A SURVEY ON k–FREENESS

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Abstract. We say that an integer n is k–free ($k \geq 2$) if for every prime p the valuation $v_p(n) < k$. If $f : \mathbb{N} \to \mathbb{Z}$, we consider the enumerating function $\mathcal{S}_f^k(x)$ defined as the number of positive integers $n \leq x$ such that $f(n)$ is k–free. When f is the identity then $\mathcal{S}_f^k(x)$ counts the k–free positive integers up to x. We review the history of $\mathcal{S}_f^k(x)$ in the special cases when f is the identity, the characteristic function of an arithmetic progression a polynomial, arithmetic. In each section we present the proof of the simplest case of the problem in question using exclusively elementary or standard techniques.

1. Introduction - The classical problem

We say that an integer $n \in \mathbb{N}$ is *square free* if for any prime $p \mid n$, one has $p^2 \nmid n$. If μ denotes the Möbius function, then μ^2 is the characteristic function of the set of square free numbers. It is a classical statement that

$$\mathcal{S}(x) := \#\{n \leq x \mid n \text{ is square free } \} = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

The proof is simple and it goes as follows: we start from the identity

$$\mu^2(n) = \sum_{d^2 \mid n} \mu(d),$$

which follows from the fact that μ is multiplicative. From this we obtain

$$\begin{aligned} \mathcal{S}(x) &= \sum_{n \leq x} \mu^2(n) = \sum_{n \leq x} \sum_{d^2 \mid n} \mu(d) \\ &= \sum_{d \leq \sqrt{x}} \mu(d) \left(\frac{x}{d^2} + O(1) \right) = x \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} + O(\sqrt{x}) \end{aligned}$$

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$$= x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\left(\sqrt{x}+x\sum_{d>\sqrt{x}}\frac{1}{d^2}\right)=\frac{x}{\zeta(2)}+O(x^{1/2})$$

where ζ denotes the Riemann ζ function.

Similarly, if $k\geq 2$, we say that an integer $n\in\mathbb{N}$ is k–free if for each prime $p\mid n$, one has $p^k\nmid n$. If $\mu^{(k)}$ denotes the characteristic function of k–free integers, then one has the identity:

$$\mu^{(k)}(n)=\sum_{d^k\mid n}\mu(d).$$

The same proof as above gives:

$$\mathcal{S}^k(x):=\#\{n\leq x\mid n\text{ is }k\text{–free}\}=\frac{x}{\zeta(k)}+O(\sqrt[k]{x}).$$

To improve the error term one can consider the generating zeta function of $\mu^{(k)}$. Indeed, it is quite simple to verify that for each $s\in\mathbb{C}$ with $\Re(s)>1$,

$$\sum_{n=1}^{\infty}\frac{\mu^{(k)}(n)}{n^s}=\frac{\zeta(s)}{\zeta(ks)}.$$

Therefore the right hand side above can be extended to a meromorphic function on $\mathbb{C}$ with poles on $s=1$ and on the complex numbers of the form ρ/k where ρ ranges over the zeros of $\zeta(s)$.

From the Perron integral (see the book of G. Tenenbaum: Introduction to analytic and probabilistic number theory, 1995 on page 130):

$$\frac{1}{2\pi i}\int_{\Re(s)=2}\frac{y^s}{s}ds=\begin{cases}0 & \text{if } y<1\\ 1/2 & \text{if } y=1\\ 1 & \text{if } y>1,\end{cases}$$

we deduce that if $x\notin\mathbb{N}$,

$$\mathcal{S}^k(x)=\frac{1}{2\pi i}\int_{\Re(s)=2}\frac{\zeta(s)}{\zeta(ks)}\frac{x^s}{s}ds.$$

Computing the residues, we obtain:

$$\mathcal{S}^k(x)=\frac{x}{\zeta(k)}+\sum_{\substack{\rho\\ \zeta(\rho)=0}}a_{\rho,k}x^{\rho/k}.$$

where $a_{\rho,k}$ can be computed in terms of the residue of $1/\zeta(s)$ at $s=\rho$.

If we assume the Riemann Hypothesis and we write $\rho = 1/2 + i\gamma$, then the above identity becomes

$$\mathcal{S}^k(x) = \frac{x}{\zeta(k)} + O(x^{1/2k} \sum_{\gamma} a_{\rho,k} x^{i\gamma/k}).$$

This argument suggests that the right error term in an asymptotic formula for the number of k–free integers corresponds to the largest real part of the zeroes of the Riemann ζ–function. Therefore the following conjecture should hold:

$$\mathcal{S}^k(x) = \frac{x}{\zeta(k)} + O(x^{1/2k+\epsilon}).$$

The above statement has a clear connection with the Riemann Hypothesis and therefore it is quite unlikely that it will be proven in the near future.

Using oscillation theorems, it is also possible to prove that

$$R_k(x) := \mathcal{S}^k(x) - \frac{x}{\zeta(k)} = \Omega(\sqrt[2k]{x}).$$

This was shown in 1968 by Vaidya [38].

