Geometrically Non-linear Analysis of Functionally Graded Material (FGM) Plates and Shells using a Four-node Quasi-Conforming Shell Element

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ABSTRACT: The four-node quasi-conforming shell element was extended in the present article to the case of geometrically non-linear behavior of the FGM plates and shells. The high stress occurring in the FGM structures will affect its integrity and the structures is susceptible to failure. Therefore, we focus on the effect of volume fraction of the constituent materials in the mechanical behavior of FGM plates and shells. The material properties are assumed to be varied in the thickness direction according to a sigmoid function in terms of the volume fraction of the constituents. The series solutions of sigmoid FGM (S-FGM) plates, based on the first-order shear deformation theory and Fourier series expansion are provided as the reference solution for the numerical results. In quasi-conforming formulation, the tangent stiffness matrix is explicitly integrated. This makes the element computationally efficient in the non-linear analysis. Several selected examples of non-linear analysis of FGM shells are included in the article for the illustration of possibilities of the presented formulation. It is seen that the present results for the non-linear behavior of FGM plates and shells can provide a useful benchmark to check the accuracy of related numerical solutions.

KEY WORDS: geometrically non-linear analysis, functionally graded material, quasi-conforming shell element, mechanical behavior.

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INTRODUCTION

The laminated composite plates and shells comprised of two different materials have been widely used to satisfy high performance demands. However, stress singularities in such composites may occur at the interface between two different materials. A functionally graded material (FGM) is a two-component composite, characterized by a compositional gradient from one component to the other. In contrast, traditional composites are homogeneous mixtures, and they involve a compromise among the desirable properties of the component materials. Since significant proportions of a FGM contain a pure form of each component, there is no need for compromise. The continuous change in the microstructure of functionally graded material (FGM) distinguishes them from the fiber–reinforced laminated composite materials, which have a mismatch of mechanical properties across an interface because of two discrete materials bonded together. As a result, the constituents of the fiber-matrix composites are prone to debond at extremely high thermal loading. Furthermore, cracks are likely to initiate at the interfaces and grow into weaker material sections. Additional problems include the presence of residual stresses due to the difference in coefficients of thermal expansion of the fiber and matrix in the composite materials. In FGM, these problems are avoided or reduced by gradual variation of the constituents’ volume fraction rather than abruptly changing it across the interface [1].

The main application of functionally gradient materials have been in high temperature environments, including a thermal shock situation that occurs when a body is subjected to a high transient heating or cooling in a short time period. But the metal–ceramic composite plates and shells are widely used in aircraft, space vehicles, reactor vessels, and other engineering applications. If a high external pressure is applied to the composite plate and shell structures, the high stresses that occurred in the structure can affect its integrity, and the structure, as a result, is susceptible to failure. For these reasons, understanding of the mechanical behavior of FGM plates and shells are very important to assess the safety of the shell and plate structure. Woo and Meguid [2] applied the von Kármán theory for large deformation to obtain the analytical solution for the plates and shells under transverse mechanical loads and a temperature field. Praveen and Reddy [3] investigated the static and dynamic responses of functionally graded ceramic–metal plates by using a plate finite element that accounts for the transverse shear strains, rotary inertia, and moderately large rotations in the von Kármán theory. However, geometrically non-linear behavior under mechanical loading of FGM plates and shells has not received adequate consideration.

Power-law function [4,5], and exponential function [6,7] are commonly used to describe the variations of material properties of FGM. However, in both power-law and exponential functions, the stress concentrations appear in one of the interfaces in which the material is continuously but rapidly changing. Therefore, Chung and Chi [8] proposed a sigmoid FGM, which was composed of two power-law functions to define a new volume fraction. Chi and Chung [9] indicated that the use of a sigmoid FGM can significantly reduce the stress intensity factors of a cracked body. Rousseau [10] introduced specific relations for out-of-plane deformation near crack tips of linearly graded materials used with the optical method of coherent gradient sensing (CGS) to interpret experimentally obtained fringes.
Tang et al. [11,12] introduced the quasi-conforming (QC) element technique. However, all of the examples were limited to linear formulations. Guan and Tang [13,14] presented a nine-node quasi-conforming degenerated shell element for linear and non-linear analysis. Many articles had been published using the QC formulation [15–20]. Shi and Voyiadjis [17,18] derived the edge displacement function of the elements using a strain energy function that included bending and shear. However, these studies had been limited only to the analysis of plates and the geometric stiffness based on the von Kármán assumption that only included membrane forces. Kim et al. [20] presented two improved formulations of four-node quasi-conforming shell elements based on the assumed strain method, using the shell element presented by Shi and Voyiadjis [17,18]. The shell elements showed very good performance in convergence, accuracy, and free locking behavior. Park et al. [21] studied the linear static and dynamic response of laminated composite plates and shells using a quasi-conforming shell element. Recently the quasi-conforming shell element has been fully extended for the large displacement of elasto-plastic analysis by Kim and Lomboy [22]. This article introduces many examples for the validation of the quasi-conforming shell element. All the results indicate very good performance in comparison with references.

The objective of this article is to extend the non-linear quasi-conforming formulation by Kim et al. [20,22] to the case of FGM structures and the associated finite element model that accounts for the mechanical behavior. The present quasi-conforming non-linear formulation is based on the updated Lagrangian method with the assumption of small strains and large displacements. To handle large displacements and rotations, the co-rotational approach proposed by Kim and Lomboy [22] is employed. Instead of a von Kármán assumption [17,18], the present geometrically non-linear formulation is derived using the full definition of the Green strain tensor that includes membrane, bending, and transverse shear stresses in the geometric stiffness. The explicit definition of stiffness matrix is used, i.e., no Gauss integration is used. Thus, the computational time is significantly reduced in the incremental non-linear analysis. In this article, the material properties of the FGM plates and shells are assumed to change continuously throughout the thickness of the plate and shell, according to the volume fraction of the constituent materials based on the sigmoid functions. The series solutions of the FGM plates are obtained by expanding the transverse load into Fourier series expansion and the analytical solutions of the FGM plates are compared with the numerical results obtained with the four-node quasi-conforming shell element.

