how much measurement error there would have to be in order to see support measures of the level that we observed if the true support level were really 100 percent. In particular, what fraction of links would have to be missing to get the observed relationships?[25] Specifically, the types of errors that are likely to arise in our data are one-sided: while people are quite likely to forget relationships, it is less likely that

[24]The variances of the random variable are likely to be different for different villages. Not taking into account heteroskedasticity biases the test against rejection of the null hypothesis.

[25]Again, this test is biased against us since we are not considering missing nodes.

they "imagine" ones given the way in which these questions were designed.[26] To address this issue, there are various ways in which one might proceed, and here we followed a fairly simple one where we simulated the survey process 100 times, proceeding as follows in each iteration:

For each village in our sample and type combination of base-context networks in question we consider the closest network to the context network that leads the base network to have full support; where closest is defined as having the least number of additional links.[27] We then remove each link in the augmented network with a measurement error probability, and calculate the support of the resulting base-context pair. Figure 11 shows for each measurement error in the x axis the mean fraction of villages in the sample that ended with a support fraction of at most the level observed in the survey. It should be noted that the number of simulations (100) is such that any differences in expectation for different measurement errors in a given base-context pair or for different base-context-pairs are statistically significant. Table 5 provides a closer look at a small segment of Figure 11.

Support Measure	**Measurement Error**			
	16%	24%	32%	40%
PF-PF	0.02	0.12	0.31	0.54
IF-IF	0.01	0.05	0.24	0.52
SR-SR	0.01	0.17	0.49	0.73
All2-All2	0.09	0.5	0.85	0.99

Table 5: A closer look at Figure 11

6.3 Comparing Support in Different Sorts of Relationships

The data also allow us to compare the support measures of favor networks (PF and IF) with those of pure social networks (SR).

We can test whether there are statistically significant differences in support among networks of different types. To do this, we compare the values of any given support measure village by village. If there were no difference in support between relationships of the types being compared, then in any given village each one has a 50% chance to be larger than the other. Thus, under the null hypothesis that there is no difference in support, the number of villages where one has a higher support than the other should have a binomial distribution.

[26]By asking questions regarding actual actions (borrowing or lending money or rice, visiting someone's home, etc.) rather than asking about perceived relationships (who is your friend), we eliminate many problems with misperceived or asymmetric sorts of relationships.

[27]We randomly draw one network from the set of closest networks

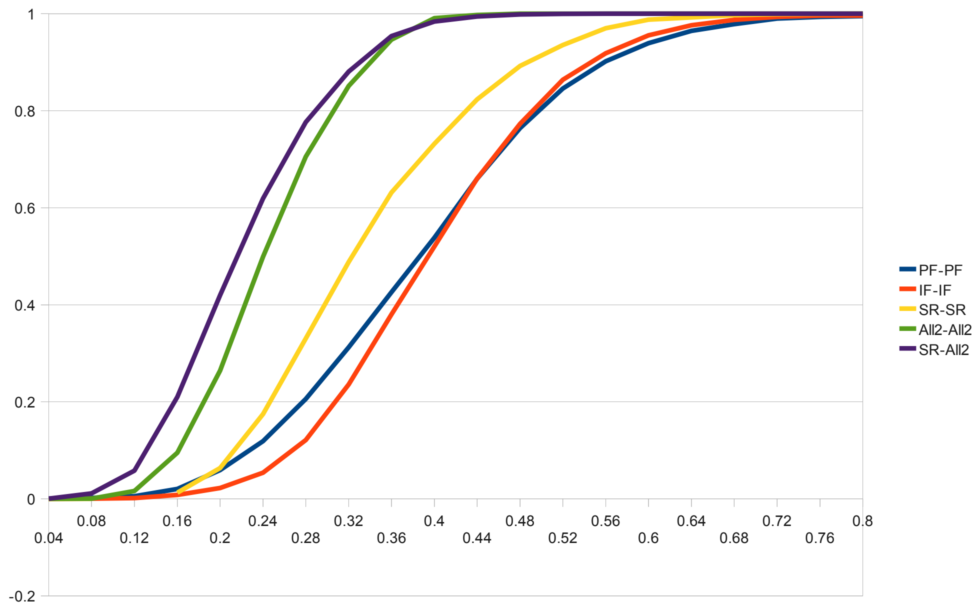


Figure 11: Fraction of villages with at most the observed support as a function of measurement error

When we examine the data we see significantly differences between the support of different types of relationships. For example, in 72 out 75 villages, the support of intangible favors $S(IF, All2)$ is higher than that of social relationships $S(SR, All2)$ (both relative to the all network). Of course the probability of a binomial random variable realizing a one in 72 out of 75 trials is effectively 0.

