Chamfer Distances with Integer Neighborhoods

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Abstract. In many applications of digital picture processing such as segmentation, skeletonizing and clustering, it is important to compute approximations of the Euclidean distance efficiently. Borgefors, Verwer, Coquin and Bolon, and others have developed a method to do so based on choosing local distances for small neighborhood sizes. These local distances are integer multiples of some fixed scaling factor. In this paper results of a systematic search are presented in the two-dimensional case for neighborhoods of sizes up to 21×21 and scaling factors up to 1000. We consider both the unrestricted case, and the case that distances along the coordinate axes are exact, and the case that all the distances exceed the Euclidean distance. Errors are compared with the minimally possible error in the sense of the maximum relative error.

1 Introduction

The theory on *chamfer distances* has been developed to approximate Euclidean distances in an efficient way. In this theory the distance to the origin w is prescribed on a small neighborhood, e.g. the *mask* $M_p := \left\{(i,j) \in \mathbb{Z}^2 : |i|, |j| \le p\right\}$, and the global distance function is induced by the local distances. We assume that $w(i,j) = w(|i|,|j|) = w(j,i)$ for all i and j. For practical use the prescribed distances are often chosen to be integer multiples of a fixed real number $\frac{1}{r}$. We denote the integer value at (i,j) by $N(i,j)$ and we call r the *scaling factor*. The function N defined on M_p is called an *integer neighborhood*.

The method was pioneered by Borgefors from 1984 onwards, in [1], [2] and [3]. She determined the optimal w for $p = 1$ and $p = 2$ with respect to the *maximum absolute error*:

$$\sup_{0 \le j \le M} \left| w\left((M,j)\right) - \sqrt{M^2 + j^2} \right|$$

for some large $M \in \mathbb{Z}_{>0}$. She also determined the optimal w for $p = 1$, $p = 2$ and $p = 3$ under the restriction that

$$(B): \qquad w(i,0) = |i| \text{ for all } i \; .$$

Borgefors presented some good integer neighborhoods for $p = 1$, $p = 2$ and $p = 3$.

In 1991 Verwer [11] computed the optimal w for all p with respect to the *maximum relative error*

$$e := \limsup_{(i,j)} \left| \frac{w(i,j)}{\sqrt{i^2+j^2}} - 1 \right| . \tag{1}$$

Besides he gave several integer neighborhoods for $p = 1$ and $p = 2$.

In his 1994 PhD thesis [9] Thiel presented numerous examples of integer neighborhoods for $p = 2$, $p = 3$ and $p = 6$.

In 1995 Coquin and Bolon [4] extended the theory to pixels on a rectangular lattice instead of a square one. All their integer neighborhoods refer to the square case, with $p = 1$, $p = 2$ and $p = 3$.

In unpublished work [5] Hajdu, Hajdu and Tijdeman have determined the optimal values of w for all p, with respect to the maximum relative error, under the restriction (B) and also under the condition

$$(D): \qquad w(i,j) \geq \sqrt{i^2+j^2} \text{ for all } (i,j) .$$

Distance functions that satisfy (D) are used in applications where it is vital that distances are not underestimated, such as collision avoidance in robotics (cf. [10]).

In the present paper we give a formula for the maximum relative error of an integer neighborhood under some mild restrictions. Moreover we introduce a general method to generate good integer neighborhoods for all p, both unconditional and under restrictions (B) and (D). By a systematic search all record holders were found for $p \leq 10$ and scaling factor $r \leq 1000$. A selection of the results is presented here and compared with results from the literature. The complete list is available at `www.math.leidenuniv.nl/~scholtus/chamfer.htm`.

2 Theory of Integer Neighborhoods

We denote $M_p \backslash \{(0,0)\}$ by M_p^*. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be vectors in M_p^*. Concatenation of these vectors yields a path $P = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$ in $\mathbb{Z}^2$. For an integer neighborhood N defined on M_p with scaling factor r, the length of this path is defined as:

$$\ell(P) := \frac{1}{r} \sum_{i=1}^{n} N(\boldsymbol{u}_i) . \tag{2}$$

The empty path gets length zero: $\ell(\emptyset) := 0$.