**FORMULATION OF FGM SHELL ELEMENT**

**Displacement Field and Strains**

A local orthogonal coordinate system \((r, s, t)\) is used to describe the geometry of the four-node shell element as shown in Figure 1. Its origin is at the geometric center of the element. The position vector of a point \(r(r, s, t)\) in the element is given by:

\[
X_p = X_c + T_r
\]

where \(X_c\) is the vector of the global coordinates of the element mid-surface center.
Having defined the local coordinate base vectors, the transformation of the incremental displacement parameters $\Delta \tilde{u}_i$, $\Delta \tilde{\varphi}_i$ at node $i$ from the element local coordinates, to the global coordinates, is done by:

$$
\begin{bmatrix}
\Delta \tilde{u} \\
\Delta \tilde{\varphi}
\end{bmatrix}_i = T_{gi}^T \begin{bmatrix}
\Delta \tilde{U} \\
\Delta \tilde{\Theta}
\end{bmatrix}_i,
$$

where:

$$
T_{gi}^T = \begin{bmatrix}
V_r & V_s & V_t \\
0 & V_r & V_s & V_t
\end{bmatrix}^T = \begin{bmatrix}
T^T & 0 \\
0 & T^T
\end{bmatrix}.
$$

It is important to note that $V_t$ is normal to the mid-surface of the element and it is independent of the top and the bottom nodal coordinates. The local kinematical relations based on the co-rotational displacement can be expressed as [20]:

$$
\begin{align*}
\Delta \tilde{u} &= \Delta \tilde{u} + t \left( \Delta \bar{\varphi}_s + \frac{1}{2} \Delta \bar{\varphi}_r \Delta \bar{\varphi}_l \right), \\
\Delta \tilde{v} &= \Delta \tilde{v} - t \left( \Delta \bar{\varphi}_r - \frac{1}{2} \Delta \bar{\varphi}_r \Delta \bar{\varphi}_l \right), \\
\Delta \tilde{w} &= \Delta \tilde{w}(r, s).
\end{align*}
$$

Detailed discussion about the adopted co-rotational formulation can be found in Refs. [22,23]. Following the non-linear thin shell theory and substituting the co-rotational kinematic relations given by Kim et al. [20], the incremental membrane, bending, and transverse shear strains can be separated into linear and non-linear parts, $\Delta \tilde{\varepsilon}_m = \Delta \tilde{\varepsilon}_m + \Delta \tilde{\varepsilon}_m^\alpha$, $\Delta \tilde{\varepsilon}_b = \Delta \tilde{\varepsilon}_b + \Delta \tilde{\varepsilon}_b^\alpha$, $\Delta \tilde{\varepsilon}_q = \Delta \tilde{\varepsilon}_q + \Delta \tilde{\varepsilon}_q^\alpha$. 

---

Figure 1. (a) Mid-surface geometry and local coordinate of four-node shell element; (b) Plan of element with derivation of local coordinates.
The linear parts are:

\[
\Delta \mathbf{\bar{e}}_m = \begin{bmatrix}
\Delta \bar{e}_r \\
\Delta \bar{e}_s \\
\Delta \bar{e}_{rs}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \Delta \bar{u}}{\partial r} \\
\frac{\partial \Delta \bar{v}}{\partial s} + \frac{\partial \Delta \bar{u}}{\partial s} \\
\frac{\partial \Delta \bar{v}}{\partial r} + \frac{\partial \Delta \bar{u}}{\partial r}
\end{bmatrix},
\Delta \mathbf{\bar{e}}_b = \begin{bmatrix}
\Delta \bar{e}_r \\
\Delta \bar{e}_s \\
\Delta \bar{e}_{rs}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \Delta \bar{\varphi}_s}{\partial r} - \frac{\partial \Delta \bar{\varphi}_r}{\partial s} \\
\frac{\partial \Delta \bar{\varphi}_s}{\partial r} - \frac{\partial \Delta \bar{\varphi}_r}{\partial s} \\
\frac{\partial \Delta \bar{\varphi}_s}{\partial r} - \frac{\partial \Delta \bar{\varphi}_r}{\partial s}
\end{bmatrix},
\]

and the non-linear part of the incremental membrane, bending, and transverse shear strain vectors are:

\[
\Delta \mathbf{\bar{e}}_m = \begin{bmatrix}
\Delta \bar{e}_m^r \\
\Delta \bar{e}_m^s \\
\Delta \bar{e}_{m,rs}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\left( \frac{\partial \Delta \bar{\varphi}_s}{\partial r} \right)^2 + \left( \frac{\partial \Delta \bar{\varphi}_s}{\partial r} \right)^2 + \frac{\partial \Delta \bar{\varphi}_s}{\partial s} \frac{\partial \Delta \bar{\varphi}_s}{\partial s}
\end{bmatrix},
\Delta \mathbf{\bar{e}}_b = \begin{bmatrix}
\Delta \bar{e}_b^r \\
\Delta \bar{e}_b^s \\
\Delta \bar{e}_{b,rs}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\left( \frac{\partial \Delta \bar{\varphi}_s}{\partial r} \right)^2 + \frac{\partial \Delta \bar{\varphi}_s}{\partial s} \frac{\partial \Delta \bar{\varphi}_s}{\partial s} + \frac{\partial \Delta \bar{\varphi}_s}{\partial r} \frac{\partial \Delta \bar{\varphi}_s}{\partial r}
\end{bmatrix},
\]

Equation of Motion

In formulating the new shell element, an updated Lagrangian formulation is adopted. The linearized equation of motion, is expressed as:

\[
\int_{V}^{t+\Delta t} \int_{V}^{t} C_{ijkl} \Delta e_{ij} \delta e_{ij} dV + \int_{V}^{t+\Delta t} \int_{V}^{t} \tau_{ij} \delta \Delta \bar{\Xi}_{ij} dV = \int_{V}^{t+\Delta t} \int_{V}^{t} \tau_{ij} \delta e_{ij} dV
\]
where $C_{i j r s}$ is the constitutive tensor, $t_{i j}$ is the component of the Cauchy stress tensor and $t_{i j}^{\Delta t}$ is the external virtual work expression. The use of the co-rotational method by Kim and Lomboy [22], where the local coordinate axes follows the rigid body motion of the element, naturally results in an updated Lagrangian approach. Equation (7) is the linearization of the virtual work equation using the second Piola–Kirchhoff stresses and Green–Lagrange strains, referred from a known equilibrium configuration at time $t$. For a detailed derivation of Equation (7) (see Bathe [24], and the references mentioned therein). Consistent with this derivation, the strain expressions in Equations (5) and (6) are derived from the full Green strain tensor and are applied to Equation (7).

