EXCHANGE GRAPHS OF ACYCLIC CALABI-YAU CATEGORIES

ALASTAIR KING AND YU QIU

Abstract. We study (the principal component of) the oriented exchange graph of hearts in the finite-dimensional derived category $\mathcal{D}(\Gamma_N Q)$ of the Calabi-Yau-N Ginzburg algebra associated to an acyclic quiver Q. We show that any such heart is induced from some heart in the bounded derived category $\mathcal{D}(Q)$ via some 'Lagrangian immersion' $\mathcal{L} : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q)$. Further, we show that the quotient graph by the Seidel-Thomas braid group is the exchange graph for $(N-1)$-clusters. As an application, we interpret Buan-Thomas' coloured quiver for an $(N-1)$-cluster in terms of the Ext-quiver of the associated hearts in $\mathcal{D}(\Gamma_N Q)$.

Key words: exchange graph, t-structure, Calabi-Yau category, higher cluster theory

1. Introduction

One key to understanding a triangulated category $\mathcal{D}$ is to understand its t-structures. Every (bounded) t-structure carries an abelian category sitting inside it, known as its heart. The classical way to understand relations between different t-structures, or their hearts, is via tilting in the sense of Happel-Reiten-Smalø. We will be especially interested in the case where the hearts are finite, i.e. are generated by finitely many simple objects, and in 'simple' tilting, i.e. where the t-structures essentially differ by one rigid simple object (see Section 3 for a more precise definition). In particular, the relationship of simple tilting makes the set of hearts in $\mathcal{D}$ into an oriented graph, called the exchange graph $\mathrm{EG}(\mathcal{D})$.

We focus on certain triangulated categories arising in representation theory, namely, the bounded derived category $\mathcal{D}(Q)$ of an acyclic quiver Q and the finite-dimensional derived category $\mathcal{D}(\Gamma_N Q)$ of the Calabi-Yau-N Ginzburg algebra $\Gamma_N Q$ associated to Q ([7]). In the Calabi-Yau-3 case, such differential graded algebras, more generally associated to quivers with potential, originally arose in studying the local geometry of Calabi-Yau 3-folds. They also arise in mirror symmetry; for example, when Q is of type A_n, Khovanov-Seidel-Thomas ([16],[21]) identify $\mathcal{D}(\Gamma_N Q)$ inside the derived Fukaya category of Lagrangian submanifolds of the Milnor fibre of a singularity of type A_n.

Note that there are canonical finite hearts $\mathcal{H}_Q$ in $\mathcal{D}(Q)$ and $\mathcal{H}_\Gamma$ in $\mathcal{D}(\Gamma_N Q)$ and we will study the 'principal' components $\mathrm{EG}^\circ(Q)$ and $\mathrm{EG}^\circ(\Gamma_N Q)$ of their respective exchange graphs that contain these canonical hearts. It is possible that these components actually consist of all finite hearts, but we do not know whether this is true in general. Nevertheless, we will prove (Theorem 5.7) that any heart in $\mathrm{EG}^\circ(Q)$ is finite, all its simples are rigid and there is a dual set of projectives (or equivalently a silting object [15]), which mutates when the heart tilts. The operation of repeated tilting with

Date: February 29, 2012.

respect to the same simple (up to shift) gives rises to the notion of lines in the exchange graph (Definition 5.9), which further determines the notions of convexity and cyclic completion (Definition 5.10) of subgraphs.

In (higher) cluster theory, associated to the cluster category $\mathcal{C}_{N-1}(Q)$, there also arises an exchange graph $\operatorname{CEG}_{N-1}(Q)$ of $(N-1)$-clusters related by mutation. When $\mathcal{C}_{N-1}(Q)$ is identified as a quotient of $\mathcal{D}(Q)$, it is possible to interpret a result of Buan-Reiten-Thomas [5] as giving an isomorphism

$$\operatorname{CEG}_{N-1}(Q) \cong \overline{\operatorname{EG}^{\circ}_{N}}(Q, \mathcal{H}_Q),$$

the cyclic completion of a certain convex subgraph $\operatorname{EG}^{\circ}_{N}(Q, \mathcal{H}_Q)$ in $\operatorname{EG}^{\circ}(Q)$ (cf. Definition 4.1).

In the same way, we define a convex subgraph $\operatorname{EG}^{\circ}_{N}(\Gamma_N Q, \mathcal{H}_\Gamma)$ in $\operatorname{EG}^{\circ}(\Gamma_N Q)$ and prove (Theorem 8.1) that it is isomorphic to $\operatorname{EG}^{\circ}_{N}(Q, \mathcal{H}_Q)$ via a canonical functor $\mathcal{I}\colon D(Q) \to \mathcal{D}(\Gamma_N Q)$ which is a 'Lagrangian immersion' (Definition 7.3) in the following sense. The 'tangent algebra', i.e. the derived endomorphism algebra, $\operatorname{Hom}^{\bullet}_{Q}(X, X)$ of any object $X \in \mathcal{D}(Q)$ is identified with a subspace of $\operatorname{Hom}^{\bullet}_{\Gamma_N Q}\big(\mathcal{I}(X), \mathcal{I}(X)\big)$ whose quotient is dual to it (up to a shift). As the isomorphism

$$\operatorname{EG}^{\circ}_{N}(Q, \mathcal{H}_Q) \cong \operatorname{EG}^{\circ}_{N}(\Gamma_N Q, \mathcal{H}_\Gamma)$$

preserves the structure of lines, it also induces an isomorphism between their cyclic completions. Moreover, we prove (Theorem 8.5) that $\operatorname{EG}^{\circ}_{N}(\Gamma_N Q, \mathcal{H}_\Gamma)$ is a fundamental domain for the action on $\operatorname{EG}^{\circ}(\Gamma_N Q)$ of the Seidel-Thomas braid group

$$\operatorname{Br} = \operatorname{Br}(\Gamma_N Q) \subset \operatorname{Aut} \mathcal{D}(\Gamma_N Q),$$

which is generated by the spherical twist functors. In the process, we see that every heart in $\operatorname{EG}^{\circ}(\Gamma_N Q)$ is induced from some heart in $\operatorname{EG}^{\circ}(Q)$ via some Lagrangian immersion and we also obtain an isomorphism

$$\overline{\operatorname{EG}^{\circ}_{N}}(\Gamma_N Q, \mathcal{H}_\Gamma) \cong \operatorname{EG}^{\circ}(\Gamma_N Q)/\operatorname{Br}.$$

Putting all the above isomorphisms together yields an isomorphism

$$\operatorname{EG}^{\circ}(\Gamma_N Q)/\operatorname{Br} \cong \operatorname{CEG}_{N-1}(Q). \tag{1.1}$$

In addition (Theorem 8.6), we exploit this circle of ideas to interpret the coloured quiver, associated to a cluster in $\operatorname{CEG}_{N-1}(Q)$ by Buan-Thomas [6], in terms of the Ext-quiver of the hearts in the corresponding Br-orbit in $\operatorname{EG}^{\circ}(\Gamma_N Q)$.

In the Calabi-Yau-3 case, the isomorphism (1.1) was obtained by Keller-Nicolás [13, Theorem 5.6], in the more general context of quivers with potential. In fact, the Calabi-Yau-3 case is in many ways more uniform and we go on to show (Theorem 9.6) that $\operatorname{EG}^{\circ}_{3}(\Gamma_3 Q, \mathcal{H})$ is a fundamental domain for the Br action, for any heart $\mathcal{H} \in \operatorname{EG}^{\circ}(\Gamma_3 Q)$. Since the oriented exchange graph $\operatorname{CEG}_2(Q)$ is obtained from the original unoriented cluster exchange graph $\operatorname{CEG}_2(Q)^*$ by replacing each edge by a two-cycle, we may therefore consider that $\operatorname{EG}^{\circ}_{3}(\Gamma_3 Q, \mathcal{H})$ is an oriented version of $\operatorname{CEG}_2(Q)^*$ and thus $\operatorname{EG}^{\circ}(\Gamma_3 Q)$ is covered by many such oriented versions of $\operatorname{CEG}_2(Q)^*$.

To illustrate several of the important ideas, we conclude the paper by explicitly describing the quotient of the exchange graph $\operatorname{EG}^{\circ}(\Gamma_N Q)$ by the shift functor, for a quiver Q of type A_2, and show how it is a rough combinatorial dual to the Farey graph. This relationship has been made more geometric, in the Calabi-Yau-3 case, by Sutherland [22], who shows that the hyperbolic disc, in which the Farey graph lives, is naturally (the $\mathbb{C}$-quotient of) the space of Bridgeland stability conditions for $\mathcal{D}(\Gamma_3 Q)$.

This paper is part of the second author's PhD thesis [19], which also provides several other applications of exchange graphs, such as to spaces of stability conditions and to quantum dilogarithm identities (cf. [20]).

Acknowledgements. The second author's Ph.D studies were supervised by the first author and supported by a U.K. Overseas Research Studentship. The second author would also like to thank Bernhard Keller for an invitation to Paris 7 in March 2011 and for sharing his insight and expertise on cluster theory, which helped to substantially improve an early version of this work. The visit to Paris was supported by the U.K.-France network RepNet.

2. Preliminaries

For simplicity, let $\mathbf{k}$ be a fixed algebraically-closed field. Let Q be an acyclic quiver with n vertices, that is, a directed graph without oriented cycles. The path algebra $\mathbf{k}Q$ is then finite dimensional. We denote by $\operatorname{mod}\mathbf{k}Q$ the category of finite dimensional $\mathbf{k}Q$-modules and let $\mathcal{D}(Q) = \mathcal{D}^b(\operatorname{mod}\mathbf{k}Q)$ be its bounded derived category, which is a triangulated category. Note that $\operatorname{mod}\mathbf{k}Q$ is hereditary, i.e. $\operatorname{Ext}^2(M,N) = 0$ for all modules M, N, and hence [8]

$$\operatorname{Ind}\mathcal{D}(Q) = \bigcup_{m\in\mathbb{Z}} \operatorname{Ind}(\operatorname{mod}\mathbf{k}Q)[m], \tag{2.1}$$

where $\operatorname{Ind}\mathcal{C}$ denotes a complete set of indecomposables in an additive category $\mathcal{C}$. In addition, $\mathcal{D}(Q)$ has Auslander-Reiten (or Serre) duality, i.e. a functor $\tau\colon \mathcal{D}(Q) \to \mathcal{D}(Q)$ with a natural duality

$$\operatorname{Ext}^1(Y,X) \cong \operatorname{Hom}(X,\tau Y)^*. \tag{2.2}$$

for all objects X, Y in $\mathcal{D}(Q)$.

Recall (e.g. from [3]) that a *t-structure* on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ and such that, if one defines

$$\mathcal{P}^\perp = \{G \in \mathcal{D} : \operatorname{Hom}_{\mathcal{D}}(F,G) = 0, \forall F \in \mathcal{P}\},$$

then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{P}$ and $G \in \mathcal{P}^\perp$. It follows immediately that we also have

$$\mathcal{P} = \{F \in \mathcal{D} : \operatorname{Hom}_{\mathcal{D}}(F,G) = 0, \forall G \in \mathcal{P}^\perp\}.$$

Any t-structure is closed under sums and summands and hence it is determined by the indecomposables in it. Note also that $\mathcal{P}^\perp[-1] \subset \mathcal{P}^\perp$.

A t-structure $\mathcal{P}$ is *bounded* if

$$\mathcal{D} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{P}^\perp[i] \cap \mathcal{P}[j],$$

or equivalently if, for every object M, the shifts $M[k]$ are in $\mathcal{P}$ for $k \gg 0$ and in $\mathcal{P}^\perp$ for $k \ll 0$. The *heart* of a t-structure $\mathcal{P}$ is the full subcategory

$$\mathcal{H} = \mathcal{P}^\perp[1] \cap \mathcal{P}$$

and any bounded t-structure is determined by its heart. More precisely, any bounded t-structure $\mathcal{P}$ with heart $\mathcal{H}$ determines, for each M in $\mathcal{D}$, a canonical filtration ([3, Lemma 3.2])

$$\begin{array}{ccccccccc} 0 = M_0 & \longrightarrow & M_1 & \longrightarrow \cdots \longrightarrow & M_{m-1} & \longrightarrow & M_m = M \\ & \nwarrow & \swarrow & & & \nwarrow & \swarrow \\ & H_1[k_1] & & & & H_m[k_m] & \end{array} \tag{2.3}$$

where $H_i \in \mathcal{H}$ and $k_1 > \ldots > k_m$ are integers. Using this filtration, one can define the k-th homology of M, with respect to $\mathcal{H}$, to be

$$\mathbf{H}_k(M) = \begin{cases} H_i & \text{if } k = k_i \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Then $\mathcal{P}$ consists of those objects with no (nonzero) negative homology, $\mathcal{P}^\perp$ those with only negative homology and $\mathcal{H}$ those with homology only in degree 0.

In this paper, we only consider bounded t-structures and their hearts, and use the phrase 'a triangulated category $\mathcal{D}$ with heart $\mathcal{H}$' to mean that $\mathcal{H}$ is the heart of a bounded t-structure on $\mathcal{D}$. Furthermore, all categories will be implicitly assumed to be $\mathbf{k}$-linear.

Let $\mathcal{H}$ be a heart with corresponding t-structure $\mathcal{P}$. We say that an object $P \in \mathcal{D}$ is a *projective* of $\mathcal{H}$ if $\operatorname{Hom}^k(P, M) = 0$, for all $k \neq 0$ and all $M \in \mathcal{H}$; or equivalently if $P \in \mathcal{P}$ and $\operatorname{Ext}^1(P, L) = 0$ for any $L \in \mathcal{P}$. Note that a projective *of* a heart is not necessary *in* the heart. We denote by $\operatorname{Proj} \mathcal{H}$ a complete set of indecomposable projectives of $\mathcal{H}$. When $\mathcal{D} = \mathcal{D}(Q)$, we see more explicitly, using Auslander-Reiten duality (2.2), that $P \in \mathcal{D}(Q)$ is a projective of a heart $\mathcal{H}$ if and only if

$$P \in \mathcal{P} \cap \tau^{-1}\mathcal{P}^\perp. \tag{2.5}$$

Proposition 2.1. *Let $\mathcal{H}$ be any heart in a triangulated category $\mathcal{D}$ and P any projective of $\mathcal{H}$. Then*

$$\operatorname{Hom}^{-k}(P, M) = \operatorname{Hom}(P, \mathbf{H}_k(M)) \tag{2.6}$$

where $\mathbf{H}_\bullet$ is as in (2.4). Thus, if $\operatorname{Proj} \mathcal{H}$ 'spans' $\mathcal{H}$, in the sense that, for any $M \in \mathcal{H}$,

$$\operatorname{Hom}(P, M) = 0, \ \forall P \in \operatorname{Proj} \mathcal{H} \implies M = 0, \tag{2.7}$$

then $\operatorname{Proj} \mathcal{H}$ determines $\mathcal{H}$, as

$$M \in \mathcal{H} \iff \operatorname{Hom}^k(P, M) = 0, \ \forall P \in \operatorname{Proj} \mathcal{H}, k \neq 0. \tag{2.8}$$

Proof. Suppose M has a filtration as in (2.3), so that $M \in \mathcal{P}[k_m] \cap \mathcal{P}^\perp[k_1 + 1]$. Then, by the definition of projective, we have

$$\operatorname{Hom}^{\geq 1}(P, M) = 0 = \operatorname{Hom}^{\leq 0}(P, L)$$

for any $M \in \mathcal{P}$ and $L \in \mathcal{P}^\perp$. Thus $\operatorname{Hom}^{-k}(P, M) = 0$, for $k > k_1$ and $k < k_m$, and $\mathbf{H}_k(M) = 0$ for the same range of k. Now, applying $\operatorname{Hom}(P, -)$ to the triangle

$$M'[-1] \to \mathbf{H}_{k_1}(M)[k_1] \to M \to M',$$

gives $\mathrm{Hom}^{-k_1}(P, M) = \mathrm{Hom}(P, \mathbf{H}_{k_1}(M))$, because $M' \in \mathcal{P}^{\perp}[k_1]$. But also

$$\mathrm{Hom}^{-k}(P, M) = \mathrm{Hom}^{-k}(P, M'),\ \forall k < k_1,$$

because $\mathbf{H}_{k_1}(M) \in \mathcal{P}^{\perp}[1]$. Thus (2.6) follows by induction. The second part is immediate, because $M \in \mathcal{H}$ if and only if $\mathbf{H}_k(M) = 0$, for all $k \neq 0$. □

Remark 2.2. Note that the heart $\mathcal{H}$ of a t-structure on $\mathcal{D}$ is always an abelian category, but $\mathcal{D}$ is not necessarily equivalent to the derived category of $\mathcal{H}$. On the other hand, any abelian category $\mathcal{C}$ is the heart of a standard t-structure on $\mathcal{D}(\mathcal{C})$. Indeed, any object in $\mathcal{D}(\mathcal{C})$ may be considered as a complex in $\mathcal{C}$ and its ordinary homology objects are the factors of the filtration (2.3) associated to this standard t-structure. Moreover, in such cases the projectives of $\mathcal{C}$ coincide with the normal definition. For instance, $\mathcal{D}(Q)$ has a canonical heart $\mathrm{mod}\,\mathbf{k}Q$, which we will write as $\mathcal{H}_Q$ from now on.

Although we easily see that $\mathcal{H}_1 \subset \mathcal{H}_2$ implies $\mathcal{H}_1 = \mathcal{H}_2$, there is a natural partial order on hearts given by inclusion of their corresponding t-structures. More precisely, for two hearts $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{D}$, with t-structures $\mathcal{P}_1$ and $\mathcal{P}_2$, we say

$$\mathcal{H}_1 \leq \mathcal{H}_2 \tag{2.9}$$

if and only if $\mathcal{P}_2 \subset \mathcal{P}_1$, or equivalently $\mathcal{H}_2 \subset \mathcal{P}_1$, or equivalently $\mathcal{P}_1^{\perp} \subset \mathcal{P}_2^{\perp}$, or equivalently $\mathcal{H}_1 \subset \mathcal{P}_2^{\perp}[1]$.

