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Quotient Hereditarily Indecomposable Banach Spaces

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Abstract. A Banach space X is said to be *quotient hereditarily indecomposable* if no infinite dimensional quotient of a subspace of X is decomposable. We provide an example of a quotient hereditarily indecomposable space, namely the space X_{GM} constructed by W. T. Gowers and B. Maurey in [GM]. Then we provide an example of a reflexive hereditarily indecomposable space $\hat{X}$ whose dual is not hereditarily indecomposable; so $\hat{X}$ is not quotient hereditarily indecomposable. We also show that every operator on $\hat{X}^*$ is a strictly singular perturbation of an homothetic map.

1 Introduction

1.1 General Setting

In [GM], W. T. Gowers and B. Maurey gave the first known example of a space X_{GM} that contains no unconditional basic sequence. Their space has even the stronger property of being *hereditarily indecomposable*, that is, no subspace of X_{GM} is decomposable (can be written as a direct sum of two infinite-dimensional subspaces). Afterwards, Gowers proved the following dichotomy theorem: every Banach space contains either a hereditarily indecomposable subspace or a subspace with an unconditional basis [G1], [G2]. This theorem is a motivation for finding general properties of hereditarily indecomposable spaces. Some were proved in [F1], [F2]. In this paper, we are interested in the properties of X^* when X is hereditarily indecomposable.

Is the hereditarily indecomposable property self-dual? A weaker question is the following: a fundamental property of a complex hereditarily indecomposable space X is that X has few operators in the sense that every operator on X is a strictly singular perturbation of an homothetic map (the $\lambda\,\mathrm{Id} + S$-property); when does this property pass to the dual? This last question is of interest in relation to the still open $\lambda\,\mathrm{Id} + K$-conjecture (does there exist a Banach space X such that every operator on X is of the form $\lambda\,\mathrm{Id} + K$, K compact?). Indeed if a space X gives a positive answer to this conjecture then both X and X^* must satisfy the $\lambda\,\mathrm{Id} + S$-property (this comes from the fact that the $\lambda\,\mathrm{Id} + K$-property is self-dual).

In the first part of this article we prove that X_{GM} is quotient hereditarily indecomposable (no subspace of a quotient is decomposable), so that X^*_{GM} is hereditarily indecomposable: the techniques used by Gowers and Maurey imply the property for the dual. In particular, all consequences of the hereditarily indecomposable property, as the $\lambda\,\mathrm{Id} + S$-property, pass to the dual. The crucial point in showing this is the following: Gowers and Maurey showed that any $Z \subset X_{\mathrm{GM}}$ contains arbitrarily many l_1^{n+}-vectors; we improve the result finding l_1^{n+}-vectors in an arbitrary quotient of subspace Z/W of X_{GM}—actually we find

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l_1^{n+}-vectors in Z whose classes in Z/W have a controled norm (Lemma 11). The proof that the space is quotient hereditarily indecomposable then follows more or less from the proof that X_{GM} is hereditarily indecomposable. For this reason, we will only sketch some parts of the proof. However, constants are different and we state the result in a more general setting (Proposition 20).

In the second part of the article, we use Proposition 20 to show by a counter-example that the hereditarily indecomposable property does not necessarily pass to the dual, even when the space is reflexive. However in this example, the $\lambda\,\mathrm{Id} + S$-property does pass to the dual. As a consequence, we find the first known example of a non hereditarily indecomposable space with the $\lambda\,\mathrm{Id} + S$-property. The construction of the space is rather technical (Proposition 25), however the proof of its properties above is based on general methods useful in the hereditarily indecomposable context (Proposition 23 and 24).

It should be mentioned that recently [AF], S. Argyros and V. Felouzis improved our duality result showing that the dual of a H.I. space may be far from being H.I.: such a dual may contain l_p for $1 \leq p < +\infty$ and other classical spaces. Their method is quite different from ours.

1.2 Notation

In the following, by *space (resp. subspace)*, we shall always mean infinite dimensional Banach space (resp. closed subspace). We shall write $Y \subset_\infty Z$ to mean that Y is a subspace of Z of infinite codimension. By QS*-space* of X we shall mean infinite dimensional quotient of a subspace of X, that is, of the form Z/Y, where Z, Y are subspaces of X such that $Y \subset_\infty Z$. We recall that two Banach spaces X and X' are *totally incomparable* if no subspace of X is isomorphic to a subspace of X'.

We now give some notation that is useful for the construction of Gowers-Maurey's space and similar spaces. Let c_{00} be the space of sequences of scalars all but finitely many of which are zero. Let $e_1, e_2, \ldots$ be its unit vector basis. If $E \subset \mathbb{N}$, then we shall also use the letter E for the projection from c_{00} to c_{00} defined by $E(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i \in E} a_i e_i$. If $E, F \subset \mathbb{N}$, then we write $E < F$ to mean that $\sup E < \inf F$. An *interval* of integers is a subset of $\mathbb{N}$ of the form $\{a, a+1, \ldots, b\}$ for some $a, b \in \mathbb{N}$. For N in $\mathbb{N}$, E_N denotes the interval $\{1, \ldots, N\}$. The *range* of a vector x in c_{00}, written $\mathrm{ran}(x)$, is the smallest interval E such that $Ex = x$. We shall write $x < y$ to mean $\mathrm{ran}(x) < \mathrm{ran}(y)$; notice that this is only defined on c_{00}. A finite or infinite sequence of vectors (x_i) is called *successive* if $x_i < x_{i+1}$ for all i. If x_1, y_1, x_2, y_2 are in c_{00}, we shall also write $(x_1, y_1) < (x_2, y_2)$ to mean that there exist intervals $F_1 < F_2$ such that for $i = 1, 2$, $\mathrm{ran}(x_i) \cup \mathrm{ran}(y_i) \subset F_i$.

Let $\mathcal{X}$ be the class of Banach sequence spaces such that $(e_i)_{i=1}^{\infty}$ is a normalized bimonotone basis. We denote by $B(l_1)$ the unit ball of $l_1 \cap c_{00}$. By a *block basis* in a space $X \in \mathcal{X}$ we mean a sequence $x_1, x_2, \ldots$ of successive non-zero vectors in X and by a *block subspace* of a space $X \in \mathcal{X}$ we mean a subspace generated by a block basis.

Let f be the function $\log_2(x+1)$. If $X \in \mathcal{X}$, and all successive vectors $x_1, \ldots, x_n$ in X satisfy the inequality $f(n)^{-1} \sum_{i=1}^{n} \|x_i\| \leq \|\sum_{i=1}^{n} x_i\|$, then we say that X satisfies an *lower f-estimate*. We denote by $\mathcal{X}(f)$ the set of such spaces.

Given X in $\mathcal{X}$, given $g\colon [1, +\infty) \to [1, +\infty)$, a functional x^* in X^* is an *(M, g)-form* if $\|x^*\|^* \leq 1$ and $x^* = \sum_{j=1}^{M} x_j^*$ for a sequence $x_1^* < \cdots < x_M^*$ of successive functionals such that $\|x_j^*\|^* \leq g(M)^{-1}$ for each j.

Let $C > 0$. An l_1^{n+}*-vector with constant* C *in* X is a vector x of the form $\sum_{i=1}^n x_i$ such that the sequence (x_i) is successive and for all i, $\|x_i\| \leq C\|x\|/n$. An l_1^{n+}*-average in* X is an l_1^{n+}-vector of norm 1 in X.

Notation We shall often refer to lemmas of [GM] (resp. [F2]), using the notation GM and F (*i.e.* "Lemma GM7" for "Lemma 7 in [GM]",...).

1.3 Some Basic Properties of Quotient Hereditarily Indecomposable Spaces

Definition 1 A Banach space X is *quotient hereditarily indecomposable* (or *Q.H.I.*) if no infinite dimensional QS-space of X is decomposable.

Remark 1 If X is quotient hereditarily indecomposable then X is hereditarily indecomposable. Indeed a subspace of X is a QS-space of X.

Proposition 2 *Let X be a Banach space. Assume that for every infinite dimensional subspace Y such that X/Y is infinite dimensional, X/Y is hereditarily indecomposable. Then X is quotient hereditarily indecomposable.*

Proof It is enough to prove that X is H.I. (then X/Y is H.I. for any finite-dimensional Y). Assume X is not H.I. Then X contains a direct sum $W \oplus Z$. Let Y be an infinite dimensional subspace of W such that W/Y is infinite dimensional (for example the space generated by the even vectors of a basic sequence in W). Then X/Y contains a space isomorphic to the sum $W/Y \oplus Z$, so X/Y is not H.I. ∎

Proposition 3 *Let X be a Banach space. If X^* is quotient hereditarily indecomposable, then X is quotient hereditarily indecomposable.*

Proof If X is not Q.H.I., then some QS-space Y/Z of X is decomposable. Then the QS-space $Z^\perp/Y^\perp \simeq (Y/Z)^*$ of X^* is decomposable, so X^* is not Q.H.I. ∎

Corollary 4 *Let X be a reflexive Banach space. Then X is quotient hereditarily indecomposable iff X^* is quotient hereditarily indecomposable.*

2 There Exists a Quotient Hereditarily Indecomposable Space

2.1 Approximating Sequences

Definition 2 Let W be a Banach space. Let $(w_n)_{n\in\mathbb{N}}$ and $(w'_n)_{n\in\mathbb{N}}$ be two non-zero sequences in W. We say that $(w'_n)_{n\in\mathbb{N}}$ approximates $(w_n)_{n\in\mathbb{N}}$ if

$$\lim_{n\to+\infty} \|w_n - w'_n\|/\|w_n\| = 0.$$

Note that approximation is an equivalence relation.