Assuming a constant thickness during deformation, Equation (7) may be expressed as:

$$
\sum_{\text{elem}} \int_{\Omega} \left( \delta \Delta \tilde{\varepsilon}_{m}^{T} A \Delta \tilde{\varepsilon}_{m} + \delta \Delta \tilde{\varepsilon}_{m}^{T} B \Delta \tilde{\varepsilon}_{b} + \delta \Delta \tilde{\varepsilon}_{b}^{T} B \Delta \tilde{\varepsilon}_{m} + \delta \Delta \tilde{\varepsilon}_{b}^{T} D \Delta \tilde{\varepsilon}_{b} + \delta \Delta \tilde{\varepsilon}_{q}^{T} \tilde{A} \Delta \tilde{\varepsilon}_{q} \right) \, dr \, ds \\
+ \sum_{\text{elem}} \int_{\Omega} \left( N^{T} \delta \Delta \tilde{\varepsilon}_{m} + M^{T} \delta \Delta \tilde{\varepsilon}_{b} + Q^{T} \delta \Delta \tilde{\varepsilon}_{q} \right) \, dr \, ds \\
= t^{\Delta t} \mathcal{Q} - \sum_{\text{elem}} \int_{\Omega} \left( \delta \Delta \tilde{\varepsilon}_{m}^{T} N + \delta \Delta \tilde{\varepsilon}_{b}^{T} M + \delta \Delta \tilde{\varepsilon}_{q}^{T} Q \right) \, dr \, ds
$$

(8)

where $A$, $B$, $D$, and $\tilde{A}$ are components of the constitutive tensor, integrated through the thickness, and $M$, $N$, and $Q$ are stress resultants.

**Constitutive Relations of FGM Structures**

The FGM can be produced by continuously varying the constituents of multi-phase materials in a predetermined profile. The most distinct features of an FGM are the non-uniform microstructures with continuously graded properties. An FGM can be defined by the variation in the volume fractions. Most researchers use the power-law function, exponential function, or sigmoid function to describe the volume fractions. This article also uses FGM plates and shells with power-law, exponential, or sigmoid function.

For analysis of FGM structures, three main functions can be employed.

**EXPONENTIAL FUNCTION: E-FGM**

To describe the material properties, the exponential function is used [6].

$$
H(t) = A e^{B(t+h/2)}
$$

(9)

with:

$$
A = H_{2} \quad \text{and} \quad B = \frac{1}{h} \ln \left( \frac{H_{1}}{H_{2}} \right),
$$

(10)

where $h$ is the thickness of the shell and $H(t)$ denotes a generic material property such as modulus, $H_{1}$ and $H_{2}$ indicate the property of the top and bottom faces of the plate, respectively.
**POWER-LAW FUNCTION: P-FGM**

The volume fraction of the P-FGM is assumed to obey a power-law function:

\[ V_f(t) = \left( \frac{t + h/2}{h/2} \right)^p, \tag{11} \]

where \( p \) is the material parameter that dictates the material variation profile through the thickness. Once the local volume fraction \( V_f(t) \) has been defined, the material properties of a P-FGM can be determined by the rule of mixture [4]:

\[ H(t) = V_f(t)H_1 + (1 - V_f(t))H_2. \tag{12} \]

Here we assume that modulus \( E, G \) and the Poisson’s ratio \( v \) vary according to Equation (12). Young’s modulus changes rapidly near the lowest surface for \( p > 1 \), and increases quickly near the top surface for \( p < 1 \) [25].

**SIGMOID FUNCTION: S-FGM**

The volume fraction using two power-law functions which ensure smooth distribution of stresses is defined [9]:

\[ V^1_f(t) = 1 - \frac{1}{2} \left( \frac{h/2 - t}{h/2} \right)^p \quad \text{for} \quad 0 \leq t \leq \frac{h}{2}, \tag{13a} \]

\[ V^2_f(t) = \frac{1}{2} \left( \frac{h/2 + t}{h/2} \right)^p \quad \text{for} \quad -\frac{h}{2} \leq t \leq 0. \tag{13b} \]

By using the rule of mixture, the material properties of the S-FGM can be calculated as:

\[ H(t) = V^1_f(t)H_1 + \left(1 - V^1_f(t)\right)H_2 \quad \text{for} \quad 0 \leq t \leq \frac{h}{2}, \tag{14a} \]

\[ H(t) = V^2_f(t)H_1 + \left(1 - V^2_f(t)\right)H_2 \quad \text{for} \quad -\frac{h}{2} \leq t \leq 0. \tag{14b} \]

Figure 2 shows that the variation of Young’s modulus in Equations (14a) and (14b) represents sigmoid distributions, and this FGM structure is thus called a sigmoid FGM structure (S-FGM structures).