In this direction Balasubramanian and Ramachandra in 1988 [3, 4] gave an effective proof that

$$R_2(x) = \Omega_{\pm}(\sqrt{x}).$$

The upper bound for $R_k(x)$ has attracted the work of many authors. Not much can be said unconditionally except that, using the classical zero–free region estimates due to Vinogradov and Korobov through exponential sums, Walfisz [40] showed that for a suitable constant c, uniformly on k,

$$R_k(x) \ll \sqrt[k]{x} \exp\{-ck^{-8/5} \log^{3/5} x \log\log^{1/5} x\}.$$

We assume the Riemann Hypothesis for the rest of this section and list the achievements on the problem of estimating $R_k(x)$:

- In 1911 Axer [1] showed that for every $\epsilon > 0$

$$R_k(x) \ll x^{2/(2k+1)+\epsilon}.$$

- In 1981 Montgomery and Vaughan [27] showed that for every $\epsilon > 0$

$$R_k(x) \ll x^{1/(k+1)+\epsilon}.$$

  In the same paper, they improved the above in the case $k = 2$ showing that for every $\epsilon > 0$

$$R_2(x) \ll x^{9/28+\epsilon}.$$

- In the same year the exponent above was improved by Graham [11] showing for every $\epsilon > 0$

$$R_2(x) \ll x^{8/25+\epsilon}.$$

- In 1985 Baker and Pintz [2] showed that for every $\epsilon > 0$

$$R_2(x) \ll x^{7/22+\epsilon}.$$

  The same result was also obtained by Jia in 1987 [22].
- In 1984 Yao [39] showed that for every $\epsilon > 0$

$$R_k(x) \ll x^{9/(9k+7)+\epsilon}.$$

- In 1988 Li [24] showed that for every $\epsilon > 0$

$$R_k(x) \ll x^{\max\{2/(2k+3), 9/(10k+8)+\epsilon\}}.$$

- In 1989 Graham and Pintz [13] showed that for every $\epsilon > 0$

$$R_k(x) \ll x^{D(k)+\epsilon},$$

where

$$D(k) = \begin{cases} 7/(8k+6), & \text{if } 2 \le k \le 5; \\ \frac{67}{514} & \text{if } k = 6; \\ 11(k-4)/(12k^2-37k-41) & \text{if } 7 \le k \le 12; \\ 23(k-1)/(24k^2+13k-37) & \text{if } 13 \le k \le 20. \end{cases}$$

For $k \ge 21$ the definition of $D(k)$ is more complicated, but it may be shown that $D(k) \sim 2\log 2/(k + \log k)$ for $k \to \infty$. These estimates could be improved for large k by appealing to Vinogradov's method. The authors obtain $R_k(x) \ll x^{E(k)}$, where $E(k) = 1/(k + ck^{1/3})$ for some constant c.
- The most recent contribution to our knowledge is due to Jia [23] in 1993. He showed that for every $\epsilon > 0$

$$R_2(x) \ll x^{17/54+\epsilon}.$$

2. k–free numbers in arithmetic progressions

Let $q \in \mathbb{N}$ and let $a \in \mathbb{Z}/q\mathbb{Z}$ be such that $\gcd(a, q)$ is k–free, then we set

$$\mathcal{S}_k(x; a, q) := \#\{n \le x \mid n \text{ is } k\text{–free and } n \equiv a \bmod q\}.$$

The first result about the distribution of k–free numbers in arithmetic progressions that we would like to mention is due to Prachar in

1958 [35]. He proved that if $(a, q) = 1$ and if

$$\delta_{k,q} := \frac{1}{q\zeta(k)} \prod_{l|q} \left(1 - \frac{1}{l^k}\right),$$

then

$$\mathcal{S}_k(x; a, q) = \delta_{k,q} x + O(\sqrt[k]{x} q^{-1/k^2} + q^{1/k} k^{\omega(q)}),$$

where $\omega(q)$ denotes the number of prime divisors of q. Prachar's proof goes as follows: using the formula

$$\mu^{(k)}(n) = \sum_{d^k|n} \mu(d),$$

and interchanging the order of summation, we write

$$\mathcal{S}_k(x; a, q) = \sum_{d \le \sqrt[k]{x}} \mu(d) \#\{m \le x/d^k \mid d^k m \equiv a \bmod q\}. \tag{1}$$

Observing that since $\gcd(a, q) = 1$, the sum above is only supported at values of d such that $\gcd(d, q) = 1$. Furthermore

$$\#\{m \le x/d^k \mid d^k m \equiv a \bmod q\} = \frac{x}{qd^k} + O(1).$$

Let us split the sum in (1) in two sums; the first with the values $d \le \sqrt[k]{x}/q^{1/k^2}$ and the second with the values $\sqrt[k]{x}/q^{1/k^2} < d \le \sqrt[k]{x}$.