The function N now induces a metric d on the whole of $\mathbb{Z}^2$, by taking $d(\boldsymbol{u}, \boldsymbol{v})$ as the minimal length over all possible paths from $\boldsymbol{u}$ to $\boldsymbol{v}$ consisting solely of steps from M_p. In particular, $w = \frac{N}{r}$ is extended to a distance function w on the whole of $\mathbb{Z}^2$, by taking $w(\boldsymbol{v})$ as the minimal length over all possible paths from the origin to $\boldsymbol{v}$ composed of steps from M_p. Yamashita and Ibaraki [12] proved that the induced distance is indeed a metric. (The proof is also given by Verwer in [10].)

35	32	29	27	26	25	26	27	29	32	35
32	28	25	22	21	20	21	22	25	28	32
29	25	21	18	16	15	16	18	21	25	29
27	22	18	14	11	10	11	14	18	22	27
26	21	16	11	7	5	7	11	16	21	26
25	20	15	10	5	0	5	10	15	20	25
26	21	16	11	7	5	7	11	16	21	26
27	22	18	14	11	10	11	14	18	22	27
29	25	21	18	16	15	16	18	21	25	29
32	28	25	22	21	20	21	22	25	28	32
35	32	29	27	26	25	26	27	29	32	35

Fig. 1. Example of a distance function. The original integer neighborhood is enclosed by the box.

As an example, Fig. 1 shows the values near $(0, 0)$ of a distance function induced by an integer neighborhood. To compare with the true Euclidean length, we have to divide by the scaling factor. This integer neighborhood has maximum relative error 0.0198 if we take $r = 5$, and 0.0179 if we choose $r = 5.0092$. In the former case, suggested by Borgefors in [2], restriction (B) is satisfied. The latter scaling factor, suggested by Verwer in [11], yields the smallest possible maximum relative error for this integer neighborhood.

3 Optimal Maximum Relative Errors

The optimal maximum relative errors have been computed by Verwer [11] in the general case and by Hajdu, Hajdu and Tijdeman [5] in all three cases. They are given by

$$e_p = \frac{\sqrt{2p^2 + 2 - 2p\sqrt{p^2+1}} - 1}{\sqrt{2p^2 + 2 - 2p\sqrt{p^2+1}} + 1} \tag{3}$$

in the general case, by

$$e_p^B = \frac{p^2 + 2 - p\sqrt{p^2+1} - 2\sqrt{p^2 + 1 - p\sqrt{p^2+1}}}{p^2} \tag{4}$$

under the condition (B), and by

$$e_p^D = \sqrt{\left(\sqrt{p^2+1} - p\right)^2 + 1} - 1 \tag{5}$$

Table 1. Approximate values of the optimal maximum relative error in the general case and under restrictions (B) and (D), for $1 \leq p \leq 10$.

p	e_p	e_p^B	e_p^D
1	0.03956613	0.05505271	0.08239220
2	0.01355683	0.01869475	0.02748630
3	0.00649823	0.00893928	0.01308146
4	0.00376031	0.00516800	0.00754900
5	0.00243927	0.00335091	0.00489047
6	0.00170657	0.00234378	0.00341897
7	0.00125948	0.00172949	0.00252214
8	0.00096713	0.00132791	0.00193614
9	0.00076570	0.00105127	0.00153258
10	0.00062112	0.00085272	0.00124302

under the condition (D). Table 1 shows rounded values of e_p, e_p^B and e_p^D for small values of p. As may be expected, the unrestricted case yields smaller optima than the (B)-case, which in turn surpasses the (D)-case, where we set the most severe restriction.