Lemma 5 *Let W be a Banach space in $\mathcal{X}$. Let $(w_i)_{i\in\mathbb{N}}$ be a successive sequence in W and let $(w'_i)_{i\in\mathbb{N}}$ approximate $(w_i)_{i\in\mathbb{N}}$. Let $n \in \mathbb{N}$. Then for every $\epsilon > 0$ there exists N such that for all subset I of $\mathbb{N}$ such that* $\mathrm{Card}(I) = n$ *and $I > E_N$ then*

$$\Big\|\sum_{i\in I} w'_i\Big\| \leq (1+\epsilon)\Big\|\sum_{i\in I} w_i\Big\|.$$

Proof For N big enough, $\|\sum_{i\in I} w'_i\| \leq \|\sum_{i\in I} w_i\| + \sum_{i\in I} \epsilon/n\|w_i\|$, and the result follows because the basis in W is bimonotone. ■

Definition 3 Let W be a Banach space, V be a subset of W. A sequence $(w_n)_{n\in\mathbb{N}}$ in W is said to be *almost in V* if it approximates a sequence of vectors in V.

Let W be a space with a basis. A sequence $(w_n)_{n\in\mathbb{N}}$ in W is said to be *almost successive* if it approximates a sequence of successive vectors in W.

Corollary 6 *Let X be a Banach space in $\mathcal{X}(f)$. Let $(x^*_i)_{i\in\mathbb{N}}$ be an almost successive sequence in X^*. Let $n \in \mathbb{N}$. Then for every $\epsilon > 0$ there exists N such that for all subset I of $\mathbb{N}$ such that* $\mathrm{Card}(I) = n$ *and $I > E_N$ then*

$$\Big\|\sum_{i\in I} x^*_i\Big\| \leq (1+\epsilon) f(n) \sup_{i\in I} \|x^*_i\|.$$

Lemma 7 *Let W be a space in $\mathcal{X}$. Let $(w_n)_{n\in\mathbb{N}}$ be a non-zero sequence in W such that $w_n/\|w_n\| \overset{w}{\to} 0$. Then $(w_n)_{n\in\mathbb{N}}$ has an almost successive subsequence.*

Proof We may assume that $(w_n)_{n\in\mathbb{N}}$ is a norm 1 sequence. Assume we have already selected $w_{n_1}, \dots, w_{n_{k-1}}$ and a successive sequence $v_1, \dots, v_{k-1}$ such that for $i = 1, \dots, k-1$, $\|v_i - w_{n_i}\| \leq 1/i$. Let E be an interval containing e_1 and the range of $v_1 + \cdots + v_{k-1}$. There exists n_k such that $\|Ew_{n_k}\| \leq 1/2k$. Let $v'_k = w_{n_k} - Ew_{n_k}$. There exists an interval F such that Fv'_k is equal to v'_k up to $1/2k$. If we let $v_k = Fv'_k$, we have that $v_k > v_{k-1}$, and $\|v_k - w_{n_k}\| \leq 1/k$. Finally, $(w_{n_k})_{k\in\mathbb{N}}$ approximates $(v_k)_{k\in\mathbb{N}}$. ■

2.2 Norming Sequences

Definition 4 Let W be a Banach space, W^* its dual. We shall say that two unit sequences $(w_n)_{n\in\mathbb{N}}$ in W and $(w^*_n)_{n\in\mathbb{N}}$ in W^* are *λ-norming* (or that (w^*_n) *λ-norms* (w_n)) if $\liminf w^*_n(w_n) \geq 1/\lambda$ and for $n \neq q$, $|w^*_n(w_q)| \leq \epsilon_{\min(n,q)}$ with $\lim_{i\to+\infty} \epsilon_i = 0$.

Two non-zero sequences $(w_n)_{n\in\mathbb{N}}$ and $(w^*_n)_{n\in\mathbb{N}}$ are λ-norming if the unit sequences $(w_n/\|w_n\|)_{n\in\mathbb{N}}$ and $(w^*_n/\|w^*_n\|)_{n\in\mathbb{N}}$ are λ-norming.

Notice that if $(w^*_n)_{n\in\mathbb{N}}$ λ-norms $(w_n)_{n\in\mathbb{N}}$, then it also λ-norms any sequence that approximates $(w_n)_{n\in\mathbb{N}}$.

Lemma 8 *Let X be in $\mathcal{X}(f)$, $Y \subset_\infty X$. Let $(z_n)_{n\in\mathbb{N}}$ in X/Y be λ-normed by an almost successive sequence in $Y^\perp$. Then for every $\epsilon > 0$, every n, there exists N such that if $I > E_N$*

and $\mathrm{Card}(I) = n$

$$\sum_{i=1}^{n} \|z_i\| \leq (1+\epsilon)\lambda f(n) \Big\| \sum_{i=1}^{n} z_i \Big\|.$$

Proof Let $(z_n^*)_{n\in\mathbb{N}}$ be an almost successive sequence in $B(Y^\perp)$ that λ-norms $(z_n)_{n\in\mathbb{N}}$. Let ϵ' be such that $1+\epsilon' < (1+\epsilon)\big(1-(n-1)\epsilon'\lambda\big)$. By Corollary 6, there is an N such that if $\epsilon_i < \epsilon'$ for $i > N$ and $I > E_N$, then

$$\Big\| \sum_{i\in I} z_i^* \Big\| \leq (1+\epsilon') f(n).$$

It follows that

$$\Big\| \sum_{i\in I} z_i \Big\| \geq \big((1+\epsilon')f(n)\big)^{-1} (\sum_{i,j\in I} (z_i^*(z_j)),$$

$$\Big\| \sum_{i=1}^{n} z_i \Big\| \geq \big((1+\epsilon')f(n)\big)^{-1} \big(1/\lambda - (n-1)\epsilon'\big) \sum_{i\in I} \|z_i\|. \qquad \blacksquare$$

Lemma 9 *Let X be in $\mathcal{X}$, $Y \subset_\infty X$. Let $(z_n)_{n\in\mathbb{N}}$ be a non-zero sequence in X/Y such that $z_n/\|z_n\|$ tends weakly to* 0. *Then some subsequence of $(z_n)_{n\in\mathbb{N}}$ has an almost successive* 2-*norming sequence in $Y^\perp$.*

Proof We may assume that $(z_n)_{n\in\mathbb{N}}$ is a norm 1 sequence. Let $(x_n'^*)_{n\in\mathbb{N}}$ be a dual sequence in $B(Y^\perp)$ such that for all n, $x_n'^*(z_n) = 1$. Passing to a subsequence, we may assume that $x_n'^* \xrightarrow{w} x^*$ (clearly x^* is in $Y^\perp$). Let $x_n^* = 1/2(x_n'^* - x^*)$. As $x_n^* \xrightarrow{w} 0$, and $z_n^* \xrightarrow{w} 0$, passing to a subsequence, we may choose (x_n^*) and (z_n) such that for $q = 1, \ldots, n-1$, $|x_n^*(z_q)| \leq 1/q$ and $|x_q^*(z_n)| \leq 1/q$. By Lemma 7, we may also assume that $(x_n^*)_{n\in\mathbb{N}}$ is almost successive. Furthermore, we have that $x_n^* \in B(Y^\perp)$ and

$$x_n^*(z_n) = 1/2\big(1 - x^*(z_n)\big) \to 1/2. \qquad \blacksquare$$

Let X be in $\mathcal{X}$, $Y \subset_\infty X$. Given x in X, we denote by $\hat{x}$ its class in X/Y. We shall say that $(x_n)_{n\in\mathbb{N}}$ in X is a *lifting* for $(\widehat{x_n})_{n\in\mathbb{N}}$. Let $\lambda \geq 1$. We shall say that $(x_n)_{n\in\mathbb{N}}$ in X is a λ-*lifting* for $(\widehat{x_n})_{n\in\mathbb{N}}$ if $\limsup \|x_n\|/\|\widehat{x_n}\| \leq \lambda$.