For the first sum note that

$$\begin{aligned}
\sum_{\substack{d \le \sqrt[k]{x}/q^{1/k^2} \\ \gcd(d,q)=1}} \mu(d) \#\{m \le x/d^k \mid d^k m \equiv a \bmod q\} &= \\
\sum_{\substack{d \le \sqrt[k]{x}/q^{1/k^2} \\ \gcd(d,q)=1}} \mu(d) \left(\frac{x}{qd^k} + O(1)\right) &= \\
\frac{x}{q} \sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} \frac{\mu(d)}{d^k} + O\left(\frac{x}{q} \sum_{d > \sqrt[k]{x}/q^{1/k^2}} \frac{1}{d^k} + \frac{\sqrt[k]{x}}{q^{1/k^2}}\right) &=
\end{aligned}$$

$$\frac{x}{q\zeta(k)}\prod_{l|q}\left(1-\frac{1}{l^k}\right)+O\left(\frac{\sqrt[k]{x}}{q^{1/k^2}}\right) \quad =$$
$$\delta_{k,q}x+O\left(\frac{\sqrt[k]{x}}{q^{1/k^2}}\right)$$

since $1-(k-1)/k^2>1/k^2$.

For the second sum note that if $\sqrt[k]{x}/q^{1/k^2}<d\leq\sqrt[k]{x}$ and $nd^k\leq x$, then necessarily $n\leq q^{1/k}$. Therefore
(2)

$$\sum_{\substack{\frac{\sqrt[k]{x}}{q^{1/k^2}}<d\leq\sqrt[k]{x}\\ \gcd(d,q)=1}}\ \sum_{\substack{n\leq x/d^k\\ nd^k\equiv a \bmod q}}\mu(d)\ll\sum_{n\leq q^{1/k}}\#\{d\leq\left(\frac{x}{n}\right)^{1/k}\ \mid d^kn\equiv a \bmod q\}.$$

Next note that for a fixed n, by the Chinese remainder Theorem the congruence $d^kn\equiv a \bmod q$ has at most $2k^{\omega(q)}$ solutions $d\in\mathbb{Z}/q\mathbb{Z}$. Hence

$$\#\{d\leq(x/n)^{1/k}\ \mid d^kn\equiv a \bmod q\}\leq 2k^{\omega(q)}\left(\left(\frac{x}{n}\right)^{1/k}\frac{1}{q}+O(1)\right).$$

Finally (2) is

$$O\left(k^{\omega(q)}\left(q^{1/k}+\frac{x^{1/k}}{q^{1+1/(k^2-k)}}\right)\right).$$

The above together with the estimate for the first sum complete the proof of Prachar's statement.

In the more general situation when $\gcd(a,q)>1$, in order to exist a k-free number $n\equiv a \bmod q$, $\gcd(a,q)$ must be k–free. In this case the density of such n depends upon q.

An asymptotic formula of the type

$$\mathcal{S}_k(x;a,q)\sim\delta_{k,a,q}x$$

in this general case appeared in the book of Landau (Handbuch der Lehre von der Verteilung der Primzahlen, 1953) for $k=2$ and Ostmann (Additive Zahlentheorie, 1956) for $k>2$. The value of the density is given by

$$\delta_{k,a,q} = \frac{\varphi(q)}{(a,q)\varphi(q/(a,q))}\frac{1}{\zeta(k)q}\prod_{l|q}\left(1-\frac{1}{l^k}\right)^{-1}.$$

An error term for the general case has been worked out by Cohen and Robinson in 1963 [9].

By comparing the main term in Prachar's Theorem with the error term, we deduce that when k is fixed, the asymptotic formula of Prachar holds uniformly for $q \leq x^{2/3-\epsilon}$ since $k^{\omega(q)} \ll q^{\epsilon}$. In 1975 Hooley [15] proves that if $\gcd(a,q) = 1$, then

$$\mathcal{S}_2(x;a,q) = \delta_{2,q}x + O(\sqrt{x}q^{-1/2} + q^{1/2+\epsilon})$$

improving Prachar's result in the range $x^{1/3} < q < x^{2/3-\epsilon}$. In the same paper Hooley proves that for every $5/8 \leq \alpha < 3/4$ there exists a constant $\eta = \eta(\alpha)$ such that

$$\mathcal{S}_2(x;a,q) = \delta_{2,q}x + O\left(\left(\frac{x}{q}\right)^{1-\eta}\right)$$

for a positive proportion of $q \in (Q, 2Q)$ provided that $x^{5/8} < Q \leq x^{\alpha}$ and $\gcd(a,q) = 1$.