Table 6 shows the comparison of support measures of various relationships based on All1, Table 7 shows the comparison of support measures of various relationships based on All2, and Table 8 shows the comparison of self support measures of various relationships. All these results are showed in a matrix M, in which $M(i, j)$ means how many times out of 75, measure i is higher than measure j.

6.4 Comparing Support to Clustering

As mentioned before, our measure of support provides a new network characteristic. More importantly, it is distinguished from the standard measure of how tightly grouped a network is: clustering.

Support	S(PF, All1)	S(IF, All1)	S(SR, All1)	S(All1, All1)
S(PF, All1)	–	57***	63***	67***
S(IF, All1)	18***	–	38	52***
S(SR, All1)	12***	37	–	46**
S(All1, All1)	8***	23***	29**	–
*** significant at 99%;	** 95%.			

Table 6: The Comparison of Support Measure (All1)

Support	S(PF, All2)	S(IF, All2)	S(SR, All2)	S(All2, All2)
S(PF, All2)	–	38	69***	56***
S(IF, All2)	37	–	72***	57***
S(SR, All2)	6***	3***	–	6***
S(All2, All2)	19***	18***	69***	–
*** significant at 99%.				

Table 7: The Comparison of Support Measure (All2)

A common definition of clustering is:

$$Clus(g) = \frac{\sum_{ij\in g, ik\in g} 1_{\{jk\in g\}}}{\sum_{ij\in g, ik\in g} 1}$$

which is the proportion of times that two friends of some agent are friends with each other.

Support and clustering can be very different as in Figure 9 where the support measure is 100% while clustering is almost zero.

As an illustration of this, we calculate the clustering in each village and compare it to the corresponding self-support measure (which is the most conservative support measure). Table 9 shows that the average clustering in village networks is substantially smaller than the corresponding self-support measure for all types of relationships.

7 Conclusion

Our analysis of favor exchange provides various insights. We have shown that renegotiation results in specific critical structures and that robustness involves social quilts and more generally in supported links. Support provides a new local characteristic of networks and insight into closure and an operationalization of a sort of social capital which emphasizes social structure.

Support	S(PF, PF)	S(IF, IF)	S(SR, SR)
S(PF, PF)	–	48**	8***
S(IF, IF)	27**	–	11***
S(SR, SR)	67***	64***	–
*** significant at 99%;	** 95%.		

Table 8: The Comparison of Self Support Measure

G	PF	IF	SR	All1	All2
Clus(G)	0.1372	0.1202	0.1399	0.1696	0.2098
S(G,G)	0.2807	0.2587	0.3795	0.5556	0.6931

Table 9: Average Clustering and Support Measure

Our empirical analysis finds high levels of support in favor networks in rural Indian villages. We also find that support levels are much higher than clustering, and that support for favor networks is higher than that of more purely social networks.

7.1 Information and Robustness

In closing, we discuss some issues regarding the information observed by the agents in the society.

We have deliberately looked at a complete information setting for two reasons. First, in many applications, including the Indian villages we look at empirically, word of mouth communication travels much faster than actions and so if someone behaves badly other people hear about it quickly. Second, much of the previous literature has focused on the information as the driver of network structure in providing incentives and so our analysis is completely complementary.

Nonetheless, there is an important observation that comes out. Our analysis ends up yielding social quilts which end up having strong informational robustness properties in addition to the properties that we have investigated. In particular, agents only need to know what the agents whom they are linked to directly are doing, and for all of those agents they also have common friends - so they are both directly, and indirectly connected at a short distance through an independent channel to all of the agents whose behavior they have to be aware of in order to best respond. Thus, even with very limited communication, the robust social networks that we have uncovered can be sustained. Our analysis ends up yielding networks that might otherwise be justified for their informational properties from a completely different perspective.