In the case of adding an FGM of a single power-law function into the multi-layered composite, stress concentrations appear on one of the interfaces where the material is continuous but changes rapidly [4,26]. In this article, the volume fraction using two power-law functions by Chung and Chi [8] is used to ensure smooth distribution of stresses among all the interfaces.
Let us consider an elastic rectangular plate and/or shell. The local coordinates \( r \) and \( s \) define the mid-plane of the plate and shell, whereas the \( t \)-axis originated at the middle surface of the plate and shell is in the thickness direction. The material properties, both, Young’s modulus and Poisson’s ratio, at the upper and lower surfaces are different but are pre-assigned according to the performance demands. However, Young’s modulus and Poisson’s ratio of the plate and/or shell vary continuously only in the thickness direction (\( t \)-axis), i.e. \( E = E(t) \), \( \nu = \nu(t) \). It is called functionally graded material (FGM) plate and/or shell:

\[
\begin{align*}
Q_{11} &= \frac{E(t)}{1 - [\nu(t)]^2}, \\
Q_{12} &= \nu(t)Q_{11}, \\
Q_{44} &= Q_{55} = Q_{66} = \frac{E(t)}{2[1 + \nu(t)]} = G(t).
\end{align*}
\]

The modulus \( E \), the shear modulus \( G \), Poisson’s ratio \( \nu \), the stresses and the coefficients \( Q_{ij} \), vary through the shell thickness according to Equations (14a) and (14b).

**Figure 2.** The variation of Young’s modulus of S-FGM plate and shell.
The stress resultants are defined by:

\[
\Delta \mathbf{N} = \begin{bmatrix} \Delta N_r \\ \Delta N_s \\ \Delta N_{rs} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \Delta \tilde{S}_r \\ \Delta \tilde{S}_s \\ \Delta \tilde{S}_{rs} \end{bmatrix} \, dt = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{bmatrix} \Delta \tilde{e}_r \\ \Delta \tilde{e}_s \\ \Delta \tilde{e}_{rs} \end{bmatrix}
\]

\[
\Delta \mathbf{M} = \begin{bmatrix} \Delta M_r \\ \Delta M_s \\ \Delta M_{rs} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \Delta \tilde{S}_r \\ \Delta \tilde{S}_s \\ \Delta \tilde{S}_{rs} \end{bmatrix} \, dt = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{bmatrix} \Delta \tilde{e}_r \\ \Delta \tilde{e}_s \\ \Delta \tilde{e}_{rs} \end{bmatrix}
\]

\[
\Delta \mathbf{Q} = \begin{bmatrix} \Delta Q_{rt} \\ \Delta Q_{st} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \Delta \tilde{S}_{rt} \\ \Delta \tilde{S}_{st} \end{bmatrix} \, dt = \begin{bmatrix} \tilde{A}_{55} & 0 \\ 0 & \tilde{A}_{44} \end{bmatrix} \begin{bmatrix} \Delta \tilde{y}_{rt} \\ \Delta \tilde{y}_{st} \end{bmatrix}
\]

where the coefficients of \( A_{ij} \) denote the extensional stiffness, \( D_{ij} \) the bending stiffness, \( B_{ij} \) the bending-extensional coupling stiffness \( \tilde{A}_{ij} \) the shear stiffness, and \( A_{ij}, B_{ij}, D_{ij}, \text{ and } \tilde{A}_{ij} \) are defined as:

\[
A_{11} = \int_{-h/2}^{h/2} \frac{E(t)}{1 - v(t)^2} \, dt, \quad A_{12} = \int_{-h/2}^{h/2} \frac{v(t)E(t)}{1 - v(t)^2} \, dt, \quad A_{66} = \int_{-h/2}^{h/2} \frac{(1 - v(t))}{2} \frac{E(t)}{1 - v(t)^2} \, dt,
\]

\[
B_{11} = \int_{-h/2}^{h/2} \frac{E(t)}{1 - v(t)^2} \, dt, \quad B_{12} = \int_{-h/2}^{h/2} \frac{tv(t)E(t)}{1 - v(t)^2} \, dt, \quad B_{66} = \int_{-h/2}^{h/2} \frac{(1 - v(t))}{2} \frac{tE(t)}{1 - v(t)^2} \, dt,
\]

\[
D_{11} = \int_{-h/2}^{h/2} \frac{E(t)}{1 - v(t)^2} \, dt, \quad D_{12} = \int_{-h/2}^{h/2} \frac{t^2E(t)}{1 - v(t)^2} \, dt, \quad D_{66} = \int_{-h/2}^{h/2} \frac{(1 - v(t))}{2} \frac{t^2E(t)}{1 - v(t)^2} \, dt
\]

\[
\tilde{A}_{44} = \tilde{A}_{55} = k_s \int_{-h/2}^{h/2} \frac{(1 - v(t))}{2} \frac{E(t)}{1 - v(t)^2} \, dt.
\]

where \( k_s \) stand for the transverse shear correction factor taking after Reissner the value of 5/6.

Formulation of the Element Tangent Stiffness Matrix

To consider large displacements, the element tangent stiffness should be composed of the linear stiffness matrix and the geometric stiffness matrix. In determining both of these matrices for the present element, the quasi-conforming technique, QCT, is used.

In the QCT, the element strain fields are approximated using polynomials and integrated using string functions or boundary displacement interpolation functions. Using this approximate definition of strain, the required stiffness matrices are determined.
In terms of the global displacement vector, the incremental strain is written as [20,21):

\[
\Delta \varepsilon = B T_g^T \left\{ \Delta \tilde{U} \begin{array}{c}
\Delta \theta \\
\end{array} \right\} = P \Delta \alpha T_g^T = P A^{-1} C T_g^T \left\{ \Delta \tilde{U} \begin{array}{c}
\Delta \theta \\
\end{array} \right\},
\]

(19)

where \( T_g \) is a diagonal matrix composed of \( T_{gi} \), \( P \) is chosen strain interpolation polynomial functions, \( \Delta \alpha \) is undetermined strain parameters:

\[
A = \int_\Omega (P^T \mathbf{P}) \, d\Omega \quad \text{and} \quad C \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi} \\
\end{array} \right\} = \int_\Omega (P^T \Delta \tilde{e}) \, d\Omega.
\]

(20)

The stiffness matrix \( K_L \) due to linear strain increments used in the present element is similar to the one presented in Kim et al. [20].