According to [5] the neighborhoods that achieve the optimal maximum relative error are given in the general case by

$$w_p(i,j) = (1 - e_p)\sqrt{i^2 + j^2} \qquad \left((i,j) \in M_p^*\right) , \tag{6}$$

under the condition (B) by

$$w_p^B(i,j) = \begin{cases} |i| & \text{if } 1 \leq |i| \leq p,\ j = 0 \\ |j| & \text{if } 1 \leq |j| \leq p,\ i = 0 \\ \left(1 - e_p^B\right)\sqrt{i^2 + j^2} & \text{for all other vectors in } M_p^* \end{cases} \tag{7}$$

and under the condition (D) by the Euclidean distance

$$w_p^D(i,j) = \sqrt{i^2 + j^2} \qquad \left((i,j) \in M_p^*\right) . \tag{8}$$

These neighborhoods can not be written in terms of an integer neighborhood and therefore have only theoretical value.

4 Computing the Maximum Relative Error

We now give upper bounds for the maximum relative error of good approximating integer neighborhoods. We omit the proofs, which are given in [8].

For $p = 1$, the integer neighborhood N has two distinct values,

$$N(i,j) = \begin{cases} n_0 & \text{if } |i| + |j| = 1 \\ n_1 & \text{if } |i| = |j| = 1 \end{cases}$$

and a scaling factor r.

Theorem 1. [8] *Let N be an integer neighborhood on M_1 with scaling factor r. Suppose that $n_0 \le n_1 \le 2n_0$. Then the maximum relative error of N is given by*

$$e = \max\left\{1 - \frac{1}{r}\min\left(n_0,\ \frac{1}{2}n_1\sqrt{2}\right),\ \frac{1}{r}\sqrt{n_0^2 + (n_1 - n_0)^2} - 1\right\}\ . \tag{9}$$

Note that $\frac{n_1}{n_0}$ should approximate $\sqrt{2}$ so that every reasonable neighborhood will satisfy the condition $n_0 \le n_1 \le 2n_0$. Similarly, every good approximation to the Euclidean distance will satisfy the first two conditions of Theorem 2. (The third condition is more restrictive.)

Theorem 2. [8] *Let $p \ge 2$ and let N be an integer neighborhood on M_p with scaling factor r. If the neighborhood satisfies*

(i) $N(p,0) \le N(p,1)$ *and* $N(p,j) \le N(p,j+1) < \frac{j+1}{j}N(p,j)$ *for* $j = 1,\ldots,p-1$,
(ii) $N(p,j-1) + N(p,j+1) \ge 2N(p,j)$ *for* $j = 1,\ldots,p-1$,
(iii) $\frac{N(i,j)}{\sqrt{i^2+j^2}} \ge \min\limits_{0\le k\le p} \frac{N(p,k)}{\sqrt{p^2+k^2}}$ *for all* $(i,j) \in M_p^*$,

then the maximum relative error satisfies

$$e \le \max\left\{1 - \frac{1}{r}\min_{0\le k\le p}\frac{N(p,k)}{\sqrt{p^2+k^2}},\ \frac{1}{r}\max_{0\le l\le p-1} H_l - 1\right\}\ , \tag{10}$$

where H_l is given by

$$H_l = \begin{cases} \sqrt{\frac{1}{p^2}\left\{(l+1)N(p,l) - lN(p,l+1)\right\}^2 + (N(p,l+1) - N(p,l))^2} & \text{if } g(l) = l \\ \frac{N(p,l)}{\sqrt{p^2+l^2}} & \text{if } g(l) < l \\ \frac{N(p,l+1)}{\sqrt{p^2+(l+1)^2}} & \text{if } g(l) > l \end{cases}$$

with $g(l) = \textbf{floor}\left(\frac{p^2(N(p,l+1)-N(p,l))}{(l+1)N(p,l)-lN(p,l+1)}\right)$.

Note that in both Theorem 1 and Theorem 2, the upper bound for the maximum relative error is of the following form:

$$\max\left\{1 - \frac{c_{\min}}{r},\ \frac{c_{\max}}{r} - 1\right\}\ . \tag{11}$$

For a given integer neighborhood N, the lowest maximum relative error is found by applying the scaling factor (cf. Verwer [11])

$$r = \frac{1}{2}\left(c_{\min} + c_{\max}\right)\ , \tag{12}$$

since the two terms of (11) then have equal value $\frac{c_{\max}-c_{\min}}{c_{\max}+c_{\min}}$. Observe however that this is only possible in the unrestricted case, since the conditions (B) and (D) are not satisfied unless $r = N(1,0)$.