An useful elementary observation is the following.

Lemma 2.3. *Given hearts $\mathcal{H}_1 \leq \mathcal{H}_2 \leq \mathcal{H}_3$, any object T in $\mathcal{H}_1$ and $\mathcal{H}_3$ is also in $\mathcal{H}_2$.*

Proof. By assumption $T \in \mathcal{P}_3 \subset \mathcal{P}_2$ and $T \in \mathcal{P}_1^{\perp}[1] \subset \mathcal{P}_2^{\perp}[1]$. □

3. Tilting Theory

A similar notion to a t-structure on a triangulated category is a torsion pair in an abelian category. Tilting with respect to a torsion pair in the heart of a t-structure provides a way to pass between different t-structures.

Definition 3.1. A *torsion pair* in an abelian category $\mathcal{C}$ is a pair of full subcategories $\langle \mathcal{F}, \mathcal{T} \rangle$ of $\mathcal{C}$, such that $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and furthermore every object $E \in \mathcal{C}$ fits into a short exact sequence $0 \longrightarrow E^{\mathcal{T}} \longrightarrow E \longrightarrow E^{\mathcal{F}} \longrightarrow 0$ for some objects $E^{\mathcal{T}} \in \mathcal{T}$ and $E^{\mathcal{F}} \in \mathcal{F}$.

Proposition 3.2 (Happel, Reiten, Smalø[9])**.** *Let $\mathcal{H}$ be a heart in a triangulated category $\mathcal{D}$. Suppose further that $\langle \mathcal{F}, \mathcal{T} \rangle$ is a torsion pair in $\mathcal{H}$. Then the full subcategory*

$$\mathcal{H}^{\sharp} = \{E \in \mathcal{D} : \mathbf{H}_1(E) \in \mathcal{F}, \mathbf{H}_0(E) \in \mathcal{T} \text{ and } \mathbf{H}_i(E) = 0 \text{ otherwise}\}$$

is also a heart in $\mathcal{D}$, as is

$$\mathcal{H}^{\flat} = \{E \in \mathcal{D} : \mathbf{H}_0(E) \in \mathcal{F}, \mathbf{H}_{-1}(E) \in \mathcal{T} \text{ and } \mathbf{H}_i(E) = 0 \text{ otherwise}\}.$$

Recall that the homology $\mathbf{H}_{\bullet}$ was defined in (2.4). We call $\mathcal{H}^{\sharp}$ the *forward tilt* of $\mathcal{H}$, with respect to the torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$, and $\mathcal{H}^{\flat}$ the *backward tilt* of $\mathcal{H}$. Note that $\mathcal{H}^{\flat} = \mathcal{H}^{\sharp}[-1]$. Furthermore, $\mathcal{H}^{\sharp}$ has a torsion pair $\langle \mathcal{T}, \mathcal{F}[1] \rangle$. With respect to this torsion pair, the forward and backward tilts are $\left(\mathcal{H}^{\sharp}\right)^{\sharp} = \mathcal{H}[1]$ and $\left(\mathcal{H}^{\sharp}\right)^{\flat} = \mathcal{H}$. Similarly with respect to the torsion pair $\langle \mathcal{T}[-1], \mathcal{F} \rangle$ in $\mathcal{H}^{\flat}$, we have $\left(\mathcal{H}^{\flat}\right)^{\sharp} = \mathcal{H}$, $\left(\mathcal{H}^{\flat}\right)^{\flat} = \mathcal{H}[-1]$.

Proposition 3.3. *Let M be an indecomposable in $\mathcal{D}$ with canonical filtration with respect to a heart $\mathcal{H}$, as in (2.3). Given a torsion pair $\langle\mathcal{F},\mathcal{T}\rangle$ in $\mathcal{H}$, the short exact sequences*

$$0 \to H_i^{\mathcal{T}} \to H_i \to H_i^{\mathcal{F}} \to 0,$$

can be used to refine the canonical filtration of M to a finer one with factors

$$\left(H_1^{\mathcal{T}}[k_1], H_1^{\mathcal{F}}[k_1], \ldots, H_m^{\mathcal{T}}[k_m], H_m^{\mathcal{F}}[k_m]\right), \tag{3.1}$$

Furthermore, if we take the canonical filtration of M with respect to the heart $\mathcal{H}^\sharp$ and refine it using the torsion pair $(\mathcal{T},\mathcal{F}[1])$, then we obtain essentially the same filtration

$$\Big(H_1^{\mathcal{T}}[k_1], \tilde{H}_1^{\mathcal{F}}[k_1-1], \ldots, H_m^{\mathcal{T}}[k_m], \tilde{H}_m^{\mathcal{F}}[k_m-1]\Big), \tag{3.2}$$

where $\tilde{H}_i = H_i[1]$.

Proof. The existence of the filtrations (3.1) and (3.2) follows by repeated use of the Octahedral Axiom. □

We observe how tilting relates to the partial ordering of hearts defined in (2.9).

Lemma 3.4. *Let $\mathcal{H}$ be a heart in $\mathcal{D}(Q)$. Then $\mathcal{H} < \mathcal{H}[m]$ for $m > 0$. For any forward tilt $\mathcal{H}^\sharp$ and backward tilt $\mathcal{H}^\flat$, we have $\mathcal{H}[-1] \le \mathcal{H}^\flat \le \mathcal{H} \le \mathcal{H}^\sharp \le \mathcal{H}[1]$.*

Proof. Since $\mathcal{P} \supsetneq \mathcal{P}[1]$, we have $\mathcal{H} < \mathcal{H}[m]$ for $m > 0$. By Proposition 3.3 we have $\mathcal{P} \supset \mathcal{P}^\sharp$, hence $\mathcal{H} \le \mathcal{H}^\sharp$. Noticing that $\left(\mathcal{H}^\sharp\right)^\sharp = \mathcal{H}[1]$ with respect to the torsion pair $(\mathcal{T},\mathcal{F}[1])$, we have $\mathcal{H}^\sharp \le \mathcal{H}[1]$. Similarly, $\mathcal{H}[-1] \le \mathcal{H}^\flat \le \mathcal{H}$. □

In fact the forward tilts $\mathcal{H}^\sharp$ can be characterized as precisely the hearts between $\mathcal{H}$ and $\mathcal{H}[1]$ (cf. [9]); similarly the backward tilts $\mathcal{H}^\flat$ are those between $\mathcal{H}[-1]$ and $\mathcal{H}$.

Recall that an object in an abelian category is *simple* if it has no proper subobjects, or equivalently it is not the middle term of any (non-trivial) short exact sequence. An object M is *rigid* if $\operatorname{Ext}^1(M,M) = 0$.

Lemma 3.5. *Let S be a rigid simple object in a Hom-finite abelian category $\mathcal{C}$. Then $\mathcal{C}$ admits a torsion pair $\langle\mathcal{F},\mathcal{T}\rangle$ such that $\mathcal{F} = \langle S\rangle$. More precisely, for any $M \in \mathcal{H}$, in the corresponding short exact sequence*

$$0 \to M^{\mathcal{T}} \to M \to M^{\mathcal{F}} \to 0 \tag{3.3}$$

we have $M^{\mathcal{F}} = S \otimes \operatorname{Hom}(M,S)^*$. Similarly, there is also a torsion pair with the torsion part $\mathcal{T} = \langle S\rangle$, obtained by setting $M^{\mathcal{T}} = S \otimes \operatorname{Hom}(S,M)$.

Proof. If we define $M^{\mathcal{F}}$ as in the lemma, then there is a canonical surjection $M \to M^{\mathcal{F}}$, whose kernel we may define to be $M^{\mathcal{T}}$, yielding the short exact sequence (3.3).

Applying $\operatorname{Hom}(-,S)$ to (3.3), we get

$$0 \to \operatorname{Hom}(M^{\mathcal{F}},S) \to \operatorname{Hom}(M,S) \to \operatorname{Hom}(M^{\mathcal{T}},S) \to \operatorname{Ext}^1(M^{\mathcal{F}},S) \to \cdots .$$

But

$$\operatorname{Hom}(M^{\mathcal{F}},S) = \operatorname{Hom}(S \otimes \operatorname{Hom}(M,S)^*,S) \cong \operatorname{Hom}(M,S),$$
$$\operatorname{Ext}^1(M^{\mathcal{F}},S) = \operatorname{Ext}^1(S \otimes \operatorname{Hom}(M,S)^*,S) = 0,$$

so we have $\mathrm{Hom}(M^{\mathcal{T}}, S) = 0$ and hence $\mathrm{Hom}(M^{\mathcal{T}}, M^{\mathcal{F}}) = 0$ as required. The proof of the second statement is similar. □

Definition 3.6. We say a forward tilt is *simple*, if the corresponding torsion free part is generated by a single rigid simple object S. We denote the new heart by $\mathcal{H}^{\sharp}_{S}$. Similarly, a backward tilt is simple if the corresponding torsion part is generated by such a simple and the new heart is denoted by $\mathcal{H}^{\flat}_{S}$.

For the standard heart $\mathcal{H}_Q$ in $\mathcal{D}(Q)$, an APR tilt ([2, p. 201]), which reverses all arrows at a sink/source of Q, is an example of a simple (forward/backward) tilt.

4. Exchange graphs

4.1. Exchange graphs and their subgraphs.

Definition 4.1. Define the *exchange graph* $\mathrm{EG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ to be the oriented graph whose vertices are all hearts in $\mathcal{D}$ and whose edges correspond to simple forward tiltings between them. For any heart $\mathcal{H}_0$ in $\mathcal{D}$ and any $N \geq 2$, define the *exchange graph with base* $\mathcal{H}_0$ to be the full subgraph of $\mathrm{EG}(\mathcal{D})$ given by

$$\mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0) = \big\{\mathcal{H} \in \mathrm{EG}(\mathcal{D}) \mid \mathcal{H}_0[1] \leq \mathcal{H} \leq \mathcal{H}_0[N-1]\big\}, \tag{4.1}$$

Define $\mathrm{EG}^{\circ}_N(\mathcal{D}, \mathcal{H}_0)$ to be the 'principal component' of the interval $\mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0)$, that is, the connected component that contains $\mathcal{H}_0[1]$.

We label each edge in $\mathrm{EG}(\mathcal{D})$ by the simple object of the corresponding tilting, that is, the edge from $\mathcal{H}$ to $\mathcal{H}^{\sharp}_{S}$ is labelled by S. Notice that, by Lemma 3.4, we have $\mathcal{H} < \mathcal{H}^{\sharp}_{S}$ for any simple tilting, which implies that there is no oriented cycle in the exchange graph.

When $\mathcal{D} = \mathcal{D}(Q)$, for a quiver Q, we will shorten $\mathcal{D}(Q)$ to Q in the notation for exchange graphs, e.g. write $\mathrm{EG}(Q)$ for $\mathrm{EG}(\mathcal{D}(Q))$. Further, we denote by $\mathrm{EG}^{\circ}(Q)$ the 'principal' component of $\mathrm{EG}(Q)$, that is, the connected component containing the heart $\mathcal{H}_Q$. Later, we will focus especially on the exchange graph $\mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$ with base the standard heart $\mathcal{H}_Q$.

Remark 4.2. Note that $\mathrm{EG}^{\circ}(Q)$ is usually a proper subgraph of $\mathrm{EG}(Q)$. However, if Q is a Dynkin quiver, then the two are equal, by a result of Keller-Vossieck [15]; an alternative proof can be found in [20, Appendix A].

Note also that, although the definition of $\mathrm{EG}^{\circ}_N(\mathcal{D}, \mathcal{H}_0)$ favours $\mathcal{H}_0[1]$ asymmetrically, we will see that, at least in the case of $\mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$ for an acyclic quiver Q, we actually have $\mathcal{H}_Q[j] \in \mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$ for $1 \leq j \leq N-1$ (Corollary 5.3), so this asymmetry disappears.

Note finally that, as defined there is *a priori* a distinction between the interval in the principal component, that is, $\mathrm{EG}^{\circ}(Q) \cap \mathrm{EG}_N(Q, \mathcal{H}_Q)$, which is not necessary connected, and the principal component of the interval, that is, $\mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$, which is connected by construction. However, we will in fact see (cf. (5.7) and Remark 5.13) that the two coincide.

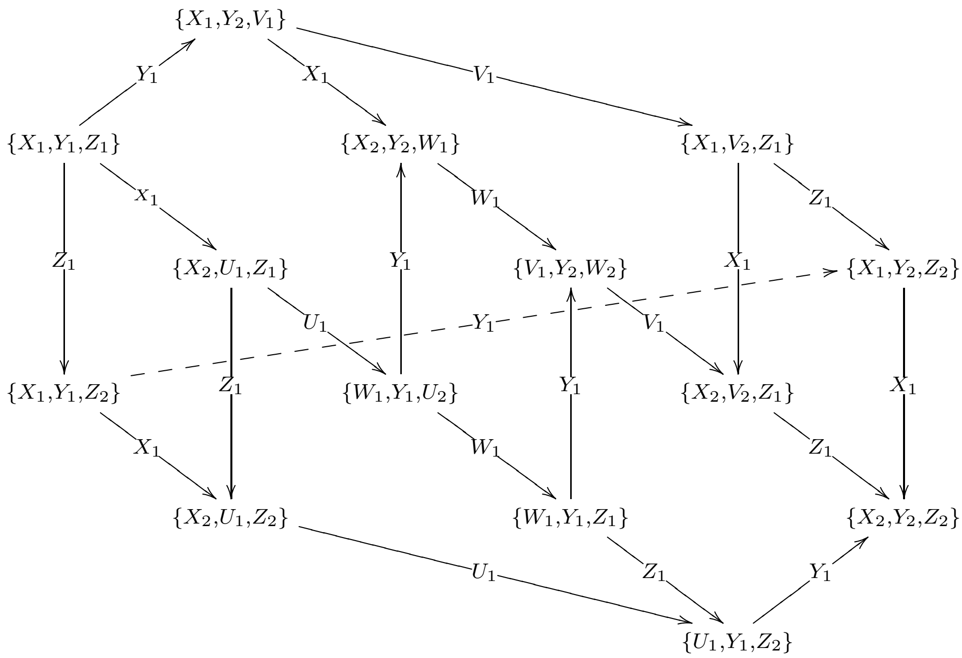
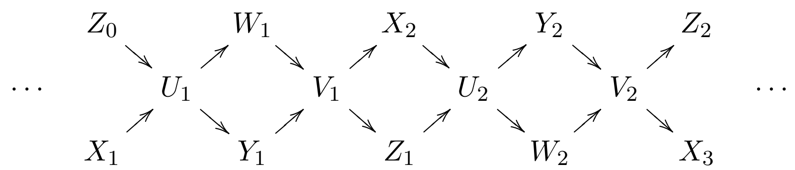


FIGURE 1. The exchange graph $\mathrm{EG}_3(Q, \mathcal{H}_Q)$ for Q of type A_3

Example 4.3. Let Q be the quiver of type A_3 with straight orientation and simple modules X, Y, Z. A piece of the Auslander-Reiten quiver of $\mathcal{D}(Q)$ is as follows



where $M_i = M[i]$ for $M \in \operatorname{Ind} \mathcal{H}_Q$. Figure 1 is the exchange graph $\mathrm{EG}_3(Q, \mathcal{H}_Q)$, where we denote each heart by a complete set of simples.

4.2. **Cluster exchange graphs.** We recall some notions from higher cluster theory and, in particular, describe the relationship between hearts and m-cluster tilting sets.

Definition 4.4 (cf. [6], [10] [24])**.** For any integer $m \geq 1$, the *m-cluster shift* is the auto-equivalence of $\mathcal{D}(Q)$ given by $\Sigma_m = \tau^{-1} \circ [m-1]$.

- The *m-cluster category* $\mathcal{C}_m(Q)$ is the orbit category $\mathcal{D}(Q)/\,\Sigma_m$ (cf. [12]), that is,
$$\begin{aligned} \operatorname{Ext}^k_{\mathcal{C}_m(Q)}(M, L) &= \operatorname{Hom}_{\mathcal{C}_m(Q)}(M, L[k]) \\ &= \bigoplus_{t \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}(Q)}(M, \Sigma_m^t\, L[k]). \end{aligned}$$
- An *m-cluster tilting set* $\{P_j\}_{j=1}^n$ in $\mathcal{C}_m(Q)$ is an Ext-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\operatorname{Ext}^k_{\mathcal{C}_m(Q)}(P_i, P_j) = 0$,

for any $1 \le k \le m-1$. Note that $n = \# Q_0$. An *almost complete cluster tilting set* in $\mathcal{C}_m(Q)$ is a subset of a cluster tilting set with $n-1$ elements.

- The *forward mutation* μ_i at the i-th object acts on an m-cluster tilting set $\{P_j\}_{j=1}^n$, by replacing P_i by
$$P_i^\sharp = \mathrm{Cone}(P_i \to \bigoplus_{j\neq i} \mathrm{Irr}(P_i, P_j)^* \otimes P_j), \tag{4.2}$$
where $\mathrm{Irr}(P_i, P_j)$ is a space of irreducible maps $P_i \to P_j$, in the additive subcategory $\mathrm{Add}\,\mathbf{P}$ of $\mathcal{C}_m(Q)$, for $\mathbf{P} = \bigoplus_{i=1}^n P_i$. When Q is acyclic, we have $\mathrm{Irr}(P_i, P_i) = 0$, that is, the Gabriel quiver of $\mathrm{End}(\mathbf{P})$ has no loops (cf. [6]). Furthermore, the *backward mutation* μ_i^{-1} replaces P_i by
$$P_i^\flat = \mathrm{Cone}(\bigoplus_{j\neq i} \mathrm{Irr}(P_j, P_i) \otimes P_j \to P_i)[-1]. \tag{4.3}$$
- The *exchange graph* $\mathrm{CEG}_m(Q)$ of m-clusters is the oriented graph whose vertices are m-cluster tilting sets and whose edges are the forward mutations. Note that $\mathrm{CEG}_m(Q)$ is connected [6, Proposition 7.1].

