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Simultaneous reconstruction and segmentation for tomography data

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We present a Mumford-Shah like approach for the inversion of CT and SPECT-data (Single Photon Emission Computerized Tomography). With this approach we aim at the simultaneous reconstruction and segmentation of activity and density distribution from given tomography data. We assume the functions to be piecewise constant with respect to a set of contours. Shape sensitivity analysis is used to find a descent direction for the cost functional which leads to an update formula for the contour in a level set framework.

1 Problem Setting

Tomography is a widely used technique in medical imaging. Suppose we are given (noisy) SPECT-data \( y_d : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \) of the attenuated Radon transform of unknown density \( \mu : \mathbb{R}^2 \rightarrow \mathbb{R} \) and activity \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and additional CT-data of the Radon transform of the density \( \mu \).

\[
y_d(s, \omega) \sim A(f, \mu)(s, \omega) = \int_{t \in \mathbb{R}} f(s \omega + t \omega^\perp) \exp \left( -\int_{\tau=t}^{\infty} \mu(s \omega + \tau \omega^\perp) \, d\tau \right) \, dt.
\]

\[
z_d(s, \omega) \sim R \mu(s, \omega) = \int_{\mathbb{R}} \mu(s \omega + t \omega^\perp) \, dt.
\]

We suppose that both functions \( f \) and \( \mu \) vanish outside a bounded domain \( D \subset \mathbb{R}^2 \). Moreover, we assume that both unknowns are piecewise constant with respect to (likewise unknown) partitions of the image domain \( \mathbb{R}^2 \). To specify this, let \( \Gamma \) be any finite collection of pairwise disjoint, closed, bounded curves and let \( \{ \Omega_i^\Gamma \}_{i=1}^n \) denote the set of all bounded connected components of \( \mathbb{R}^2 \setminus \Gamma \). We define the space of piecewise constant functions with respect to the geometry \( \Gamma \) as

\[
PC(\mathbb{R}^2 \setminus \Gamma) = \left\{ \sum_{i=1}^n \alpha_i \chi_{\Omega_i^\Gamma} : \alpha_i \in \mathbb{R} \right\} \subset L^2(D)
\]

where \( \chi_\Omega \) denotes the characteristic function of the set \( \Omega \). The characteristic functions \( \{ \chi_{\Omega_i^\Gamma} \}_{i=1}^n \) form a basis in \( PC(\mathbb{R}^2 \setminus \Gamma) \).

With this definition of piecewise constant functions, we formulate the objective of the reconstruction problem as to find simultaneously the singularity sets \( \Gamma^f, \Gamma^\mu \) and the functions \( f \in PC(\mathbb{R}^2 \setminus \Gamma^f) \) and \( \mu \in PC(\mathbb{R}^2 \setminus \Gamma^\mu) \) such that the given data \( y_d \) and \( z_d \) are fitted best possible in a least-squares sense. We therefore consider the Mumford-Shah like functional \( (1) \)

\[
J(f, \mu, \Gamma^f, \Gamma^\mu) = \|A(f, \mu) - y_d\|_{L^2(\mathbb{R} \times S^1)}^2 + \beta \|R \mu - z_d\|_{L^2(\mathbb{R} \times S^1)}^2 + \alpha (|\Gamma^f| + |\Gamma^\mu|),
\]

where \( |\Gamma| \) is the 1-dimensional Hausdorff measure of \( \Gamma \). Note that it is not necessary to add regularization terms for \( f \) and \( \mu \) since — for fixed \( \Gamma^f, \Gamma^\mu \)— the density \( \mu \) and the activity \( f \) are elements in the finite (usually low) dimensional space \( PC(\mathbb{R}^2 \setminus \Gamma^f/\mu) \). It follows that the identification of \( (f, \mu) \) from the data for fixed \( (\Gamma^f, \Gamma^\mu) \) is well-posed.

2 Minimization Algorithm

An algorithm for the minimization of the functional \( (1) \) which updates both variables \( \Gamma = (\Gamma^f, \Gamma^\mu) \) and \( z = (f, \mu) \) independently is difficult to formulate. This is mainly due to the fact that the geometry \( \Gamma \) defines the domain of definition for the functional variable \( z \) and thus does not allow to treat \( z \) and \( \Gamma \) as independent. We therefore choose a reduced formulation where we alternately fix the geometric variable and minimize with respect to the functional variable and vice versa. In the following we present an overview of the algorithm and its realization.

\textbf{Step 1:} Choose an initial estimate \( \Gamma_0 = (\Gamma^f_0, \Gamma^\mu_0) \) for the geometries.