A further contribution to this problem in the case $\gcd(a,q) = 1$ was proposed by McCurley in 1982 [26] when he proved that there exist absolute computable constants c_1 and c_2 such that if q is such that none of the L-functions associated to real characters modulo q have a Siegel zero, then

$$\mathcal{S}_k(x;a,q) = \delta_{k,q}x + O\left(\frac{(xq)^{1/k}}{\exp(c_1\sqrt{\log x/k^3})}\right) \text{ if } x \geq \exp(c_2 k \log^2 q).$$

Furthermore let q and k be such that $L(s,\chi^k)$ has no Siegel zeros for all χ modulo q. Then there exist constants c_3 and c_4 such that

$$\mathcal{S}_k(x;a,q)=\delta_{k,q}x+O\left(\frac{x^{1/k}}{\exp(c_3\sqrt{\log x/k^3})}\right)\quad \begin{array}{l}\text{if } x\geq \exp(c_4 k\log^2 q)\\ \text{and } (a,q)=1.\end{array}$$

Average type results for k–free numbers in arithmetic progressions have also been considered. Orr in 1971 [31] proved the following two statements.

For any constant A

$$\sum_{q\leq x^{2/3}/\log^{A+1}x}\ \max_{\substack{a\in\mathbb{Z}/q\mathbb{Z}\\ \gcd(a,q)\text{ square free}}} |\mathcal{S}_2(x;a,q)-\delta_{2,a,q}x|\ll\frac{x}{\log^A x}$$

and for $y\leq x$

$$\sum_{q\leq y}\ \sum_{\substack{a\in\mathbb{Z}/q\mathbb{Z}\\ \gcd(a,q)\text{ square free}}} |\mathcal{S}_2(x;a,q)-\delta_{2,a,q}x|^2\ll xy+x^{8/5}\log^5 x.$$

Warlimont in 1969 [41] proved some similar statements showing that if $y\leq x$

$$\sum_{q\leq y}\ \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*} |\mathcal{S}_2(x;a,q)-\delta_{2,q}x|^2\ll\begin{cases} xy & \text{if } 1\leq y\leq x^{\frac{1}{3}}\\ & \text{or } y\geq x^{1/3}\log^{10/3}x;\\ y^{\frac{4}{3}}x^{\frac{2}{3}+\epsilon} & \text{if } y=x^{\alpha},\ \frac{1}{2}\leq\alpha\leq 1.\end{cases}$$

To the author's knowledge, the above results have not been generalized to the case of k–free numbers.

We conclude this section mentioning the results of Suryanarayana [37] obtained in 1969 where he showed that the number of k-free integers $\leq x$ which are relatively prime to m equals

$$\varphi(m)\delta_{k,m}x + O\left(\frac{\varphi(m)2^{\omega(m)}}{m}x^{1/k}\right),$$

saving a factor of $\frac{2^{\omega(m)}}{m}$ with respect to the direct estimate.

In the same paper Suryanarayana also shows that

$$\sum_{\substack{n\le x\\ \gcd(n,m)=1\\ n\ k\text{–free}}} n = \varphi(m)\delta_{k,m}\frac{x^2}{2} + O\left(\frac{\varphi(m)2^{\omega(m)}}{m}x^{1+1/k}\right)$$

and that

$$\sum_{n\le x}\sum_{\substack{d\mid n\\ (d,n/d)=1\\ d\ k\text{–free}}} d = \prod_l \left(1-\frac{1}{l^k}+\frac{1}{l^2+l}\right)\cdot\frac{x^2}{2} + O(x^{1+1/k}).$$

3. The general problem of k–freeness

If $f:\mathbb{N}\to\mathbb{Z}$ is any function, we define

$$\mathcal{S}_f^k(x) := \#\{n\le x \mid f(n) \text{ is } k\text{–free}\}.$$

We have the following clear identity:

$$\mathcal{S}_f^k(x) = \sum_{d=1}^{\infty}\mu(d)\#\{n\le x \mid d^k \mid f(n)\}$$

which is suggesting that if the probability that $f(n)$ is divisible by an integer D is given by a number $\mathbb{P}_f(D)$, then one might expect to hold an asymptotic formula of the type

$$\mathcal{S}_f^k(x) \sim \prod_l \left(1-\mathbb{P}_f(l^k)\right)x.$$

This problems turns out to be quite challenging in general and the above formula is not often known to hold.

A natural variation to the problem is to restrict the argument of the function f to prime values. Therefore we set:

$$\tilde{\mathcal{S}}_f^k(x) := \#\{p \le x \mid p \text{ is prime and } \; f(p) \text{ is } k\text{–free}\}.$$

We mention that a result of Mirsky of 1949 [25] deals with the case when $f(t) = t + a$ with $a \in \mathbb{Z}$. In this case for any constant A,

$$\tilde{\mathcal{S}}_f^k(x) := \#\{p \le x \,|\, p \text{ prime, } p{+}a \text{ is } k\text{–free}\} = \beta_a \pi(x) {+} O\left(\frac{x}{\log^A x}\right)$$

where $\pi(x)$ is the number of primes $p \le x$ and

$$\beta_a = \prod_{l \text{ prime}} \left(1 - \frac{1}{l^k(l-1)}\right)$$

is the so called *Artin k–dimensional constant*.