In Sections 5, 6 and 7 we present tables of parameters determining integer neighborhoods in the general case, the (B)-case and the (D)-case, respectively. For each integer neighborhood the maximum relative error is given. These tables enable one to select a neighborhood with maximum relative error smaller than a prescribed bound down to about 0.1%.

```
μ := n;
for j = 0, 1, ..., p
    N(±p, ±j) := round (n * (1 − e_p) * sqrt(p^2 + j^2));
    a := N(p, j)/sqrt(p^2 + j^2);
    if a < μ
        μ := a;
    end
end
for all other (i, j) ∈ M_p^*
    N(i, j) := ceil (μ * sqrt(i^2 + j^2));
end
```

Fig. 2. A pseudocode for constructing ${}_nN_p$.

```
μ := n;
for j = 0, 1, ..., p
    if j = 0, 1
        N(±p, ±j) := round (n * (1 − e_p) * sqrt(p^2 + j^2));
    else
        N(±p, ±j) := ceil (n * (1 − e_p) * sqrt(p^2 + j^2));
    end
    a := N(p, j)/sqrt(p^2 + j^2);
    if a < μ
        μ := a;
    end
end
for all other (i, j) ∈ M_p^*
    N(i, j) := ceil (μ * sqrt(i^2 + j^2));
end
```

Fig. 3. A pseudocode for constructing ${}_n^*N_p$.

5 Integer Neighborhoods: the Unrestricted Case

We now define two classes of integer neighborhoods ${}_nN_p$ and ${}_n^*N_p$ that satisfy the conditions of the previous section. Theorem 1 and Theorem 2 may then be used to compute (for $p = 1$), resp. bound (for $p \geq 2$) the maximum relative error of these neighborhoods.

For $p = 1$ and a positive integer n, we define ${}_nN_1$ by choosing

$$ {}_nN_1(1,0) := \mathbf{round}\,(n * (1 - e_1))\,,\ {}_nN_1(1,1) := \mathbf{round}\,(n * (1 - e_1) * \mathbf{sqrt}(2)) $$

and ${}_n^*N_1$ by choosing

$$ {}_n^*N_1(1,0) := \mathbf{round}\,(n * (1 - e_1))\,,\ {}_n^*N_1(1,1) := \mathbf{ceil}\,(n * (1 - e_1) * \mathbf{sqrt}(2))\ . $$

It is established in [8] that $n_0 \leq n_1 \leq 2n_0$ holds in both cases for every $n \geq 1$, i.e. we may apply Theorem 1 to these neighborhoods.

Table 2. A selection of good integer neighborhoods. Plain values of n refer to the class ${}_nN_p$ and starred values refer to ${}^*_nN_p$. For each p we write the optimal maximum relative error on the first line and the values of n and $*n$ for which Theorem 1 or Theorem 2 is valid on the second.

