# ON GEOMETRIC INTERSECTION OF CURVES ON SURFACES

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*This is a pre-preprint. Please give comments!*

Abstract. In order to better understand the geometric intersection number of curves, we introduce a new tool, the *smoothing lemma*. This lets us write $i(\alpha, \beta)$, where $\alpha$ is a simple curve and $\beta$ is any curve, canonically as a maximum of various $i(\alpha, \beta')$, where $\beta'$ is now also simple. We can use this for several purposes, including a new derivation for the behaviour of Dehn-Thurston coordinates under an elementary change in the pair of pants decomposition. Both the proofs and the resulting formulas are simpler than those found earlier by Penner.

## Contents



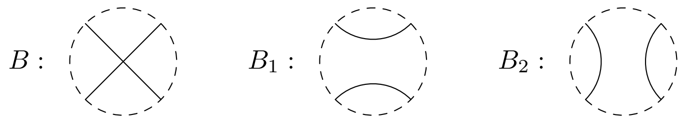
## 1. Introduction

In 1922, Dehn [1] introduced what he called the *arithmetic field of a surface*, by which he meant coordinates for the space of simple curves[1] on a surface, together with a description for how to perform a change of coordinates (in particular, how the mapping class group of the surface acts). He presumably chose the name "arithmetic field" because in the simplest non-trivial cases, namely the torus, the once-punctured torus, and the 4-punctured sphere, the result is closely related to continued fractions and Euler's method for finding the greatest common divisor. Although these coordinates were later rediscovered (XXX: cite), and now go under the name Dehn-Thurston coordinates, the question of how they change under change of coordinates has been largely neglected. It was not until 1982 that Penner [7, 8] gave explicit formulas for the change of coordinates, but probably due to the intimidating complexity of his formulas, little further work has been done with this.

[1] In this paper, "curve" without further qualification always includes the case of multi-curves, unions of distinct curves; "simple" means that there are no intersections between a component and itself or between different components.

In this paper, we study geometric intersection numbers $i(\cdot,\cdot)$ of two curves to give a simpler and more conceptual answer to the question of how Dehn-Thurston coordinates change under a change of the marking on this surface. The principal tool is the following lemma:

**Lemma 1** (Smoothing)**.** *Let $A$ be a simple closed curve and $B$ an arbitrary curve (with self-intersections), embedded so that $A \cup B$ is taut (has a minimal number of intersections). Pick an intersection of $B$, and let $B_1$ and $B_2$ be the two different ways to smooth that intersection:*




*Then*

$$i(A,B) = \max(i(A,B_1), i(A,B_2)).$$

This lemma has several applications beyond the change of coordinates question above. In particular, by applying the smoothing lemma repeatedly, we can canonically reduce $i(A,B)$, for *arbitrary* $B$, to a maximum of terms of the form $i(A,B')$, where each $B'$ has no self-intersections. Another corollary is the following:

**Corollary 2.** *Fix a pair of pants decomposition $\{C_i\}$ of a surface $\Sigma$ and a curve $B$. Then $i(A,B)$ (as a function of the simple closed curve $A$) is a* convex*, piece-wise linear function of the Dehn coordinates of $A$.*

## 2. Preliminaries on curves

Let us start with some preliminary definitions and results on curves on surfaces. These are mostly standard, but slightly idiosyncratic.

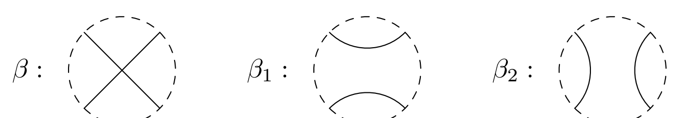
- A *loop* is an immersed circle in $\Sigma$.
- An *arc* is an immersed arc in $\Sigma$, with endpoints at punctures of $\Sigma$.
- A *curve* is finite collection of loops and arcs. (Most authors would call this a multi-curve, but since we almost always allow multiple components this notation is more convenient for us.)
- A *closed curve* is a curve with no arc components.
- A *simple curve* has no self-intersections.
- A *taut curve* has the minimal number of self-intersections among all curves in the same isotopy class. Taut curves also minimise the number of self-intersections of any given component or the number of intersections between any pair of components.
- The *geometric intersection number* $i(\alpha,\beta)$ of two curves $\alpha$, $\beta$ is the number of times $\alpha$ crosses $\beta$. Usually of more interest is $i([\alpha],[\beta])$, which is the minimum of $i(\alpha',\beta')$ as $\alpha'$ and $\beta'$ range over the homotopy class. This is equal to $i(\alpha,\beta)$ if $\alpha\cup\beta$ is taut.