In the 'classical' case $m = 2$, the exchange graph is often presented as an unoriented graph $\mathrm{CEG}(Q)^*$, from which $\mathrm{CEG}_2(Q)$ is obtained by replacing each unoriented edge by an oriented two-cycle. For instance, for Q of type A_3, $\mathrm{CEG}(Q)^*$ is the underlying unoriented graph of Figure 1 (cf. [4, Figure 4]). We will explain why this should be the case in Section 9.

To relate hearts in $\mathcal{D}(Q)$ and cluster tilting sets in $\mathcal{C}_m(Q)$, we consider the restriction of the quotient map $\pi_m : \mathcal{D}(Q) \to \mathcal{C}_m(Q)$ to a fundamental domain (cf. [25, Proposition 2.2]) to obtain a bijection
$$\pi_m \colon \mathrm{Proj}\,\mathcal{H}_Q[m] \cup \bigcup_{j=1}^{m-1} \mathrm{Ind}\,\mathcal{H}_Q[j] \longrightarrow \mathrm{Ind}\,\mathcal{C}_m(Q). \tag{4.4}$$

Lemma 4.5. *Let $\mathcal{H} \in \mathrm{EG}_N(Q, \mathcal{H}_Q)$. Then $\#\,\mathrm{Proj}\,\mathcal{H} \le \# Q_0$ and, when these are equal, $\pi_m\big(\mathrm{Proj}\,\mathcal{H}\big)$ is an m-cluster tilting set, for any $m \ge N-1$. In addition, if $m \ge N$ and $\mathbf{P} = \bigoplus_{P \in \mathrm{Proj}\,\mathcal{H}} P$, then*
$$\mathrm{End}_{\mathcal{D}(Q)}(\mathbf{P}) = \mathrm{End}_{\mathcal{C}_m(Q)}(\pi_m(\mathbf{P})). \tag{4.5}$$

Proof. By (2.5),
$$\mathrm{Proj}\,\mathcal{H} \ \subset\ \mathcal{P} \cap \tau^{-1}\mathcal{P}^\perp \ \subset\ \mathcal{P}_Q[1] \cap \tau^{-1}\mathcal{P}_Q^\perp[N-1], \tag{4.6}$$
which is a subset of the LHS of (4.4) for $m \ge N-1$. This means that (the images of) the projectives in $\mathrm{Proj}\,\mathcal{H}$ are distinct, i.e. non-isomorphic, in $\mathcal{C}_m(Q)$. For all $P_i, P_j \in \mathrm{Proj}\,\mathcal{H}$, as they are projective in $\mathcal{D}(Q)$, we have $\mathrm{Ext}^k_{\mathcal{D}(Q)}(P_i, P_j) = 0$, for all $k > 0$, and hence ([23, Lemma 1.1]) we deduce $\mathrm{Ext}^k_{\mathcal{C}_m(Q)}(P_i, P_j) = 0$, for all $1 \le k \le m$. Thus $\pi_m\big(\mathrm{Proj}\,\mathcal{H}\big)$ is a partial m-cluster tilting set, and so $\#\,\mathrm{Proj}\,\mathcal{H} \le \# Q_0$. When we have equality, $\pi_m\big(\mathrm{Proj}\,\mathcal{H}\big)$ is a m-cluster tilting set ([5]).

To prove (4.5), observe that by (4.6) we have, for any $t \ge 1$,
$$\Sigma_m^t\, P_i \in \Sigma_m^t\, \mathcal{P}_Q[1] \subset \Sigma_m^{t-1}\, \tau^{-1}\mathcal{P}_Q[m] \subset \tau^{-1}\mathcal{P}_Q[m] \subset \tau^{-1}\mathcal{P}_Q[N],$$

which implies $\operatorname{Hom}(\Sigma_m^t P_i, P_j) = 0$, as $\operatorname{Hom}(\mathcal{P}_Q[N], \mathcal{P}_Q^{\perp}[N-1]) = 0$. On the other hand, for $t \le -1$,

$$\Sigma_m^t P_i \in \Sigma_m^t \tau^{-1} \mathcal{P}_Q^{\perp}[N-1] \subset \Sigma_m^{t+1} \mathcal{P}_Q^{\perp}[N-m] \subset \mathcal{P}_Q^{\perp},$$

which implies $\operatorname{Hom}(\Sigma_m^t P_i, P_j) = 0$, as $\operatorname{Hom}(\mathcal{P}_Q^{\perp}, \mathcal{P}_Q[1]) = 0$, because $\mathcal{H}_Q$ is hereditary. □

5. Structure of exchange graphs

5.1. **Finite hearts.** For any heart $\mathcal{H}$ in a triangulated category $\mathcal{D}$, we denote by $\operatorname{Sim}\mathcal{H}$ a complete set of non-isomorphic simples in $\mathcal{H}$.

Definition 5.1. We say that a heart $\mathcal{H}$ is

- *finite*, if $\operatorname{Sim}\mathcal{H}$ is a finite set which generates $\mathcal{H}$ by means of extensions, i.e. every object M in $\mathcal{H}$ has a finite filtration with simple factors. Note that, by the Jordan-Hölder Theorem, these factors are uniquely determined up to reordering.
- *rigid* if every simple S in $\mathcal{H}$ is rigid, i.e. $\operatorname{Ext}^1(S, S) = 0$.

For example, when Q is acyclic, the standard heart $\mathcal{H}_Q$ in $\operatorname{EG}^\circ(Q)$ is finite and rigid. Our main interest is in the part of the exchange graph that contains finite hearts. We also know, from Lemma 3.5, that we can tilt with respect rigid simples, so a rigid heart is one for which we can tilt with respect to all simples. Thus, to see how adjacent hearts in the exchange graph are related, we begin by determining how the simples of a finite heart change under simple tilting.

Proposition 5.2. *In any triangulated category $\mathcal{D}$, let S be a rigid simple in a finite heart $\mathcal{H}$. Then after a forward or backward simple tilt (Definition 3.6) the new simples are*

$$\operatorname{Sim}\mathcal{H}_S^{\sharp} = \{S[1]\} \cup \{\psi_S^{\sharp}(X) \mid X \in \operatorname{Sim}\mathcal{H}, X \neq S\}, \tag{5.1}$$

$$\operatorname{Sim}\mathcal{H}_S^{\flat} = \{S[-1]\} \cup \{\psi_S^{\flat}(X) \mid X \in \operatorname{Sim}\mathcal{H}, X \neq S\}, \tag{5.2}$$

where

$$\psi_S^{\sharp}(X) = \operatorname{Cone}\left(X \to S[1] \otimes \operatorname{Ext}^1(X, S)^*\right)[-1], \tag{5.3}$$

$$\psi_S^{\flat}(X) = \operatorname{Cone}\left(S[-1] \otimes \operatorname{Ext}^1(S, X) \to X\right). \tag{5.4}$$

Thus $\mathcal{H}_S^{\sharp}$ and $\mathcal{H}_S^{\flat}$ are also finite.

Proof. We only deal with the case for forward tilting; the backwards case is similar. Let $\langle \mathcal{F}, \mathcal{T} \rangle$ be the torsion pair in $\mathcal{H}$ whose forward tilt yields $\mathcal{H}_S^{\sharp}$. Any simple in $\mathcal{H}_S^{\sharp}$ is either in $\mathcal{T}$ or $\mathcal{F}[1]$. Since S has no self extension, we have $\mathcal{F} = \{S^m \mid m \in \mathbb{N}\}$. Furthermore, choose any simple quotient S_0 of $S[1]$ in $\mathcal{H}_S^{\sharp}$. S_0 cannot be in $\mathcal{T}$ since $\operatorname{Hom}(\mathcal{F}[1], \mathcal{T}) = 0$. Thus $S_0 \in \mathcal{F}[1]$ which implies $S[1] = S_0$, i.e. $S[1] \in \operatorname{Sim}\mathcal{H}_S^{\sharp}$.

Let $X \ncong S$ be any other simple in $\mathcal{H}$, so that $X \in \mathcal{T}$, and set $T = \psi_S^{\sharp}(X)$. Let T' be a simple submodule of X in $\mathcal{H}_S^{\sharp}$ and $f : T' \to X$ be a non-zero map. Since

$\operatorname{Hom}(S[1], X) = 0$, T' is in $\mathcal{T}$ instead of $\mathcal{F}[1]$. Because S is simple in $\mathcal{H}$ and T' is simple in $\mathcal{H}^\sharp_S$, there are short exact sequences

$$0 \to L \to T' \xrightarrow{f} X \to 0 \tag{5.5}$$

$$0 \to T' \xrightarrow{f} X \to M \to 0$$

in $\mathcal{H}$ and $\mathcal{H}^\sharp_S$ respectively. Thus $L = M[-1]$. On the other hand $\mathcal{H}^\sharp[-1] \cap \mathcal{H} = \mathcal{F}$, which implies $L \in \mathcal{F}$. Hence we have $L = S^m$ for some integer m, and indeed there is a canonical isomorphism

$$L \cong S \otimes \operatorname{Hom}(L, S)^*.$$

Now applying $\operatorname{Hom}(-, S)$ to (5.5), we get

$$0 = \operatorname{Hom}(T', S) \to \operatorname{Hom}(L, S) \xrightarrow{g} \operatorname{Hom}^1(X, S) \to \operatorname{Hom}^1(T', S) = 0.$$

which implies g is an isomorphism and hence we have an isomorphism between triangles

$$\begin{array}{ccccccc}
X[-1] & \longrightarrow & S \otimes \operatorname{Hom}^1(X,S)^* & \longrightarrow & T & \longrightarrow & X \\
\| & & \cong \downarrow \operatorname{id} \otimes g^* & & \downarrow & & \| \\
X[-1] & \longrightarrow & S \otimes \operatorname{Hom}(L,S)^* & \longrightarrow & T' & \longrightarrow & X
\end{array}$$

showing that $T \cong T'$, so that T is a simple in $\mathcal{H}^\sharp_S$, as required.

Now, if $\mathcal{H}$ is finite, with $\#\operatorname{Sim}\mathcal{H} = n$, then the RHS of (5.1) contains n simples in $\mathcal{H}^\sharp_S$, whose classes form a basis of the Grothendieck group $\mathcal{K}(\mathcal{D}) \cong \mathcal{K}(\mathcal{H}) \cong \mathbb{Z}^n$. Hence these new simples of $\mathcal{H}^\sharp_S$ are non-isomorphic and must be a complete set of simples as also $\mathcal{K}(\mathcal{D}) \cong \mathcal{K}(\mathcal{H}^\sharp_S)$. □

A first useful consequence is the following.

Corollary 5.3. *If Q is an acyclic quiver, then $\mathcal{H}_Q[k] \in \operatorname{EG}^\circ(Q)$, for all $k \in \mathbb{Z}$. Furthermore, $\mathcal{H}_Q[k] \in \operatorname{EG}^\circ_N(Q, \mathcal{H}_Q)$, for $1 \le k \le N-1$.*

Proof. Since Q is acyclic, we can order the simples in $\mathcal{H}_Q$ so that

$$\operatorname{Ext}^1(S_j, S_i) = 0, \quad \text{for } 1 \le i < j \le n. \tag{5.6}$$

Then Proposition 5.2 implies that forward tilting $\mathcal{H}_Q$ by S_1, gives a heart $\mathcal{H}$ with $\operatorname{Sim}\mathcal{H} = \{S_2, \ldots, S_n, S_1[1]\}$, whose simples still satisfy (5.6), in this new order. Hence, by iterated forward tilting $\mathcal{H}_Q$ with respect to $S_1, \ldots, S_n$ we obtain $\mathcal{H}_Q[1]$. A similar result applies for backward tilting and hence we obtain the first claim. The second claim follows because this iterated tilting from $\mathcal{H}_Q[1]$ to $\mathcal{H}_Q[k]$, etc., remains within the interval $\operatorname{EG}_N(Q, \mathcal{H}_Q)$, for $k \le N-1$. □

Note that this corollary also implies that

$$\operatorname{EG}^\circ_N(Q, \mathcal{H}_Q) \subset \operatorname{EG}^\circ(Q) \cap \operatorname{EG}_N(Q, \mathcal{H}_Q). \tag{5.7}$$

We next identify, again for a general triangulated category $\mathcal{D}$, elementary criteria for when a heart $\mathcal{H}$ is in the interval $\operatorname{EG}_N(\mathcal{D}, \mathcal{H}_0)$, that is, $\mathcal{H}_0[1] \le \mathcal{H} \le \mathcal{H}_0[N-1]$ (Definition 4.1), and also when its forward and backward tilts remain in this interval.

Lemma 5.4. *Let $\mathcal{H}_0$ and $\mathcal{H}$ be finite hearts in* $\mathrm{EG}(\mathcal{D})$. *Then $\mathcal{H} \in \mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0)$ if and only if*

$$\mathbf{H}_m(S) = 0, \quad \forall S \in \operatorname{Sim} \mathcal{H},\ m \notin (0, N),$$

where the homology $\mathbf{H}_\bullet$ is with respect to $\mathcal{H}_0$. Moreover, for any rigid $S \in \operatorname{Sim} \mathcal{H}$, we have

1°. $\mathcal{H}^\flat_S \in \mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0)$ if and only if $\mathbf{H}_1(S) = 0$,

2°. $\mathcal{H}^\sharp_S \in \mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0)$ if and only if $\mathbf{H}_{N-1}(S) = 0$.

Proof. The conditions $\mathbf{H}_m(S) = 0$, for $m \leq 0$, are equivalent to $S \in \mathcal{P}_0[1]$ and as $\mathcal{H}$ is generated by its simples, this is equivalent to $\mathcal{H} \subset \mathcal{P}_0[1]$, i.e. $\mathcal{H}_0[1] \leq \mathcal{H}$. The other inequality is similar. The necessity in the second assertions is then immediate.

As $\mathcal{H}$ is finite, (5.1) and (5.2) in Proposition 5.2 determine the simples in $\mathcal{H}^\sharp_S$ and $\mathcal{H}^\flat_S$ and thus the sufficiency in the second assertions also follows from the first one. □

Corollary 5.5. *Let $\mathcal{H}_0$ and $\mathcal{H}$ be finite hearts in* $\mathrm{EG}(\mathcal{D})$. *Then, for any rigid $S \in \operatorname{Sim} \mathcal{H}$, we have*

1°. $\mathcal{H}^\flat_S \in \mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0)$ if and only if $\operatorname{Hom}(S, T) = 0$ for every $T \in \operatorname{Sim} \mathcal{H}_0[1]$,

2°. $\mathcal{H}^\sharp_S \in \mathrm{EG}_N(\mathcal{D}, \mathcal{H}_0)$ if and only if $\operatorname{Hom}(T, S) = 0$ for every $T \in \operatorname{Sim} \mathcal{H}_0[N-1]$.

Proof. By the first part of Lemma 5.4, we have $\mathbf{H}_m(S) = 0$ for $m \notin (0, N)$, where $\mathbf{H}_\bullet$ is with respect to $\mathcal{H}_0$. Let $\mathbf{H}_1(S) = H \in \mathcal{H}_0$ which induces a triangle

$$H \to S' \to S \to H[1],$$

where $\mathbf{H}_{\leq 1} S' = 0$, i.e. $S' \in \mathcal{P}_0[2]$. Hence, for any $T \in \operatorname{Sim} \mathcal{H}_0[1] \subset \mathcal{P}_0^\perp[2]$, we have $\operatorname{Hom}^{\leq 0}(S', T) = 0$ and so $\operatorname{Hom}(S, T) = \operatorname{Hom}(H[1], T)$. This implies that $\operatorname{Hom}(S, T) = 0$ for any $T \in \operatorname{Sim} \mathcal{H}_0[1]$ if and only if $H = 0$. Then the first claim follows from the second part of Lemma 5.4.

For the second claim, let $\mathbf{H}_{N-1}(S) = H \in \mathcal{H}_0$ which induces a triangle

$$H[N-1] \to S \to S' \to H[N].$$

Now, for any $T \in \operatorname{Sim} \mathcal{H}_0[N-1]$, we have $\operatorname{Hom}^{\leq 0}(T, S') = 0$ and so $\operatorname{Hom}(T, S) = \operatorname{Hom}(T, H[N-1])$ and the claim follows as before. □

5.2. **Tilting and mutation of projectives.** Given a finite heart $\mathcal{H}$ with simples $\operatorname{Sim} \mathcal{H} = \{S_1, \ldots, S_n\}$, it is natural to seek a set of projectives $\{P_1, \ldots, P_n\}$, which is *dual*, in the sense that

$$\dim \operatorname{Hom}(P_i, S_j) = \delta_{ij} \tag{5.8}$$

and further to expect that these are a complete set of indecomposable projectives. This property holds, of course, for the standard heart $\mathcal{H}_Q$ in $\mathcal{D}(Q)$ and we will show that it is preserved under simple tilting. We first show that this duality implies the familiar relationship (e.g. in a module category) between irreducible maps of projectives and extensions of simples.

Proposition 5.6. *Let $\mathcal{H}$ be a finite heart in a triangulated category $\mathcal{D}$, with simples $\operatorname{Sim} \mathcal{H} = \{S_1, \ldots, S_n\}$ and a dual set of projectives $\{P_1, \ldots, P_n\}$, in the sense of (5.8). Then the P_j are non-isomorphic, indecomposable and span $\mathcal{H}$. Furthermore*

$$\operatorname{Irr}(P_i, P_j) \cong \operatorname{Ext}^1(S_j, S_i)^*, \tag{5.9}$$

where $\operatorname{Irr}(P_i, P_j)$ *are the irreducible maps in the additive subcategory* $\operatorname{Add}\{P_1, \ldots, P_n\}$.