8 References

Abreu, D., D. Pearce and E. Stacchetti (1993): “Renegotiation and Symmetry in Repeated Games”, Journal of Economic Theory, 60, 217-240.

Ali, S. N., and D. A. Miller (2009): “Cooperation and Collective Enforcement in Networked Societies”, working paper.

Attanasio, O., A. Barr, J. Camilo Cardenas, G. Genicot, and C. Meghir (2009): “Risk Pooling, Risk Preferences, and Social Networks,” mimeo: UCL.

Banerjee, A., A. Chandrasekhar, E. Duflo, and M.O. Jackson (2010) “Microfinance and Social Networks,” mimeo.

Benoit, J., and V. Krishna (1993): “Renegotiation in Finitely Repeated Games,” Econometrica, 61(2), 303-23.

Bernheim, B. D., B. Peleg, and M. D. Whinston (1987): “Coalition-Proof Nash Equilibria I. Concepts,” Journal of Economic Theory, 42(1), 1-12.

Bernheim, B.D., and D. Ray (1989): “Collective dynamic consistency in repeated games”, Games and Economic Behavior, 1, 295-326.

Bloch, F., G. Genicot, and D. Ray (2008): “Informal Insurance in Social Networks,” Journal of Economic Theory

Bloch, F., G. Genicot, and D. Ray (2007): “Reciprocity in Groups and the Limits to Social Capital,” American Economic Review, 97(2), 6569.

Bourdieu, P. (1986): “Forms of Capital, in Handbook of Theory and Research for the Sociology of Education,” J.G. Richardson, ed., Westport, Conn.: Greenwood Press.

Bramoullé Y., and R. Kranton (2007): “Risk-sharing networks”, Journal of Economic Behavior and Organization, 64(34): 275 - 94

Coleman, J.S. (1988): “Social Capital in the Creation of Human Capital”, American Journal of Sociology 94(Supplement: Organizations and Institutions: Sociological and Economic Approaches to the Analysis of Social Structure):S95S120.

Coleman, J.S. (1990): Foundations of Social Theory, Cambridge Mass.: Harvard University Press.

Dasgupta, P. (2000): Social capital : a multifaceted perspective. World Bank, Washington D.C.

De Weerdt J. and S. Dercon (2006): "Risk-sharing networks and insurance against illness", Journal of Development Economics, 81(2) 337-356.

Ellison G. (1994): "Cooperation in the prisoner's dilemma with anonymous random matching", The Review of Economic Studies 61, 567-588.

Fafchamps M., and S. Lund (2003): "Risk-sharing networks in rural Philippines", Journal of Development Economics. 71: 261 - 87

Farrell, J., and E. Maskin (1989): "Renegotiation in repeated games", Games and Economic Behavior, 1(4), 327-360.

Glaeser, E. L., D. Laibson, and B. Sacerdote (2002): "An Economic Approach to Social Capital", Economic Journal, 112(483), 437458.

Grief A. (1989): "Reputation and coalitions in medieval trade: evidence on the Maghribi traders", The Journal of Economic History,

Guiso, L., P. Sapienza, and L. Zingales (2004), The Role of Social Capital in Financial Development, American Economic Review, 94, 526556.

Haag, M. and R. Lagunoff (2006): "Social Norms, Local Interaction, and Neighborhood Planning," *International Economic Review*,

Homans, C.G. (1958): "Social Behavior as Exchange", American Journal of Sociology 62:597606.

Jackson M.O. (2008) *Social and Economic Networks*, Princeton, NJ: Princeton Univ. Press

Kandori, M. (1992): "Social Norms and Community Enforcement", Review of Economic Studies, 59: 63-80.

Karlan D., M. Mobius, T. Rosenblat and A. Szeidl (2009): "Trust and Social Collateral," Quarterly Journal of Economics, 124(3), 1307-1361.

Lippert, S. and G. Spagnolo (2009): "Networks of Relations and Word-of-Mouth Communication," SSE/EFI Working Paper in Economics and Finance No 570.