We would like here to prove this statement with a much weaker error term. Indeed if $\pi(x; -a, q)$ counts the number of primes $p \le x$, $p \equiv -a \bmod q$, then using some prime number Theorem for arithmetic progressions,

$$\begin{aligned}
\tilde{\mathcal{S}}_f^k(x) &= \sum_{d \le x^{1/k}} \mu(d) \pi(x; -a, d^k) \\
&= \sum_{d \le \log x} \mu(d) \left(\frac{\pi(x)}{\varphi(d^k)} + O\left(\frac{x}{\log^3 x}\right)\right) + \left(\sum_{\substack{d \le x^{1/k} \\ d > \log x}} \pi(x; -a, d^k)\right) \\
&= \beta_a \pi(x) + O\left(\sum_{d > \log x} \frac{\pi(x)}{\varphi(d^k)} + \frac{x}{\log^2 x} + \sum_{d > \log x} \frac{x}{d^k}\right) \\
&= \beta_a \pi(x) + O\left(\frac{x}{\log^2 x}\right)
\end{aligned}$$

where we used the clear estimate $\pi(x;-a,q) \ll x/q$. This concludes the proof.

The last result that we mention in this section is due in 1998 to Brüdern, Granville, Perelli, Vaughan and Wooley [8]. They consider L^1 and L^2 means of exponential sums supported on k–free numbers. As a corollary of various results, they show that

$$\#\{n \le x \mid n = a^3+b^3+c^3, a, b, c \in \mathbb{N} \text{ and } n \text{ is square free}\} \gg x^{11/12}.$$

4. The case of polynomials in one variable

In this section we consider the case when $f(t) \in \mathbb{Z}[t]$ is a polynomial without multiple roots, primitive and such that the greatest common divisors of all the $f(n)$'s is k–free (to avoid pathologies like $f(t) = t(t+1)(t+2)(t+3)$ whose values are always divisible by 8).

This interesting case has also attracted several authors.

After Nagell in 1922 [28] proved that a polynomial of degree $\deg f$ less or equal then k assumes infinitely many k–free values, the Italian mathematician Ricci in 1933 [36] proved that if

$$\delta_{f,k} = \prod_{l \text{ prime}} \left(1 - \frac{\varrho_f(l^k)}{l^k}\right)$$

where once again $\varrho_f(d)$ is the number of roots of f in $\mathbb{Z}/d\mathbb{Z}$, then

$$\mathcal{S}_f^k(x) \sim \delta_{f,k} x$$

provided that $\deg f \le k$. The asymptotic formula above is conjectured to hold also when $\deg f > k$ and this is the main open question in this part of the subject.

Here we would like to prove the statement of Ricci's Theorem and more precisely that for any $\epsilon > 0$

$$\mathcal{S}_f^k(x) = x\delta_{f,k} + O\left(\frac{x}{\log^{1-\epsilon} x}\right) \qquad \text{if } \deg f \le k.$$

Indeed write r for $\deg f$, let $z = \log x$ and $P(z)$ denote the product of all primes up to z, write

$$\begin{aligned}
\mathcal{S}_f^k(x) &= \sum_{n\le x} \mu^{(k)}(f(n)) \\
&= \sum_{d\ge 1} \mu(d)\#\{n \le x \mid d^k \mid f(n)\} \\
&= \sum_{d\mid P(z)} \mu(d)\#\{n \le x \mid d^k \mid f(n)\} + O(\sum_{p>z}\#\{n \le x \mid p^k \mid f(n)\}). \qquad (3)
\end{aligned}$$

Now note that

$$\#\{n \le x \mid d^k \mid f(n)\} = \varrho_f(d^k)\left(\frac{x}{d^k} + O(1)\right).$$

Furthermore we have that

a) if $p^k \mid f(n)$, then $p \le cx^{r/k}$ for a suitable constant $c = c(k, f)$ (Note that the hypothesis $r/k \le 1$ will be crucial in the sequel);
b) ϱ_f is a multiplicative function;
c) $\varrho_f(p^k)$ is uniformly bounded in term of k and f for all primes p. So there exists $\tilde{c} = \tilde{c}(k, f)$ such that $\varrho_f(p^k) \le \tilde{c}$

Hence (3) equals:

$$\begin{aligned}
&\sum_{d\mid P(z)} \mu(d)\varrho_f(d^k)\left(\frac{x}{d^k} + O(1)\right) + O\left(\sum_{p>z}\frac{x}{p^k} + \sum_{p\le cx^{r/k}} 1\right) \\
&= x\sum_{d\mid P(z)} \frac{\mu(d)\varrho_f(d^k)}{d^k} + O\left((1+\tilde{c})^{\pi(z)} + \frac{x}{z^{k-1}\log z} + \frac{x^{r/k}}{\log x}\right) \\
&= x\sum_{d=1}^{\infty} \frac{\mu(d)\varrho_f(d^k)}{d^k} + O\left(x\sum_{d>z}\frac{\tilde{c}^{\omega(d)}}{d^k} + \frac{x}{\log x\log\log x}\right) \\
&= x\delta_{f,k} + O\left(\frac{x}{\log^{1-\epsilon} x}\right),
\end{aligned}$$

where we used the estimate $\tilde{c}^{\omega(d)} = d^\epsilon$ and this completes the proof.