Tautness is an important property, since it allows us to compute intersection numbers of homotopy classes. The following reformulation is easier to check than the global property of being minimal.

**Proposition 3.** *$C$ is taut if and only if there are no smoothly embedded monogons or bigons $P$ with $\partial P \subset C$, turning corners at the self-intersections or punctures of $C$.*

## 3. The smoothing lemma

**Lemma 4.** *(Single crossing version) Let $\alpha$ be a simple closed curve and $\beta$ be an arbitrary curve so that $\alpha\cup\beta$ is taut. Fix a self-interesection $C$ of $\beta$. Then if $\beta_1$ and $\beta_2$ are identical to $\beta$ outside of a neighborhood of $C$ and inside the neighborhood are arranged like*




*then*

$$i([\alpha],[\beta]) = \max(i([\alpha],[\beta_1]), i([\alpha],[\beta_2]))$$

*Proof.* Clearly $i([\alpha],[\beta]) = i(\alpha,\beta) = i(\alpha,\beta_1) \geq i([\alpha],[\beta_1])$ and similarly $i([\alpha],[\beta]) \geq i([\alpha],[\beta_2])$, so we must only show that equality is acheived. If $i([\alpha],[\beta_1]) \neq i([\alpha],[\beta])$, then there is an $\alpha\beta_1$-bigon, or equivalently (since $\alpha \cup \beta$ is taut) a trigon with two sides labelled $\beta$ and one side labelled $\alpha$, with one vertex at $C$ and exiting in quadrant I or III. Since $\alpha$ has no self-intersection, any $\alpha$ curve that intersects this trigon must intersect the two $\beta$ sides; this then forms a smaller trigon. Thus if we look at the smallest such trigon, one vertex is at $C$ and the other two are at the first crossing of $\beta$ with $\alpha$ that you reach as you travel along the respective strands from $C$. (Note that, since $\alpha$ is closed, these corners are at transversal crossings, not at punctures.) Similarly if $i([\alpha],[\beta_2]) \neq i([\alpha],[\beta])$, then there must be a $\beta\beta\alpha$-trigon in quadrant II or IV. If both these inequalities held, then the two trigons would abut each other, and could then be glued together to give a $\beta\alpha$-bigon, a contradiction.

□

By repeatedly using Lemma 4, we can write $i(\alpha,\beta)$, where $\alpha$ is simple closend, in terms of a maximum over some $i(\alpha,\beta')$, where each $\beta'$ is simple: pick a crossing, smooth it, pull the resulting curve taut, and iterate. In general, $\beta$ must be taut before applying Lemma 4, but the following lemma lets us skip it in this case.

**Lemma 5.** *Let $\alpha$ be a simple closed curve and $\beta$ any curve with $n$ self-intersections so that $\alpha \cup \beta$ is taut. Then*

$$i(\alpha,\beta) = \max_{\substack{\beta' \text{ a smoothing} \\ \text{of } \beta}} i(\alpha,\beta')$$

*where the maximum runs over all the $2^n$ different ways to smooth the crossings of $\beta$ to get a simple curve $\beta'$.*

Lemma 4 is remarkably simple; this makes it rather suprising that it has not (as far as I know) been noticed before.

*Remark* 6. Although we will not need this, the collection of simple curves $\{\beta'\}$ obtained by repeatedly resolving $\beta$ is canonical once you remove components of each $\beta'$ which are null-homotopic circles (and therefore have 0 intersection number with each $\alpha$. That is, any two different ways of smoothing the crossings of $\beta$ give the same collection of curves $\{\beta'\}$. Any two such collections must have the same geometric intersection number with any $\alpha$, but this is not enough to determine $\{\beta'\}$ (even after taking the convex hull): for instance, if $\beta$ is supported in a pair of pants, each $\beta'$ consists of a number of parallel copies of the border of the pair of pants, but, as Leininger [2] points out, the intersection with all $\alpha$ are determined by the intersections with just 6 different arcs, which is much less information.