n	r	max.rel.err.
$p = 1$		$e_1 \approx 0.03956613$
$n \geq 1;\ *n \geq 1$		
2	2.11803399	0.05572809
5	5.16745614	0.04213072
∗12	12.50000000	0.04000000
30	30.18804384	0.03964039
73	72.88469348	0.03957887
425	424.80817467	0.03956650
$p = 2$		$e_2 \approx 0.01355683$
$n \geq 1;\ *n \geq 1$		
∗3	3.10078106	0.03250183
5	5.00918453	0.01793405
∗8	8.12310563	0.01515500
∗13	13.15542917	0.01415650
26	25.84377437	0.01364350
73	72.98986348	0.01356166
∗846	845.96880353	0.01355701
$p = 3$		$e_3 \approx 0.00649823$
$n \geq 2;\ *n = 1, \geq 3$		
∗6	6.07318149	0.01204994
7	7.00241013	0.00980486
13	13.05828538	0.00724912
31	30.86913428	0.00655890
∗62	62.06831521	0.00650736
701	700.88791323	0.00649830
$p = 4$		$e_4 \approx 0.00376031$
$n \geq 1;\ *n = 2, \geq 4$		
9	8.99982852	0.00617307
17	17.04390276	0.00430301
40	39.90661564	0.00392455
∗57	56.96504416	0.00377502
106	105.89896518	0.00376741
∗530	529.99295103	0.00376033
$p = 5$		$e_5 \approx 0.00243927$
$n = 2, 3, \geq 5;\ *n = 2, 4, \geq 6$		
∗20	20.05742666	0.00334964
∗31	31.05805001	0.00279390
62	61.93358078	0.00253494
∗80	80.00000000	0.00250000
∗101	101.04740862	0.00244844
∗982	981.99537787	0.00243930
$p = 6$		$e_6 \approx 0.00170657$
$n = 4, 6, \geq 8;\ *n = 2, \geq 5$		
∗24	24.04159908	0.00211365
∗37	37.04553981	0.00204836
62	61.92458975	0.00178557
73	72.95668374	0.00175354
∗98	97.98863635	0.00174754
932	931.92386027	0.00170671
$p = 7$		$e_7 \approx 0.00125948$
$n = 3\text{–}5, \geq 7;\ *n = 2, 6, 7, 9, \geq 11$		
∗44	44.02876730	0.00166679
∗46	46.02748689	0.00152594
71	70.94781656	0.00134912
82	81.96694322	0.00133957
99	98.98053814	0.00128318
∗310	309.96175937	0.00125954
$p = 8$		$e_8 \approx 0.00096713$
$n \geq 2;\ *n = 6\text{–}8, 10\text{–}12, \geq 14$		
∗37	37.00818277	0.00123843
∗54	54.01858061	0.00119752
73	72.96066260	0.00117409
81	80.95002953	0.00098064
97	96.96587816	0.00097715
∗611	610.96583216	0.00096716
$p = 9$		$e_9 \approx 0.00076570$
$n = 2, 5\text{–}8, 10\text{–}13, 15, \geq 17;$		
$*n = 3, 8, 9, \geq 11$		
∗60	60.01735720	0.00097506
82	81.96514082	0.00093030
90	89.96587281	0.00085570
109	108.96760745	0.00079347
127	126.98382399	0.00077176
524	523.95682907	0.00076585
$p = 10$		$e_{10} \approx 0.00062112$
$n = 4\text{–}6, 9\text{–}11, \geq 13;\ *n = 3, 8\text{–}10, \geq 12$		
∗66	66.02065351	0.00075261
100	99.96448551	0.00066583
∗161	160.99813609	0.00062499
∗963	962.99811719	0.00062113

Similarly for $p \geq 2$ and a positive integer n, the programs given in Fig. 2 and Fig. 3 compute the elements of the integer neighborhoods ${}_nN_p$ and ${}_n^*N_p$, respectively. Note that the outer values are calculated first, and that the other values are rounded upwards to ensure that condition (*iii*) of Theorem 2 is satisfied.

It turns out (cf. [8]) that condition (*ii*) of Theorem 2 is not satisfied by ${}_nN_p$ and ${}_n^*N_p$ for certain small values of n, particularly as p becomes larger. (Condition (*i*) is actually true for every $n \geq 1$ in both cases.) For $p \leq 10$ the values of n where Theorem 2 may be applied are noted in Table 2.

In both classes, and both for $p = 1$ and for $p \geq 2$, we choose the scaling factor r according to (12), since this leads to the lowest maximum relative error. Thus for a given p, ${}_nN_p$ and ${}_n^*N_p$ are described entirely by one parameter, viz. n. Good integer approximations to the optimal neighborhood w_p defined by (6) may now be found by varying n and evaluating the maximum relative errors of ${}_nN_p$ and ${}_n^*N_p$ for each choice.

In Table 2 a selection of good neighborhoods with $1 \leq p \leq 10$ and $1 \leq n \leq 1000$ is given. The main focus is on values of n below 100, for higher values only marginally improve the maximum relative error. The best choice of n below 1000 is also listed, for completeness.