The next case, degree of f equals $k+1$, had to wait until 1953, when Erdős [10] under this hypothesis showed that $\mathcal{S}_f^k(x) \to \infty$ as $x \to \infty$.

In 1976 Hooley [18], improving his results of 1967 [16, 17], applies the large sieve to Ricci's proof showing that if f is irreducible, then

$$\mathcal{S}_f^k(x) = x\delta_{f,k} + O\left(\frac{x}{\log^{k/(k+2)} x}\right) \qquad \text{if } \deg f = k+1.$$

The next conquest is due to Nair [29] who in 1976 proves that if $\lambda = \sqrt{2} - 1/2 = 0.9142\cdots$, and f is irreducible, then

$$\mathcal{S}_f^k(x) = x\delta_{f,k} + O\left(\frac{x}{\log^{k-1} x}\right) \qquad \text{if } \deg f \geq \lambda k.$$

In two subsequent papers [21, 30], the second in collaboration with Huxley, Nair improves his results proving that for some $\sigma = \sigma(k, f) > 1$ that can be explicitly computed, one has

$$\mathcal{S}_f^k(x) = x\delta_{f,k} + O\left(x^{1-\sigma}\right) \quad \text{if } k \geq \sqrt{2(\deg f)^2 + 1} - (\deg f + 1)/2.$$

Furthermore in the same paper Nair also proves that there exists a suitable positive $\alpha \leq 70/71$ such that

$$\mathcal{S}_f^k(x) = x\delta_{f,k} + O\left(x^{\alpha}\right) \qquad \text{if } \deg f = k+1 \geq 7.$$

Nair's approach was also employed by Hinz [14] in 1982 to prove generalizations of these results in the algebraic number fields settings.

More recently in 1998, Granville [12] proves that the *abc*–conjecture of Œsterle, Masser and Szpiro implies that if $f \in \mathbb{Z}[t]$ does not have repeated roots and if the greatest common divisor of all values $f(n)$ is square free, then the conjecture always holds for f when $k = 2$:

$$\textbf{abc} \quad \Longrightarrow \quad \mathcal{S}_f^2(x) \sim \delta_{f,2} x.$$

Granville's statement is indeed more general and does not assume that the greatest common divisor of all values $f(n)$ is square free. Furthermore his paper [12] deals also with other problems involving square-free numbers.

The first ones to use the *abc*–conjecture to the problem of counting square free values of polynomials were Browkin, Filaseta, Greaves and Schinzel [7], who in 1995 proved that the *abc*–conjecture implies that the cyclotomic polynomial $\Phi_m(n)$ is square free for infinitely many values of n.

For a given integer d, we set $\tilde{\varrho}_f(d)$ to be the number of roots of $f(t)$ in $(\mathbb{Z}/d\mathbb{Z})^*$. Furthermore we define

$$\tilde{\delta}_{f,k} = \prod_{l \text{ prime}} \left(1 - \frac{\tilde{\varrho}_f(l^k)}{l^{k-1}(l-1)}\right).$$

It is reasonable to conjecture that

$$\tilde{\mathcal{S}}_f^k(x) \sim \tilde{\delta}_{f,k} \frac{x}{\log x}.$$

Indeed if the degree of f is less or equal to k, this can be proven along the lines of Ricci's Theorem. However the above is considered to be in general a more difficult problem with respect to the previous one.

It is a result of Hooley [19] of 1977 that if $\deg f = k+1 \geq 55$ and f is irreducible, then there exists a constant $\Delta_k > 0$ such that

$$\tilde{\mathcal{S}}_f^k(x) = \tilde{\delta}_{f,k} \frac{x}{logx} + O\left(\frac{x}{\log^{1+\Delta_k} x}\right) \quad \text{if } \deg f = k+1 \geq 55.$$

Hooley [20] later improved the above range for k to $k \geq 41$, under some technical conditions. Nair [29, 30] and later Huxley and Nair [21] also consider this problem.

5. The case of classical arithmetical functions

As we will see, the behavior of $\mathcal{S}_f^k$ in the case of classical arithmetical functions is rather different from the case of polynomials.

We recall that the Carmichael function $\lambda(n)$ is defined as the largest possible order of any element in the unit group of the residues modulo

$n \geq 1$. More explicitly, for a prime power p^ν, we have

$$\lambda(p^\nu) = \begin{cases} p^{\nu-1}(p-1), & \text{if } p \geq 3 \text{ or } \nu \leq 2; \\ 2^{\nu-2}, & \text{if } p = 2 \text{ and } \nu \geq 3 \end{cases}$$

and for arbitrary $n \geq 2$,

$$\lambda(n) = \operatorname{lcm}(\lambda(p_1^{\nu_1}), \dots, \lambda(p_s^{\nu_s})),$$

where $n = p_1^{\nu_1} \cdots p_s^{\nu_s}$ is the prime factorization of n. Clearly $\lambda(1) = 1$. Furthermore for every integer n we have that $\lambda(n) \mid \varphi(n)$ and the equality holds when n is a prime power.