Loury, G (1977). A Dynamic Theory of Racial Income Differences. Chapter 8 of Women, Minorities, and Employment Discrimination, Ed. P.A. Wallace and A. Le Mund. Lexington, Mass.: Lexington Books.

Mihm, M., R. Toth and C. Lang "What Goes Around Comes Around: A Theory of Indirect Reciprocity in Networks," CAE Working Paper 09-07

Okuno-Fujiwara M. and A. Postlewaite (1995): "Social norms and random matching games", Games and Economic Behavior, 9(1), 79-109.

Putnam, R. (1993): "The prosperous community: social capital and public life", American Prospect, 13, 42, 35.

Putnam, R. D. (1995): "Bowling Alone: America's Declining Social Capital", Journal of Democracy, 6(1), 6578, Volume 6, Number 1, January 1995.

Putnam, R. (2000): "Bowling Alone: The Collapse and Revival of American Community" (Simon and Schuster).

Raub, W. and J. Weesie (1990): "Reputation and Efficiency in Social Interactions: An Example of Network Effects," American Journal of Sociology, 96:3, 626-654.

Simmel, G. (1950) The Sociology of George Simmel. Trans. K. Wolf. Glencoe: Free Press.

Sobel, J. (2002): "Can We Trust Social Capital?" Journal of Economic Literature 40: 139-154.

Tabellini, G. (2009), Culture and Institutions: Economic Development in the Regions of Europe, *Journal of the European Economic Association*, 6, 255294.

Woolcock, M. (1998): "Social Capital and Economic Development: Toward a Theoretical Synthesis and Policy Framework," Theory and Society, 27(2): 151-208.

Appendix A: Proofs of Results

Proof of Proposition 2: Let g' be a smallest nonempty network (in the sense of set inclusion) that is a subset of g and lies in $G(m)$. Such a network exists (possibly g itself) and is thus minimally critical by definition. The second statement is easily verified by construction, but also follows as a corollary to Theorem 1 which is proven below. ∎

Proof of Lemma 1: Let us first argue that if there is a critical node, then there are at least two nodes that are path-connected but that do not lie on a simple cycle. Suppose that there is a critical node i in the path connected network g, such that deleting i results in at least two separate components. Pick nodes j in one of those components and k in another component. It follows that i lies on all paths connecting j and k or else deleting i would not have resulted in these nodes falling in separate components. Thus, there could not have been a simple cycle containing these two nodes in the original network.

For the other direction of the lemma, we consider any two path-connected nodes i and j that are embedded in a network with at least 3 nodes that has no critical nodes. We show that there exists a simple cycle containing i and j. We proceed by induction on the distance between i and j (with the standard definition of distance being the number of links of the shortest path between them).

For the base case let the distance between i and j be 1 so that i and j are direct neighbors. There must exist some other node k that is a neighbor of either i or j since the network involves at least 3 nodes and is path connected. Without loss of generality assume that it is adjacent to i. Since there are no critical nodes in the graph, k and j remain path-connected if we delete node i. Thus, let P be a path that goes from k to j without passing through i. There is a simple cycle containing i and j given by $j - i - k$, and then taking P from k to j.

For the inductive step, suppose that the claim is true for any pair of nodes of distance n or less, and consider some pair of nodes i and j at distance $n + 1$, and let S be a path of length $n + 1$ between i and j. There is a unique node k adjacent to i on S, at distance n from j and by the inductive hypothesis there exists a simple cycle containing k and j. Let P_1 be a path from k to j contained in this simple cycle, and P_2 a path from j to k disjoint from P_1. Since the graph has no critical nodes there exists some path P_3 from i to j that does not go through k. If P_3 is disjoint from P_1 or P_2 we are done, since we then have a simple cycle given either by $i - k - P_1$ to j and then back to i via P_3, or by $i - k - P_2$ to j and then back to i via P_3. So assume that P_3 intersects both P_1 and P_2 and without loss of generality that it intersects P_1 first, at some node m (since P_1 and P_2 are disjoint, this first-to-be-intersected order is strict). We now have a simple cycle including i and j given by $i \to m$ (via P_3), then $m \to j$ (via P_1) and then from j to k (via P_2), and then $k - i$ finally back to i. ∎

Proof of Proposition 3: The proof proceeds by induction on the size of the tree union.