Lemma 10 *Let X be in $\mathcal{X}$, $Y \subset_\infty X$. Let $(z_n)_{n\in\mathbb{N}}$ be a non-zero sequence in X/Y such that $z_n/\|z_n\|$ tends weakly to* 0. *Then some subsequence of $(z_n)_{n\in\mathbb{N}}$ has an almost successive* 2-*lifting.*

Proof We may assume that $(z_n)_{n\in\mathbb{N}}$ is a norm 1 sequence. Let $(x_n')_{n\in\mathbb{N}}$ be a lifting for $(z_n)_{n\in\mathbb{N}}$ such that $\|x_n'\| \to 1$. The sequence $(x_n')_{n\in\mathbb{N}}$ is bounded, so, passing to a subsequence, we may assume that x_n' converges weakly. Let y be the weak limit of (x_n'). The vector y has norm 1, and belongs to Y, because for every y^* in $Y^\perp$, $y^*(y) = \lim y^*(x'_n) = \lim y^*(z_n) = 0$.

Let $x_n = x'_n - y$. Then $x_n \xrightarrow{w} 0$ so passing to a further subsequence, we may assume by Lemma 7 that $(x_n)_{n\in\mathbb{N}}$ is almost successive; clearly $\widehat{x_n} = z_n$, and $\limsup \|x_n\|/\|z_n\| = \limsup \|x_n\| \leq 2$. ■

2.3 Norming of l_1^{n+} Vectors

Lemma 11 *Let X be reflexive in $\mathcal{X}(f)$, Y, Z be subspaces of X such that $Y \subset_\infty Z$. Let $N \in \mathbb{N}$. Let $\epsilon > 0$. Then there is a successive sequence of l_1^{N+}-averages with constant $2+\epsilon$ almost in Z that is $4+\epsilon$-normed by a successive sequence almost in $Y^\perp$.*

Proof Let $\epsilon' > 0$ be such that $2(1+\epsilon')^4 \leq 2+\epsilon$. Let C be such that $(1+\epsilon')^C > (2+\epsilon')f(N^C)$. As Z/Y is reflexive, there exists a basic sequence $(z_n)_{n\in\mathbb{N}}$ of unit vectors in Z/Y such that $z_n \xrightarrow{w} 0$. By Lemma 7 and Lemma 9, we may assume that $(z_n)_{n\in\mathbb{N}}$ is 2-normed by some almost successive sequence in $Y^\perp$. We shall denote $(z_n)_{n\in\mathbb{N}}$ by $\big(z_n(0)\big)_{n\in\mathbb{N}}$.

Now consider the sequence

$$\big(z_n(1)\big)_{n\in\mathbb{N}} = \Big(\sum_{i=0}^{N-1} z_{Nn+i}\Big)_{n\in\mathbb{N}}$$

obtained by making packs of N z_i's. The sequence $\big(z_n(1)\big)_{n\in\mathbb{N}}$ converges weakly to 0 and by Lemma 8, it is bounded below for n large enough, so by Lemma 7 and Lemma 9, passing to a subsequence, we may assume that it is 2-normed by some almost successive sequence in $Y^\perp$.

We now repeat the procedure above for $j = 2, \dots, C$ defining

$$\big(z_n(j)\big)_{n\in\mathbb{N}} = \Big(\sum_{i=0}^{N-1} z_{Nn+i}(j-1)\Big)_{n\in\mathbb{N}}.$$

Passing to a subsequence at each step, we may assume that for every $j \in [0, C]$, $\big(z_n(j)\big)_{n\in\mathbb{N}}$ is 2-normed by some almost successive sequence in $Y^\perp$.

We now prove that there exists a sequence $(u_i)_{i\in\mathbb{N}}$ in Z/Y such that $(U_n)_{n\in\mathbb{N}} = (\sum_{i=0}^{N-1} u_{Nn+i})_{n\in\mathbb{N}}$ is 2-normed by an almost successive sequence $(U_n^*)_{n\in\mathbb{N}}$ in $Y^\perp$, and $\sup_{0\leq i\leq n-1} \|u_{Nn+i}\| \leq (1+\epsilon')/N\|U_n\|$ for all n.

Indeed, otherwise, for n large enough, and $j \in [0, C]$, we have the inequality $\|z_n(j)\| \leq N/(1+\epsilon') \sup_{0\leq i\leq n-1} \|z_{Nn+i}(j-1)\|$; it follows by induction that

$$\|z_n(j)\| \leq \big(N/(1+\epsilon')\big)^j,$$

so that

$$\|z_n(C)\| \leq \big(N/(1+\epsilon')\big)^C.$$

But on the other hand,

$$\|z_n(C)\| = \|z_{N^C n} + \cdots + Z_{N^C n+N^C-1}\| \geq N^C/(2+\epsilon')f(N^C),$$

by Lemma 8, a contradiction by choice of N.

We now deduce the existence of successive l_1^{N+}-vectors almost in Z, well-normed in $Y^\perp$. Applying Lemma 10, passing to a subsequence at each step of the previous induction, we may assume that the sequence $(u_i)_{i\in\mathbb{N}}$ we obtained has an almost successive 2-lifting $(x_i')_{i\in\mathbb{N}}$ in Z. Let $(x_i)_{i\in\mathbb{N}}$ be a successive sequence approximating $(x_i')_{i\in\mathbb{N}}$. Then $(x_i)_{i\in\mathbb{N}}$ is almost in Z. Let $X_n = \sum_{i=0}^{N-1} x_{Nn+i}$ and let $X_n' = \sum_{i=0}^{N-1} x_{Nn+i}'$. Clearly $(X_n')_{n\in\mathbb{N}}$ is a lifting for $(U_n)_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}}$ approximates $(X_n')_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}}$ is successive, and $(X_n')_{n\in\mathbb{N}}$ in Z.

All the following estimates are for n large enough. For such n's, and i in $[0, N-1]$,

$$\|x_{Nn+i}'\| \leq 2(1+\epsilon')\|u_{Nn+i}\| \leq 2(1+\epsilon')^2/N\|U_n\| \leq 2(1+\epsilon')^2/N\|X_n'\|.$$

It follows that

$$\|x_{Nn+i}\| \leq 2(1+\epsilon')^4/N\|X_n\| \leq (2+\epsilon)/N\|X_n\|,$$

and so X_n is a l_1^{N+}-vector with constant $2+\epsilon$. Now

$$\|X_n'\| \leq \sum_{i=0}^{N-1} \|x_{Nn+i}'\| \leq 2(1+\epsilon') \sum_{i=0}^{N-1} \|u_{Nn+i}\| \leq 2(1+\epsilon')^2\|U_n\|,$$

so

$$\|U_n^*\|\,\|X_n'\| \leq 4(1+\epsilon')^3 U_n^*(U_n) \leq 4(1+\epsilon')^3 U_n^*(X_n'),$$

and so $(X_n')_{n\in\mathbb{N}}$ is $4+\epsilon$-normed by some almost successive sequence in $Y^\perp$. It follows that it is also $4+\epsilon$-normed by some successive sequence almost in $Y^\perp$, and that $(X_n)_{n\in\mathbb{N}}$ shares the same property. ■

2.4 Rapidly Increasing Sequences

Following Gowers and Maurey, we now define R.I.S.-vectors in a Banach space X in $\mathcal{X}(f)$. In fact, the properties of R.I.S. are not interesting in all spaces in $\mathcal{X}(f)$, but they are in spaces that have, in a sense, Gowers-Maurey's type; we give a meaning to this expression in Definition 6, and then state several lemmas true in those spaces.

Let J be a set of integers $\{j_n, n \in \mathbb{N}\}$, such that $f(j_1) > 256$ and such that for all n, $\log\log\log j_{n+1} \geq 4j_n^2$. Let $K = \{j_1, j_3, j_5, \ldots\}$ and let $L = \{j_2, j_4, j_6, \ldots\}$.

Definition 5 An *L-sequence* is a successive sequence $x_1^* < \cdots < x_k^*$ with $k \in K$, such that for all i, x_i^* is a (M_i, f)-form where M_i is an element in L greater than j_{2k}. An *L-sum* is a vector of the form $1/\sqrt{f(k)}\sum_{i=1}^k x_i^*$, where $x_1^*, \ldots, x_k^*$ is an L-sequence.

In the same way, one can define *L'-sequences* and *L'-sums* for any subset L' of L.

Definition 6 A space X in $\mathcal{X}$ has *pre GM-type* if there is a set $\mathcal{S}$ of L-sums such that X is the completion of c_{00} under a norm $\|\cdot\|$ satisfying the following equation for all $x \in c_{00}$:

$$\|x\| = \|x\|_\infty \vee \sup_{n\geq 2, F_1<\cdots<F_n} \frac{1}{f(n)} \sum_{j=1}^n \|F_j x\| \vee \sup_{x^*\in\mathcal{S}, E\subset\mathbb{N}} |\langle x^*, Ex\rangle|,$$

where E and the F_j's are intervals of integers. Notice that a space of pre GM-type belongs to $\mathcal{X}(f)$.

Definition 7 We recall that a *R.I.S. of length N with constant C in X* is a successive sequence $(x_i)_{i=1}^N$ of $l_1^{n_i+}$-averages with constant C in X such that $n_1 \geq 4(1+\epsilon)M_f(N/\epsilon')/\epsilon'$ and $\epsilon'/2f(n_i)^{1/2} \geq |\operatorname{ran}(x_{i-1})|$ for $i = 2, \ldots, N$, where $\epsilon' = \min\{\epsilon, 1\}$ and $M_f(x) = f^{-1}(36x^2)$. A R.I.S.-vector is a non-zero multiple of the sum of a R.I.S.