*Remark* 7. These smoothing lemmas extend naturally to the case when $\alpha$ is a measured foliation. Local picture: foliation goes one way or the other, determines which smoothing is valid.

3.1. **Relations to hyperbolic geometry.** A pair $(\Sigma,\alpha)$ of a surface with a simple closed curve on it can be thought of as the limit of a sequence of hyperbolic structures in which the curves in $\alpha$ shrink to zero length and the hyperbolic length of any curve $\beta$ approaches $k \cdot i(\alpha,\beta)$ for growing constants $k$. Thus $i([\alpha],\cdot)$ behaves like the length function on geodesics in a hyperbolic surface, and it is not surprising that Lemma 4 is a polarised version of an identity relating the lengths of hypebolic geodesics. *Polarisation* is the process of taking a polynomial with only positive coefficients and replacing addition by maximum, multiplication by addition, and constants by nothing.

Definition of length fcn for:

- loops (the trace);
- arcs; and
- unions

**Lemma 8.** *Smoothing, geometric version. TBW.*

For instance, consider the closed curves within a four-punctured sphere. Geometric [3] vs. topological version [4, 5].

## 4. Twisting around a loop

As we saw in the last section, the smoothing lemma lets us completely solve the problem of finding the arithmetic field of a punctured surface. As usual, however, closed surfaces are more complicated. To parameterize simple curves on a closed surface, Dehn and Thurston independently proposed to pick a maximal collection of disjoint loops measure the intersection number with each loop. (Need twisting numbers)

Multiplication/earthquakes; convexity.

Asympototics.

Definition of twisting number as extrapolated center.

Height of extrapolated center.

**Lemma 9.** *(Smoothing lemma: loop version) Let $\alpha$ be a simple closed curve, $\beta$ a loop, and $\gamma$ a simple curve, so that $\alpha \cup \beta \cup \gamma$ is taut. Then*

$$i(\alpha,\beta) + i(\alpha,\gamma) = i(\alpha,\beta\cup\gamma) = |t_\alpha(\gamma;\beta)| + i([\tilde{\alpha}],[\tilde{\gamma}]) \vee \max_{\substack{\beta' \text{ a smoothing of } \beta\cup\gamma \\ \beta'\neq\beta\cdot\gamma,\beta'\neq\gamma\cdot\beta}} i(\alpha,\beta')$$

*where $\tilde{\alpha}$ and $\tilde{\gamma}$ are the intersections of $\alpha$ and $\gamma$ with $\Sigma \setminus \beta$, respectively.*

*Remark* 10. Unlike the earlier smoothing lemmas, this version of the smoothing lemma is not the polarized version of a lemma that is true for hyperbolic lengths. Thus although the Dehn-Thurston coordinates can be thought of as polarised versions of the Fenchel-Nielsen coordinates on Teichmüller space, the formulas that we find below for the change of coordinates are somewhat simpler than the corresponding answer for Fenchel-Nielsen coordinates [6].

### 4.1. Twist parameters for triangulations.

There is an alternate set of parameters to describe curves given by a triangulation. Draw canonical transversal to each edge and measure its twist.

### 4.2. Dehn-Thurston coordinates.

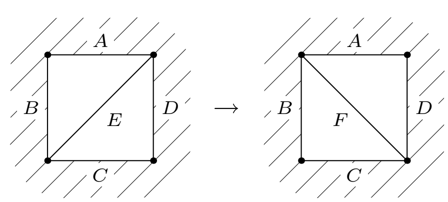
## 5. The elementary moves

We now proceed with the elementary moves. For each of the elementary moves, we draw the move, state the formulas relating the right hand side to the left hand side (and vice versa, if necessary), and then give the proof. There are two choices of the parameters on the arcs (measure and twist). We give both sets of formulas, but typically only prove the result in terms of the measure parameters, since the twist-parameter formulas are easily derived from the measure-parameter ones.

On the boundary of any of these figures, the measure parameters do not change (since they depend only on the homotopy class of the boundary arc or circle). The twist parameters on the boundary do generally change, but only by an additive constant.