Some of these neighborhoods have been published before. Verwer [11] suggested ${}_2N_1$, ${}_5N_1$, ${}_{12}^*N_1$ and ${}_5N_2$, all of which are listed in Table 2. He also suggested ${}_4^*N_2$ and ${}_9^*N_2$, but a lower maximum relative error is achieved by ${}_4N_2$ and ${}_8^*N_2$, respectively. Finally, he suggested a neighborhood with $r = 17.2174$ which does not correspond to ${}_{17}N_2$ or ${}_{17}^*N_2$, but ${}_{13}^*N_2$ already yields a lower maximum relative error. Furthermore, Coquin and Bolon [4] suggested ${}_{25}N_1$, but ${}_{12}^*N_1$ achieves the same maximum relative error. Thiel [9] suggested ${}_{73}N_2$, which is listed in Table 2.

Integer neighborhoods have also been published that do not fall within one of the classes ${}_nN_p$ and ${}_n^*N_p$. However, we found no examples in the literature that achieve a better maximum relative error than the values given in Table 2.

6 Integer Neighborhoods with Restriction (B)

We now construct two classes of integer neighborhoods ${}_nN_p^B$ and ${}_n^*N_p^B$ that approximate the optimal neighborhoods w_p^B given by (7). The construction works completely analogously to the general case, but we now impose condition (B).

For $p = 1$ and a positive integer n, we define ${}_nN_1^B$ by

$$ {}_nN_1^B(1,0) := n, \ {}_nN_1^B(1,1) := \mathbf{round}\left(n * \left(1 - e_1^B\right) * \mathbf{sqrt}(2)\right) $$

and ${}_n^*N_1^B$ by

$$ {}_n^*N_1^B(1,0) := n, \ {}_n^*N_1^B(1,1) := \mathbf{ceil}\left(n * \left(1 - e_1^B\right) * \mathbf{sqrt}(2)\right) \ . $$

Theorem 1 is valid for these neighborhoods for all $n \geq 1$, as is established in [8].

For $p \geq 2$, ${}_nN_p^B$ and ${}_n^*N_p^B$ are constructed by adding the following lines to the programs of Fig. 2 and Fig. 3, and replacing e_p by e_p^B:

Table 3. A selection of good integer neighborhoods with restriction (B). Plain values of n refer to the class ${}_nN_p^B$ and starred values refer to ${}_n^*N_p^B$. For each p we also note the values of n and $*n$ for which Theorem 1 or Theorem 2 is valid.

n	max.rel.err.	n	max.rel.err.
$p = 1$	$e_1^B \approx 0.05505271$	$p = 6$	$e_6^B \approx 0.00234378$
$n \geq 1$		$n = 4, 6, \geq 8$	
$*n \geq 1$		$*n = 2, 5, 6, \geq 8$	
3	0.05719096	*25	0.00319490
35	0.05714286	26	0.00295422
110	0.05505474	*30	0.00277393
993	0.05505468	*44	0.00239705
$p = 2$	$e_2^B \approx 0.01869475$	*73	0.00234587
$n \geq 1$		*978	0.00234387
$*n \geq 1$		$p = 7$	$e_7^B \approx 0.00172949$
4	0.03077641	$n = 3\text{–}5, \geq 7$	
5	0.01980390	$*n = 2, 6, 7, 9, \geq 11$	
*26	0.01957019	18	0.00219376
*31	0.01901534	*33	0.00183486
36	0.01872893	*51	0.00173160
314	0.01869518	$p = 8$	$e_8^B \approx 0.00132791$
$p = 3$	$e_3^B \approx 0.00893928$	$n \geq 2$	
$n \geq 2$		$*n = 6\text{–}8, 11, 12, \geq 14$	
$*n = 1, \geq 3$		*37	0.00145985
8	0.01178823	*58	0.00133680
*15	0.00915300	*97	0.00132858
52	0.00901997	*931	0.00132820
*67	0.00898172	$p = 9$	$e_9^B \approx 0.00105127$
97	0.00894079	$n = 2, 5\text{–}8, \geq 10$	
*888	0.00893933	$*n = 3, 7\text{–}9, \geq 11$	
$p = 4$	$e_4^B \approx 0.00516800$	*60	0.00126463
$n \geq 1$		*64	0.00115824
$*n = 2, \geq 4$		*65	0.00106452
9	0.00619201	*87	0.00105638
19	0.00552490	*109	0.00105155
*29	0.00533653	$p = 10$	$e_{10}^B \approx 0.00085272$
*39	0.00524594	$n = 4\text{–}6, 9\text{–}11, \geq 14$	
49	0.00519268	$*n = 3, 8, 10, \geq 12$	
167	0.00516824	*67	0.00100195
$p = 5$	$e_5^B \approx 0.00335091$	*68	0.00097271
$n = 2, 3, \geq 5$		69	0.00094473
$*n = 2, 4, \geq 7$		*73	0.00087362
13	0.00460478	*97	0.00085956
*24	0.00360461	*121	0.00085340
*49	0.00340984	*581	0.00085282
*61	0.00335369		
*476	0.00335099		

```
for i = 1, 2, ..., p
    N(±i, 0) := n * i =: N(0, ±i);
end
```