We now show some lemmas very similar to those of [GM]; we have to state them because we shall use different constants, and because they can be applied to any pre GM-type space, which will be useful in the last part of the article. From now on we set $\epsilon_0 = 1/40$.

Lemma 12 *Let X have pre GM-type. Let $\epsilon > 0$, let $\epsilon' = \min\{\epsilon, 1\}$. Let N be in L, let n be in $[\log N, \exp N]$, let $(x_i)_{i=1}^n$ be a R.I.S. of length n with constant $1+\epsilon$ in X. Then*

$$\Big\|\sum_{i=1}^n x_i\Big\| \leq (1+\epsilon+\epsilon')nf(n)^{-1}.$$

Proof Apply Lemma GM7 and Lemma GM9. ■

Lemma 13 *Let X have pre GM-type. Let $N \in L$. Let $M = N^{\epsilon_0}$. Let $x_1, \ldots, x_N$ be a R.I.S. in X with constant $2+\epsilon_0$. Then $\sum_{i=1}^N x_i$ is an l_{1+}^M-vector with constant 4.*

Proof Follow the proof of Lemma GM11 using Lemma 12 instead of Lemma GM10. ■

Lemma 14 *Every pre GM-type space is reflexive.*

Proof Follow the proof that Gowers-Maurey's space is reflexive (end of Part GM3), using Lemma 12 instead of Lemma GM10. ■

Definition 8 Let X have pre GM-type. Let $x_1^*, \ldots, x_k^*$ be an L-sequence of length k, and for $i = 1, \ldots, k$, let M_i be the element of L greater than j_{2k} such that x_i^* is an (M_i, f)-form. A sequence of successive vectors $x_1 < \cdots < x_k$ in X is said to be *a R.I.S. associated to* $x_1^*, \ldots, x_k^*$ if for every i, x_i is a normalized R.I.S. of length M_i and constant $2+\epsilon_0$, and for $i \geq 2$, $1/2f\big((M_i)^{1/40}\big)^{1/2} \geq |\operatorname{ran}(x_{i-1})|$.

Because of the choice of the increasing condition in Definition 8 and by Lemma 13, a R.I.S. associated to an L-sequence of length k is a R.I.S. with constant 4.

Lemma 15 *Let X have pre GM-type. Let x be a norm 1 R.I.S.-vector in X of length $N_1 \in L$ and constant $2+\epsilon_0$ and let x^* be an (N_2, f)-form in X^* with $N_2 \in L$, and assume $N_1 \neq N_2$. Let $k \in K$ be such that $N_1 \geq j_{2k}$, $N_2 \geq j_{2k}$. Then for every interval E, $|x^*(Ex)| \leq 1/k^2$.*

Proof First, by Lemma 13, x is a $l_1^{N_1'+}$-average with constant 4, where $N_1' = N_1^{1/40}$. Just as in the middle of Lemma GM12, we then apply Lemma GM4 if $N_2 < N_1$ and Lemma GM5 if $N_2 > N_1$ to obtain the result. ■

Lemma 16 *Let X have pre GM-type. Let $k \in K$. Let $x_1 < \cdots < x_k$ in X be a R.I.S. associated to some L-sequence. Let $x = \sum_{i=1}^k x_i$. Assume that for every L-sum z^* in $\mathcal{S}$, every interval E,*

$|z^*(Ex)| \leq 1/4$. *Then*

$$\|x\| \leq 5k/f(k).$$

Proof As in the end of Lemma GM12, apply Lemma GM9 to $K_0 = K \setminus \{k\}$ and Lemma GM7. ∎

Lemma 17 *Let X have pre* GM*-type. Let L' and L'' be subsets of L such that $L' \cap L'' = \varnothing$. Let $x_1^*, \dots, x_k^*$ be an L'-sequence in X^*. Let $x_1 < \cdots < x_k$ in X be a R.I.S. associated to $x_1^*, \dots, x_k^*$. Let $x = x_1 + \cdots + x_k$. Then for every L''-sum z^* of length k in X^*, every interval E, $|z^*(Ex)| \leq 1/4$.*

Proof Let z^* be an L''-sum, E be an interval. Then there are (l_i, f)-forms z_i^*, with l_i in L'', such that $z^* = 1/\sqrt{f(k)} \sum_{i=1}^k z_i^*$. For every j, x_j has length in L', and $L' \cap L'' = \varnothing$ are disjoint, so it follows from Lemma 15 that $|z_i^*(Ex_j)| \leq 1/k^2$. Finally,

$$|z^*(Ex)| \leq 1/\sqrt{f(k)} \sum_{i,j=1}^{k} |z_i^*(Ex_j)| \leq 1/\sqrt{f(k)} \leq 1/4. \qquad \blacksquare$$

Definition 9 A pre GM-type space has GM-*type* if there are subsets L' and L'' of L with L' infinite and $L' \cap L'' = \varnothing$, and an injection σ from the collection of finite sequences of vectors in $\mathbf{Q}$ into L' such that the set $\mathcal{S}$ in the definition of the pre GM-type space is of the form $\mathcal{S}' \cup \mathcal{S}''$ where $\mathcal{S}''$ is some set of L''-sums and $\mathcal{S}'$ is the set of L'-sums of the form $1/\sqrt{f(k)} \sum_{i=1}^k x_i^*$, where the L'-sequence $x_1^* < \cdots < x_k^*$ satisfies the additional condition that $M_i = \sigma(x_1^*, \dots, x_{i-1}^*)$ for $i = 2, \dots, k$.

Here we added a set $\mathcal{S}''$ in the definition of Gowers-Maurey's space. As in [GM], the elements of $\mathcal{S}'$ are called *special functionals*. The condition $L' \cap L'' = \varnothing$ makes sure that the action of elements of $\mathcal{S}''$ is small on the R.I.S. used in Gowers-Maurey's construction (see Lemma 17), so in a GM-type space, one can more or less repeat Gowers-Maurey's proofs. In Part 3, we will carefully choose $\mathcal{S}''$ to get additional properties. Of course, we have in particular:

Remark 18 Gowers-Maurey's space has GM-type (with $L = L'$, $\mathcal{S}'' = \varnothing$, and $\mathcal{S}'$ the set of special sums).

2.5 GM-Type Spaces are Quotient Hereditarily Indecomposable

We first show a lemma similar to Lemma GM12.

Lemma 19 *Let X have* GM*-type. Let $x_1^*, \dots, x_k^*$ be a special sequence in X. Let $x_1 < \cdots < x_k$ be a R.I.S associated to $x_1^*, \dots, x_k^*$. Let $x = \sum_{i=1}^k x_i$. Assume that for every interval E, $|(\sum_{i=1}^k x_i^*)(Ex)| \leq 2$, then*

$$\|x\| \leq 5k/f(k).$$

Proof By Lemma 16, it is enough to prove that for any function z^* in $\mathcal{S}$, every interval E, $|z^*(Ex)| \leq 1/4$, and by Lemma 17, it is enough to prove it for $z^* = f(k)^{-1/2} \sum_{i=1}^{k} z_i^*$ in $\mathcal{S}'$. Following the proof of Lemma GM12, using Lemma 15, we obtain that $|(\sum_{i=1}^{k} z_i^*)(Ex)| \leq 4$, and that $|z^*(Ex)| \leq 4f(k)^{-1/2} < 1/4$. ■

Proposition 20 *Every* GM*-type space is reflexive, quotient hereditarily indecomposable.*

Proof The reflexive part is Lemma 14. Let X have GM-type, let $Y \subset_\infty X$. Let Z/Y and W/Y be two subspaces of X/Y. We want to prove that their sum is not direct. Let $\delta > 0$, let $k \in K$ be such that $150/\sqrt{f(k)} \leq \delta$ and $\epsilon > 0$ be such that $182k\epsilon \leq 1$.

First we show that given $\eta > 0$ and $M \in L$, there is a R.I.S. z of length M and constant $2 + \epsilon_0$ such that $\mathrm{dist}(z, Z) < \eta$, and an (M, f)-form z^* such that $\mathrm{dist}(z^*, Y^\perp) < \eta$ with $z^*(z) \geq 1/\big((4 + \epsilon_0)(3 + \epsilon_0)\big)$.