Proof. Note first that if a projective P satisfies $\operatorname{Hom}(P, S) = 0$ for all $S \in \operatorname{Sim}\mathcal{H}$, then it satisfies $\operatorname{Hom}(P, M) = 0$ for all $M \in \mathcal{H}$, as $\operatorname{Sim}\mathcal{H}$ generates $\mathcal{H}$. But then $\operatorname{Hom}^k(P, M) = 0$ for all $M \in \mathcal{H}$ and all k, as P is projective and so, using the canonical filtration (2.3), $\operatorname{Hom}(P, X) = 0$ for all $X \in \mathcal{D}$, which means $P = 0$.

Thus, as the duality (5.8) means that at most one indecomposable summand of P_j can have a non-zero map to S_j, any other summand would be trivial., i.e. P_j is indecomposable. The duality immediately implies that the P_j are pairwise non-isomorphic and they span $\mathcal{H}$, because it is generated by $\operatorname{Sim}\mathcal{H}$.

To prove (5.9), we start by defining $\Omega_j = \operatorname{Cone}(P_j \to S_j)[-1]$, so that we have a triangle

$$S_j[-1] \xrightarrow{h} \Omega_j \to P_j \to S_j, \tag{5.10}$$

Applying $\operatorname{Hom}(-, S_i)$ to this triangle, for any S_i, yields an isomorphism

$$h^*\colon \operatorname{Hom}(\Omega_j, S_i) \xrightarrow{\cong} \operatorname{Ext}^1(S_j, S_i) \tag{5.11}$$

and tells us that $\operatorname{Hom}(\Omega_j, S_i[-1]) = 0$, and so $\operatorname{Hom}(\Omega_j, M[-1]) = 0$, for all $M \in \mathcal{H}$. Since (5.10) immediately gives $\operatorname{Hom}(\Omega_j, M[-k]) = 0$, for $k > 1$ and all $M \in \mathcal{H}$, we deduce that $\Omega_j \in \mathcal{P}$, the t-structure associated to $\mathcal{H}$. Hence, applying $\operatorname{Hom}(P_i, -)$ to (5.10) yields a short exact sequence

$$0 \to \operatorname{Hom}(P_i, \Omega_j) \to \operatorname{Hom}(P_i, P_j) \to \operatorname{Hom}(P_i, S_j) \to 0. \tag{5.12}$$

Next define $\Omega_j^i = \operatorname{Cone}(\Omega_j \to S_i \otimes \operatorname{Hom}(\Omega_j, S_i)^*)[-1]$, so that we have a triangle

$$S_i \otimes \operatorname{Hom}(\Omega_j, S_i)^*[-1] \to \Omega_j^i \to \Omega_j \to S_i \otimes \operatorname{Hom}(\Omega_j, S_i)^*. \tag{5.13}$$

Applying $\operatorname{Hom}(-, S_k)$ to this triangle, for any k, yields again $\operatorname{Hom}(\Omega_j^i, S_k[-1]) = 0$ and thus that $\Omega_j^i \in \mathcal{P}$ as before. Hence applying $\operatorname{Hom}(P_k, -)$ to (5.13) yields the exact sequence

$$0 \to \operatorname{Hom}(P_k, \Omega_j^i) \to \operatorname{Hom}(P_k, \Omega_j) \to \operatorname{Hom}(P_k, S_i) \otimes \operatorname{Hom}(\Omega_j, S_i)^* \to 0. \tag{5.14}$$

Combining this, when $k = i$, with (5.11) gives

$$\operatorname{Ext}^1(S_j, S_i)^* \cong \operatorname{Hom}(P_i, \Omega_j) / \operatorname{Hom}(P_i, \Omega_j^i) \tag{5.15}$$

To see that the RHS is $\operatorname{Irr}(P_i, P_j)$, note that it would be had Ω_j and Ω_j^i been constructed in the analogous way inside $\operatorname{mod}\operatorname{End}(\mathbf{P})$, where $\mathbf{P} = \bigoplus_{i=1}^n P_i$. However, the duality (5.8) means that $\operatorname{Hom}(\mathbf{P}, S_j)$ are the simple $\operatorname{End}(\mathbf{P})$-modules, while $\operatorname{Hom}(\mathbf{P}, P_j)$ are the corresponding projectives. Then the short exact sequences (5.12) and (5.14) mean that $\operatorname{Hom}(\mathbf{P}, \Omega_j)$ and $\operatorname{Hom}(\mathbf{P}, \Omega_j^i)$ are precisely the analogous $\operatorname{End}(\mathbf{P})$-modules and so the result follows by the Yoneda Lemma. □

We now prove the first important result about the hearts in the exchange graph $\operatorname{EG}^\circ(Q)$, observing in the process that, when a heart tilts, its projectives mutate in a way precisely analogous to cluster tilting sets, as in (4.2) and (4.3).

Theorem 5.7. *Let Q be an acyclic quiver with n vertices and $\mathcal{H}$ be any heart in* $\mathrm{EG}^\circ(Q)$*. Then $\mathcal{H}$ is finite and rigid, with exactly n simples* $\operatorname{Sim}\mathcal{H} = \{S_1, \ldots, S_n\}$ *and exactly n indecomposable projectives* $\operatorname{Proj}\mathcal{H} = \{P_1, \ldots, P_n\}$*, which are dual in the sense of* (5.8)*.*

If we tilt $\mathcal{H}$ with respect to any simple $S_i \in \operatorname{Sim}\mathcal{H}$, then the simples change according to the formulae (5.1) *and* (5.2) *and the projectives change according to the formulae*

$$\operatorname{Proj}\mathcal{H}^\sharp_{S_i} = \operatorname{Proj}\mathcal{H} - \{P_i\} \cup \{P_i^\sharp\}, \tag{5.16}$$

$$\operatorname{Proj}\mathcal{H}^\flat_{S_i} = \operatorname{Proj}\mathcal{H} - \{P_i\} \cup \{P_i^\flat\}, \tag{5.17}$$

where

$$P_i^\sharp = \operatorname{Cone}(P_i \to \bigoplus_{j\neq i} \operatorname{Irr}(P_i, P_j)^* \otimes P_j), \tag{5.18}$$

$$P_i^\flat = \operatorname{Cone}(\bigoplus_{j\neq i} \operatorname{Irr}(P_j, P_i) \otimes P_j \to P_i)[-1], \tag{5.19}$$

with $\operatorname{Irr}(P_i, P_j)$ *as in* (5.9)*.*

Proof. We use induction starting from the standard heart $\mathcal{H}_Q$, which we know is finite and rigid, with standard simples and projectives satisfying (5.8). We will only give the proof for forward tilting, but a very similar proof works for backward tilting and thus we can reach all hearts in $\mathrm{EG}^\circ(Q)$.

So, suppose $\mathcal{H}$ is a finite and rigid heart satisfying (5.8) and $S_i \in \operatorname{Sim}\mathcal{H}$. By Proposition 5.2, we know that $\mathcal{H}^\sharp_{S_i}$ is finite with new simples given by (5.1), that is, they are $S_i[1]$ together with $S_j^\sharp = \psi^\sharp_{S_i}(S_j)$, for $j \neq i$, occurring in the triangle

$$S_j[-1] \xrightarrow{u} E_j^* \otimes S_i \to S_j^\sharp \to S_j, \tag{5.20}$$

where $E_j = \operatorname{Ext}^1(S_j, S_i)$ and u is the universal map.

We claim that the new projectives are given by (5.16) and that they satisfy (5.8) with respect to the new simples. First, to see that P_k, for $k \neq i$, remains projective, note that $\operatorname{Ext}^1(P_k, M) = 0$ for any $M \in \mathcal{P} \supset \mathcal{P}^\sharp_{S_i}$, so we just need to show P_k is in $\mathcal{P}^\sharp_{S_i} = \mathcal{P} \cap {}^\perp S_i$, and this follows from (5.8). Next, applying $\operatorname{Hom}(P_k, -)$ to (5.20) gives $\operatorname{Hom}(P_k, S_j^\sharp) \cong \operatorname{Hom}(P_k, S_j)$ and, as also $\operatorname{Hom}(P_k, S_i[1]) = 0$, most of the new duality (5.8) holds and it remains to consider the case of $P_i^\sharp$.

Applying $\operatorname{Hom}(-, S_i)$ to (5.18) gives $\operatorname{Hom}^k(P_i^\sharp, S_i[1]) \cong \operatorname{Hom}^k(P_i, S_i)$, for all k, so it remains to show that $\operatorname{Hom}^k(P_i^\sharp, S_j^\sharp) = 0$, for all k and all $j \neq i$. As well as completing the duality, this will mean that $\operatorname{Hom}^k(P_i^\sharp, M) = 0$, for all $k \neq 0$ and all $M \in \mathcal{H}^\sharp_{S_i}$, so that $P_i^\sharp$ is a new projective.

Since $S_j^\sharp \in \mathcal{H}$, the required vanishing follows immediately from applying $\operatorname{Hom}(-, S_j^\sharp)$ to (5.18), except in the cases $k = 0, 1$. These two cases appear in the long exact sequence

$$0 \to \operatorname{Hom}(P_i^\sharp, S_j^\sharp) \to \operatorname{Irr}(P_i, P_j) \xrightarrow{\delta_*} \operatorname{Hom}(P_i, S_j^\sharp) \to \operatorname{Hom}(P_i^\sharp, S_j^\sharp[1]) \to 0, \tag{5.21}$$

where $\delta \in \operatorname{Hom}(P_j, S_j^\sharp)$ is any non-zero map in this 1-dimensional space. Thus what we must show is that δ_* is an isomorphism.

For this, we recall from the proof of Proposition 5.6 that we associated to S_j the syzygy Ω_j, defined by the triangle (5.10), with a map $h\colon S_j[-1] \to \Omega_j$ inducing an isomorphism $h^*\colon \operatorname{Hom}(\Omega_j, S_i) \cong E_j$. Hence we may factor the universal map u in (5.20) through h and the universal map

$$\alpha : \Omega_j \to \operatorname{Hom}(\Omega_j, S_i)^* \otimes S_i \cong E_j^* \otimes S_i.$$

Applying the Octahedral Axiom to this factorisation $u = \alpha \circ h$ gives the following commutative diagram of triangles

$$\begin{array}{ccccccc}
 & & \Omega_j^i & = & \Omega_j^i & & \\
 & & \downarrow & & \downarrow & & \\
S_j[-1] & \xrightarrow{h} & \Omega_j & \longrightarrow & P_j & \longrightarrow & S_j \\
\| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \| \\
S_j[-1] & \xrightarrow{u} & E_j^* \otimes S_i & \longrightarrow & S_j^\sharp & \longrightarrow & S_j \\
 & & \downarrow & & \downarrow & & \\
 & & \Omega_j^i[1] & = & \Omega_j^i[1] & &
\end{array} \tag{5.22}$$

where Ω_j^i is as defined in (5.13) and hence in particular $\Omega_j^i \in \mathcal{P}$. Notice that the right square ensures that δ is nonzero and so provides the map required in (5.21). In fact, we can also observe that Ω_j^i is the new syzygy $\Omega_j^\sharp$, and hence is actually in $\mathcal{P}^\sharp_{S_i} \subset \mathcal{P}$, although this is more than we need to know. But now we can apply $\operatorname{Hom}(P_i, -)$ to the right hand vertical triangle in (5.22) and deduce, as required, that δ_* is an isomorphism, because, as explained after (5.15), $\operatorname{Irr}(P_i, P_j)$ is a complement to $\operatorname{Hom}(P_i, \Omega_j^i)$ in $\operatorname{Hom}(P_i, \Omega_j) \cong \operatorname{Hom}(P_i, P_j)$ and also $\operatorname{Hom}(P_i, \Omega_j^i[1]) = 0$.

Thus we have found n new projectives of the new heart $\mathcal{H}^\sharp_{S_i}$, which are dual to the new simples. Hence these new projectives are non-isomorphic and indecomposable, by the first part of Proposition 5.6, and so they must form $\operatorname{Proj}\mathcal{H}^\sharp_{S_i}$, as Lemma 4.5 implies there can be no other projectives, because we can shift the finite heart $\mathcal{H}$ so that $\mathcal{H}[k] \in \operatorname{EG}_N(Q, \mathcal{H}_Q)$, for some sufficiently large N.

To complete the inductive step, we must show that the new heart $\mathcal{H}^\sharp_{S_i}$ is rigid. By Proposition 5.6, this amounts to showing that $\operatorname{Irr}(P_k, P_k) = 0$, for all k, i.e. that the Gabriel quiver of $\operatorname{End}_{\mathcal{D}(Q)}(\mathbf{P})$ has no loops, for $\mathbf{P} = \bigoplus_{k=1}^n P_k$. However, by shifting $\mathcal{H}^\sharp_{S_i}$ if necessary, we may assume that it is in $\operatorname{EG}_N(Q, \mathcal{H}_Q)$, for some sufficiently large N. Then, by Lemma 4.5, we can choose $m \geq N$ and get

$$\operatorname{End}_{\mathcal{D}(Q)}(\mathbf{P}) = \operatorname{End}_{\mathcal{C}_m(Q)}(\pi_m(\mathbf{P}))$$

and we know that there is no loop in the Gabriel quiver of $\operatorname{End}_{\mathcal{C}_m(Q)}(\pi_m(\mathbf{P}))$, by [6, Section 2]. □

This result enables us to begin to relate the exchange graphs for $\mathcal{D}(Q)$ and $\mathcal{C}_m(Q)$. In particular, Theorem 5.7 tells us that any heart in $\operatorname{EG}^\circ_N(Q, \mathcal{H}_Q)$ has $\#Q_0$ projectives and so, by Lemma 4.5, there is a map

$$\mathcal{J}_{N,m} : \mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q) \to \mathrm{CEG}_m(Q), \tag{5.23}$$

for $m \geq N-1$, sending a heart $\mathcal{H}$ to $\pi_m\big(\mathrm{Proj}\,\mathcal{H}\big)$, with π_m as in (4.4).

Corollary 5.8. *The map $\mathcal{J}_{N,m}$ is injective on vertices, for all $m \geq N-1$, and preserves edges, when $m \geq N$.*

Proof. The map is injective on vertices, because $\mathrm{Proj}\,\mathcal{H}$ spans $\mathcal{H}$, by Proposition 5.6 and hence determines $\mathcal{H}$, by Proposition 2.1.

For the map to preserve edges, we need to show that, within $\mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q)$, the cluster tilting set $\pi_m(\mathrm{Proj}\,\mathcal{H}^{\sharp}_{S_i})$ is the forward mutation of $\pi_m(\mathrm{Proj}\,\mathcal{H})$ at $\pi_m(P_i)$, for the corresponding projective P_i. When $m \geq N$, this follows from (4.5) in Lemma 4.5, as the mutation formula (4.2) for $\pi_m(\mathrm{Proj}\,\mathcal{H})$ then agrees with the mutation formula (5.18) for $\mathrm{Proj}\,\mathcal{H}$. □

In the next subsection, a more careful analysis (Corollary 5.12) will show that $\mathcal{J}_{N,N-1}$ also preserves edges.

5.3. **Convexity of exchange graphs with base.** For $S \in \mathrm{Sim}\,\mathcal{H}$, inductively define

$$\mathcal{H}^{m\sharp}_{S} = \left(\mathcal{H}^{(m-1)\sharp}_{S}\right)^{\sharp}_{S[m-1]}$$

for $m \geq 1$ and similarly for $\mathcal{H}^{m\flat}_{S}, m \geq 1$. We will write $\mathcal{H}^{m\sharp}_{S} = \mathcal{H}^{-m\flat}_{S}$ for $m < 0$.

Definition 5.9. A *line* $l = l(\mathcal{H}, S)$ in $\mathrm{EG}(\mathcal{D})$, for some triangulated category $\mathcal{D}$, is the full subgraph consisting of the vertices $\{\mathcal{H}^{m\sharp}_{S}\}_{m\in\mathbb{Z}}$, for some heart $\mathcal{H}$ and a simple $S \in \mathrm{Sim}\,\mathcal{H}$. We say an edge in $\mathrm{EG}(\mathcal{D})$ has *direction* T if its label is $T[m]$ for some integer m; we say a line l has *direction-T* if some (and hence every) edge in l has direction T.

A *line segment* of length m in $\mathrm{EG}(\mathcal{D})$ is the full subgraph consisting of vertices $\{\mathcal{H}^{i\flat}_{S}\}_{i=0}^{m-1}$ of some line $l(\mathcal{H}, S)$ in $\mathrm{EG}(\mathcal{D})$ and for some positive integer m. Notice that any line segment inherits a direction from the corresponding line and in particular line segments of length one consisting of the same vertex may differ by their directions.

Definition 5.10. A subgraph EG_0 of $\mathrm{EG}(\mathcal{D})$ is *convex* if any line in $\mathrm{EG}(\mathcal{D})$ that meets EG_0 meets it in a single line segment. Define the *cyclic completion* of a convex subgraph EG_0 to be the oriented graph $\overline{\mathrm{EG}_0}$ obtained from EG_0 by adding an edge $e_l = \left(\mathcal{H} \to \mathcal{H}^{(m-1)\flat}_{S}\right)$ with direction S for each line segment $l \cap \mathrm{EG}_0 = \{\mathcal{H}^{i\flat}_{S}\}_{i=0}^{m-1}$ of direction S, in EG_0. Call the line segment $l \cap \mathrm{EG}_0$ together with e_l a *basic cycle* (induced by l with direction S) in EG_0.

Proposition 5.11. $\mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q)$ *is a convex subgraph in* $\mathrm{EG}^{\circ}(Q)$*. Moreover, any basic cycle in* $\overline{\mathrm{EG}}_N(Q, \mathcal{H}_Q)$ *is an* $(N-1)$*-cycle. Further, there are a unique source* $\mathcal{H}_Q[1]$ *and a unique sink* $\mathcal{H}_Q[N-1]$ *in* $\mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q)$*.*

Proof. Let $\mathcal{H} \in \mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q)$ and $S \in \mathrm{Sim}\,\mathcal{H}$. Then S is indecomposable in $\mathcal{D}(Q)$ and hence in $\mathcal{H}_Q[m]$ for some integer $1 \leq m \leq N-1$, by the first part of Lemma 5.4 and (2.1). By the second part of Lemma 5.4, we have

$$l(\mathcal{H}, S) \cap \mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q) = \{\mathcal{H}^{i\sharp}_{S}\}_{i=1-m}^{N-1-m}$$

which implies the first two statements.