Just as in the general case, there exist exceptional values of n for which we may not apply Theorem 2 to these neighborhoods. For $p \leq 10$ all values of n for which the theorem is valid are given in Table 3 (cf. [8]).

Condition (B) implies that for both ${}_nN_p^B$ and ${}_n^*N_p^B$ the scaling factor must be set to $r = N(1,0) = n$. By varying n and using Theorem 1 and Theorem 2 to evaluate the maximum relative error, we find good neighborhoods. Table 3 shows a selection of neighborhoods for $1 \leq p \leq 10$ and $n \leq 1000$.

Some of these integer neighborhoods have been published previously. Borgefors [2] suggested neighborhoods corresponding to ${}_3N_1^B$, ${}_5N_2^B$ and ${}_{12}N_3^B$. The first two of these are very good neighborhoods (they are listed in Table 3), but the third one is surpassed in maximum relative error by ${}_8N_3^B$. Coquin and Bolon [4] suggested ${}_{67}^{\;*}N_3^B$, which is also listed in Table 3. No examples were found in the literature that achieve better maximum relative errors than the neighborhoods given in Table 3.

7 Integer Neighborhoods with Restriction (*D*)

Finally we construct a class of integer neighborhoods ${}_nN_p^D$ that approximate the optimal neighborhoods w_p^D given by (8). This time we have to take condition (D) into account.

For $p = 1$ and a positive integer n, ${}_nN_1^D$ is defined by:

$$ {}_nN_1^D(1,0) := n, \; {}_nN_1^D(1,1) := \textbf{ceil}\,(n * \textbf{sqrt}(2)) \ . $$

Theorem 1 is valid for this type of neighborhood for every $n \geq 1$ (cf. [8]).

For $p \geq 2$, ${}_nN_p^D$ is constructed by adding the same lines to the pseudocode of Fig. 2 as in the (B)-case, and replacing the third line by:

$$ N(\pm p, \pm j) := \textbf{ceil}\left(n * \textbf{sqrt}(p^2 + j^2)\right) \ ; $$

Table 4 lists a selection of good neighborhoods under restriction (D), with $1 \leq p \leq 10$ and $n \leq 1000$. For each p, we also give all values of n for which Theorem 2 is valid in this case (cf. [8]). Just as in the (B)-case, the scaling factor is set automatically to $r = n$.

No previous neighborhoods have been suggested under condition (D), other than the classical *city block* distance transformation (cf. [7]), which corresponds to ${}_1N_1^D$ (listed in Table 4).