Indeed, adding $l_1^{n_i+}$-averages given by Lemma 11, we may obtain a successive sequence of R.I.S. vectors almost in Z, of length M and constant $2 + \epsilon_0$. Write $z = \sum_{i=1}^{M} z_i$ a R.I.S. vector in this sequence. Then by Lemma 12, $\|z\| \leq (3+\epsilon_0)M/f(M)$. Let y_i^* be a successive norm 1 sequence close to $Y^\perp$ satisfying $y_i^*(z_i) \geq 1/(4+\epsilon_0)$ and let $y^* = f(M)^{-1} \sum_{i=1}^{M} y_i^*$; then y^* is an (M, f)-form arbitrarily close to $Y^\perp$ when $\min\big(\mathrm{ran}(z)\big)$ increases and

$$y^*(z) = f(M)^{-1} \sum_{i=1}^{M} y_i^*(z_i) \geq M/\big((4 + \epsilon_0)f(M)\big) \geq \|z\|/\big((3 + \epsilon_0)(4 + \epsilon_0)\big).$$

Then starting from $M_1 = j_{2k}$, and repeating by induction as in Gowers-Maurey's construction, build for $i = 1, \ldots, k$, vectors z_i such that z_i is in Z up to ϵ if i is odd, in W up to ϵ if i is even, and (M_i, f)-forms z_i^* in $Y^\perp$ up to ϵ, such that $1/2f\big((M_i)^{1/40}\big)^{1/2} \geq |\,\mathrm{ran}(z_{i-1})|$, $|z_i^*(z_i) - 1/13| \leq \epsilon$, $(z_i, z_i^*) > (z_{i-1}, z_{i-1}^*)$, and $M_i = \sigma(z_1^*, \ldots, z_{i-1}^*)$ for $i \geq 2$; $z_1^*, \ldots, z_k^*$ is a special sequence, and $z_1, \ldots, z_k$ is a R.I.S. associated to $z_1^*, \ldots, z_k^*$. Let $y_1^*, \ldots, y_k^*$ be an ϵ-perturbation of $z_1^*, \ldots, z_k^*$ in $Y^\perp$.

It follows that that $\|\sum_{i=1}^{k} y_i^*\| \leq \sqrt{f(k)} + k\epsilon \leq 2\sqrt{f(k)}$, so

$$\Big\|\sum_{i=1}^{k} \hat{z}_i\Big\| \geq (1/2)f(k)^{-1/2}\Big(\sum_{i=1}^{k} y_i^*\Big)\Big(\sum_{i=1}^{k} z_i\Big),$$

$$\Big\|\sum_{i=1}^{k} \widehat{z_i}\Big\| \geq (1/2)f(k)^{-1/2}\Big(\sum_{i=1}^{k} z_i^*(z_i) - k\epsilon\Big\|\sum_{i=1}^{k} z_i\Big\|\Big),$$

$$\Big\|\sum_{i=1}^{k} \widehat{z_i}\Big\| \geq (1/2)f(k)^{-1/2}\big(k(1/13 - \epsilon) - k^2\epsilon\big) \geq (1/30)kf(k)^{-1/2}.$$

On the other hand, we have $|(\sum_{i=1}^{k} z_i^*)E\big(\sum_{i=1}^{k}(-1)^i z_i\big)| \leq 2$ for all interval E, so by Lemma 19,

$$\Big\|\sum_{i=1}^{k}(-1)^i\widehat{z_i}\Big\| \leq \Big\|\sum_{i=1}^{k}(-1)^i z_i\Big\| \leq 5kf(k)^{-1}.$$

If z denotes the sum of the odd vectors, w the sum of the even vectors, we have that $\hat{z} \in Z/Y$, $\hat{w} \in W/Y$, and

$$\|\hat{z} - \hat{w}\| \leq 150 f(k)^{-1/2} \|\hat{z} + \hat{w}\| \leq \delta \|\hat{z} + \hat{w}\|.$$

As δ is arbitrary, it follows that the sum of Z/Y and W/Y is not direct, and finally, that X/Y is H.I., so by Proposition 2, X is Q.H.I. ■

Corollary 21 *By Remark 18,* X_{GM} *is quotient hereditarily indecomposable.*

Corollary 22 *By Remark 1 and Corollary 4, if* X *has* GM*-type then* X^* *is hereditarily indecomposable. In particular,* X^*_{GM} *is hereditarily indecomposable.*

3 There Exists a Hereditarily Indecomposable Space Which is Not Quotient Hereditarily Indecomposable

In this section, we build a H.I. space $\hat{X}$ which is not Q.H.I. as a quotient of a direct sum of two GM-type spaces X_1 and X_2. The space $\hat{X}$ is reflexive, and we show that the space $\hat{X}^*$ contains a direct sum of two subspaces, which means that $\hat{X}^*$ is not H.I., and implies that $\hat{X}$ is not Q.H.I. (Corollary 4). The result stated clearly follows from Propositions 23 and 25 below.

Proposition 23 *For* $i = 1, 2$*, let* X_i *be a hereditarily indecomposable Banach space, let* Z_i *be a subspace of* X_i*. Assume that* Z_1 *and* Z_2 *are isometric, and that* X_1/Z_1 *and* X_2/Z_2 *are infinite dimensional and totally incomparable. By abuse of notation, we identify both* Z_1 *and* Z_2 *with a same space* Z*. Let* $\hat{X}$ *be the quotient space* $(X_1 \oplus X_2)/\{(z, -z), z \in Z\}$*. Then* $\hat{X}$ *is hereditarily indecomposable and* $\hat{X}^*$ *is not hereditarily indecomposable.*

In fact, it is possible to prove that in the complex case, every operator on $\hat{X}^*$ is a strictly singular perturbation of an homothetic map, which proves that this property does not characterize H.I. spaces. This result clearly follows from Proposition 24 and Proposition 25 below.

Proposition 24 *For* $i = 1, 2$*, let* X_i, Z_i *and* $\hat{X}$ *be complex spaces as in Proposition 23. Assume furthermore that* X_1^* *and* X_2^* *are totally incomparable hereditarily indecomposable Banach space. Then every operator on* $\hat{X}^*$ *is a strictly singular perturbation of an homothetic map.*

Proposition 25 *For* $i = 1, 2$*, there exist* X_i *complex reflexive quotient hereditarily indecomposable Banach space,* Z_i *subspace of* X_i*, such that* Z_1 *and* Z_2 *are isometric,* X_1/Z_1 *and* X_2/Z_2 *are totally incomparable, and* X_1^* *and* X_2^* *are totally incomparable.*

By a simple generalization explained in the Appendix, it is even possible to build for any n a H.I. space $\hat{X}$ such that $\hat{X}^*$ contains a direct sum of n subspaces, and every operator on $\hat{X}^*$ is a strictly singular perturbation of an homothetic map.

3.1 Proof of Proposition 23

Some Definitions Let X be a Banach space. Let Y be a subspace of X. We shall denote by i_Y (resp. I_X) the identity map from Y (resp. X) to X. Following [GM], we will say that an operator from Y to X is *infinitely singular* if its restriction to a finite codimensional subspace is never an isomorphism into. An operator S from Y to X is said to be *strictly singular* if the restriction of S to a subspace is never an isomorphism into (see [LT, 75–80]). This is equivalent to saying that for any $\epsilon > 0$, any Z, there exists z in Z such that $\|S(z)\| \leq \epsilon\|z\|$. We denote by $\mathcal{S}(Y, X)$ the space of strictly singular operators from Y to X.

Two subspaces Y and Z of X are said to be Id $+S$*-isomorphic* if there exists an isomorphism of the form $I_Y + S$ from Y onto Z, with $S \in \mathcal{S}(Y, X)$. It is proved easily that this is an equivalence relation.

The subspace Y is said to be *quasi-maximal* if Y and any subspace W of X have Id $+S$-isomorphic subspaces. By Corollary F1, X is hereditarily indecomposable if and only if every subspace of X is quasi-maximal; it follows easily that if X has a quasi-maximal hereditarily indecomposable subspace then X is hereditarily indecomposable. By Lemma F2, if the restriction of $S \in \mathcal{L}(X)$ to some quasi-maximal subspace of X is strictly singular, then S is strictly singular.

Proof For x_i in X_i, $i = 1, 2$, we denote by $\widehat{x_i}$ the class of x_i in X_i/Z_i, by $\widehat{(x_1, x_2)}$ the class of (x_1, x_2) in $\hat{X}$. By definition,

$$\|\widehat{(x_1, x_2)}\| = \inf_{z \in Z}(\|x_1 + z\| + \|x_2 - z\|).$$

It follows that the space $\widehat{X_1} = \{\widehat{(x_1, 0)}, x_1 \in X_1\}$ is isometric to X_1, the space $\widehat{X_2} = \{\widehat{(0, x_2)}, x_2 \in X_2\}$ is isometric to X_2, and the space $\hat{Z} = \{\widehat{(z, 0)}, z \in Z\} = \{\widehat{(0, z)}, z \in Z\}$ is isometric to Z. As an easy consequence, we have the relation

$$\hat{X}/\hat{Z} = \widehat{X_1}/\hat{Z} \oplus \widehat{X_2}/\hat{Z} \simeq X_1/Z_1 \oplus X_2/Z_2,$$

so

$$\hat{Z}^{\perp} = \widehat{X_2}^{\perp} \oplus \widehat{X_1}^{\perp} \simeq (X_1/Z_1)^* \oplus (X_2/Z_2)^*,$$

and this proves that $\hat{X}^*$ is not H.I. Now for $i = 1, 2$, we define a linear operator $\phi_i \colon \hat{X} \to X_i/Z_i$ by $\phi_i\big(\widehat{(x_1, x_2)}\big) = \widehat{x_i}$. It is easy to check that ϕ_i is well defined. Now let W be a subspace of $\hat{X}$. There exists an i such that $\phi_{i/W}$ is infinitely singular: indeed, if $\phi_{1/W}$ and $\phi_{2/W}$ are both not infinitely singular, then there exists a subspace V of W on which ϕ_1 and ϕ_2 are isomorphisms into, so that X_1/Z_1 and X_2/Z_2 have isomorphic subspaces, a contradiction.