For easy of drawing, the part of the surface containing the elementary move is drawn in the plane when possible. Crosshatching indicates holes in this piece, boundaries that are attached to other portions of the complete surface.

### 5.1. Triangle + triangle.




Measure parameters:

$$\mathrm{mF} = (\mathrm{mA} + \mathrm{mC} - \mathrm{mE}) \vee (\mathrm{mB} + \mathrm{mD} - \mathrm{mE})$$

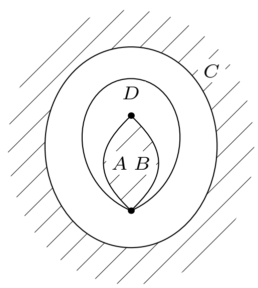
Twist parameters:

$$\begin{aligned} \mathrm{tF} &= -\mathrm{tE} \\ \Delta\mathrm{tA} = \Delta\mathrm{tC} &= \mathrm{tE} \vee 0 \\ \Delta\mathrm{tB} = \Delta\mathrm{tD} &= \mathrm{tE} \wedge 0 \end{aligned}$$

*Proof.* Consider the union of curves $\mathrm{E} \cup \mathrm{F}$. By Lemma 4, $\mathrm{mE} + \mathrm{mF} = \mathrm{m}(\mathrm{E} \cup \mathrm{F})$ is $(\mathrm{mA} + \mathrm{mC}) \vee (\mathrm{mB} + \mathrm{mD})$, from which the result in measure coordinates follows. ☐

This result is well-known, and very useful by itself. If you are interested exclusively in surfaces with at least one puncture (as in braid groups), it is probably most convenient to use only this one.

### 5.2. **Triangle + cusped annulus.**




Measure parameters:

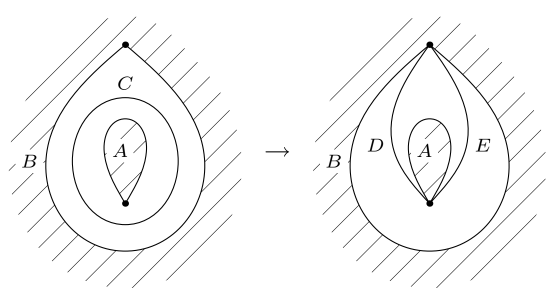
$$\begin{aligned} \mathrm{mE} &= (\mathrm{mA} + \mathrm{mB} + \mathrm{mC} - \mathrm{mD}) \vee (2\mathrm{mA} - \mathrm{mD}) \vee (2\mathrm{mB} - \mathrm{mD}) \\ \Delta\mathrm{tC} &= (-\mathrm{mC}) \vee (\mathrm{mA} - \mathrm{mB}) \wedge \mathrm{mC} \end{aligned}$$

Twist parameters:

$$\begin{aligned} \mathrm{tE} &= -\mathrm{tD} \\ \Delta\mathrm{tA} &= \mathrm{tD} + (\mathrm{mC} \vee |\mathrm{tD}|) \\ \Delta\mathrm{tB} &= \mathrm{tD} - (\mathrm{mC} \vee |\mathrm{tD}|) \\ \Delta\mathrm{tC} &= (-\mathrm{mC}) \vee \mathrm{tD} \wedge \mathrm{mC} \end{aligned}$$

*Proof.* Consider the curve system $\mathrm{D} \cup \mathrm{E}$. There are two intersections, so by Lemma 4, $\mathrm{m}(\mathrm{D} \cup \mathrm{E})$ is the maximum of the measures of the four ways of resolving the crossings. One of these resolutions yields a trivial curve, and the other three yield curves homotopic to $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$, 2A, and 2B. The formula for mE follows. How do you compute $\Delta\mathrm{tC}$? ☐

### 5.3. **Cusped annulus + cusped annulus.**



Measure parameters:

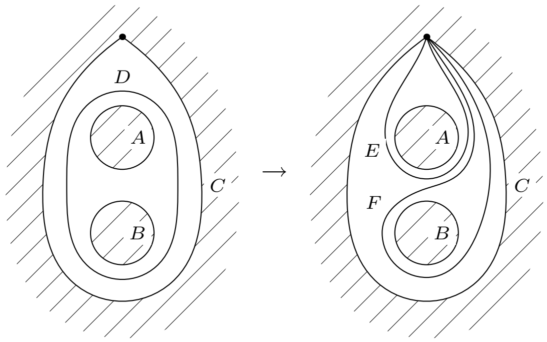
$$\begin{aligned}
\mathrm{mD} &= -\mathrm{mC} + \frac{\mathrm{mA}+\mathrm{mB}}{2} + \frac{|\mathrm{mC}+\mathrm{tC}|}{2} \\
\mathrm{mE} &= -\mathrm{mC} + \frac{\mathrm{mA}+\mathrm{mB}}{2} + \frac{|\mathrm{mC}-\mathrm{tC}|}{2} \\
\mathrm{mC} &= (\mathrm{mD}-\mathrm{mE}) \vee^{\dagger} (\mathrm{mA}+\mathrm{mB}-\mathrm{mD}-\mathrm{mE}) \vee^{\dagger} (-\mathrm{mD}+\mathrm{mE}) \\
\mathrm{tC} &= \big(2\mathrm{mD}-\mathrm{mA}-\mathrm{mB} \,|^{\dagger}\, 0 \,|^{\dagger}\, -2\mathrm{mE}+\mathrm{mA}+\mathrm{mB}\big) + \mathrm{mD}-\mathrm{mE}
\end{aligned}$$

Twist parameters:

$$\begin{aligned}
\mathrm{tD} &= 2\mathrm{mC} - |\mathrm{mC}-\mathrm{tC}| \\
\mathrm{tE} &= -2\mathrm{mC} + |\mathrm{mC}+\mathrm{tC}| \\
\mathrm{mC} &= \frac{\mathrm{tD}+\mathrm{tE}}{2} \vee^{\dagger} \frac{\mathrm{tD}-\mathrm{tE}}{2} \vee^{\dagger} \frac{-\mathrm{tD}-\mathrm{tE}}{2} \\
\mathrm{tC} &= (\mathrm{tD} \,|^{\dagger}\, 0 \,|^{\dagger}\, \mathrm{tE}) + \frac{\mathrm{tD}+\mathrm{tE}}{2} \\
&= \mathrm{tD}+\mathrm{tE} - \mathrm{sign}(\mathrm{tD}+\mathrm{tE})\left(\frac{|\mathrm{tD}+\mathrm{tE}|}{2} \wedge \frac{|\mathrm{tD}-\mathrm{tE}|}{2}\right) \\
\Delta\mathrm{tA} &= -\Delta\mathrm{tB} = \mathrm{tE}-\mathrm{tD}
\end{aligned}$$

*Proof.* The arc D can be put in a standard form, in which it runs from the inner vertex to the dividing circle C (with minimal intersections), runs nearly parallel to C as necessary, and then runs to the outer vertex. Running from the inner vertex to C crosses $\frac{\mathrm{mA}-\mathrm{mC}}{2}$ strands; running along C crosses $\frac{|\mathrm{mC}+\mathrm{tC}|}{2}$ strands, by the choice of the twist parameter; and running to the outer vertex crosses $\frac{\mathrm{mB}-\mathrm{mC}}{2}$ strands, for the total above. Similarly for mE. mC and tC may be deduced by inverting the equations for mD and mF. More directly, if E jogs right as it crosses C, we may smooth out the crossing in $\mathrm{E} \cup \mathrm{C}$ to obtain D, so $\mathrm{mE}+\mathrm{mC} = \mathrm{mD}$ in this case; if D jogs left as it crosses C, we deduce $\mathrm{mD}+\mathrm{mC} = \mathrm{mE}$; and if D jogs right and E jogs left, then smoothing both crossings in $\mathrm{C} \cup \mathrm{D} \cup \mathrm{E}$ yields $\mathrm{A} \cup \mathrm{B}$. These are the three cases in the expression for mC. These three cases also determine which of the cases in the formulas for mD and mE we are in, and so allow us to determine tC by linear algebra. □