If $\mathcal{H} \in \mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$, with $S \in \mathrm{Sim}\,\mathcal{H}$, and $\mathcal{H}^{\flat}_S \notin \mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$, then by Lemma 5.4, $\mathbf{H}_1(S) \neq 0$ and so, by (2.1), $S \in \mathcal{H}_Q[1]$. Thus, if $\mathcal{H}$ is a source, then $\mathcal{H} \subset \mathcal{H}_Q[1]$ and so $\mathcal{H} = \mathcal{H}_Q[1]$, i.e. $\mathcal{H}_Q[1]$ is the unique source. Similarly for the uniqueness of the sink. □

Corollary 5.12. *We have a canonical isomorphism*

$$\overline{\mathcal{J}} : \overline{\mathrm{EG}^{\circ}_N}(Q, \mathcal{H}_Q) \cong \mathrm{CEG}_{N-1}(Q). \tag{5.24}$$

between oriented graphs, which is induced by $\mathcal{J} = \mathcal{J}_{N,N-1}$ in (5.23). Moreover, this induces a bijection between basic cycles in $\overline{\mathrm{EG}^{\circ}_N}(Q, \mathcal{H}_Q)$ and almost complete cluster tilting sets in $\mathcal{C}_{N-1}(Q)$.

Proof. We write $\pi = \pi_{N-1} \colon \mathcal{D}(Q) \to \mathrm{CEG}_{N-1}(Q)$. For any maximal line segment

$$l(\mathcal{H}, S_i) \cap \mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q) = \{\mathcal{H}^{j\flat}_{S_i}\}_{j=0}^{N-2}$$

in $\mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q)$, let $\mathcal{H}_j = \mathcal{H}^{j\flat}_{S_i}$, $A_l = \bigcap_{j=0}^{N-2} \mathrm{Proj}\,\mathcal{H}_j$ and $P^j_i = \mathrm{Proj}\,\mathcal{H}_j - A_l$. By formula (5.17), we have $\#A_l = n - 1$ which implies $\pi(A_l)$ is an almost complete cluster tilting set. By [24, Theorem 4.3], any almost complete cluster tilting set has precisely $N-1$ completions, and hence $\{\mathcal{J}(\mathcal{H}_k)\}_{k=0}^{N-2}$ are all the completions of $\pi(A_l)$.

We claim that

$$\mathcal{J}(\mathcal{H}_{j-1}) = \mu_i \mathcal{J}(\mathcal{H}_j), \tag{5.25}$$

for $j = 2, \ldots, N-2$, where μ_i is mutation at $\pi(P^j_i)$. Assuming this for the moment, we deduce that (5.25) also holds for $j = 1$ and

$$\mathcal{J}(\mathcal{H}_{N-2}) = \mu_i \mathcal{J}(\mathcal{H}_0),$$

since $\{\mathcal{J}(\mathcal{H}_k)\}_{k=0}^{N-2}$ forms a $(N-1)$-cycle in $\mathrm{CEG}_{N-1}(Q)$ (cf. [10]). Therefore $\mathcal{J}$ preserves edges and can be extended to the required map $\overline{\mathcal{J}}$ that sends each new edge $e_l = \left(\mathcal{H} \to \mathcal{H}^{(N-2)\flat}_{S_i}\right)$ in any basic cycle to the mutation μ_i on $\mathcal{J}(\mathcal{H})$ at $\pi(P^0_i)$. The surjectivity of $\overline{\mathcal{J}}$ follows by a direct induction, using the fact that both graphs are connected and the fact that each graph is $2n$-regular and $\overline{\mathcal{J}}$ is a local isomorphism. We already know, from Corollary 5.8, that $\overline{\mathcal{J}}$ is injective, so we have the required isomorphism and, moreover, $l \longmapsto A_l$ gives the canonical bijection between basic cycles and almost complete cluster tilting sets.

To see that (5.25) does hold, we first show that

$$\mathrm{Hom}_{D(Q)}(P^j_i, P) = \mathrm{Hom}_{\mathcal{C}_{N-1}(Q)}(\pi(P^j_i), \pi(P)) \tag{5.26}$$

for any $P \in A_l$. By the first part of Lemma 5.4 and (2.1), we know that the simple $S_i \in \mathcal{H}_Q[N-1]$. Further, by (5.8), we have

$$\mathrm{Hom}^{\bullet}(P^j_i, S_i[-j]) = \mathrm{Hom}(P^j_i, S_i[-j]) \neq 0. \tag{5.27}$$

But $\mathrm{Hom}(M, S_i[-j]) = 0$ for $M \in \mathcal{P}_Q[N-j]$ as $S_i[-j] \in \mathcal{H}_Q[N-1-j]$, so we have $P^j_i \in \mathcal{P}_Q[1] \cap \mathcal{P}_Q^{\perp}[N-j]$. Since $j \geq 2$, we have $\mathrm{Hom}(\Sigma^t_{N-1} P^j_i, P) = 0$, for any $t \neq 0$ and $P \in A_l$, by the same calculation as in the proof of Lemma 4.5, which implies (5.26).

Then we deduce that $V'_P = \mathrm{Irr}(\pi(P_i^j), \pi(P))$ is induced from $V_P \subset \mathrm{Irr}(P_i^j, P)$, for any $P \in A_l$. Hence the triangle

$$\pi(P_i^j) \to \bigoplus_{P\in A_l} (V'_P)^* \otimes \pi(P) \to \mu_i(\pi(P_i^j)) \to \pi(P_i^j)[1]$$

in $\mathcal{C}_{N-1}(Q)$ is induced from some triangle

$$P_i^j \to \bigoplus_{P\in A_l} V_P^* \otimes P \to X \to P_i^j[1]. \tag{5.28}$$

Note that $\mu_i(\pi(P_i^j))$ is one of the complements of A_l and thus $X = P_i^k$ for some $\leq k \leq N-2$. Applying $\mathrm{Hom}(-, S_i)$ to (5.28) gives

$$\mathrm{Hom}(X, S_i[1-j]) = \mathrm{Hom}(P_i^j, S_i[-j]) \neq 0,$$

which implies $X = P_i^{j-1}$ because (5.27), and hence (5.25). □

Remark 5.13. The isomorphism (5.24) is an interpretation of the result of Buan-Reiten-Thomas [5, Theorem 2.4]. Moreover, we can use this isomorphism to show that

$$\mathrm{EG}^\circ(Q, \mathcal{H}_Q) = \mathrm{EG}^\circ(Q) \cap \mathrm{EG}(Q, \mathcal{H}_Q). \tag{5.29}$$

To see this, note from (5.7) that the LHS of (5.29) is contained in the RHS. Conversely, any heart $\mathcal{H}$ in the RHS is finite with $\#Q_0$ projectives, by Theorem 5.7, and so $\pi_{N-1}(\mathrm{Proj}\,\mathcal{H})$ is in $\mathrm{CEG}_{N-1}(Q)$, by Lemma 4.5. But then Corollary 5.12 implies that $\mathcal{H}$ is in the LHS of (5.29) as required.

6. Coloured quivers and Ext-quivers

Recall from [6], that any cluster tilting set $\mathbf{T} = \{T_1, \dots, T_n\}$ in $\mathrm{CEG}_m(Q)$ determines a coloured quiver $\mathcal{Q}(\mathbf{T})$. The multiplicity of the colour-zero arrows in $\mathcal{Q}(\mathbf{T})$ is given by $\mathrm{Irr}(T_i, T_j)$. Furthermore, $\mathcal{Q}(\mathbf{T})$ is monochromatic and skew-symmetric, in the sense that the arrows from T_i to T_j all have one colour, c say, where $0 \leq c \leq m-1$, and then the arrows from T_j to T_i all have colour $m-1-c$, with the same multiplicity. We will denote this situation by

$$V \cdot (T_i \underset{m-1-c}{\overset{c}{\rightleftarrows}} T_j),$$

where V is the multiplicity.

Definition 6.1. Given such a coloured quiver $\mathcal{Q}(\mathbf{T})$, we will define the *augmented graded quiver*, denoted by $\mathcal{Q}^+(\mathbf{T})$, with the same vertex set, containing the arrows of $\mathcal{Q}(\mathbf{T})$, with degree equal to their colour plus one (so that the degrees of opposite arrows now sum to $m+1$), together with an additional loop of degree $m+1$ at each vertex.

Definition 6.2. Let $\mathcal{H}$ be a finite heart in a triangulated category $\mathcal{D}$ and $\mathbf{S} = \bigoplus_{S\in\mathrm{Sim}\,\mathcal{H}} S$. The Ext-quiver $\mathcal{Q}(\mathcal{H})$ is the (positively) graded quiver whose vertices are the simples of $\mathcal{H}$ and whose graded edges correspond to a basis of $\mathrm{End}^\bullet(\mathbf{S}, \mathbf{S})$. Further, define the *CY-N double* of a graded quiver $\mathcal{Q}$, denoted by ${}^N\overline{\mathcal{Q}}$, to be the quiver obtained from $\mathcal{Q}$ by adding an arrow $T \to S$ of degree $N-k$ for each arrow $S \to T$ of degree k and adding a loop of degree N at each vertex.

The objective of this section is to relate the coloured quiver of a cluster tilting set $\mathbf{T}$, or more precisely its augmented graded quiver, to the double of the Ext-quiver of a heart in $\mathcal{D}(Q)$ that corresponds to $\mathbf{T}$. In the process, we uncover another important property of such hearts.

Definition 6.3. We say that a heart $\mathcal{H}$ is

- *monochromatic* (cf. [6]) if, for any simples $S \neq T$ in $\operatorname{Sim}\mathcal{H}$, $\operatorname{Hom}^\bullet(S,T)$ is concentrated in a single (positive) degree;
- *strongly monochromatic* if it is monochromatic and in addition, for any simples $S \neq T$ in $\operatorname{Sim}\mathcal{H}$, $\operatorname{Hom}^\bullet(S,T) = 0$ or $\operatorname{Hom}^\bullet(T,S) = 0$;

Proposition 6.4. *Any heart in* $\operatorname{EG}^\circ(Q)$ *is strongly monochromatic. Moreover, for any heart* $\mathcal{H} \in \operatorname{EG}^\circ_N(Q,\mathcal{H}_Q)$ *and any* $m \geq N$,

$$\mathcal{Q}^+(\mathcal{J}_{N,m}(\mathcal{H})) = {}^{m+1}\overline{\mathcal{Q}(\mathcal{H})}. \tag{6.1}$$

where $\mathcal{J}_{N,m} = \pi_m \circ \operatorname{Proj}$ *is as in* (5.23).

Proof. Note first that the loops on both sides match by construction and so (6.1) just needs to be checked between two vertices.

Choose any simples $T_i, T_j \in \mathcal{H}$. By the first part of Lemma 5.4 and (2.1), we have $T_i, T_j \in \bigcup_{t=1}^{N-1} \mathcal{H}_Q[t]$ and hence $\operatorname{Hom}^{\geq N}(T_i,T_j) = 0$, since $\mathcal{H}_Q$ is hereditary. Thus the maximum degree of any arrow in $\mathcal{Q}(\mathcal{H})$ is $N-1$.

Consider the maximal line segment

$$l(\mathcal{H},T_i) \cap \operatorname{EG}^\circ_{m+1}(Q,\mathcal{H}_Q) = \{\mathcal{H}_k\}_{k=0}^{m-1}, \tag{6.2}$$

where $\mathcal{H}_k = (\mathcal{H}_0)^{k\sharp}_{S_i}$ and $\mathcal{H}_0$ has simples S_i, S_j, corresponding to T_i, T_j, with associated projectives P_i, P_j. Note that $\mathcal{H}_0 \in \operatorname{EG}^\circ_N(Q,\mathcal{H}_Q)$ and that $\mathcal{H} = \mathcal{H}_h$ for some $0 \leq h \leq N-2$, with $T_i = S_i[h]$ and $S_i \in \mathcal{H}_Q[1]$. Therefore $\operatorname{Hom}^{\geq 2}(S_i,S_j) = 0$, because $S_j \in \bigcup_{t=1}^{N-1} \mathcal{H}_Q[t]$ and $\mathcal{H}_Q$ is hereditary. Thus

$$\operatorname{Hom}^\bullet(S_i,S_j) = \operatorname{Hom}^1(S_i,S_j). \tag{6.3}$$

Suppose that, in the coloured quiver $\mathcal{Q}(\pi_m(\operatorname{Proj}\mathcal{H}_0))$, the full sub-quiver between $\pi_m(P_i)$ and $\pi_m(P_j)$ is

$$V \cdot \big(\pi_m(P_j) \underset{m-1-c}{\overset{c}{\rightleftarrows}} \pi_m(P_i)\big) \tag{6.4}$$

for some $0 \leq c \leq m-1$ and multiplicity V. Then, by the mutation rule in [6], in the coloured quivers $\mathcal{Q}(\pi_m(\operatorname{Proj}\mathcal{H}_k))$, we have the following full sub-quivers:

$$V \cdot \big(\pi_m(P_j) \underset{m-1-c+k}{\overset{c-k}{\rightleftarrows}} \pi_m(P_i^k)\big) \qquad k = 0,\ldots,c \tag{6.5}$$

$$V \cdot \big(\pi_m(P_j) \underset{k-c-1}{\overset{m-k+c}{\rightleftarrows}} \pi_m(P_i^k)\big) \qquad k = c+1,\ldots,m-1 \tag{6.6}$$

where P_i^k is the projective in $\mathcal{H}_k$ corresponding to $S_i[k]$ and hence $\pi_m(P_i^k)$ is the replacement of P_i in $\pi_m(\operatorname{Proj}\mathcal{H}_k)$. For $0 \leq k \leq m-2$, the hearts $\mathcal{H}_k$ satisfy (4.5) and so, by (5.9), the number of colour-zero arrows in (6.5) or (6.6) equals $\dim \mathrm{Ext}^1$ between the corresponding simples. Arguing inductively, using Proposition 5.2, these are S_j and $S_i[k]$ for (6.5) and $S_j^\sharp$ and $S_i[k]$ for (6.6), where $S_j^\sharp = \psi^\sharp_{S_i[c]}(S_j)$.

Consider the case when $c \neq m-1$. We have

$$\begin{gathered}\mathrm{Ext}^1(S_j, S_i[k]) = 0 = \mathrm{Ext}^1(S_i[k], S_j), \quad k = 0, \dots, c-1,\\ \dim \mathrm{Ext}^1(S_j, S_i[c]) = V, \quad \mathrm{Ext}^1(S_i[c], S_j) = 0,\\ \dim \mathrm{Ext}^1(S_i[c+1], S_j^\sharp) = V, \quad \mathrm{Ext}^1(S_j^\sharp, S_i[c+1]) = 0, \quad \text{if } c < m-2\\ \mathrm{Ext}^1(S_j^\sharp, S_i[k]) = 0 = \mathrm{Ext}^1(S_i[k], S_j^\sharp), \quad k = c+2, \dots, m-2.\end{gathered}$$

In particular $\mathrm{Ext}^1(S_i, S_j) = 0$ and so, by (6.3), $\mathrm{Hom}^\bullet(S_i, S_j) = 0$. Also S_i is exceptional, because it is rigid and $\mathcal{D}(Q)$ is hereditary. Then, applying $\mathrm{Hom}(S_i, -)$ and $\mathrm{Hom}(-, S_i)$ to (5.3), a direct calculation shows that

$$\begin{gathered}\mathrm{Hom}^\bullet(S_i, S_j^\sharp) = \mathrm{Hom}^{-c}(S_i, S_j^\sharp) \cong \mathrm{Ext}^1(S_j, S_i[c])^*,\\ \mathrm{Hom}^k(S_j, S_i) \cong \mathrm{Hom}^k(S_j^\sharp, S_i), \quad \forall k \neq c, c+1,\\ \mathrm{Hom}^c(S_j^\sharp, S_i) = \mathrm{Hom}^{c+1}(S_j^\sharp, S_i) = 0.\end{gathered}$$

Since the degree of any arrow in the Ext-quiver $\mathcal{Q}(\mathcal{H}_0)$ is between 1 and $N-1$ and $m \geq N$, we have

$$\mathrm{Hom}^\bullet(S_j, S_i) = \mathrm{Hom}^{c+1}(S_j, S_i), \quad \mathrm{Hom}^\bullet(S_j^\sharp, S_i) = 0.$$

Therefore, the full sub-quiver between T_j, T_i in $\mathcal{Q}(\mathcal{H})$ is

$$\begin{aligned} &V \cdot \big(S_j \xrightarrow{c-h+1} S_i[h]\big) && \text{if } 0 \leq h \leq c,\\ &V \cdot \big(S_j^\sharp \xleftarrow{h-c} S_i[h]\big) && \text{if } c+1 \leq h \leq N-2,\end{aligned}$$

as required.

On the other hand, in the case $c = m-1$, we have

$$\begin{gathered}\dim \mathrm{Ext}^1(S_i, S_j) = V, \quad \mathrm{Ext}^1(S_j, S_i) = 0,\\ \mathrm{Ext}^1(S_i[k], S_j) = 0 = \mathrm{Ext}^1(S_j, S_i[k]), \quad k = 2, \dots, m-2.\end{gathered}$$

As before we deduce that the full sub-quiver between T_j, T_i in $\mathcal{Q}(\mathcal{H})$ is

$$V \cdot \big(S_j \xleftarrow{h+1} S_i[h]\big),$$

as required.