Now assume for example that $\phi_{1/W}$ is infinitely singular. Then there exists a norm 1 basic sequence $(w_n)_{n\in\mathbb{N}}$ in W such that $\phi_1(w_n) \xrightarrow{+\infty} 0$. By definition of ϕ_1, this means that $d(w_n, \widehat{X_2}) \xrightarrow{+\infty} 0$. It follows easily that W and $\widehat{X_2}$ have Id $+S$-isomorphic subspaces. As $\widehat{X_2}$

is isometric to X_2, it is H.I.; it follows that $\hat{Z}$ is quasi-maximal in $\widehat{X_2}$, so W and $\hat{Z}$ also have Id $+S$-isomorphic subspaces.

We have now proved that for every subspace W of $\hat{X}$, W and $\hat{Z}$ have Id $+S$-isomorphic subspaces. This means that $\hat{Z}$ is quasi-maximal in $\hat{X}$. As $\hat{Z}$ is H.I., this implies that $\hat{X}$ is H.I. ■

3.2 Proof of Proposition 24

More definitions An operator on X is *Fredholm* if TX is closed, and the kernel and cokernel of T are finite dimensional. According to [GM], every operator on a hereditarily indecomposable space is either Fredholm or strictly singular. Also, if T is Fredholm then T^* is Fredholm.

We also recall a definition and some results from [F2]: a Banach space is said to be HD_n if the maximum number of subspaces in a direct sum is finite and equal to n. Clearly, any subspace of a HD_n space is HD_m for some $m \leq n$. By Corollary F1, every direct sum of n subspaces is quasi-maximal in a HD_n space. By Corollary F2, the direct sum of n H.I. spaces is HD_n. Finally if Y is complex HD_m, included in X complex HD_n, the dimension of $\mathcal{L}(Y,X)/\mathcal{S}(Y,X)$ is finite and there exists an upper estimate (smaller than mn) for it (Proposition F4).

Proof If $T^* \in \mathcal{L}(\hat{X}^*)$ then there exists some scalar λ such that $T - \lambda I_{\hat{X}} = S$, strictly singular, and $T^* - \lambda I_{\hat{X}^*} = S^*$, so it is enough to prove that if $S \in \mathcal{L}(\hat{X})$ is strictly singular, then $S^* \in \mathcal{L}(\hat{X}^*)$ is strictly singular.

So assume $S^* \in \mathcal{L}(\hat{X}^*)$ is not strictly singular. First notice that $\widehat{X_1}^{\perp}$ is H.I., since $\widehat{X_1}^{\perp} \simeq Z_2^{\perp} \subset X_2^*$. Likewise, $\widehat{X_2}^{\perp}$ is H.I. It follows that $\hat{X}^*$ is HD_2: indeed it is included in the HD_2 space $X_1^* \oplus X_2^*$ and contains the HD_2 space $\widehat{X_1}^{\perp} \oplus \widehat{X_2}^{\perp}$. It follows that $\widehat{X_1}^{\perp} \oplus \widehat{X_2}^{\perp}$ is quasi-maximal in $\hat{X}^*$, and so that the restriction of S^* to $\widehat{X_1}^{\perp} \oplus \widehat{X_2}^{\perp}$ is not strictly singular (Lemma F2). So the restriction of S^* to say $\widehat{X_1}^{\perp}$ is not strictly singular. Now by Proposition F4,

$$\dim \mathcal{L}(\widehat{X_1}^{\perp}, \hat{X}^*)/\mathcal{S}(\widehat{X_1}^{\perp}, \hat{X}^*) \leq \dim \mathcal{L}(\widehat{X_1}^{\perp}, \widehat{X_1}^{\perp} \oplus \widehat{X_2}^{\perp})/\mathcal{S}(\widehat{X_1}^{\perp}, \widehat{X_1}^{\perp} \oplus \widehat{X_2}^{\perp}),$$

and this last dimension is equal to

$$\dim \mathcal{L}(\widehat{X_1}^{\perp})/\mathcal{S}(\widehat{X_1}^{\perp}) + \dim \mathcal{L}(\widehat{X_1}^{\perp}, \widehat{X_2}^{\perp})/\mathcal{S}(\widehat{X_1}^{\perp}, \widehat{X_2}^{\perp}) = 1 + 0 = 1,$$

because $\widehat{X_1}^{\perp}$ is H.I. and $\widehat{X_1}^{\perp} \hookrightarrow X_2^*$ and $\widehat{X_2}^{\perp} \hookrightarrow X_1^*$ are totally incomparable. So for some non zero scalar λ, $S^*_{|\widehat{X_1}^{\perp}} - \lambda i_{\widehat{X_1}^{\perp}}$ is strictly singular. This means that the restriction of $S^* - \lambda I_{\hat{X}^*}$ to $\widehat{X_1}^{\perp}$ is strictly singular. So $S^* - \lambda I_{\hat{X}^*}$ is not Fredholm, and $S - \lambda I_{\hat{X}}$ is not Fredholm. As $\lambda \neq 0$, it follows that S is not strictly singular. ■

3.3 Construction of Spaces Satisfying Proposition 25

Following the Gowers-Maurey's method, we shall equip c_{00} with two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$, forcing however these norms to be equal on Z_{00}, the algebraic subspace of c_{00} generated by $\{e_{2n+1}, n \in \mathbb{N}\}$. Then we shall take the completions of c_{00} under these norms to obtain the Banach spaces X_1 and X_2 in Proposition 25 (and the closure of Z_{00} in those spaces to obtain their subspaces Z_1 and Z_2).

Let $\mathbf{Q}$ be the set of sequences with finite range, rational coordinates and maximum at most one in modulus. We recall that J is a set of integers $\{j_n, n \in \mathbb{N}\}$, such that $f(j_1) > 256$ and for all n, $\log\log\log j_{n+1} \geq 4j_n^2$, that $K = \{j_1, j_3, j_5, \ldots\}$, and $L = \{j_2, j_4, j_6, \ldots\}$. Furthermore, we let $L_1 = \{j_2, j_6, j_{10}, \ldots\}$, $L_2 = \{j_4, j_8, \ldots\}$. For $i = 1, 2$, let σ_i be an injection from the collection of finite sequences of successive elements of $\mathbf{Q}$ to L_i. We now need some definitions.

Definition 10 A *dual couple* is a couple (G, H) of balanced bounded convex subsets of c_{00}.

Let (G, H) be a dual couple. A vector in c_{00} is an *N-Schlumprecht sum in G* if it is of the form $1/f(N)\sum_{i=1}^N y_i^*$, where the y_i^*'s are in G and $y_1^* < \cdots < y_N^*$. A *Schlumprecht sum in G* is a N-Schlumprecht sum in G for some N. The set of Schlumprecht sums in G is denoted by $\Sigma(G)$. In the same way, we define *Schlumprecht sums in H*.

A *special sequence in G* is a sequence of successive vectors $x_1^* < \cdots < x_k^*$, with $k \in K$, such that for $i = 1, \ldots, k$, x_i^* is an M_i-Schlumprecht sum in G with $M_i \geq j_{2k}$, and $M_i = \sigma_1(x_1^*, \ldots, x_{i-1}^*)$ for $i = 2, \ldots, k$. A *special sum in G* is a sum of the form $1/\sqrt{f(k)}\sum_{i=1}^k x_i^*$, where $x_1^* < \cdots < x_k^*$ is a special sequence in G. The set of special sums in G is denoted by $S(G)$. We similarly define *special sequences in H* and *special sums in H* replacing σ_1 by σ_2 in the above definitions.

So far, we just defined the notions needed for a usual Gowers-Maurey procedure in G and in H separately. We now need to add elements to link the two procedures. To do this, we define an *associated dual couple* as a dual couple (G, H) such that there exist two multivalued functions $a\colon G \to H$ and $b\colon H \to G$ satisfying the following four properties.

(a) for all $x^* \in G$, all $y^* \in a(x^*)$, $y^* - x^*$ is in $Z_{00}^{\perp}$;
(b) for all $x^* \in G$, all $y^* \in a(x^*)$, $\mathrm{ran}(y^*) \subset \mathrm{ran}(x^*)$;
(c) for all $x^* \in G \cap Z_{00}^{\perp}$, $a(x^*) = \{0\}$;
(d) for all N-Schlumprecht sum x^* in G with x^* in G, $a(x^*)$ contains an N-Schlumprecht sum in H,

and the similar four properties for b.