\[
K_L = T_g \int_A \left( B_m^T \mathbf{A} B_m + B_m^T \mathbf{B}_b + B_b^T \mathbf{B}_b + B_b^T \mathbf{B}_b + B_q^T \mathbf{A} B_q \right) \, ds \, T_g^T
\]

\[
= T_g \left\{ C_m A^{-1} T_m \left[ \int \mathbf{P}_m \mathbf{A} \mathbf{P}_m \, ds \right] A_m^{-1} C_m + C_m A^{-1} T_m \left[ \int \mathbf{P}_m \mathbf{B} \mathbf{P}_m \, ds \right] A_b^{-1} C_b \\
+ C_b A^{-1} T_b \left[ \int \mathbf{P}_b \mathbf{B} \mathbf{P}_b \, ds \right] A_m^{-1} C_m + C_b A^{-1} T_b \left[ \int \mathbf{P}_b \mathbf{B} \mathbf{P}_b \, ds \right] A_b^{-1} C_b \\
+ C_q A^{-1} T_q \left[ \int \mathbf{P}_q \mathbf{A} \mathbf{P}_q \, ds \right] A_q^{-1} C_q \right\} T_g^T.
\]

(21)

The torsional rotation of the normal from the mid-surface is assumed to have the governing strain energy:

\[
\pi_t = k_i A_{66} h \int_\Omega \left[ \Delta \tilde{\varphi}_t - \frac{1}{2} \left( \frac{\partial \Delta \tilde{v}}{\partial s} - \frac{\partial \Delta \tilde{u}}{\partial s} \right) \right]^2 \, ds,
\]

(22)

where \( A_{66} \) is taken from the extensional stiffness matrix \( A \), and \( k_i \), by numerical trials, is chosen to be equal to 10. In applying the QCT, a constant strain field is assumed, \( P_d = 1 \). Using Greens theorem, the torsional stiffness will be:

\[
K_d = k_i A_{66} h T_g \left\{ C_d A_d^{-1} T_d \left[ \int \mathbf{P}_d^T \mathbf{P}_d \, ds \right] A_d^{-1} C_d \right\} T_g^T.
\]

(23)

The above formulation of the drilling stiffness takes its roots from the formulation of Kanok-Nukulchai [27], where the rotation of the normal from the mid-surface is assumed to have governing strain energy. It is noted here that numerous formulations have been put forward to incorporate drilling stiffness. One is the use of the ‘vertex connectors’, introduced by Allman [28], which was also adopted in Kim et al. [20] for a quasi-conforming shell element. In the work of Hughes and Brezzi [29], the drilling degree of freedom was studied by using a number of variational principles. As a result, several formulations based on modified variational principles were proposed. Simo et al. [30] and Ibrahimbegovic [31] also presented drilling D.O.F formulations on stress-resultant-based geometrically non-linear shell models. Sansour and Bednarczyk [32] have formulated a shell element using the Cosserat continuum wherein the drilling dof was inherent in the theoretical framework itself.
Shi and Voyiadjis [18] presented a geometrically non-linear formulation of the QCT for four-node and three-node shell elements. They used the von Kármán assumption. The present formulation uses the full expression of the Green strain tensor. Aside from membrane stress resultants, the bending and transverse shear stress resultants are included in the geometric stiffness formulation, which make the present formulation more suitable for buckling analysis of shell structures. Another geometrical non-linear formulation of the QCT was presented by Guan and Tang [13], on a nine-node element. The geometrical stiffness matrix uses all the non-linear strain terms, however, numerical integration was required. For a purely geometrically non-linear analysis, the present formulation has the advantage of being computationally more efficient due to the explicit form of tangent stiffness matrix.

The non-linear term of Equation (14) can be written in matrix form so that the QCT can be applied:

\[
\int_\Omega (N^T \delta \Delta \vec{\varepsilon}_m + M^T \delta \Delta \vec{\varepsilon}_b + Q^T \delta \Delta \vec{\varepsilon}_q) \, dr \, ds = \int_\Omega \delta \Delta \eta^T F_g \Delta \eta \, dr \, ds. \tag{24}
\]

The strain vector as \(\Delta \eta\) and the resultant force matrix as \(F_g\) are:

\[
\Delta \eta = \begin{bmatrix}
\frac{\partial \Delta \ddot{u}}{\partial r} & \frac{\partial \Delta \ddot{u}}{\partial s} & \frac{\partial \Delta \ddot{v}}{\partial r} & \frac{\partial \Delta \ddot{v}}{\partial s} & \frac{\partial \Delta \ddot{w}}{\partial r} & \frac{\partial \Delta \ddot{w}}{\partial s} & \frac{\partial \Delta \ddot{\phi}_x}{\partial r} & \frac{\partial \Delta \ddot{\phi}_x}{\partial s} & \frac{\partial \Delta \ddot{\phi}_y}{\partial r} & \frac{\partial \Delta \ddot{\phi}_y}{\partial s}
\end{bmatrix}^T,
\]

\[
F_g = \begin{bmatrix}
F_{g1} & F_{g2} & F_{g3} \\
0 & F_{g4} & \text{symm.} \\
\text{symm.} & F_{g5} & \text{symm.}
\end{bmatrix}, \tag{26}
\]

where:

\[
F_{g1} = \begin{bmatrix}
N_r & N_{rs} & 0 & 0 & 0 & 0 \\
N_s & 0 & 0 & 0 & 0 & 0 \\
N_r & N_{rs} & 0 & 0 & 0 & 0 \\
N_s & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad F_{g2} = \begin{bmatrix}
0 & 0 & M_r & M_{rs} & 0 & 0 \\
0 & 0 & M_{rs} & M_s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\text{symm.} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
F_{g3} = \begin{bmatrix}
0 & Q_{rt} & 0 & 0 & 0 \\
0 & Q_{st} & 0 & 0 & 0 \\
-Q_{rt} & 0 & 0 & 0 & 0 \\
-Q_{st} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad F_{g4} = \begin{bmatrix}
0 & 0 & M_r/2 & 0 \\
0 & 0 & M_{rs}/2 & 0 \\
0 & 0 & M_{rs}/2 & 0 \\
M_r/2 & M_{rs}/2 & 0 & 0 \\
M_{rs}/2 & M_s/2 & 0 & 0
\end{bmatrix},
\]

\[
F_{g5} = \begin{bmatrix}
0 & 0 & Q_{rt}/2 & 0 & 0 \\
0 & 0 & Q_{st}/2 & 0 & 0 \\
\text{symm.} & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{27}
\]
A constant strain field is assumed for the non-linear strains components:

\[
\Delta \eta = P_g \Delta x_g = \begin{bmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
\Delta \alpha_{g1} \\
\vdots \\
\Delta \alpha_{g15}
\end{bmatrix}.
\] (28)