Thus we have proved that (6.1) holds and, in the process, seen that $\mathcal{H}$ is strongly monochromatic. By Lemma 5.4, we can shift any heart in $\mathrm{EG}^\circ(Q)$ into $\mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ for some $N \gg 1$, which implies that any heart in $\mathrm{EG}^\circ(Q)$ is strongly monochromatic. □

Now that we know that every heart in $\mathrm{EG}^\circ(Q)$ is strongly monochromatic, a more careful analysis shows that (6.1) also holds for $m = N-1$. We write $\pi = \pi_{N-1} \colon \mathcal{D}(Q) \to \mathrm{CEG}_{N-1}(Q)$ and recall that $\mathcal{J} = \mathcal{J}_{N,N-1}$ in (5.23). The key observation is the following.

Lemma 6.5. *Let $\mathcal{H} \in \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ with simples S, S' and corresponding projectives P, P'. If $\mathcal{H}^\sharp_S \in \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$, then $\dim \mathrm{Ext}^1(S', S) = \dim \mathrm{Irr}(\pi(P), \pi(P'))$.*

Proof. By (5.9), we only need to show that

$$\dim \mathrm{Irr}(P, P') = \dim \mathrm{Irr}(\pi(P), \pi(P')). \tag{6.7}$$

Denote by $P^\sharp$ the new projective of $\mathcal{H}^\sharp_S$ that replaces P. Since $\mathcal{H}^\sharp_S \in \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$, Corollary 5.12 implies that $\mathcal{J}(\mathcal{H}^\sharp_S)$ is the forward mutation at $\pi(P)$ of $\mathcal{J}(\mathcal{H})$, so that $\pi(P^\sharp)$ replaces $\pi(P)$. The proof rests on showing that, for any $R \in \mathrm{Proj}\,\mathcal{H}$, we have $\mathrm{Irr}(\pi(P), \pi(R)) = \pi(V(P, R))$ for some subspace $V(P, R)$ of $\mathrm{Irr}(P, R)$. Indeed, if this is so, then the triangle defining $\pi(P^\sharp)$ in $\mathcal{C}_{N-1}(Q)$ is induced, via π, from the triangle

$$P \to \bigoplus_{X \in \mathrm{Proj}\,\mathcal{H}} V(P, X)^* \otimes X \to P^\sharp \to P[1]$$

in $\mathcal{D}(Q)$. Then by comparing this to the triangle

$$P \xrightarrow{\alpha} \bigoplus_{X \in \mathrm{Proj}\,\mathcal{H}} \mathrm{Irr}(P, X)^* \otimes X \to P^\sharp \to P[1] \tag{6.8}$$

defining $P^\sharp$ in $\mathcal{D}(Q)$, we deduce that $V(P, X) = \mathrm{Irr}(P, X)$, because $\mathrm{Proj}\,\mathcal{H}$ is a basis for the Grothendieck group of $\mathcal{D}(Q)$ (by Theorem 5.7). Hence, in particular, (6.7) holds.

To see that $\mathrm{Irr}(\pi(P), \pi(R)) = \pi(V(P, R))$, we may suppose that $\mathrm{Irr}(\pi(P), \pi(R)) \neq 0$, otherwise it is trivially true. Since $\mathcal{H}_Q$ is hereditary, $\mathrm{Hom}(P, \Sigma^t_{N-1} R)$ vanishes for all but one $t \in \mathbb{Z}$, where Σ_{N-1} is the cluster shift, and thus

$$\mathrm{Hom}(\pi(P), \pi(R)) = \pi(\mathrm{Hom}(P, \Sigma^t_{N-1} R)). \tag{6.9}$$

By the calculation in Lemma 4.5, we know that $t \geq 0$. Applying $\mathrm{Hom}(-, \Sigma^t_{N-1} R)$ to the triangle (6.8) gives the exact sequence

$$\bigoplus_{X \in \mathrm{Proj}\,\mathcal{H}} \mathrm{Irr}(P, X) \otimes \mathrm{Hom}(X, \Sigma^t_{N-1} R) \xrightarrow{\alpha^*} \mathrm{Hom}(P, \Sigma^t_{N-1} R) \to \mathrm{Hom}(P^\sharp, \Sigma^t_{N-1} R[1]).$$

If $t > 0$, then the last term is zero by another calculation as in Lemma 4.5, so no map in $\mathrm{Hom}(\pi(P), \pi(R))$ can be irreducible. So $t = 0$ and then (6.9) implies that $\mathrm{Irr}(\pi(P), \pi(R))$ is induced via π from some subspace $V(P, R)$ of $\mathrm{Irr}(P, R)$. □

Remark 6.6. In the setting of Lemma 6.5, if further $\mathrm{Ext}^1(S', S)$ and $\mathrm{Irr}(\pi(P), \pi(P'))$ are non-zero, then, because the Ext-quiver $\mathcal{Q}(\mathcal{H})$ is strongly monochromatic and the augmented graded quiver $\mathcal{Q}^+(\mathcal{J}(\mathcal{H}))$ is monochromatic and skew-symmetric, the full sub-quiver between S and S' in the CY-N double ${}^N\overline{\mathcal{Q}(\mathcal{H})}$ is equal to the sub-quiver between $\pi(P)$ and $\pi(P')$ in $\mathcal{Q}^+(\mathcal{J}(\mathcal{H}))$.

Proposition 6.7. *For any heart $\mathcal{H} \in \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$, we have*

$$\mathcal{Q}^+(\mathcal{J}(\mathcal{H})) = {}^N\overline{\mathcal{Q}(\mathcal{H})}. \tag{6.10}$$

Proof. As before, i.e. in the proof of Proposition 6.4, the loops match and we only need to check that (6.10) holds between two simples $T_i, T_j \in \mathcal{H}$. Also as before, we choose a maximal line segment (6.2), with $m = N - 1$, so that $\mathcal{H} = \mathcal{H}_h$, for some $0 \leq h \leq N - 2$, and $\mathcal{H}_0$ has simples S_i, S_j and projectives P_i, P_j.

Thus it is sufficient to show that the full sub-quiver between $\pi(P_j)$ and $\pi(P_i^k)$ in $\mathcal{Q}^+(\mathcal{J}(\mathcal{H}_k))$ is equal to the full sub-quiver between the corresponding simples in ${}^N\overline{\mathcal{Q}(\mathcal{H}_k)}$, for $0 \le k \le N-2$. Using the mutation rule for coloured quivers (cf. (6.5) and (6.6)) and the change of simples formulae in Proposition 5.2, once we know that the equality holds for one $\mathcal{H}_k$, a direct calculation will show that it holds for all $\mathcal{H}_k$.

If there are no arrows in both the sub-quivers of $\mathcal{Q}^+(\mathcal{J}(\mathcal{H}_0))$ and ${}^N\overline{\mathcal{Q}(\mathcal{H}_0)}$, the equality holds for $\mathcal{H}_0$. Suppose that there are arrows between S_i and S_j in ${}^N\overline{\mathcal{Q}(\mathcal{H}_0)}$. Since $\mathcal{H}_0$ is strongly monochromatic, there are three cases.

1°. $\operatorname{Hom}^\bullet(S_i, S_j) = \operatorname{Hom}^c(S_i, S_j) \neq 0$ and $c = 1$, by (6.3). Then applying Remark 6.6 to $\mathcal{H}_0$ with simples $S = S_j$ and $S' = S_i$ gives the equality for $\mathcal{H}_0$.

2°. $\operatorname{Hom}^\bullet(S_j, S_i) = \operatorname{Hom}^{c+1}(S_j, S_i) \neq 0$ for some $0 \le c \le N-3$. Then applying Remark 6.6 to $\mathcal{H}_c$ with simples $S = S_i[c]$ and $S' = S_j$ gives the equality for $\mathcal{H}_c$.

3°. $\operatorname{Hom}^\bullet(S_j, S_i) = \operatorname{Hom}^{N-1}(S_j, S_i) \neq 0$ and $S_j \in \mathcal{H}_Q[N-1]$, because $S_i \in \mathcal{H}_Q[1]$. By Proposition 5.2, we deduce that $S_j, S_i[N-2] \in \operatorname{Sim} \mathcal{H}_{N-2}$. Since $S_j \notin \mathcal{H}_Q[1]$, we have $(\mathcal{H}_{N-2})^\flat_{S_j} \in \operatorname{EG}^\circ_N(Q, \mathcal{H}_Q)$ by Proposition 5.11. Then applying Remark 6.6 to $(\mathcal{H}_{N-2})^\flat_{S_j}$ with simples $S = S_j[-1]$ and $S' = \psi^\flat_{S_j}(S_i[N-2])$ gives the equality for $(\mathcal{H}_{N-2})^\flat_{S_j}$ and hence for $\mathcal{H}_{N-2}$.

On the other hand, suppose that there are arrows between $\pi(P_j)$ and $\pi(P_i)$ in $\mathcal{Q}^+(\mathcal{J}(\mathcal{H}_0))$. Similar to above, there are three cases and we obtain the required equality for an appropriate $\mathcal{H}_k$, by applying Remark 6.6. □

7. Calabi-Yau categories

We now bring in the Calabi-Yau categories $\mathcal{D}(\Gamma_N Q)$, whose relationship with the derived category $\mathcal{D}(Q)$ and the cluster category $\mathcal{C}_{N-1}(Q)$ is the main focus of the paper.

7.1. **Ginzburg algebras for quivers.** Let $N > 1$ be an integer and Q any quiver. The *Calabi-Yau-N Ginzburg algebra* $\Gamma_N Q$, is the differential graded (dg) algebra

$$\mathbf{k}Q_0\big\langle x, x^*, e^* \mid x \in Q_1, e \in Q_0 \big\rangle$$

with the degrees of the generators being

$$\deg e = \deg x = 0, \quad \deg x^* = N-2, \quad \deg e^* = N-1$$

and the only nontrivial differentials determined by

$$\mathrm{d} \sum_{e \in Q_0} e^* = \sum_{x \in Q_1} [x, x^*].$$

Write $\mathcal{D}(\Gamma_N Q)$ for $\mathcal{D}_{fd}(\operatorname{mod} \Gamma_N Q)$, the derived category of finite dimensional dg modules for $\Gamma_N Q$.

Recall that a triangulated category $\mathcal{D}$ is called *Calabi-Yau-N* if, for any objects L, M in $\mathcal{D}$ we have a natural isomorphism

$$\mathfrak{S} : \operatorname{Hom}^\bullet_{\mathcal{D}}(L, M) \xrightarrow{\sim} \operatorname{Hom}^\bullet_{\mathcal{D}}(M, L)^\vee[N]. \tag{7.1}$$

Further, an object S is *N-spherical* when $\operatorname{Hom}^\bullet(S, S) = \mathbf{k} \oplus \mathbf{k}[-N]$.

By [11] (see also [16],[21]), we know that $\mathcal{D}(\Gamma_N Q)$ is a Calabi-Yau-N category, which admits a standard heart $\mathcal{H}_\Gamma$ generated by simple $\Gamma_N Q$-modules S_e, for $e \in Q_0$, each of which is N-spherical. We denote by $\mathrm{EG}^\circ(\Gamma_N Q)$ the principal component of the exchange graph $\mathrm{EG}(\Gamma_N Q) = \mathrm{EG}(\mathcal{D}(\Gamma_N Q))$, that is, the component containing $\mathcal{H}_\Gamma$.

7.2. **Twist functors and braid groups.** We recall (cf. [16],[21]) a distinguished family of auto-equivalences of $\mathcal{D}(\Gamma_N Q)$.

Definition 7.1. The *twist functor ϕ of a spherical object S* is defined by

$$\phi_S(X) = \mathrm{Cone}\left(X \to S \otimes \mathrm{Hom}^\bullet(X,S)^\vee\right)[-1] \tag{7.2}$$

with inverse

$$\phi_S^{-1}(X) = \mathrm{Cone}\left(S \otimes \mathrm{Hom}^\bullet(S,X) \to X\right) \tag{7.3}$$

Note that the graded dual of a graded $\mathbf{k}$-vector space $V = \oplus_{i\in\mathbb{Z}} V_i[i]$ is

$$V^\vee = \bigoplus_{i\in\mathbb{Z}} V_i^*[-i].$$

where V_i is an ungraded $\mathbf{k}$-vector space and V_i^* is its usual dual. The *Seidel-Thomas braid group*, denoted by $\mathrm{Br}(\Gamma_N Q)$, is the subgroup of $\mathrm{Aut}\,\mathcal{D}(\Gamma_N Q)$ generating by the twist functors of the simples in $\mathrm{Sim}\,\mathcal{H}_\Gamma$.

By [21, Lemma 2.11], if S_1, S_2 are spherical, then so is $S = \phi_{S_2}(S_1)$. Moreover, we have

$$\phi_S = \phi_{S_2} \circ \phi_{S_1} \circ \phi_{S_2}^{-1}. \tag{7.4}$$

Furthermore, $S_2 = \phi_{S_1}(S)$ and the corresponding conjugation formula holds. Hence the generators $\{\phi_S \mid S \in \mathrm{Sim}\,\mathcal{H}_\Gamma\}$ satisfy the braid group relations ([16],[21]) and so $\mathrm{Br}(\Gamma_N Q)$ is a quotient group of the braid group Br_Q associated to the quiver Q.

Remark 7.2. We will call a heart *spherical*, if all its simples are spherical. Observe that, if a finite heart $\mathcal{H}$, as in Proposition 5.2, is also monochromatic (Definition 6.3) and spherical, then the change of simples formulae (5.3) and (5.4) can be expressed in terms of the spherical twists, as follows.

$$\psi_S^\sharp(X) = \begin{cases} \phi_S^{-1}(X) & \text{if } \mathrm{Ext}^1(X,S) \neq 0 \\ X & \text{if } \mathrm{Ext}^1(X,S) = 0 \end{cases}$$

$$\psi_S^\flat(X) = \begin{cases} \phi_S(X) & \text{if } \mathrm{Ext}^1(S,X) \neq 0 \\ X & \text{if } \mathrm{Ext}^1(S,X) = 0 \end{cases}$$

Hence, formulae (5.1) and (5.2) are equivalent to

$$\mathrm{Sim}\,\mathcal{H}_{S_i}^\sharp = \{S_i[1]\} \cup \{S_k\}_{k\in K_i^\sharp} \cup \{\phi_S^{-1}(S_j)\}_{j\in J_i^\sharp} \tag{7.5}$$

$$\mathrm{Sim}\,\mathcal{H}_{S_i}^\flat = \{S_i[-1]\} \cup \{S_k\}_{k\in K_i^\flat} \cup \{\phi_S(S_j)\}_{j\in J_i^\flat} \tag{7.6}$$

where

$$J_i^\sharp = \{j \mid \mathrm{Ext}^1(S_j,S_i) \neq 0 \mid S_j \in \mathrm{Sim}\,\mathcal{H}\}, \qquad K_i^\sharp = \{1,\dots,n\} - \{i\} - J_i^\sharp,$$
$$J_i^\flat = \{j \mid \mathrm{Ext}^1(S_i,S_j) \neq 0 \mid S_j \in \mathrm{Sim}\,\mathcal{H}\}, \qquad K_i^\flat = \{1,\dots,n\} - \{i\} - J_i^\flat.$$

7.3. **Lagrangian immersions.** We now introduce the main tool for relating hearts in $\mathcal{D}(Q)$ and $\mathcal{D}(\Gamma_N Q)$.

Definition 7.3. An exact functor $\mathcal{L} : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q)$ is called a *Lagrangian immersion* (L-immersion) if for any pair of objects $(\widehat{S}, \widehat{X})$ in $\mathcal{D}(Q)$ there is a short exact sequence

$$0 \to \operatorname{Hom}^\bullet(\widehat{S}, \widehat{X}) \xrightarrow{\mathcal{L}} \operatorname{Hom}^\bullet(\mathcal{L}(\widehat{S}), \mathcal{L}(\widehat{X})) \xrightarrow{\mathcal{L}^\dagger} \operatorname{Hom}^\bullet(\widehat{X}, \widehat{S})^\vee[N] \to 0, \tag{7.7}$$

where $\mathcal{L}^\dagger = \mathcal{L}^\vee[N] \circ \mathfrak{S}$ is the following composition

$$\operatorname{Hom}^\bullet(\mathcal{L}(\widehat{S}), \mathcal{L}(\widehat{X})) \xrightarrow{\mathfrak{S}} \operatorname{Hom}^\bullet(\mathcal{L}(\widehat{X}), \mathcal{L}(\widehat{S}))^\vee[N] \xrightarrow{\mathcal{L}^\vee[N]} \operatorname{Hom}^\bullet(\widehat{X}, \widehat{S})^\vee[N]$$

Further, we say a L-immersion is *strong* if (7.7) has a natural splitting.

Definition 7.4. Let $\mathcal{H}$ be a finite heart in $\mathcal{D}(\Gamma_N Q)$ with $\operatorname{Sim}\mathcal{H} = \{S_1, \ldots, S_n\}$. If there is a L-immersion $\mathcal{L} : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q)$ and a finite heart $\widehat{\mathcal{H}} \in \operatorname{EG}^\circ(Q)$ with $\operatorname{Sim}\widehat{\mathcal{H}} = \{\widehat{S}_1, \ldots, \widehat{S}_n\}$, such that $\mathcal{L}(\widehat{S}_i) = S_i$, then we say that $\mathcal{H}$ is *induced* via $\mathcal{L}$ from $\widehat{\mathcal{H}}$ and write $\mathcal{L}_*(\widehat{\mathcal{H}}) = \mathcal{H}$.

In particular, an L-immersion ensures that the Ext-quivers of the two hearts are related precisely by the doubling of Definition 6.2.

Proposition 7.5. *Let $\mathcal{H} = \mathcal{L}_*(\widehat{\mathcal{H}})$ be a heart in $\operatorname{EG}(\Gamma_N Q)$ induced by a heart $\widehat{\mathcal{H}}$ in $\operatorname{EG}^\circ(Q)$. Then $\mathcal{H}$ is monochromatic and*

$$\mathcal{Q}(\mathcal{H}) = {}^N\overline{\mathcal{Q}(\widehat{\mathcal{H}})}. \tag{7.8}$$

Proof. The fact that $\mathcal{H}$ is monochromatic follows from (7.7) and the fact that $\widehat{\mathcal{H}}$ is strongly monochromatic (Proposition 6.4). Then (7.8) also follows directly from (7.7). □

Under appropriate conditions, tilting preserves the property of being induced.