The multifunction a from G to H allows us to define so-called "shadows" in H of elements in G (and likewise for b). Actually, we will only define shadows in G (resp. H) of *special sequences* in H (resp. G).

Definition 11 Let (G, H) be an associated dual couple.

A *shadow sequence in G* is a sequence of successive vectors $x_1^* < \cdots < x_k^*$ such that there exists a special sequence $y_1^* < \cdots < y_k^*$ in H such that for all i, x_i^* is an M_i-Schlumprecht sum in G belonging to $b(y_i^*)$, where M_i is the integer associated to y_i^* in the definition of the special sequence. A *shadow sum in G* is a sum of the form $1/\sqrt{f(k)}\sum_{i=1}^k x_i^*$, where

$x_1^* < \cdots < x_k^*$ is a shadow sequence in G. The set of shadow sums in G is denoted by $s(G,H)$.

We similarly define *shadow sequences* and *shadow sums in H*, and denote the set of shadow sums in H by $s(H,G)$.

To define the norms, we shall now build by induction an associated dual couple (C,D) where C (resp. D) is meant to be almost the dual unit ball of X_1 (resp. X_2). We shall build C as $\bigcup_{n\in\mathbb{N}} C_n$, building the increasing sequence C_n by induction. We shall also build a by induction, defining a function a_n from C_n to D_n at each step n; but to simplify the notation, we shall denote all the terms of the sequence by a (and we shall do symmetrically the same for D and b).

In this situation, Property (a) ensures that the subspaces Z_1 and Z_2 are isometric. Properties (b) and (d) allow us to give convenient properties to the images by a of the special sequences, that is the shadow sequences. Property (c) allows the quotient spaces X_1/Z_1 and X_2/Z_2 (resp. the dual spaces X_1^* and X_2^*) to be totally incomparable. As pointed out at the end of 2.4, the action of shadow sums will be small, so that adding them allows new properties but doesn't prevent the Q.H.I. property for X_1 or X_2.

Construction At the first step, we define $C_0 = B(l_1)$ and $D_0 = B(l_1)$, a and b by $a(\sum_{i\in\mathbb{N}} \lambda_i e_i^*) = b(\sum_{i\in\mathbb{N}} \lambda_i e_i^*) = \sum_{i \text{ odd}} \lambda_i e_i^*$. It is easy to check that (C_0, D_0) is an associated dual couple.

Now assume we are given an associated dual couple (C_{n-1}, D_{n-1}), with functions a: $C_{n-1} \to D_{n-1}$ and b: $D_{n-1} \to C_{n-1}$. We define C'_{n-1} to be $\Sigma(C_{n-1}) \cup S(C_{n-1}) \cup s(C_{n-1}, D_{n-1})$, and C_n to be the set of elements of the form $E(\sum_{i=1}^M \lambda_i x_i^*)$, where E is an interval projection, $\sum_{i=1}^M |\lambda_i| = 1$, and for all i, x_i^* is in C'_{n-1}. We define D_n in a similar way.

We now extend a to C_n. If $x^* \in C_n \cap Z_{00}^{\perp}$, then we let $a(x^*) = \{0\}$. We now define a construction if x^* is in C_n and not in $Z_{00}^{\perp}$.

The set $a(x^*)$ may be already defined or not (it is when x^* is in C_{n-1}); if not we may assume $a(x^*) = \varnothing$. Then we add new values to the set $a(x^*)$ in each of the following cases (notice that at least one of the possibilities happens, so that a is well defined on the whole of C_n, but that the possibilities are not exclusive).

- If x^* is a Schlumprecht sum of the form $f(N)^{-1}\sum_{i=1}^N x_i^*$ with $x_i^* \in C_{n-1}$ then we add to $a(x^*)$ the set $f(N)^{-1}\sum_{i=1}^N a(x_i^*)$.

- If x^* is a special sum of the form $f(k)^{-1/2}\sum_{i=1}^k x_i^*$ where x_i^* is an (M_i, f)-form in C_{n-1} then we add to the set $a(x^*)$ the set of all sums of the form $f(k)^{-1/2}\sum_{i=1}^k y_i^*$, where y_i^* is an (M_i, f)-form in $a(x_i^*)$.

- If x^* is a shadow sum of the form $f(k)^{-1/2}\sum_{i=1}^k x_i^*$ with $x_i^* \in b(y_i^*)$ and $y_1^*, \ldots, y_k^*$ is a special sum in D_{n-1}, then we add to the set $a(x^*)$ the singleton $\{f(k)^{-1/2}\sum_{i=1}^k E y_i^*\}$, where $E = \operatorname{ran}(x^*)$.

- If x^* is the projection of a convex combination of elements of the three previous forms, that is, $x^* = \operatorname{ran}(x^*)(\sum_i \lambda_i x_i^*)$, then we add to the set $a(x^*)$ the set $\operatorname{ran}(x^*)\big(\sum_i \lambda_i a(x_i^*)\big)$, $a(x_i^*)$ being defined as above whether x_i^* is a Schlumprecht sum, a special sum, or a shadow sum in C_{n-1}. It is important to remember that we only use this construction when x^* is not in $Z_{00}^{\perp}$.

It is then easy to check that a (resp. b) takes its values in D_n (resp. in C_n) and that it still satisfies the four properties (a)–(d), so (C_n, D_n) is an associated dual couple. Define C as $\bigcup_{n\in\mathbb{N}} C_n$ and D as $\bigcup_{n\in\mathbb{N}} D_n$; the multifunction a (resp. b) is defined on C (resp. D), so (C, D) is an associated dual couple as well. Then define $\|\cdot\|_1 = \sup_{x^*\in C}\langle x^*, \cdot\rangle$ (resp. $\|\cdot\|_2 = \sup_{y^*\in D}\langle y^*, \cdot\rangle$), X_1 (resp. X_2) as the completion of c_{00} under $\|\cdot\|_1$ (resp. $\|\cdot\|_2$) and Z_1 (resp. Z_2) as the closure of Z_{00} in X_1 (resp. X_2).

Remark 26 With Definition 5, a special sequence in X_1 is an L_1-sequence, a shadow sequence in X_1 is an L_2-sequence. It follows that the space X_1 has GM-type, the set $\mathcal{S}'$ being the set of special sequences in X_1^* and the set $\mathcal{S}''$ being the set of shadow sums in X_1^*. The symmetric facts are of course true for X_2.

Lemma 27 *The spaces Z_1 and Z_2 are isometric.*

Proof Let z be an element of Z_{00}. Then

$$\|z\|_1 = \sup_{x^*\in C}(\langle x^*, z\rangle) = \sup_{x^*\in C, y^*\in a(x^*)}(\langle x^* - y^*, z\rangle + \langle y^*, z\rangle).$$

Now by definition of a, for $x^* \in C$ and $y^* \in a(x^*)$, $x^* - y^*$ is in $Z_{00}^{\perp}$, so $\langle x^* - y^*, z\rangle = 0$; and as y^* is in D, $(\langle y^*, z\rangle) \le \|z\|_2$. It follows that $\|z\|_1 \le \|z\|_2$, and by symmetry, $\|z\|_1 = \|z\|_2$. ■

Lemma 28 *Let $y_1^*, \dots, y_k^*$ be a special sequence in $Z_2^{\perp}$. Let $x_1 < \cdots < x_k$ in X_1 be associated to $y_1^*, \dots, y_k^*$. Let $x = \sum_{i=1}^k x_i$. Then*

$$\|x\| \le 5k/f(k).$$

Proof The space X_1 has GM-type, so it is enough to prove the hypothesis of Lemma 16. By Lemma 17, it is true for every special sum, so now consider z^* be a shadow sum in X_1^* and E an interval.

For every i, let M_i be such that y_i^* is an (M_i, f)-form. There exists a special sequence $v_1^*, \dots, v_k^*$ in X_2^* such that $z^* = 1/\sqrt{f(k)}\sum_{i=1}^k z_i^*$ with for every i, $z_i^* \in b(v_i^*)$; let N_i be such that v_i^* is an (N_i, f)-form; by definition of a shadow sum, z_i^* is also an (N_i, f)-form. Let $I = \sup\{i/M_i = N_i\}$, or 0 if no such I exists. For $i < I$, because σ_2 is injective, we have that $v_i^* = y_i^*$. It follows that v_i^* is in $Z_2^{\perp}$, so $b(v_i^*) = \{0\}$, and $z_i^* = 0$. For $i > I$, the now usual application of Lemma 15 shows that $|z_i^*(Ex_j)| \le 1/k^2$. Finally,

$$|z^*(Ex)| \le 1/\sqrt{f(k)}\big(0 + |z_I^*(x_I)| + k^2.k^{-2}\big) \le 2/\sqrt{f(k)} \le 1/4. \quad ■$$

Lemma 29 *The spaces X_1/Z_1 and X_2/Z_2 are totally incomparable.*

Proof We now assume that there exists an isomorphism α between a subspace W_1/Z_1 of X_1/Z_1 and a subspace W_2/Z_2 of X_2/Z_2 and we intend to find a contradiction.