Another cognate function is the order function: if $a \in \mathbb{Z} \setminus \{0, \pm 1\}$, then for each n coprime to a, we define

$$\operatorname{ord}_a(n) := \min\{e \in \mathbb{N} \mid a^e \equiv 1 \bmod n\}.$$

Clearly $\operatorname{ord}_a(n) \mid \lambda(n)$. It is convenient to extend the definition of $\operatorname{ord}_a(n)$ setting it to be equal to 0 when $\gcd(a, n) \neq 1$.

In [34], Saidak, Shparlinski and the author prove that

$$\mathcal{S}_\lambda^k(x) = (\kappa_k + o(1)) \frac{x}{\log^{1-\alpha_k} x},$$

where

$$\kappa_k := \frac{2^{k+2}-1}{2^{k+2}-2} \cdot \frac{\eta_k}{e^{\gamma\alpha_k}\Gamma(\alpha_k)}, \quad \alpha_k := \prod_{l \text{ prime}} \left(1 - \frac{1}{l^{k-1}(l-1)}\right)$$

and η_k is defined by

$$\eta_k := \lim_{T\to\infty} \frac{1}{\log^{\alpha_k} T} \prod_{\substack{l \leq T \\ l-1 \text{ is } k\text{–free}}} \log\left(1 + \frac{1}{l} + \ldots + \frac{1}{l^k}\right).$$

For example we have $k_2 = 0.80328\ldots$ and $\alpha_2 = 0.37395\ldots$.

As for the order function, in [35] the author proves that

$$\mathcal{S}_{\operatorname{ord}_a}(x) = (\iota_{a,k} + o(1)) \frac{x}{\log^{1-\beta_{a,k}} x}.$$

The constants $\iota_{a,k}$ and $\beta_{a,k}$ have a complicated definition. However in the case when a is square free and $k \geq 3$, the definition for $\beta_{a,k}$

simplifies in

$$\beta_{a,k} := \left[\prod_l \left(1 - \frac{1}{l^{k-2}(l^2-1)}\right)\right] \cdot \left[1 - \frac{1}{2}\prod_{l \mid [2,a]} \frac{1}{1 - l^{k-2}(l^2-1)}\right].$$

The proof of both results follows the same framework using a classical statement, due to Wirsing [42]:

Theorem. *Assume that $g(n)$ is multiplicative, $0 < g(p^\nu) \leq c_5 c_6^\nu$, $c_6 < 2$ and*

$$\sum_{p \leq x} g(p) = (1 + o(1))\tau\pi(x)$$

for some $\tau \neq 0$. Let γ denote the Euler constant, and Γ the gamma-function. Then

$$\sum_{n \leq x} g(n) \sim \frac{1}{e^{\gamma\tau}\Gamma(\tau)} \frac{x}{\log x} \prod_{l \leq x} \sum_{\nu=0}^{\infty} \frac{g(l^\nu)}{l^\nu}.$$

It is immediate to check that both

$$\mu^{(k)}(\lambda(n)) \quad \text{and} \quad \mu^{(k)}(\mathrm{ord}_a(n))$$

are multiplicative functions. In order to apply Wirsing's Theorem, one needs to find an asymptotic formula for

$$\tilde{\mathcal{S}}_\lambda^k(x) = \#\{p \leq x \mid p-1 \text{ is } k\text{-free}\}$$

in the case of λ, and for

$$\tilde{\mathcal{S}}_{\mathrm{ord}_a}^k(x) = \#\{p \leq x \mid p \nmid a,\ \mathrm{ord}_p(a) \text{ is } k\text{–free}\}$$

in the case of ord_a.

The first of these asymptotic formulas is due to Mirsky [25] and we already referred to it in the last part of Section 3. For any constant $A > 0$ we have

$$\tilde{\mathcal{S}}_\lambda^k(x) = \alpha_k \pi(x) + O\left(\frac{x}{\log^A x}\right).$$

The second asymptotic formula is proven in [35] by using the Chebotarev density Theorem in an unconditional version:

$$\tilde{\mathcal{S}}^k_{\mathrm{ord}_a}(x) = \beta_{a,k}\pi(x) + O\left(\frac{x}{\log^\sigma x}\right),$$

where $\sigma = \sigma(a,k)$ can be explicitly computed.

In general if f in any function such that $\mu^{(k)}(f(n))$ is multiplicative, then Wirsing's Theorem gives that

$$\tilde{\mathcal{S}}^k_f(x) \sim \tau_{f,k}\pi(x) \quad \Longrightarrow \quad \mathcal{S}^k_f(x) \sim \frac{\pi(x)}{e^{\gamma\tau_{f,k}}\Gamma(\tau_{f,k})}\prod_{p\le x}\sum_{\nu=0}^{\infty}\frac{\mu^{(k)}(p^\nu)}{p^\nu}.$$

Unfortunately Wirsing's Theorem does not provide any error term.