When $k = 1$, it is a single simply minimally critical network, and so it is renegotiation proof. Suppose it is true for all $k' < k$. We show that a tree union of k simply minimally critical networks is renegotiation proof.

To establish the proposition, we show that tree unions of simply minimally critical networks and some nonempty strict subnetworks of simply minimally critical networks cannot be renegotiation-proof. This is enough to establish that tree unions of simply minimally critical networks are renegotiation-proof, simply by deleting all links in any particular simply minimally critical subnetwork of the tree union if some agent fails to perform a favor in that subnetwork.

Begin with a tree union of k simply minimal critical networks, $g_1, \dots g_k$.

Let $g^0 = \left(\bigcup_{h=1\dots m_0-1} g_h\right) \cup \left(\bigcup_{h=m_0\dots k} g_h^0\right)$, with $m_0 \leq k$, $g_h^0 \subset g_h$, $g_h^0 \neq g_h$ $\forall h \geq m_0$ and at least one g_h^0 in the union is nonempty. So this is the tree union of simply minimally critical networks and some nonempty strict subnetworks of simply minimally critical networks. Suppose to the contrary that it is renegotiation-proof.

Note that $\bigcup_{h=m_0\dots k} g_h^0$ is a tree union of networks, and it must therefore have some leafs. Pick one such leaf and denote it $g_{h^*}^0$. Since $g_{h^*}^0$ is a strict subset of the simply minimally critical network g_{h^*} and a leaf of the subtree, there is some agent i_0 who has a positive number of links, less than m, in the subtree. Suppose this agent were to fail to provide a favor on a link i_0j_0 in $g_{h^*}^0$. Since by supposition $g^0 \in RPN$, agent i_0 would have to lose at least m links if he or she failed to provide a favor on any link i_0j_0 in the subtree. Since the agent does not have enough links to lose in the subtree, he or she would have to lose links in $\bigcup_{h=1\dots m_0-1} g_h$. Denote the continuation by g^1 which must be renegotiation-proof. Note that g^1 cannot be a strict subset of $\bigcup_{h=1\dots m_0-1} g_h$, since by the inductive hypothesis $\bigcup_{h=1\dots m_0} g_h \in RPN$. Therefore g^1 must have some links from $\bigcup_{h=m_0\dots k} g_h^0$. In particular $g^1 = \left(\bigcup_{h=1\dots m_1-1} g_h\right) \cup \left(\bigcup_{h=m_1\dots k} g_h^1\right)$, where $g_h^1 \subset g_h$, $g_h^1 \neq g_h$ $\forall h \geq m_1$ and $m_1 < m_0$. This last inequality results from the fact that i_0 lost links in $\bigcup_{h=1\dots m_0} g_h$. Again, any agent who has fewer than m links in $\bigcup_{h=m_1\dots k} g_h^1$ must have links in $\bigcup_{h=1\dots m_1-1} g_h$. We the derive a subnetwork g^2 from g^1 analogously to the way we derived g^1 from g^0. Proceeding in this fashion we produce a finite sequence of renegotiation proof networks $g^0, g^1, ..., g^\ell$, with $m_x < m_{x-1}$ at each iteration and there is always at least one link in $\bigcup_{h=m_x\dots k} g_h^x$. Continue until $m_\ell = 0$. Using the same argument with which we found i_0, it can be seen that we would find some node with less than m links in total, contradicting $g^\ell \in RPN$. ∎

Proof of Theorem 1:

We first show that if $g \in RPN_k$ then $g \in TC_k$. Given a network $g \in RPN_k$, by the definition it follows that g is sustainable on the equilibrium path. So for any i and $ij \in g$, if i is called upon to do a favor for j and does not, then at least one possible continuation[28]

[28]Even though players use pure strategies, nature picks which favors are asked in the future, so there are

must lead to a network $g' \subseteq g - ij$ such that $g' \in RPN_{k'} = TC_{k'}$, $d_i(g') \le d_i(g) - m$, and there is no $g'' \subset g - ij$ such that $g'' \in RPN_{k''}$ and $D(g'') > D(g')$. If this were not the case, then if i did not perform the favor, he or she would save the cost c and lose at most $m - 1$ links in any continuation. Thus, i would benefit from deviating and not performing the favor since by the definition of m, (1) holds and so the cost of the favor outweighs the loss in future payoffs from losing no more than $m - 1$ links, which contradicts the fact that g is sustained as an equilibrium. Thus, for every i and ij, there exists $g' \subset g - ij$ such that $g' \in TC_{k'}$ for some k', $d_i(g') \le d_i(g) - m$, and there is no $g'' \in TC_{k''}$ such that $g'' \subset g - ij$ and $D(g'') > D(g')$. Therefore $g \in TC_k$.