5.4. **Cusped annulus + pair of pants.**




Measure parameters:

$$\begin{aligned}
\mathrm{mE} &= \mathrm{mC} - \mathrm{mD} + \big(\mathrm{mA} \vee (|\mathrm{tD}| + g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD}))\big)\\
\mathrm{mF} &= \mathrm{mC} - \mathrm{mD} + \big(\mathrm{mB} \vee (|\mathrm{tD} - \mathrm{mD}| + g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD}))\big)\\
\mathrm{mD} &= \big((\mathrm{mF} - \mathrm{mE}) \vee (\mathrm{mA} + \mathrm{mC} - \mathrm{mE}) \vee (2\mathrm{mC} - \mathrm{mE} - \mathrm{mF})\big) \vee^{\dagger} \big((\mathrm{mB} + \mathrm{mC} - \mathrm{mF}) \vee (\mathrm{mE} - \mathrm{mF})\big)\\
\mathrm{tD} &= \big(\mathrm{mC} - \mathrm{mF} + g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD})\big) \,|^{\dagger}\, \big(\mathrm{mD} - \mathrm{mC} + \mathrm{mE} - g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD})\big)\\
\Delta\mathrm{tA} &= -l(\mathrm{mA}, \mathrm{mD}; \mathrm{mB}) \vee \mathrm{tD} \wedge l(\mathrm{mA}, \mathrm{mD}; \mathrm{mB})\\
\Delta\mathrm{tB} &= -l(\mathrm{mB}, \mathrm{mD}; \mathrm{mA}) \vee (\mathrm{tD} - \mathrm{mD}) \wedge l(\mathrm{mB}, \mathrm{mD}; \mathrm{mA})
\end{aligned}$$

Twist parameters:

$$\begin{aligned}
\mathrm{tE} &= \mathrm{mD} - \big(\mathrm{mA} \vee (|\mathrm{tD}| + g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD}))\big)\\
\mathrm{tF} &= -\mathrm{mD} + \big(\mathrm{mA} \vee (|\mathrm{tD}| + g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD}))\big)\\
\mathrm{mD} &= \big((-\mathrm{tE} - \mathrm{tF}) \vee (\mathrm{mA} - \mathrm{tF}) \vee (\mathrm{tE} - \mathrm{tF})\big) \vee^{\dagger} \big((\mathrm{mB} + \mathrm{tE}) \vee (\mathrm{tF} + \mathrm{tE})\big)\\
\mathrm{tD} &= \big(\mathrm{tE} + g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD})\big) \,|^{\dagger}\, \big(\mathrm{mD} + \mathrm{tF} - g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD})\big)\\
\Delta\mathrm{tA} &= -l(\mathrm{mA}, \mathrm{mD}; \mathrm{mB}) \vee \mathrm{tD} \wedge l(\mathrm{mA}, \mathrm{mD}; \mathrm{mB})\\
\Delta\mathrm{tB} &= -l(\mathrm{mB}, \mathrm{mD}; \mathrm{mA}) \vee (\mathrm{tD} - \mathrm{mD}) \wedge l(\mathrm{mB}, \mathrm{mD}; \mathrm{mA})
\end{aligned}$$

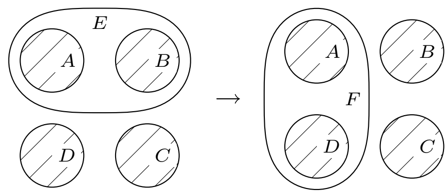
*Proof.* To compute mE, consider what happens as E crosses D. If it jogs left and then right, then smoothing out the two crossings in $\mathrm{E} \cup \mathrm{D}$ yields $\mathrm{A} \cup \mathrm{C}$, for the first case in the maximum.

Otherwise, connecting E to D (at both ends) costs $\mathrm{mC} - \mathrm{mD}$; jogging right or left along D costs $|\mathrm{tD}|$; and connecting E from D to D costs $g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD})$. mF is the same, except that we first do a negative half-Dehn twist along D to get A and B in the right position.

To find $\Delta\mathrm{tA}$, we need to locate the position around the boundary of A from which we can attach an arc directly to the vertex of C. This position must be among the $l(\mathrm{mA}, \mathrm{mD}; \mathrm{mB})$ endpoints on A which attach to arcs between A and D. ...