Similarly to the derivation for the linear strains, the strain vector can be expressed as:

\[
\Delta \eta = B_g T_g^T \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} = P_g A_g^{-1} C_g T_g^T \left\{ \begin{array}{c}
\Delta \tilde{U} \\
\Delta \tilde{\theta}
\end{array} \right\}.
\] (29)

where \( A_g \) is a \( 15 \times 15 \) diagonal matrix of the inverse of the mid-surface area, and \( C_g \) is derived using the previously defined string functions. Since only a constant approximation is assumed for the strain vector, the components of \( A_g \) and \( C_g \) would be the same as some of the components of \( A \) and \( C \) for the linear strain terms. These components do not have to be re-derived or re-computed during implementation, e.g., \( C_g(1,j) = C_m(1,j) \) \( (j = 1, 7, 13, 19) \) and \( C_g(8,j) = -C_m(5,j) \) \( (j = 3 - 5, 9 - 11, 15 - 17, 21 - 23) \).

Rows \( C_g(5,j) \) to \( C_g(7,j) \), \( C_g(10,j) \) are solved by converting the area integral to line integrals:

\[
\begin{align*}
\mathbf{C}_g(5,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \frac{\partial \tilde{w}}{\partial r} \, dr \, ds = \oint \tilde{w} n_r \, dL, \\
\mathbf{C}_g(6,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \frac{\partial \tilde{w}}{\partial s} \, dr \, ds = \oint \tilde{w} n_s \, dL, \\
\mathbf{C}_g(7,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \frac{\partial \tilde{\phi}}{\partial r} \, dr \, ds = \oint \tilde{\phi} n_r \, dL = \oint \left[ \tilde{\phi} n_r - \Delta \tilde{\phi} n_r \right] n_r \, dL, \\
\mathbf{C}_g(10,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \frac{\partial \tilde{\phi}}{\partial s} \, dr \, ds = \oint \tilde{\phi} n_s \, dL \quad (j = 1, 2, \ldots, 24),
\end{align*}
\] (30)

where \( n_r \) and \( n_s \) are the side normal and tangent.

Rows 13 to 15 of \( C_g \) are determined with the following:

\[
\begin{align*}
\mathbf{C}_g(13,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \tilde{\phi}_r \, dr \, ds \simeq \int_{\Omega} \left[ \lambda_s \frac{\partial \tilde{w}}{\partial s} + (1 - \lambda_s) \tilde{\phi}_r \right] \, dr \, ds \\
&= \lambda_s \oint \tilde{w} n_s \, dL + (1 - \lambda_s) \int_{\Omega} \tilde{\phi}_r \, dr \, ds, \\
\mathbf{C}_g(14,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \tilde{\phi}_s \, dr \, ds = -\lambda_r \oint \tilde{w} n_r \, dL + (1 - \lambda_r) \int_{\Omega} \tilde{\phi}_s \, dr \, ds, \\
\mathbf{C}_g(15,j) \left\{ \begin{array}{c}
\Delta \tilde{u} \\
\Delta \tilde{\phi}
\end{array} \right\} &= \int_{\Omega} \tilde{\phi}_l \, dr \, ds \quad (j = 1, 2, \ldots, 24),
\end{align*}
\] (31)
with:

$$\lambda = \frac{1}{1 + 12(D_{11}/A_{11}L^2)},$$

where \(D_{11}, A_{11},\) and \(L\) are: the first component of the flexural rigidity, the first component of the transverse shear rigidity and the side length, respectively.

The remaining area integral is integrated by isoparametric mapping. The stress resultants in matrix \(F_g\) are defined as:

$$N = A\tilde{e}_m + B\tilde{e}_b = [A P_m A^{-1}_m C_m + B P_b A^{-1}_b C_b] T_g \begin{bmatrix} \Delta \tilde{U} \\ \Delta \tilde{\theta} \end{bmatrix},$$

(33)

$$M = B\tilde{e}_m + D\tilde{e}_b = [B P_m A^{-1}_m C_m + D P_b A^{-1}_b C_b] T_g \begin{bmatrix} \Delta \tilde{U} \\ \Delta \tilde{\theta} \end{bmatrix},$$

(34)

$$Q = \tilde{A}\tilde{e}_q = \tilde{A} P_q A^{-1}_q C_q T_g \begin{bmatrix} \Delta \tilde{U} \\ \Delta \tilde{\theta} \end{bmatrix}.$$  

(35)

The resulting geometric stiffness is

$$K_g = T_g \left\{ C_g A^{-1}_g \left[ \int_{\Omega} P_g^T F_g P_g \, dr \, ds \right] A_g^{-1} C_g \right\} T_g^T.$$  

(36)

**Internal Forces and Updating Shell Configuration**

The resultant stresses are defined in Equations (33)–(35) and the linear strain terms are in Equation (19). Substituting these into the second term on the right-hand side of Equation (8), the internal force vector is calculated as:

$$F = T_g \left\{ C_m A^{-1}_m \left[ \int_{\Omega} P_m^T A P_m \, dr \, ds \right] A_m^{-1} C_m + C_b A^{-1}_b \left[ \int_{\Omega} P_b^T B P_b \, dr \, ds \right] A_b^{-1} C_b \right\} + C_b A^{-1}_b \left[ \int_{\Omega} P_b^T D P_b \, dr \, ds \right] A_b^{-1} C_b + C_q A^{-1}_q \left[ \int_{\Omega} P_q^T \tilde{A} P_q \, dr \, ds \right] A_q^{-1} C_q \right\} T_g \begin{bmatrix} \Delta \tilde{U} \\ \Delta \tilde{\theta} \end{bmatrix}.$$  