Proposition 7.6. *Let $\mathcal{H} = \mathcal{L}_*(\widehat{\mathcal{H}})$ be a heart in $\operatorname{EG}(\Gamma_N Q)$ induced by a heart $\widehat{\mathcal{H}}$ in $\operatorname{EG}(Q)$. If $\widehat{\mathcal{H}}$ is rigid, then $\mathcal{H}$ is spherical. Moreover, suppose $S = \mathcal{L}(\widehat{S})$ is a simple in $\mathcal{H}$, induced by $\widehat{S} \in \operatorname{Sim}\widehat{\mathcal{H}}$. Then*

1°. if $\operatorname{Hom}^{N-1}(\widehat{S}, \widehat{X}) = 0$ for any $\widehat{X} \in \operatorname{Sim}\widehat{\mathcal{H}}$, then $\mathcal{H}^\sharp_S = \mathcal{L}_*(\widehat{\mathcal{H}}^\sharp_{\widehat{S}})$.

2°. if $\operatorname{Hom}^{N-1}(\widehat{X}, \widehat{S}) = 0$ for any $\widehat{X} \in \operatorname{Sim}\widehat{\mathcal{H}}$, then $\mathcal{H}^\flat_S = \mathcal{L}_*(\widehat{\mathcal{H}}^\flat_{\widehat{S}})$.

Proof. Since $\mathcal{D}(Q)$ is hereditary, any rigid object $\widehat{M} \in \mathcal{D}(Q)$ is exceptional and hence, by (7.7), $\mathcal{L}(\widehat{M})$ is spherical.

For any $\widehat{X}(\ncong \widehat{S})$ in $\operatorname{Sim}\widehat{\mathcal{H}}$, let $X = \mathcal{L}(\widehat{X})$. Since $\operatorname{Hom}^{N-1}(\widehat{S}, \widehat{X}) = 0$, the short exact sequence (7.7) gives an isomorphism $\mathcal{L} : \operatorname{Hom}^1(\widehat{S}, \widehat{X}) \xrightarrow{\sim} \operatorname{Hom}^1(S, X)$. Since $\mathcal{L}$ is exact, we have

$$\mathcal{L}(\psi^\sharp_{\widehat{S}}(\widehat{X})) = \psi^\sharp_S(X),$$

where $\psi^\sharp$ is defined as in (5.3). Then, by Proposition 5.2, we have $\mathcal{L}_*(\widehat{\mathcal{H}}^\sharp_{\widehat{S}}) = \mathcal{H}^\sharp_S$. Similarly for 2°. □

8. Main results

8.1. **Inducing hearts.** The natural quotient morphism $\Gamma_N Q \to Q$ induces a functor

$$\mathcal{I} : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q). \tag{8.1}$$

For more general dg algebras, this functor was considered by Keller, who showed ([14, Lemma 4.4 (b)]) that $\mathcal{I}$ is a strong L-immersion (see Definition 7.3).

Consider the subgraph $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$ in $\mathrm{EG}^\circ(\Gamma_N Q)$ with standard heart $\mathcal{H}_\Gamma$ as base. Observe that $\mathcal{I}$ sends the simples in $\mathcal{H}_Q$ to the corresponding simples in $\mathcal{H}_\Gamma$ and hence we have $\mathcal{I}_*(\mathcal{H}_Q) = \mathcal{H}_\Gamma$.

Theorem 8.1. *Any heart in $\mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ induces a heart in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$ via the natural L-immersion $\mathcal{I}$ in (8.1), i.e. we have a well-defined map*

$$\mathcal{I}_* : \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q) \to \mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma). \tag{8.2}$$

Moreover, it is an isomorphism between oriented graphs and can be extended to an isomorphism $\overline{\mathcal{I}_*} : \overline{\mathrm{EG}^\circ_N}(Q, \mathcal{H}_Q) \to \overline{\mathrm{EG}^\circ_N}(\Gamma_N Q, \mathcal{H}_\Gamma)$.

Proof. We prove that $\mathcal{I}_*$ is well-defined by induction starting from $\mathcal{I}_*(\mathcal{H}_Q[1]) = \mathcal{H}_\Gamma[1]$. Thus, if $\mathcal{I}_*(\widehat{\mathcal{H}}) = \mathcal{H}$, for some $\widehat{\mathcal{H}}, \widehat{\mathcal{H}}^\sharp_{\widehat{S}} \in \mathrm{EG}(Q, \mathcal{H}_Q)$, $\widehat{S} \in \mathrm{Sim}\,\widehat{\mathcal{H}}$ and $\mathcal{H} \in \mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$, then we need to show that $\widehat{\mathcal{H}}^\sharp_{\widehat{S}}$ induces a heart in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$.

For any $\widehat{X} \in \mathrm{Sim}\,\widehat{\mathcal{H}}$, by the first part of Lemma 5.4 and (2.1), we know that $\widehat{X} \in \mathrm{Ind}\,\mathcal{H}_Q[m]$ for some $1 \le m(\widehat{X}) \le N-1$. By the second part of Lemma 5.4, we have $\mathbf{H}_{N-1}(\widehat{S}) = 0$ which implies $m(\widehat{S}) \le N-2$, where the homology $\mathbf{H}_\bullet$ is with respect to $\mathcal{H}_Q$. Then $\mathrm{Hom}^{N-1}(\widehat{S}, \widehat{X}) = 0$, since $\mathbf{k}Q$ is hereditary. By Proposition 7.6 we have $\mathcal{L}_*(\widehat{\mathcal{H}}^\sharp_{\widehat{S}}) = \mathcal{H}^\sharp_S$. Since $S = \mathcal{L}(\widehat{S}) \in \mathcal{H}_\Gamma[m(\widehat{S})]$ for some $m(\widehat{S}) \le N-2$, by the second part of Lemma 5.4, we know that $\mathcal{H}^\sharp_S$ is in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$.

The injectivity of $\mathcal{I}_*$ follows from the facts that a heart is determined by its simples and $\mathcal{I}$ is injective.

For surjectivity of $\mathcal{I}_*$, we consider the line segments. By the first part of Lemma 5.4, any line segment in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$ has length less or equal than $N-1$. Notice that, by Proposition 5.11, any maximal line segment in $\mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ has length $N-1$, and hence its image under $\mathcal{I}_*$ is a maximal line segment in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$. This implies that, if a heart $\mathcal{H}$ in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$ is induced from some heart $\widehat{\mathcal{H}} \in \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ via $\mathcal{I}$, then the maximal line segment $l(\mathcal{H}, S) \cap \mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$ is induced from the line segment $l(\widehat{\mathcal{H}}, \widehat{S}) \cap \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ via $\mathcal{I}$, where $S \in \mathrm{Sim}\,\mathcal{H}$, and $\widehat{S} \in \mathrm{Sim}\,\widehat{\mathcal{H}}$ such that $\mathcal{I}(\widehat{S}) = S$. Hence any simple tilt of an induced heart via $\mathcal{I}$ is also induced via $\mathcal{I}$, provided this tilt is still in $\mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$. Thus, inductively, we deduce that $\mathcal{I}_*$ is surjective.

The last assertion follows from the facts that we can cyclic complete $\mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ (Proposition 5.11) and $\mathcal{I}_*$ preserves the structure of line segments. $\square$

Proposition 8.2. $\mathrm{Br}(\Gamma_N Q) \cdot \mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma) = \mathrm{EG}^\circ(\Gamma_N Q)$.

Proof. We use induction starting from Theorem 8.1. Suppose $\mathcal{H}' \in \mathrm{EG}^\circ(\Gamma_N Q)$ such that $\mathcal{H}' = \phi(\mathcal{H})$ for $\phi \in \mathrm{Br}(\Gamma_N Q)$ and $\mathcal{H} \in \mathrm{EG}^\circ_N(\Gamma_N Q, \mathcal{H}_\Gamma)$. Choose any simple $S' \in \mathrm{Sim}\,\mathcal{H}'$ and let $S = \phi^{-1}(S')$.

If $\mathcal{H}_S^\sharp$ is still in $\mathrm{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$, then by Theorem 8.1 we have $(\mathcal{H}')_{S'}^\sharp = \phi(\mathcal{H}_S^\sharp)$.

Now suppose that $\mathcal{H}_S^\sharp \notin \mathrm{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$. By Theorem 8.1, the maximal line segment $l(\mathcal{H}, S) \cap \mathrm{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ is induced from $l(\widehat{\mathcal{H}}, \widehat{S}) \cap \mathrm{EG}_N^\circ(Q, \mathcal{H}_Q)$, where $\mathcal{H} = \mathcal{I}_*(\widehat{\mathcal{H}})$ and $\mathcal{I}(\widehat{S}) = S$. By Proposition 5.11, we know these maximal line segments are $\{\mathcal{H}_S^{m\flat}\}_{m=0}^{N-2}$ and $\{\widehat{\mathcal{H}}_{\widehat{S}}^{m\flat}\}_{m=0}^{N-2}$. Write $\mathcal{H}^- = \mathcal{H}_S^{(N-2)\flat}$ and $S^- = S[2-N]$. Since $\mathcal{H}_S^{m\flat}$ is induced, it is finite, spherical and monochromatic by Proposition 7.6 and Proposition 7.5, for $0 \le m \le N-2$. Applying Proposition 5.2 to the simple forward tilt of $\mathcal{H}_S^{m\flat}$ with respect to $S[-m]$, for $m = N-2, N-1, \ldots, 0$, we have formula (7.6) and hence deduce that the changes of simples from $\mathcal{H}^-$ to $\mathcal{H}_S^\sharp$ are as follows:

- for $S^- \in \mathrm{Sim}\,\mathcal{H}^-$, it becomes $S[1]$ which equals $\phi_S^{-1}(S^-)$;
- for $X \in \mathrm{Sim}\,\mathcal{H}^-$ such that $\mathrm{Hom}^\bullet(X, S) = 0$, it remains in $\mathcal{H}_S^\sharp$. Observe that $X = \phi_S^{-1}(X)$.
- for $X \in \mathrm{Sim}\,\mathcal{H}^-$ such that $\mathrm{Hom}^\bullet(X, S) \neq 0$, the monochromaticity of $\mathcal{H}^-$ implies $\mathrm{Hom}^\bullet(X, S) = \mathrm{Hom}^m(X, S^-)$, for some integer $m > 0$. Notice that $\mathcal{H}^-$ is induced from $\widehat{\mathcal{H}}_{\widehat{S}}^{(N-2)\flat}$, let $\widehat{X}$ and $\widehat{S}^-$ be the corresponding simples in $\mathrm{Sim}\,\widehat{\mathcal{H}}_{\widehat{S}}^{(N-2)\flat}$. By (2.1), $\widehat{X} \in \mathcal{H}_Q[x], \widehat{S}^- \in \mathcal{H}_Q[s]$ for some integer $1 \le x, s \le N-1$. Since $\mathbf{k}Q$ is hereditary, we know that
$$\mathrm{Hom}^{\ge N}(\widehat{X}, \widehat{S}^-) = \mathrm{Hom}^{\ge N}(\widehat{S}^-, \widehat{X}) = 0.$$
By (7.7), this implies $1 \le m \le N-1$. Then X is in $\mathrm{Sim}\,(\mathcal{H}^-)_{S^-}^{j\sharp}$ for $j = 0, \ldots, m-1$, and becomes $\phi_S^{-1}(X)$ in $\mathrm{Sim}\,(\mathcal{H}^-)_{S^-}^{j\sharp}$ for $j = m, \ldots, N-1$.

Since the simples determine the heart, we have $\mathcal{H}_S^\sharp = \phi_S^{-1}(\mathcal{H}^-)$ which implies
$$(\mathcal{H}')_{S'}^\sharp = \phi(\mathcal{H}_S^\sharp) = \phi \circ \phi_S^{-1}(\mathcal{H}^-)$$
as required. □

Corollary 8.3. *Every heart in $\mathrm{EG}^\circ(\Gamma_N Q)$ is induced and hence finite, spherical and monochromatic. Moreover, for any heart $\mathcal{H}$ in $\mathrm{EG}^\circ(\Gamma_N Q)$, the set of twist functors of its simples is a set of generators of $\mathrm{Br}(\Gamma_N Q)$. Further, for any $S \in \mathrm{Sim}\,\mathcal{H}$, we have*
$$\mathcal{H}_S^{\pm(N-1)\sharp} = \phi_S^{\mp 1}(\mathcal{H}). \tag{8.3}$$

Proof. Proposition 8.2 shows that every heart is induced via the L-immersion which is the composition of the natural L-immersion $\mathcal{I}$ with some twist functors. Then every heart is finite, spherical and monochromatic by Proposition 7.6 and Proposition 7.5. Moreover, Proposition 5.2 applies to any such heart, with formulae (7.5) and (7.6). Hence the new simples in any simple tilt of such a heart are either the shift or the twist of the old simples. Thus the second assertion follows inductively by (7.4).

Further, we know that (8.3) is true for any heart $\mathcal{H}^- \in \mathcal{I}_*(\mathrm{EG}_N^\circ(Q, \mathcal{H}_Q))$ with simple S^- as in Proposition 8.2. Hence it is true for any hearts in $l(\mathcal{H}^-, S^-)$, which implies it is also true for any heart induced via $\mathcal{I}_*$, by Proposition 5.11. Notice that the autoequivalences preserve (8.3), thus this equation holds for any heart in $\mathrm{EG}^\circ(\Gamma_N Q)$ by Proposition 8.2. □

Corollary 8.4. *Let $\mathcal{H}$ and $\mathcal{H}'$ be hearts in $\mathrm{EG}^\circ(\Gamma_N Q)$ in the same braid group orbit, i.e. $\phi(\mathcal{H}) = \mathcal{H}'$ for some $\phi \in \mathrm{Br}(\Gamma_N Q)$. Then there exists a sequence of spherical objects $T_0, \ldots, T_{m-1}$ in hearts $\mathcal{H}_0, \ldots, \mathcal{H}_{m-1}$ (for some integer $m \geq 0$) together with signs $\epsilon_i \in \{\pm 1\}, i = 0, \ldots, m-1$, such that $\mathcal{H}_0 = \mathcal{H}$,*

$$\mathcal{H}_{i+1} = (\mathcal{H}_i)_{T_i}^{\epsilon_i (N-1)\sharp}, \quad i = 0, 1, \ldots, m-1, \tag{8.4}$$

and $\mathcal{H}_m = \mathcal{H}'$.

Proof. Fix $\mathcal{H}$ and let $\mathrm{Sim}\,\mathcal{H} = \{S_1, \ldots, S_n\}$, $\phi_k = \phi_{S_k}$ for $1 \leq k \leq n$. Since $\phi_1, \ldots, \phi_n$ generate $\mathrm{Br}(\Gamma_N Q)$ by Corollary 8.3, we have

$$\phi = \phi_{t_{m-1}}^{\lambda_{m-1}} \circ \cdots \circ \phi_{t_0}^{\lambda_0}$$

for some $t_j \in \{1, \ldots, n\}$ and $\lambda_j \in \{\pm 1\}$. Use induction on m. If $m = 0$, i.e. $\mathcal{H} = \mathcal{H}'$, there is nothing to prove. Suppose the statement holds for $m \leq s$ and consider the case when $m = s + 1$. Write $\varphi = \phi_{t_s}^{\lambda_s}$. For hearts $\mathcal{H}$ and

$$\varphi^{-1}(\mathcal{H}') = \left(\phi_{t_{s-1}}^{\lambda_{s-1}} \circ \cdots \circ \phi_{t_0}^{\lambda_0} \right) (\mathcal{H}),$$

by inductive hypothesis, there are spherical objects $R_0, R_2, \ldots, R_{s-1}$ together with $\varepsilon_i \in \{\pm 1\}$, such that $\mathcal{H}'_0 = \mathcal{H}$,

$$\mathcal{H}'_{i+1} = (\mathcal{H}'_i)_{R_i}^{\varepsilon_i (N-1)\sharp}, \quad i = 0, 1, \ldots, s-1$$

and $\mathcal{H}'_s = \varphi^{-1}(\mathcal{H}')$. Let $T_0 = S_{t_m}, \epsilon_0 = \lambda_m$ and $T_i = \varphi(R_{i-1}), \epsilon_i = \varepsilon_{i-1}$ for $i = 1, \ldots, s$. Then we have $\mathcal{H}_0 = \mathcal{H}$, $\mathcal{H}_1 = \varphi(\mathcal{H}_0)$ and (inductively)

$$\mathcal{H}_{i+1} = (\mathcal{H}_i)_{T_i}^{\epsilon_i (N-1)\sharp} = \left(\varphi(\mathcal{H}'_{i-1})\right)_{\varphi(R_{i-1})}^{\varepsilon_{i-1}(N-1)\sharp} = \varphi(\mathcal{H}'_i)$$

for $i = 1, \ldots, s$. In particular, we have $\mathcal{H}_{s+1} = \varphi(\mathcal{H}'_s) = \mathcal{H}'$ as required. □

8.2. **Cyclically completing.** By Proposition 8.2, there is a surjection on vertex sets

$$p_0 : \mathrm{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) \twoheadrightarrow \mathrm{EG}^\circ(\Gamma_N Q)/\,\mathrm{Br}\,.$$

Moreover, by the proof of Proposition 8.2, we can extend p_0 to a surjection (between oriented graphs)

$$\overline{p_0} : \overline{\mathrm{EG}_N^\circ}(\Gamma_N Q, \mathcal{H}_\Gamma) \twoheadrightarrow \mathrm{EG}^\circ(\Gamma_N Q)/\,\mathrm{Br}\,.$$

sending the new edge e_l in each basic cycle c_l to the edge in $\mathrm{EG}^\circ(\Gamma_N Q)/\,\mathrm{Br}$ induced by $(\mathcal{H} \xrightarrow{S} \mathcal{H}_S^\sharp)$, where c_l is induced by the line $l = l(\mathcal{H}, S)$ such that

$$l(\mathcal{H}, S) \cap \mathrm{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma) = \{\mathcal{H}_S^{i\flat}\}_{i=0}^{N-2}.$$

We can now bring everything together and show that $\mathrm{EG}_N^\circ(\Gamma_N Q, \mathcal{H}_\Gamma)$ is actually a fundamental domain for the braid group action on $\mathrm{EG}^\circ(\Gamma_N Q)$ and, simultaneously, that the quotient is the cluster exchange graph.