Next, we show that if $g \in TC_k$ then $g \in RPN_k$. We do this by induction on the number of links in a network. In order to establish the result, we also need to be careful about what happens starting at subgames that are off the equilibrium path. As such, we work with a stronger induction hypothesis, with the induction indexed by k.

The induction hypothesis is that starting from any node and any $g_0 \in G_k$, there exists a pure strategy subgame perfect equilibrium continuation such that

- (i) there is a unique network $g_1 \in RPN_{k_1}$ for some $k_1 \le k$ that is reached in the continuation, with $g_1 = g_0$ if $g_0 \in TC_k$,
- (ii) on the equilibrium continuation path a favor is performed if and only if it corresponds to a link in g_1, and
- (iii) in any subgame starting with some network $g' \in G_{k'}$ with $k' \le k$ if g'' is played in perpetuity with some probability in the continuation then $g'' \in RPN_{k''}$ for some k'' and there does not exist any $g''' \subset g'$ such that $g''' \in RPN_{k'''}$ and $u_i(g''') \ge u_i(g'')$ for all i with strict inequality for some i.

As a first step in the induction note that it follows directly from the definitions that $RPN_0 = \{\emptyset\} = TC_0$. Note also that starting from $g_0 = \emptyset$ there is a unique subgame perfect equilibrium continuation (no favors can be supplied and no links can be maintained) and so it follows directly that conditions (i)-(iii) are satisfied. So, let us presume that the induction hypothesis holds for all $k' < k$. We show that the same is true for k.

Begin with the case such that $g_0 = g \in TC_k$. On the equilibrium path, have all agents maintain all links (so $L_i(g_t) = N_i(g_t)$ whenever $g_t = g_0 = g$) and perform all favors. The off the equilibrium path strategies are described as follows. If an agent i is called upon to provide a favor for an agent j such that $ij \in g$ and does not do the favor, then the continuation is as follows. Given that $g \in TC_k$, by the definition of transitive criticality, there exists $g' \subseteq g - ij$ such that $g' \in TC_{k'} = RPN_{k'}$, $d_i(g') \le d_i(g) - m$ and there is no

potentially many continuations from any given node.

$g'' \in TC_{k''} = RPE_{k''}$for any k'' such that $g'' \subset g - ij$ and $D(g'') > D(g')$. Denote this network by $g(i,j) = g'$. Following i's failure to provide a favor to j, have the continuation be such that $L_\ell(g - ij) = N_k(g(i,j))$ for all ℓ. This results in the network $g(i,j) \in RPN_{k'}$ following the link announcement phase, and so from then on there is a pure strategy subgame perfect equilibrium sustaining $g(i,j)$ and satisfying (i) - (iii) by the induction step, and so have players play the strategies corresponding to such an equilibrium in that continuation. At all other nodes off the equilibrium path for which strategies are not already specified we are necessarily at a network with fewer links, and so pick a pure strategy equilibrium continuation that satisfies (i) - (iii), which is possible by the induction hypothesis.

This satisfies (i)-(iii) by construction. To check that this a subgame perfect equilibrium, by the specification of the strategies above, we only need to check that no agent wants to deviate from the equilibrium path, and also that following some i's failure to provide a favor to j, no agent ℓ wants to deviate from $L_\ell(g - ij) = N_\ell(g(i,j))$. We only need to check these sorts of deviations since all other continuations were specified to be pure strategy subgame perfect equilibrium continuations. By construction, an agent i who is called upon to do a favor for an agent j who deviates will end up losing at least m links, and so by (1) this cannot be an improving deviation. Next, consider, some agent ℓ's incentive to deviate from $L_\ell = N_\ell(g_0)$ if g_0 is still in play, or else from $L_\ell(g - ij) = N_\ell(g(i,j))$ following some i's failure to provide a favor to j. By not deviating the agent gets the payoff from g_0 or $g(i,j)$ in perpetuity. By deviating, the agent ℓ will end up with a continuation starting from a network $g'' \subset g_0$ or $g'' \subset g(i,j)$, respectively, where the agent has not gained any links and may have lost some links. Since each link has a positive future expected value, this cannot be an improving deviation.