The last classical function that we will discuss is the Euler function. It has a rather different behavior with respect to the others. Indeed Banks and the author in [5] showed that for a fixed integer $k \ge 3$, the asymptotic relation

$$\mathcal{S}^k_\varphi(x) = \frac{3\alpha_k}{2}\frac{x\,(\log\log x)^{k-2}}{(k-2)!\,\log x}\left(1 + O_k\left(\frac{(\log\log\log x)^{2(k+1)^{2k-4}-1}}{(\log\log x)^{1-1/k}}\right)\right)$$

holds as $x \to \infty$ with the constant α_k defined by

$$\alpha_k := \frac{1}{2^{k-1}}\prod_{l>2}\left(1 - \frac{1}{l^{k-1}}\sum_{i=0}^{k-2}\sum_{j=0}^{k-2-i}\binom{k-1}{i}\binom{k-1+j}{j}\frac{(l-2)^j}{(l-1)^{i+j+1}}\right).$$

The above statement reminds Landau's Theorem for the number of integers up to x with at most $k-1$ prime factors. Indeed if $\varphi(n)$ is k–free, then $\omega(n) \le k-1$.

6. k–free numbers with restricted digits.

Let $g \in \mathbb{N}$, $\mathcal{D} \subset \{0,1,\dots,g-1\}$ be a subset with d elements and let e be the number of elements in $\mathcal{D}$ which are coprime with g. Furthermore denote by $\mathcal{A}_g(\mathcal{D})$ the set of those integers which are coprime with g and such that all their digits in base g are in $\mathcal{D}$.

It is natural to consider the counting function

$$\mathcal{S}^k_{\mathcal{D},g}(x) = \#\{n \le x \mid n \text{ is } k\text{–free and } n \in A_g(\mathcal{D})\}.$$

Banks and Shparlinski [4] prove that there exists a constant $c > 0$ depending on $\mathcal{D}$ and k such that if $x = g^m$ and $m \to \infty$, then

$$\mathcal{S}^k_{\mathcal{D},g}(x) = \delta_{\mathcal{D},g} x^{\log_g d} + O\left(\frac{x^{\log_g d}}{\exp(c\sqrt{\log x})}\right),$$

where

$$\delta_{\mathcal{D},g} = \frac{e}{\zeta(k)d} \prod_{l|g} \left(1 - \frac{1}{l^k}\right)^{-1}.$$

Furthermore if r is an integer, $\mathcal{B}_r$ denotes the set of those odd integers having exactly r ones in their binary expansion and

$$\mathcal{S}^k_{\mathcal{B}_r}(x) = \#\{n \le x \mid n \text{ is } k\text{–free and } n \in \mathcal{B}_r\}.$$

Then there exists a constant $d > 0$ such that, if $x = 2^m$ and $m \to \infty$, then

$$\mathcal{S}^k_{\mathcal{B}_r}(x) = \frac{\binom{m-1}{k-1}}{\zeta(k)(1-2^{-k})} \left(1 + O\left(\frac{1}{\exp(d\delta\sqrt{\log x})}\right)\right),$$

provided that $r/m \in [(\log 2 + \delta)/k, 1/2]$.

7. Conclusion

Many variations of the results in the literature could be considered and would provide a good project for young mathematicians. We list a few here. Some might be easier, some might be very difficult, others might already be known.

(1) Find an asymptotic formula for $\mathcal{S}^k_f(x)$ when f is some other classical arithmetic functions: e.g. $f = \sigma, \tau, \omega, \Omega$;

(2) Find an asymptotic formula for $\mathcal{S}_f^k(x)$ when $f(n)$ is the n–th coefficient of a modular form;
(3) Find an asymptotic formula for $\mathcal{S}_{\mathrm{ind}_a}^k(x)$, where the index function is defined as $\mathrm{ind}_a(n) = \varphi(n)/\operatorname{ord}_a(n)$. Some partial results already appeared in [33];
(4) Find an asymptotic formula for the values of an arithmetic function on numbers in arithmetic progressions;
(5) Given a finitely generated subgroup $\Gamma \subset \mathbb{Q}^*$, we know that for all but finitely many primes p we can consider the reduction $\langle \Gamma \bmod p \rangle \subset (\mathbb{Z}/p\mathbb{Z})^*$. Furthermore there exists an integer M_Γ such that if $\gcd(n, M_\Gamma) = 1$, then the reduction $\langle \Gamma \bmod n \rangle$ is a well defined subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$. Find an asymptotic formula for the number of integers n up to x such that $(n, M_\Gamma) = 1$ and $\#\langle \Gamma \bmod n \rangle$ is k–free.

We realize that this survey is far from complete. For example we completely left out important topics like k–free values of multivariate polynomials or binomial coefficients, gaps between k–free numbers and surely many others.

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