First notice that X_2/Z_2 has a basis (namely the basis $(e'_{2n})_{n\in\mathbb{N}}$ dual to the basis $(e^*_{2n})_{n\in\mathbb{N}}$ of $Z_2^\perp$). By Lemma 11, there exists a sequence $(w_n)_{n\in N}$ of $l_1^{n_i+}$ vectors almost in W_1, $4+\epsilon_0$-normed in $Z_1^\perp$, and up to perturbations on α and W_1, we may assume that $(w_n)_{n\in\mathbb{N}}$ is actually in W_1 and that the sequence $\big(\alpha(\widehat{w_n})\big)$ is a sequence of unit vectors, successive with respect to (e'_{2n}).

Now let $k\in K$. We may find a unit R.I.S vector $x_1=\sum_{i=1}^{M_1}x_1^i$ in W_1, and x_1^{i*} in $Z_1^\perp$ such that $x_1^{i*}(x_1^i)\geq(4+\epsilon_0)^{-1}\|x_1^i\|$, so that

$$\|\widehat{x_1^i}\|\geq x_1^{i*}(\widehat{x_1^i})=x_1^{i*}(x_1^i)\geq(4+\epsilon_0)^{-1}\|x_1^i\|.$$

For $i=1,\ldots,M_1$, let $y_1^{i*}\in Z_2^\perp$ be a functional that norms $\alpha(\widehat{x_1^i})$ and such that $\operatorname{ran}(y_1^{i*})\subset\operatorname{ran}\big(\alpha(\widehat{x_1^i})\big)$, and let y_1^* be the (M_1,f)-form $f(M_1)^{-1}\sum_{i=1}^{M_1}y_1^{i*}$. As $y_1^*\big(\alpha(\widehat{x_1})\big)=f(M_1)^{-1}\sum_{i=1}^{M_1}\|\alpha(\widehat{x_1^i})\|\geq\big((4+\epsilon_0)\|\alpha^{-1}\|f(M_1)\big)^{-1}\sum_{i=1}^{M_1}\|x_1^i\|$, by Lemma 12,

$$y_1^*\big(\alpha(\widehat{x_1})\big)\geq\big((4+\epsilon_0)(3+\epsilon_0)\|\alpha^{-1}\|\big)^{-1},$$

and by a perturbation, if we only ask that $y_1^*\big(\alpha(\widehat{x_1})\big)\geq(13\|\alpha^{-1}\|)^{-1}$, we may assume that y_1^* is in $\mathbf{Q}$, and that $\operatorname{ran}(y_1^*)\subset\operatorname{ran}\big(\alpha(\widehat{x_1})\big)$.

Repeating this procedure, we obtain vectors x_i in W_1, and y_i^* in $B(Z_2^\perp)$, such that $x_1,\ldots,x_k$ is associated to the special sequence $y_1^*,\ldots,y_k^*$ in $Z_2^\perp$. It follows that

$$\Big\|\alpha\Big(\sum_{i=1}^k\widehat{x_i}\Big)\Big\|\geq f(k)^{-1/2}\sum_{i=1}^k y_i^*\big(\alpha(\widehat{x_i})\big)\geq(13\|\alpha^{-1}\|)^{-1}kf(k)^{-1/2},$$

while as (x_i) is associated to (y_i^*), by Lemma 28,

$$\Big\|\sum_{i=1}^k\widehat{x_i}\Big\|\leq\Big\|\sum_{i=1}^k x_i\Big\|\leq 5kf(k)^{-1}.$$

It follows that $\|\alpha\|\,\|\alpha^{-1}\|\geq 65^{-1}\sqrt{f(k)}$, and this for any k in K, contradicting the boundedness of α. ■

Lemma 30 *The spaces X_1^* and X_2^* are totally incomparable.*

Proof As they are hereditarily indecomposable, if X_1^* and X_2^* had isomorphic subspaces, passing to further subspaces which Id $+S$-embed in $Z_1^\perp$ and $Z_2^\perp$ respectively, we would find an isomorphism β between a subspace W_{1*} of $Z_1^\perp$ and a subspace W_{2*} of $Z_2^\perp$.

By Lemma 11 and a perturbation on W_{1*} and β, find a successive sequence of $l_1^{n_i+}$ vectors in X_1 $4+\epsilon_0$-normed by $(w_n^*)_{n\in N}$, successive in W_{1*}, such that $\big(\beta(w_n^*)\big)$ is a sequence of unit vectors, successive with respect to (e^*_{2n}). Applying the usual method, get for any $k\in K$, vectors x_i in X_1, x_i^* in W_{1*}, such that $x_i^*(x_i)\geq(13\|\beta\|)^{-1}$ and $x_1,\ldots,x_k$ is associated to the special sequence $\beta(x_1^*),\ldots,\beta(x_k^*)$ in W_{2*}. By Lemma 28, it follows that

$$\Big\|\sum_{i=1}^k x_i\Big\|\leq 5kf(k)^{-1},$$

so

$$\Big\|\sum_{i=1}^{k} x_i^*\Big\| \geq k/\Big(13\|\beta\|\,\Big\|\sum_{i=1}^{k} x_i\Big\|\Big) \geq (65\|\beta\|)^{-1} f(k),$$

while

$$\Big\|\beta\Big(\sum_{i=1}^{k} x_i^*\Big)\Big\| \leq \sqrt{f(k)}.$$

It follows that $\|\beta\|\,\|\beta^{-1}\| \geq 65^{-1}\sqrt{f(k)}$, and this for any k in K, contradicting the boundedness of β. ■

4 Appendix

We give a sketch of the proof of the existence of a hereditarily indecomposable space $\hat{X}$ such that $\hat{X}^*$ contains a direct sum of n subspaces, and every operator on $\hat{X}^*$ is a strictly singular perturbation of an homothetic map.

Proposition A1 *Let $n \in \mathbb{N}$. For $i = 1, \ldots, n$, let X_i be a hereditarily indecomposable Banach space, let Z_i be a subspace of X_i. Assume that the spaces Z_i are all isometric to a same space Z, and that for any $i \neq j$, X_i/Z_i and X_j/Z_j are infinite dimensional and totally incomparable. Let $Z_{[1,n]} = \{(z_1, \ldots, z_n) \in Z_1 \times \cdots \times Z_n / \sum_{i=1}^{n} z_i = 0\}$. Let $\hat{X}$ be the quotient space $(X_1 \times \cdots \times X_n)/Z_{[1,n]}$. Then $\hat{X}$ is hereditarily indecomposable and $\hat{X}^*$ contains a direct sum of n subspaces.*

Proof (sketch) We use the same notation as in the case $n = 2$, in particular we let $\hat{Z} = \{\widehat{(z, 0, \ldots, 0)}, z \in Z\}$, and we show that

$$\hat{Z}^{\perp} \simeq \bigoplus_{i=1}^{n} (X_i/Z_i)^*.$$

Now we consider W a subspace of $\hat{X}$. There exists at most one value i_W of i such that $\phi_{i/W}$ is not infinitely singular, otherwise two quotient spaces X_i/Z_i and X_j/Z_j would have isomorphic subspaces. It follows easily that W and $\hat{X}_{i_W}$ have $\mathrm{Id} + S$-isomorphic subspaces, and finally that $\hat{X}$ is H.I. ■

Proposition A2 *Let $n \in \mathbb{N}$. For $i = 1, \ldots, n$, let X_i, Z_i and $\hat{X}$ be complex spaces as in Proposition A1. Assume furthermore that for any i, X_i^* is hereditarily indecomposable, and that for any $i \neq j$, X_i^* and X_j^* are totally incomparable. Then every operator on $\hat{X}^*$ is a strictly singular perturbation of an homothetic map.*

Proof It follows exactly the case $n = 2$. ■

Proposition A3 *Let $n \in \mathbb{N}$. For $i = 1, \ldots, n$, there exists X_i complex quotient hereditarily indecomposable reflexive Banach space, Z_i subspace of X_i, such that all Z_i are isometric, and for any $i \neq j$, X_i/Z_i and X_j/Z_j (resp. X_i^* and X_j^*) are totally incomparable.*

Proof (sketch) We make a construction similar to the case $n = 2$, using a partition of L in n subsets $L_1, \dots, L_n$. We build n balanced bounded convex subsets $C_1, \dots, C_n$ of c_{00}, and multifunctions $a_{ij} : C_i \to C_j$ for $i \neq j$, such that for all $i \neq j$, (C_i, C_j) is an associated dual couple. The difference is that we have $n - 1$ kinds of shadow sequences in each C_i (those coming from special sequences in C_j for all $j \neq i$). ■

Notice that $\hat{X}^*$ is not decomposable, otherwise $\hat{X}$ reflexive would be decomposable. It follows:

Corollary A4 *Let $n \in \mathbb{N}^*$. Then there exists a non decomposable* HD_n *space.*

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