Next, let us show that from any node in the continuation from some initial $g_0 \notin TC_k$ there exists a pure strategy subgame perfect equilibrium continuation satisfying (i)-(iii). There are two types of nodes to consider. One is a node at which some player i is called upon to provide a favor for an agent j such that $ij \in g_0$, and another is a node where agents announce the links they wish to sustain.

First, consider starting at g_0 and a node where agents announce the links that they wish to sustain. Find some g' that has the maximal $k' < k$ of links such that $g' \in RPN_{k'}$ and $g' \subset g_0$. For each ℓ set $L'_\ell = N_\ell(g')$ and then from g' play a continuation satisfying (i) - (iii) (by the induction step). If any agent deviates, to L such that $L'_\ell \subset L$, then play the same continuation as this will not affect the network formed. Otherwise, the continuation will lead to some g'' with strictly fewer links for ℓ and the continuation will necessarily result in a lower expected continuation payoff. This establishes the claim for this sort of node.

Next, let us consider a node at which some player i is called upon to provide a favor for an agent j such that $ij \in g_0$. There are two cases that can follow: one where i performs the favor and so the resulting network is then g_0. In that case, we have just shown that there is a pure strategy subgame perfect equilibrium continuation satisfying (i) to (iii). Let g' be the network sustained on the equilibrium path in one of these that has the most links for i. If i does not perform the favor, then $g - ij \in G_{k-1}$ is reached. By the induction hypothesis again there is a pure strategy subgame perfect equilibrium continuations satisfying (i)-(iii), and let g'' be a network sustained by one of these that has the most links for i. Now, based on those two continuations, have i choose a pure strategy best response. The claim follows. ∎

Proof of Proposition 6: First, we analyze the fraction of subgame perfect equilibria. Denote by SPE_n the set of subgame perfect networks on n nodes. A network is a subgame perfect network if and only if no node in the network has at least 1 and no more than $m-1$ links. We set a very loose upper bound on the number of networks in which at least one node has at least 1 and no more than $m-1$ links by first picking any node and its links and then allowing the remaining $n-1$ nodes to link among themselves however they want, and then multiplying by n to allow for any starting node. A lower bound on the cardinality of SPE_n is $2^{\frac{n(n-1)}{2}}$ minus this bound. Therefore: $|SPE_n| \geq 2^{\frac{n(n-1)}{2}} - n\sum_{k=1}^{m-1}\binom{n-1}{k}2^{\frac{(n-1)(n-2)}{2}} \geq$
$2^{\frac{n(n-1)}{2}} - (m-1)n\binom{n-1}{m-1}2^{\frac{(n-1)(n-2)}{2}}$,

where the inequality on the right holds for any m such that $n-1 > 2(m-1)$. This implies that

$$\begin{aligned}
\frac{|SPE_n|}{|G_n|} &\geq 1 - \frac{(m-1)n\binom{n-1}{m-1}2^{\frac{(n-1)(n-2)}{2}}}{2^{\frac{n(n-1)}{2}}} \\
&= 1 - \frac{(m-1)n\binom{n-1}{m-1}2^{\frac{(n-1)(n-2)}{2}}}{\sum_{k=0}^{n-1}\binom{n-1}{k}2^{\frac{(n-1)(n-2)}{2}}} \\
&= 1 - \frac{(m-1)n\binom{n-1}{m-1}}{\sum_{k=0}^{n-1}\binom{n-1}{k}} \\
&= 1 - \frac{(m-1)^2\binom{n}{m-1}}{\sum_{k=0}^{n-1}\binom{n-1}{k}} \to 1 \text{ as n goes to infinity}
\end{aligned}$$

Next, we find an upper bound on the fraction of social quilts that goes to 0 as n grows. There are two possible sets of degrees that an agent can have in a network: $A = \{km < n | k$ is a nonnegative integer$\}$ and $B = \{k < n | k$ is an integer$\} - A$. In a social quilt g, every agent i has a degree $d_i(g) \in A$. We show that the number of networks such that all agents have degrees in A is a fraction of no more than $\frac{1}{2}^{n-1}$ of all possible networks.