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Choosing the parameter for given error level such that \( \alpha \)

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\section{Numerics and Parameter Choice Rule}

\subsection*{3 Numerics and Parameter Choice Rule}

Figure 1 shows a reconstruction from synthetically generated CT-data [2]. The original density has 5 regions with constant value, Fig. 1 a). The data was contaminated with 18\% Gaussian noise. The reconstruction of the density distribution and its contour is shown in Fig. 1 b) and c). A comparison of the exact values of \( \mu \) and its reconstruction is given in Table 1.

We wish to remark that the quality of the reconstruction depends heavily on the choice of the regularization parameter \( \alpha \) and its reconstruction \( \mu_{\text{MS}} \) is given in Table 1.

We wish to remark that the quality of the reconstruction depends heavily on the choice of the regularization parameter \( \alpha \) and its reconstruction \( \mu_{\text{MS}} \). We choose the parameter for given error level such that \( \alpha(\delta) \to 0 \) and \( \delta^2/\alpha(\delta) \to 0 \) as \( \delta \to 0 \), we can show that the corresponding reconstructions –functional as well as geometric variable– converge to the true solution.

\begin{table}[h]
\centering
\caption{Values of density distribution: true and reconstructed.}
\begin{tabular}{|c|c|c|}
\hline
Region & \( \mu \) & \( \mu_{\text{MS}} \) \\
\hline
1 & 7 & 6.9 \\
2 & 2 & 1.65 \\
3 & 4 & 4.2 \\
4 & 11 & 11.05 \\
5 & 8 & 8.15 \\
\hline
\end{tabular}
\end{table}

Step 2: For fixed \((\Gamma^f, \Gamma^\mu)\) minimize \( J \) with respect to the pair \((f, \mu) \in PC(\mathbb{R}^2 \setminus \Gamma^f) \times PC(\mathbb{R}^2 \setminus \Gamma^\mu)\) by solving the respective optimality system. I.e., for fixed geometric variables \((\Gamma^f, \Gamma^\mu)\) we solve the variational problem

\begin{equation}
\min_{(f, \mu)} \| A(f, \mu) - y_d \|_{L^2(\mathbb{R} \times S)} + \beta \| R \mu - z_d \|_{L^2(\mathbb{R} \times S)} \quad \text{with} \quad f \in PC(\mathbb{R}^2 \setminus \Gamma^f), \mu \in PC(\mathbb{R}^2 \setminus \Gamma^\mu).
\end{equation}

Assume that \( f = \sum f_i \chi_{\Omega_i} \) and \( \mu = \sum \mu_k \chi_{\Omega_k} \) is the solution to (2). The corresponding vectors of coefficients, \((f_i)\) and \((\mu_k)\), solve a system of nonlinear equations on \( \mathbb{R}^{n(\Gamma^f) \times n(\Gamma^\mu)} \) and are computed by a Newton-type method.

Step 3: Consider the reduced functional \( \hat{J}(\Gamma^f, \Gamma^\mu) = J(f(\Gamma^f), \mu(\Gamma^\mu), \Gamma^f, \Gamma^\mu) \). Find a descent direction for the functional \( J \) with respect to the geometric variable \((\Gamma^f, \Gamma^\mu)\) using techniques from shape sensitivity analysis [3, 4, 2].

The relevant terms for the update of the geometry are the directional derivatives of \( J \) w.r.t. \( \Gamma^f \) and \( \Gamma^\mu \). Let \( A_1^\ast \) and \( (A_2^\ast) \) denote the adjoint of the Frechet derivative of the attenuated Radon transform \( A \) w.r.t. the first and second argument respectively. With \( g = A(f, \mu) - y_d \) and \( h = R \mu - z_d \) we get

\begin{align*}
\frac{d_{\Gamma^f} \hat{J}(\Gamma^f; F)}{\Gamma^f} &= \sum_{i=1}^{n(\Gamma^f)} s_i^f f_i \int_{\partial \Omega_i^f} A_1^\ast(g, \mu)(x) F \, dS + \alpha \int_{\Gamma^f} \kappa F \, dS, \\
\frac{d_{\Gamma^\mu} \hat{J}(\Gamma^\mu; G)}{\Gamma^\mu} &= \sum_{k=1}^{n(\Gamma^\mu)} s_k^\mu \mu_k \int_{\partial \Omega_k^\mu} -((A_2^\ast)^\ast g)(x) + \beta R^\ast h(x) G \, dS + \alpha \int_{\Gamma^\mu} \kappa G \, dS.
\end{align*}

Step 4: Update \( \Gamma \) by moving it in the chosen descent direction according to a chosen line-search rule. Use a level-set formulation for the update of the geometry [5, 6].


\section{References}