(37)

**NUMERICAL EXAMPLES**

In order to validate the numerical performance of the formulation, the results of linear and geometrically non-linear static problems are presented. The FGM material formulation of the nonlinear four-node quasi-conforming shell element is implemented into the general purpose non-linear dynamic finite element Package, XFINAS [33], developed in AIT and Konkuk University. XFINAS is an extended version of the
non-linear finite element package FINAS, developed in Imperial College, London. Since the linear stiffness formulation of the present nonlinear element is the same as the in Refs. [20,21], the present element passes the membrane, bending and transverse shear patch tests and gives the same results for isotropic materials. Delale and Erdogan [6] indicated that the effect of Poisson’s ratio on the deformation was much less essential than that of Young’s modulus. Thus, Poisson’s ratio of the plate and shell is assumed to be constant. However, Young’s modulus in the thickness direction of the FGM plates and shells can vary with power-law functions (P-FGM), exponential functions (E-FGM), or with sigmoid functions (S-FGM). In every case, the bottom face of the shell is assumed to be metal rich and the top face is assumed to be pure ceramic. In this analysis, linear and geometrically non-linear elastic behavior under mechanical loads of the FGM plates and shells was considered.

Simply Supported Rectangular FGM Plate

First, the results of S-FGM plates (see Figure 3) using the classical plate theory [34] are compared with present solutions using the first-order shear deformation theory for validation. The material properties are \( E_2 = 2.1 \times 10^6 \text{kg/cm}^2, \nu = 0.3, a = 100 \text{cm}, h = 2 \text{cm}, \) and \( q_0 = 1.0 \text{kg/cm}^2. \)

The results of classical and first-order theory by Chi and Chung [34] are plotted in Figure 4. It shows that more \( E_1/E_2 \) brings larger deflection, because larger \( E_1/E_2 \) decreases the stiffness of the FGM plate. The present and reference results agree very well.

In order to investigate the effects of the aspect ratio \( a/b, \) the center deflection of the FGM plate is shown in Figure 5. The center deflection increases upon raising the aspect ratio for \( a/b \) is less than 3. In Figures 4 and 5, it is clear that the results show very good agreement, because of the large side-to-thickness ratio. As expected, the less the side-to-thickness ratio is, the error becomes is larger. In this study, all discussions are based on the first-order shear deformation theory.

Second, in order to show the applicability of the present shell element and FGM formulation, S-FGM plate is solved. Here we present some representative results of the Navier solutions obtained for a simply supported square plate under uniformly distributed

\[ z \]

\[ a \]

\[ b \]

\[ h \]

\[ E(z), \nu(z) \]

\[ x \]

\[ y \]

\[ x \]

\[ y \]

\[ E(z), \nu(z) \]

Figure 3. Geometry of FGM plate.
Figure 4. The deflection of S-FGM plate along the x direction for different \( E_1/E_2 \).

Figure 5. Normalized center deflection of S-FGM plate vs the aspect ratio for different \( E_1/E_2 \).
load of intensity \( q_0 \). The boundary condition is simply supported and the following material and geometrical properties are used. In all cases of FGM plates and shells, the material properties of Equation (38) are used in computing the numerical values:

\[
E_1 = 151 \times 10^9 \text{ Pa}, \quad E_2 = 70 \times 10^9 \text{ Pa}, \quad v = 0.3, \quad a/h = 100, \quad q_0 = 1.0 \text{ N/m}^2, \quad (38)
\]

where \( E_1 \) and \( E_2 \) denote the property of the top and bottom faces of the plate, respectively. The results are presented in non-dimensional form using the equation:

\[
\bar{w} = w \times \frac{E_1 h^3}{q_0 a^4}.
\]

The non-dimensional center deflections are shown in Table 1, which shows that the present shell element and FGM formulation obtain excellent agreement with the result by analytical solution.

Table 1. Normalized center deflection of S-FGM plate (power-law index: \( p = 10 \)).

<table>
<thead>
<tr>
<th>Full mesh</th>
<th>Normalized deflection</th>
<th>Ratio (present/exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 x 4</td>
<td>0.060309</td>
<td>0.897</td>
</tr>
<tr>
<td>8 x 8</td>
<td>0.065504</td>
<td>0.975</td>
</tr>
<tr>
<td>16 x 16</td>
<td>0.066787</td>
<td>0.994</td>
</tr>
<tr>
<td>32 x 32</td>
<td>0.067104</td>
<td>0.998</td>
</tr>
<tr>
<td>64 x 64</td>
<td>0.067180</td>
<td>0.999</td>
</tr>
<tr>
<td>Analytical solution(^a)</td>
<td>0.067217</td>
<td>–</td>
</tr>
</tbody>
</table>

\(^a\)Results are computed using the Navier method with first-order shear deformation theory, independently.

In Figure 7, it can be observed that the pure metal plate has the largest central deflection and the pure ceramic plate has the smallest one. The deflection curves of the functionally graded material plates are located between those of the metal and ceramic plates.

### Hinged FGM Shell

In this section, the snap through behavior of two cylindrical shells subjected to a point load at the center is analyzed. The straight edges are hinged and the curved edges are free from any support. Figure 8 presents the geometry of a shell. Only one quarter of the problem is modeled using 8 x 8 meshes:

\[
E = 3.103 \text{ kN/mm}^2, \quad v = 0.3, \quad P = 1 \text{ kN}.
\]

Two different thickness values are considered, i.e., \( h = 12.7 \text{ mm} \) and \( h = 6.35 \text{ mm} \). The curves of the central deflection vs. load are given in Figure 9, showing good
Figure 6. Normalized center deflection vs S-FGM power-law index.