For the other direction, consider the four intersections in $\mathrm{E} \cup \mathrm{F} \cup \mathrm{D}$. D may jog right or left as it passes through each intersection, but with restrictions: starting from the part of D on the left of the figure and proceeding clockwise, D must jog left some number of times and then jog right. (Otherwise we create a null-homotopic arc after smoothing.) So there are five possibilities, depending on whether D jogs left 0, 1, 2, 3, or 4 times. We can compute mD by, respectively, adding E and smoothing to get F; adding E and smoothing to get $\mathrm{A} \cup \mathrm{C}$; adding E and F and smoothing to get 2 copies of C; adding F to get $\mathrm{B} \cup \mathrm{C}$, and adding F to get E. This corresponds to the 5 cases in the formula for mD. tD is most easily computed by inverting the formulas for mE and mF: depending on which choice was chosen for mD, we can distinguish which of E or F jogs twice in the same direction, and so which depends linearly on tD. Note that in the middle of the 5 cases for mD, the two choices for tD are equal, since $g(\mathrm{mA}, \mathrm{mB}; \mathrm{mD})$ is 0 in this case. ☐

### 5.5. **Pair of pants + pair of pants.**




$$
\begin{aligned}
g(x,y;z) &= 0 \vee \frac{x+y-z}{2} \vee x-z \vee y-z \\
l(x,y;z) &= 0 \vee \left(\frac{x+y-z}{2} \wedge x \wedge y\right) \\
\mathrm{mF} &= \big((\mathrm{mA}+\mathrm{mC}-\mathrm{mE}) \vee (\mathrm{mB}+\mathrm{mD}-\mathrm{mE})\big) \vee^{\dagger} \big(|\mathrm{tE}| + g(\mathrm{mA},\mathrm{mB};\mathrm{mE}) + g(\mathrm{mC},\mathrm{mD};\mathrm{mE})\big) \\
\mathrm{tF} &= -\mathrm{tE}\,|^{\dagger} - \operatorname{sign}\mathrm{tE} \times \big(\mathrm{mE} - g(\mathrm{mA},\mathrm{mD};\mathrm{mF}) - g(\mathrm{mB},\mathrm{mC};\mathrm{mF})\big) \\
l_a &= l(\mathrm{mA},\mathrm{mE};\mathrm{mB}) \\
l_b &= l(\mathrm{mB},\mathrm{mE};\mathrm{mA}) \\
l_c &= l(\mathrm{mC},\mathrm{mE};\mathrm{mD}) \\
l_d &= l(\mathrm{mD},\mathrm{mE};\mathrm{mC}) \\
\Delta\mathrm{tA} &= \operatorname{sign}\mathrm{tE} \times l(l_a, |\mathrm{tE}|; l_d) \\
&= \operatorname{sign}\mathrm{tE} \times \big(0 \vee (|\mathrm{tE}| \wedge |\mathrm{tF}| \wedge \frac{\mathrm{mE}+\mathrm{mA}-\mathrm{mB}}{2} \wedge \frac{\mathrm{mF}+\mathrm{mA}-\mathrm{mD}}{2} \wedge \frac{\mathrm{mE}+\mathrm{mF}-\mathrm{mB}-\mathrm{mD}}{2}))\big) \\
\Delta\mathrm{tB} &= \operatorname{sign}\mathrm{tE} \times l(l_b, |\mathrm{tE}|; l_c) \\
\Delta\mathrm{tC} &= \operatorname{sign}\mathrm{tE} \times l(l_c, |\mathrm{tE}|; l_b) \\
\Delta\mathrm{tD} &= \operatorname{sign}\mathrm{tE} \times l(l_d, |\mathrm{tE}|; l_a)
\end{aligned}
$$

*Proof.* TBW. □

### 5.6. **Punctured torus.**

$$
\begin{aligned}
\mathrm{mC} &= \frac{|\mathrm{tB}|}{2} + \big(0 \vee (\frac{\mathrm{mA}}{2} - \mathrm{mB})\big) \\
\mathrm{tC} &= -2\operatorname{sign}\mathrm{tB}\Big(\mathrm{mB} - \big(0 \vee (\frac{\mathrm{mA}}{2} - \mathrm{mC})\big)\Big) \\
\Delta\mathrm{tA} &= -\frac{\mathrm{mA}}{2} + \operatorname{sign}\mathrm{tB}\Big(\frac{|\mathrm{tB}|}{2} \wedge \frac{\mathrm{mA}}{2} \wedge \mathrm{mB}\Big)
\end{aligned}
$$

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