8 Conclusion

The present paper shows that very good approximations of the Euclidean distance can be obtained by a uniform choice of the distance function on the mask

Table 4. A selection of good integer neighborhoods from the class ${}_nN_p^D$. For each p we also note the values of n for which Theorem 1 or Theorem 2 is valid.

n	max.rel.err.	n	max.rel.err.
$p = 1$	$e_1^D \approx 0.08239220$	$p = 6$	$e_6^D \approx 0.00341897$
$n \geq 1$		$n \geq 5$	
1	0.41421356	33	0.00469513
2	0.11803399	36	0.00427755
7	0.08796759	46	0.00377360
12	0.08333333	70	0.00366675
41	0.08255322	95	0.00353944
408	0.08239301	882	0.00341930
$p = 2$	$e_2^D \approx 0.02748630$	$p = 7$	$e_7^D \approx 0.00252214$
$n \geq 2$		$n \geq 4$	
3	0.06718737	42	0.00295736
8	0.03077641	53	0.00292725
21	0.02795396	82	0.00272940
38	0.02766443	110	0.00264114
72	0.02749621	999	0.00252237
987	0.02748651	$p = 8$	$e_8^D \approx 0.00193614$
$p = 3$	$e_3^D \approx 0.01308146$	$n = 6, \geq 9$	
$n \geq 2$		60	0.00236050
6	0.02439383	64	0.00207305
11	0.01639454	141	0.00204172
18	0.01379376	787	0.00193639
43	0.01316376	$p = 9$	$e_9^D \approx 0.00153258$
80	0.01311710	$n = 5, \geq 8$	
228	0.01308194	53	0.00191660
$p = 4$	$e_4^D \approx 0.00754900$	90	0.00162496
$n \geq 3$		141	0.00160828
14	0.01157207	159	0.00160071
16	0.00971840	993	0.00153272
23	0.00847074	$p = 10$	$e_{10}^D \approx 0.00124302$
40	0.00778222	$n \geq 8$	
73	0.00757126	110	0.00152776
528	0.00754906	111	0.00147768
$p = 5$	$e_5^D \approx 0.00480947$	115	0.00140582
$n \geq 3$		120	0.00129916
28	0.00650943	217	0.00128398
30	0.00596002	822	0.00124315
49	0.00519268		
59	0.00515763		
70	0.00498756		
919	0.00489059		

M_p. This distance function is completely determined by a parameter n which equals or approximates the scaling factor. The tables given here make it possible to select mask size p and parameter n to guarantee that the approximation of the Euclidean distance does not exceed a prescribed maximum relative error down to about 0.1%.

References

1. Borgefors, G.: Distance transformations in arbitrary dimensions. *Computer Vision, Graphics, and Image Processing* **27** (1984), 321–345.
2. Borgefors, G.: Distance transformations in digital images. *Computer Vision, Graphics, and Image Processing* **34** (1986), 344–371.
3. Borgefors, G.: Another comment on "A note on 'Distance transformations in digital images' ". *CVGIP: Image Understanding* **54** (1991), 301–306.
4. Coquin, D., Bolon, Ph.: Discrete distance operator on rectangular grids. *Pattern Recognition Letters* **16** (1995), 911–923.
5. Hajdu, A., Hajdu, L., Tijdeman, R.: Approximation of Euclidean distances by chamfer distances. Unpublished.
6. Montanari, U.: A method for obtaining skeletons using a quasi-Euclidean distance. *Journal of the Association for Computing Machinery* **15** (1968), 600–624.
7. Rosenfeld, A., Pfaltz, J.L.: Distance functions on digital pictures. *Pattern Recognition* **1** (1968), 33–61.
8. Scholtus, S.: Master Thesis, 2006, Universiteit Leiden. Will be downloadable from: `www.math.leidenuniv.nl/~scholtus/chamfer.htm`
9. Thiel, E.: *Les distances de chanfrein en analyse d'images: fondements et applications.* PhD Thesis, 1994, Université Joseph Fourier de Grenoble. Downloadable from: `www.lis.inpg.fr/stages_dea_theses/theses/manuscript/thiel.pdf`
10. Verwer, B.H.J.: *Distance Transforms: Metrics, Algorithms and Applications.* PhD Thesis, 1991, Technische Universiteit Delft.
11. Verwer, B.H.J.: Local distances for distance transformations in two and three dimensions. *Pattern Recognition Letters* **12** (1991), 671–682. (Corresponds with Chapter 3 of [10].)
12. Yamashita, M., Ibaraki, T.: Distances defined by neighborhood sequences. *Pattern Recognition* **19** (1986), 237–246.