Figure 7. Normalized deflection vs S-FGM power-law index.
agreement between the present solution and the reference solutions by Sansour and Bednarczyk [32], Crisfield [35], Rhiu and Lee [36], and Kim et al. [37]. Also, the present quasi-conforming solution gives better accuracy than the quasi-conforming solution by Shi and Voyiadjis [18] featured in Figure 9.

As the above figures prove, the results of the present theory agree very well with those of Sansour and Bednarczyk [32], Crisfield [35], Rhiu and Lee [36], and Kim et al. [37]. The results by Shi and Voyiadjis [18] show some discrepancy before and after the limit point since their theory is based on the von Kármán plate theory.
Figures 10 and 11 show the central deflection due to a mechanically applied load $F$ for hinged cylindrical shells. The analysis of the FGM shells is conducted for two types. The first shell has thickness $h = 12.6$ mm and the second one has thickness $h = 6.3$ mm. The stiffness of the shells, or the slope of the load–deflection curve, decreases with increasing load. At the limit point, the load–deflection curve reaches a zero slope. The snap-through buckling of the shell occurs at this limit point. Also, the load–deflection curve for the shell of $h = 12.6$ mm shows the upper and lower limit load but the load–deflection curve of the second shell ($h = 6.3$ mm) shows the upper limit load and snap-back behavior in the range of the negative loading.

Hemispherical FGM Shell with $18^\circ$ Hole

The hemispherical shell with two inward and two outward forces $90^\circ$ apart is considered in the following example. Figure 12 presents geometry and material properties of the shell. There are two issues that are crucial for an element to yield good results in this problem. First, an inextensional-bending mode must be allowed; and second, rigid-body motion must be well expressed. Because the present problem can undergo large displacements, it may be difficult to produce the result fully (Figure 13). Symmetry conditions with $16 \times 16$ meshes are used in this example. The loads are increased by a factor of 490 to compare with results by Saleeb et al. [38], Simo et al. [39], and Lee and Kanok-Nukulchai [40]. Figure 13 shows the good performance of the large displacement analysis.

Figure 14 shows the deflection of loading point of the hemispherical shell with $18^\circ$ hole under a mechanically applied load $F$. As one could expect, the load–deflection curves
of the functionally graded material shells are located between those of the metal and ceramic shells. Even though the FGM shell \( p = 10.0 \) contains a small volume fraction of ceramic, it is much stiffer than the pure metal shell.

**Non-linear Analysis of Stretched Cylinder**

The problem definition of the stretching of a cylinder with free ends is shown Figure 15. The shell is subjected to a pair of concentrated forces. The analysis considers large
Figure 13. Center deflection vs load parameter for isotropic hemispherical shell under point load.

Figure 14. Center deflection vs load parameter for a aluminum–zirconia S-FGM hemispherical shell under point load.
displacements and rotations. One-octant of the structure is analyzed with $24 \times 8$ regular mesh. The following material properties are assumed: the elasticity modulus, $E = 2068.5$ and the Poisson ratio, $\nu = 0.3$. As presented in Figure 15, the cylinder length is $L = 10.35$ and the radius is $R = 4.953$ with thickness $h = 0.1$. The results of the present analysis are compared in Figure 16 with the reference solutions of Sansour et al. [41], and Fontes Valente et al. [42].

![Figure 15. Pinched cylinder with free edge.](image)

![Figure 16. Deflection vs load parameter for isotropic stretched cylinder under point load.](image)
Figure 17 shows the load–deflection curves of the loading point of the S-FGM cylindrical shells due to a mechanically applied load $F$. When the deflection of loading point reaches the level of 2.5, the equilibrium path shows snap-through phenomena. This behavior can be explained by the geometrical shape of shell. The combination of metal and ceramic in the FGM shells can increase the limit load by 52% as compared with the pure metal shell.

**Pinching of a Fixed-Free Cylinder**

Here, the cylindrical shell which is clamped at one end and subjected to a pair of concentrated loads at the other free end is studied. The two forces act in opposite directions as shown in Figure 18. Material properties are: Young modulus, $E = 2.0685 \times 10^7$, Poisson ratio, $v = 0.3$. The cylinder length is $L = 3.048$ and the radius $R = 1.016$ with thickness $h = 0.03$. One-quarter of the structure is analyzed with $28 \times 28$ element mesh. The load–deflection curve is shown in Figure 19 and compared with the reference solution of Fontes Valente et al. [42]. Figure 19 shows that the load–deflection curve goes beyond the highest physically possible deflection of the loading point. The main purpose of the presented example was to test the behavior of the numerical model, although the authors are aware that the result of a real structure can differ from the presented solution due to a possible contact between two deformed regions of the shell in the post-buckling region.

The load–deflection curves for the loading point of the S-FGM cylindrical shell subjected to point load are presented in Figure 20 for a varying value of the S-FGM power-law index ($p = 0.2, 2.0, 10.0$). As one could expect, the load–deflection curve of the pure ceramic shell exhibits the highest value.

![Figure 17. Center deflection vs load parameter for a aluminum–zirconia S-FGM stretched cylinder under point load.](image-url)
CONCLUSIONS

The non-linear formulation of the four-node FGM shell element is developed through the quasi-conforming technique. The geometric stiffness formulation uses the full definition of the Green strain tensor. Without carrying out the numerical integration
which is required for stresses taken from Gauss points, the stresses are directly and accurately taken at the nodes by the explicit integration. Thus, both the explicit integration and elimination of unnecessary extrapolation to the nodal points during the post processing make computational work very efficient. The present non-linear formulation shows good performances in many non-linear benchmark tests for isotropic material. The Navier solutions for simply supported plates based on the linear first-order theory is presented to show how the volume fractions and modulus ratio of the constituents can affect the deflections. The linear and non-linear deflection curves of the FGM plates and shells are intermediate to those of the metal and ceramic plates and shells. Results of FGM plates and shells presented here may be used as the benchmark tests for the large deformation analysis of a FGM shell element.

REFERENCES