To do this we catalog networks g by first forming a network among the agents 1 to $n-1$, and then considering links between those agents and agent n. We show that regardless of the starting network g_0, there is at most one configuration of links for agent n (out of 2^{n-1} possible) that will allow all agents to have degrees in A. The result then follows directly.

Beginning with the network g_0 among agents 1 to $n-1$, if $d_i(g_0) \in A$ then it must be that there is no link between i and n in g. In contrast, if $d_i(g_0) \notin A$ then in order for $d_i(g)$ to be in A it would have to be that i and n are linked in g. Thus, if there is some configuration of links between n and the other agents that results in the correct configuration of degrees, there is at most one such configuration out of the 2^{n-1} possible configurations. ∎

Proof of Theorem 3: Suppose to the contrary that g is robust against social contagion and ij is not supported. Consider $h \notin \{i, j\}$ and delete a link of h (and there is at least one such h who has a link, as the single link ij is not sustainable independently as part of a subgame perfect equilibrium). This leads to a continuation g' such that $ij \in g'$, as otherwise by robustness both i and j would have to be neighbors of h which would contradict the fact that ij is not supported. Iterate on this argument. Eventually, we reach the empty network which is a contradiction since ij cannot be deleted. ∎

Proof of Theorem 4: We only prove that robustness implies that a network must be a social quilt, since the converse is an easy analog of the proof of Proposition 3, adapted to strong tree unions.

Suppose that g is robust against social contagion. If there is a minimally critical clique $g_c \subset g$ that has at most one node i connected with nodes outside of the clique and m_i is the smallest in g_c, then $g - g_c$ is also robust against social contagion. This follows since if any agent $j \neq i$ who is in g_c deletes a link, he or she must lose all of his or her links, and then so must all other agents except i in the clique, but by robustness no other links can be deleted. So, eliminate g_c and continue with the network $g - g_c$. If repeating this process leads to an empty network, then g must have been a social quilt. Suppose instead, that this elimination process leads to some nonempty g' (and note that g' is robust and hence sustainable) such that g' contains no minimally critical cliques where at most one agent has links outside of the clique and that agent has the smallest m_i.

By the above process, any remaining cliques that are subnetworks of g' and are such that each agent i in the clique has at least m_i links, must belong to at least one of the following sets:

(1) Cliques that not minimally critical.

(2) Cliques that are minimally critical but such that some agent j connected to another part of the network is not the agent i with the smallest m_i in the clique.

(3) Cliques that have at least two agents who have links outside of the clique.

So, identify some remaining clique that is a subnetwork of g' and is such that each agent i in the clique has at least m_i links, and identify the first case of (1) to (3) that applies.

Next, do the following depending on which case applied: In case (1) delete a link between the agents with the two smallest m_is. In case (2) delete the link ij. In case (3), delete a link between a pair of agents who have links outside of the clique.

Keep repeating this process until there are no "valid" cliques (where every agent i has at least m_i links in the clique) left in the resulting network g''. After this process we are left with a nonempty g'' that lies in

$$G(m) = \{g \mid \forall i, d_i(g) \geq m_i \text{ or } d_i(g) = 0\}.$$

Thus, a direct extension (literally word for word) of the proof of Proposition 2 implies that there is a subnetwork of g'' that is minimally critical and nonempty. Let that network be g'''. It follows from our derivation of g'' that g''' cannot be a clique. However, since g''' is minimally critical, then by robustness and the weak renegotiation-proofness of g, there is an equilibrium where in any subgame starting from g''', it is sustained. However, starting from g''' if any link is deleted then all links must be deleted in any equilibrium, by the definition of minimal criticality. This contradicts the robustness of g, since g''' is not a clique. ∎