Theorem 8.5. *Let Q be an acyclic quiver. As oriented exchange graphs we have a canonical isomorphism*

$$\overline{p_0} : \overline{\mathrm{EG}_N^\circ}(\Gamma_N Q, \mathcal{H}_\Gamma) \cong \mathrm{EG}^\circ(\Gamma_N Q)/\,\mathrm{Br}(\Gamma_N Q), \tag{8.5}$$

and hence

$$\mathrm{EG}^\circ(\Gamma_N\, Q)/\,\mathrm{Br}(\Gamma_N\, Q) \cong \mathrm{CEG}_{N-1}(Q). \tag{8.6}$$

Proof. There is an exact sequence of triangulated categories (cf. [1])

$$0 \to \mathcal{D}(\Gamma_N\, Q) \to \mathrm{per}(\Gamma_N\, Q) \to \mathcal{C}_{N-1}(Q) \to 0,$$

where $\mathrm{per}(\Gamma_N\, Q)$ is the perfect derived category of $\Gamma_N\, Q$. By [1, Section 2], every heart $\mathcal{H}$ in $\mathrm{EG}^\circ(\Gamma_N\, Q)$ induces a t-structure on $\mathrm{per}(\Gamma_N\, Q)$ and determines a silting object in $\mathrm{per}(\Gamma_N\, Q)$, which induces a tilting object in $\mathcal{C}_{N-1}(Q)$. Thus we have a map

$$\upsilon : \mathrm{EG}^\circ(\Gamma_N\, Q) \to \mathrm{CEG}_{N-1}(Q). \tag{8.7}$$

Moreover, via υ, $\mathcal{H}_\Gamma$ corresponds to the initial cluster tilting set and the simple tilting of a heart corresponds to the mutation of a tilting/silting object.

By Corollary 8.4, if two hearts $\mathcal{H}, \mathcal{H}' \in \mathrm{EG}^\circ(\Gamma_N\, Q)$ are in the same braid group orbit, then $\mathcal{H}'$ can be obtained from $\mathcal{H}$ by a sequence of simple tiltings as in (8.4). Then $\upsilon(\mathcal{H}) = \upsilon(\mathcal{H}')$ because repeating the same mutation $N-1$ times returns every cluster tilting object back to itself. Hence we have a map $\overline{\upsilon} : \mathrm{EG}^\circ(\Gamma_N\, Q)/\,\mathrm{Br} \to \mathrm{CEG}_{N-1}(Q)$.

Inductively, we know that the simple tilting of a heart in $\mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$ or $\mathrm{EG}^\circ(\Gamma_N\, Q)$ corresponds to the mutation of a tilting/silting object. Thus we obtain the following commutative diagram

$$\begin{array}{ccc}
\overline{\mathrm{EG}^\circ_N}(Q, \mathcal{H}_Q) & \xrightarrow[\cong]{\overline{\mathcal{J}}} & \mathrm{CEG}_{N-1}(Q) \\
{\scriptstyle\cong}\big\downarrow{\scriptstyle\overline{\mathcal{I}_*}} & & \big\uparrow{\scriptstyle\overline{\upsilon}} \\
\overline{\mathrm{EG}^\circ_N}(\Gamma_N\, Q, \mathcal{H}_\Gamma) & \xrightarrow{\overline{p_0}} & \mathrm{EG}^\circ(\Gamma_N\, Q)/\,\mathrm{Br}
\end{array} \tag{8.8}$$

which implies the theorem. □

This also enables us to interpret Buan-Thomas' coloured quiver for cluster tilting sets via hearts in $\mathcal{D}(\Gamma_N\, Q)$.

Theorem 8.6. *For any heart $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N\, Q)$, the Ext-quiver $\mathcal{Q}(\mathcal{H})$ (Definition 6.2) is equal to the augmented graded quiver $\mathcal{Q}^+(\upsilon(\mathcal{H}))$ (Definition 6.1) of the corresponding cluster tilting set, where υ is defined in (8.7).*

Proof. By Theorem 8.1, any heart $\mathcal{H}$ in $\mathrm{EG}^\circ_N(\Gamma_N\, Q, \mathcal{H}_\Gamma)$ is induced from a heart $\widehat{\mathcal{H}}$ in $\mathrm{EG}^\circ_N(Q, \mathcal{H}_Q)$. Hence, combining (7.8), (6.10) and (8.8), we see that

$$\mathcal{Q}(\mathcal{H}) = {}^N\overline{\mathcal{Q}(\widehat{\mathcal{H}})} = \mathcal{Q}^+(\mathcal{J}(\widehat{\mathcal{H}})) = \mathcal{Q}^+(\upsilon(\mathcal{H})).$$

But Theorem 8.5 also tells us that $\mathrm{EG}^\circ_N(\Gamma_N\, Q, \mathcal{H}_\Gamma)$ is a fundamental domain for the action (by automorphisms) of $\mathrm{Br}(\Gamma_N\, Q)$, while υ is invariant under this action. Hence, we deduce that the equality holds for all $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N\, Q)$. □

Remark 8.7. We need the standard heart as base on the left-hand-side to ensure the isomorphism (8.5) holds. Example 8.8 illustrates this phenomenon. However, if $N = 3$, the isomorphism (8.5) holds for any heart (see Section 9). Further, for $N = 3$, Keller-Nicolás (cf. [13, Theorem 5.6]) prove (8.6) in full generality, that is, when Q is a loop-free, 2-cycle-free quiver with a polynomial potential W.

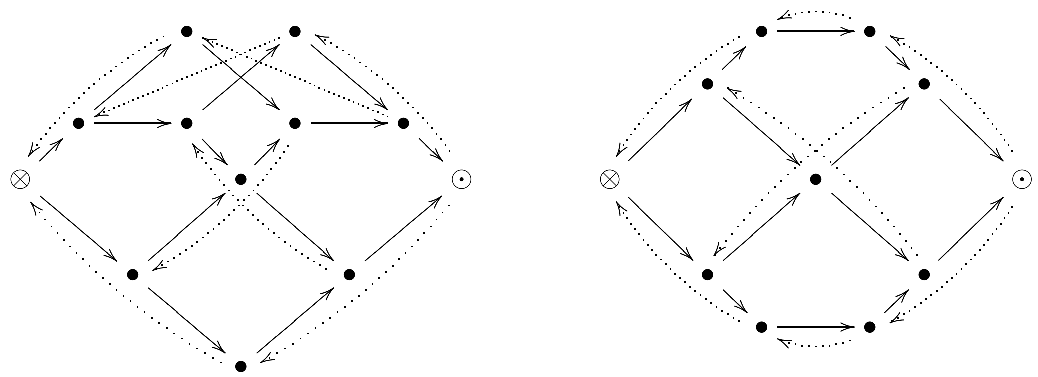


FIGURE 2. Two cyclic completions of CY-4 exchange graphs of type A_2

Example 8.8. Let Q be a quiver of type A_2 with corresponding $\operatorname{Sim}\mathcal{H}_\Gamma = \{S, X\}$ such that $\operatorname{Hom}^1(S, X) = \mathbf{k}$. Figure 2 shows the cyclic completions of two exchange graphs: $\overline{\mathrm{EG}}_4(\Gamma_4 Q, \mathcal{H}_\Gamma)$ on the left and $\overline{\mathrm{EG}}_4(\Gamma_4 Q, (\mathcal{H}_\Gamma)^\sharp_S)$ on the right. The solid arrows are the edges in $\mathrm{EG}^\circ(\Gamma_4 Q)$ and the dotted arrows are the extra edges in the cyclic completions. The vertices $\otimes$ and $\odot$ represent the source and sink (i.e. $\mathcal{H}[1]$ and $\mathcal{H}[3]$ in fact) in the exchange graph $\mathrm{EG}_4(\Gamma_4 Q, \mathcal{H})$ with base $\mathcal{H}$. Notice that $\overline{\mathrm{EG}}_4(\Gamma_4 Q, (\mathcal{H}_\Gamma)^\sharp_S)$ cannot be isomorphic to $\mathrm{EG}^\circ(\Gamma_4 Q)/\operatorname{Br}$, because it has the wrong number of vertices.

Remark 8.9. By Corollary 5.12, each almost complete cluster tilting set in $\mathcal{C}_{N-1}(Q)$ can be identified with a basic cycle in $\mathrm{CEG}_{N-1}(Q) \cong \overline{\mathrm{EG}^\circ_N}(Q, \mathcal{H}_Q)$, which can be identified with a basic cycle in $\overline{\mathrm{EG}^\circ_N}(\Gamma_N Q, \mathcal{H}_\Gamma)$ by Theorem 8.1. By Theorem 8.5, these basic cycles also can be interpreted as braid group orbits of lines of $\mathrm{EG}^\circ(\Gamma_N Q)$ in $\mathrm{EG}^\circ(\Gamma_N Q)/\operatorname{Br}$.

9. Orientations of cluster exchange graphs

We now consider just the case $N = 3$. Recall that, by Theorem 8.5, we have the following three descriptions of the same oriented graph

$$\overline{\mathrm{EG}^\circ_3}(\Gamma_3 Q, \mathcal{H}_\Gamma) \cong \mathrm{EG}^\circ(\Gamma_3 Q)/\operatorname{Br}_3 \cong \mathrm{CEG}_2(Q).$$

In this graph, every basic cycle is a 2-cycle and hence we have an induced isomorphism of the oriented graph $\mathrm{EG}^\circ(\Gamma_3 Q, \mathcal{H}_\Gamma)$ with the usual unoriented cluster exchange graph $\mathrm{CEG}(Q)^*$, that is, the graph obtained from $\mathrm{CEG}_2(Q)$ by replacing each basic 2-cycle with an unoriented edge. For example, for Q of type A_2, Figure 3 shows the cyclic completion $\overline{\mathrm{EG}^\circ_3}(\Gamma_3 Q, \mathcal{H}_\Gamma)$ on the left and corresponding unoriented cluster exchange graph $\mathrm{CEG}(Q)^*$ on the right.

Thus $\mathrm{EG}^\circ(\Gamma_3 Q, \mathcal{H}_\Gamma)$ induces an orientation of $\mathrm{CEG}(Q)^*$. In this section, we will explain how this property extends to all hearts in $\mathrm{EG}^\circ(\Gamma_3 Q)$. To do this, we first observe that $\mathrm{EG}^\circ(\Gamma_3 Q, \mathcal{H})$ divides naturally into two pieces, using the following result.

Proposition 9.1. *Let $\mathcal{H}$ and $\mathcal{H}_0$ be hearts with $\mathcal{H}[1] \le \mathcal{H}_0 \le \mathcal{H}[2]$ and $S \in \operatorname{Sim}\mathcal{H}$ be such that the simple tilts of $\mathcal{H}$ with respect to S exist. Then*

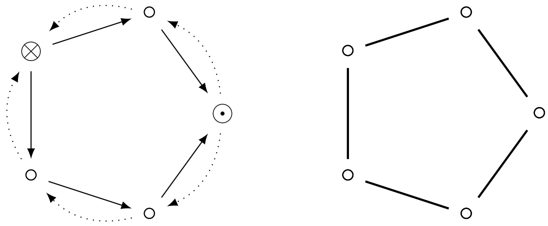


FIGURE 3. $\overline{\mathrm{EG}_3^\circ}(\Gamma_3\, Q, \mathcal{H}_\Gamma)$ and $\mathrm{CEG}(Q)^*$ for a quiver Q of type A_2

1°. $S[1] \notin \mathcal{H}_0$ if and only if $\mathcal{H}_S^\sharp[1] \le \mathcal{H}_0$ if and only if $S[2] \in \mathcal{H}_0$,
2°. $S[2] \notin \mathcal{H}_0$ if and only if $\mathcal{H}_0 \le \mathcal{H}_S^\flat[2]$ if and only if $S[1] \in \mathcal{H}_0$.

Thus, in particular,

$$S \in \mathcal{H}_0[-1] \sqcup \mathcal{H}_0[-2]. \tag{9.1}$$

Proof. Since, in each case, the third condition follows from the second, by Lemma 2.3, and also clearly implies the first, we only need to prove that the first implies the second.

For the first implication in 1°, note that $S[1] \in \mathcal{P}^\perp[2] \subset \mathcal{P}_0^\perp[1]$, so that $S[1] \notin \mathcal{H}_0$ implies $S[1] \notin \mathcal{P}_0$. Suppose, for contradiction, that there is an object $M \in \mathcal{P}_0$, but with $M \notin \mathcal{P}_S^\sharp[1]$. Consider the filtrations (3.1) and (3.2) of M, with respect to $\mathcal{H}$, and the torsion pair corresponding to $\mathcal{H}_S^\sharp$. Since $M \in \mathcal{P}_0 \subset \mathcal{P}[1]$, we have $k_m \ge 1$. But $M \notin \mathcal{P}_S^\sharp[1]$ forces $k_m = 1$ and $H_m^{\mathcal{F}} = S^t \neq 0$. In this case, there is a triangle $M' \to M \to S^t[1] \to M'[1]$ with $M' \in \mathcal{P}[1]$. Hence we have $M'[1] \in \mathcal{P}[2] \subset \mathcal{P}_0$ and, as $M \in \mathcal{P}_0$, this implies $S[1] \in \mathcal{P}_0$, contradicting the initial observation. Thus $\mathcal{P}_S^\sharp[1] \supset \mathcal{P}_0$, that is, $\mathcal{H}_S^\sharp[1] \le \mathcal{H}_0$.

Similarly, for the first implication in 2°, $S[2] \in \mathcal{P}[2] \subset \mathcal{P}_0$, so $S[2] \notin \mathcal{H}_0$ implies $S[2] \notin \mathcal{P}_0^\perp[1]$. If there is an object $M \notin \mathcal{P}_S^\flat[2]^\perp$, but with $M \in \mathcal{P}_0^\perp \subset \mathcal{P}^\perp[2]$, we deduce as before that $k_1 = 1$ with $H_1^{\mathcal{T}} = S^t \neq 0$ in (3.2). Hence there is a triangle $M'[-1] \to S^t[1] \to M \to M'$ with $M'[-1] \in \mathcal{P}^\perp[1] \subset \mathcal{P}_0^\perp$, which implies $S[1] \in \mathcal{P}_0^\perp$, contradicting the initial observation. Thus $\mathcal{P}_S^\flat[2]^\perp \supset \mathcal{P}_0^\perp$, that is, $\mathcal{H}_0 \le \mathcal{H}_S^\flat[2]$. □

Now, for $S \in \operatorname{Sim}\mathcal{H}$, we can define

$$\mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H})_S^- = \{\mathcal{H}' \in \mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H}) \mid S[1] \in \mathcal{H}'\} \tag{9.2}$$

$$\mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H})_S^+ = \{\mathcal{H}' \in \mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H}) \mid S[2] \in \mathcal{H}'\} \tag{9.3}$$

and observe that Proposition 9.1 means that

$$\mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H}) \;=\; \mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H})_S^- \;\sqcup\; \mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H})_S^+.$$

Another immediate consequence of Proposition 9.1 is the following.

Corollary 9.2. *For any $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_3\, Q)$, $\mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H})$ has a unique source $\mathcal{H}[1]$ and a unique sink $\mathcal{H}[2]$.*

Proof. For any $\mathcal{H}_0 \in \mathrm{EG}_3^\circ(\Gamma_3\, Q, \mathcal{H})$, we have $\mathcal{H}_0 \le \mathcal{H}[2] \le \mathcal{H}_0[1]$. Hence, for any simple $S_0 \in \operatorname{Sim}\mathcal{H}_0$, we have $S_0 \in \mathcal{H}[1] \sqcup \mathcal{H}[2]$, by (9.1).

Now if $\mathcal{H}_0$ is a source, we have $(\mathcal{H}_0)^\sharp_{S_0} \in \mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})$ for any $S_0 \in \mathrm{Sim}\,\mathcal{H}_0$. By the second part of Lemma 5.4, we must have $S_0 \in \mathcal{H}[1]$ and not $S_0 \in \mathcal{H}[2]$. Thus $\mathcal{H}_0 \subset \mathcal{H}[1]$ which implies $\mathcal{H}_0 = \mathcal{H}[1]$, or equivalently, $\mathcal{H}[1]$ is the unique source. Similarly for the uniqueness of the sink. □

From Proposition 9.1, it immediately follows that every edge in $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})$ labelled by $S[1]$ connects $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^-_S$ to $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^+_S$. In fact, the converse is also true.

Lemma 9.3. *Let $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_3 Q), S \in \mathrm{Sim}\,\mathcal{H}$ and e be an edge connecting $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^\pm_S$. Then the tail of e is in $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^-_S$ and the label of e is $S[1]$.*

Proof. Let $\mathcal{H}_1 \in \mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^-_S$ and $\mathcal{H}_2 \in \mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^+_S$ be the vertices of e. Since $S[1] \in \mathcal{P}_1$ and $S[1] \notin \mathcal{P}_2$, we mush have $\mathcal{P}_1 \supset \mathcal{P}_2$, i.e. $\mathcal{H}_1$ is the tail of e. Let T be the label of e and $\langle \mathcal{F}, \mathcal{T} \rangle$ is the torsion pair in $\mathcal{H}_1$ corresponding to e. By (9.1) we have $T \in \mathcal{H}[1]$. Suppose $T \neq S[1]$. Noticing that $\mathcal{T} \in \mathcal{H}_2$ but $S[1] \notin \mathcal{H}_2$, we have $S[1] \notin \mathcal{T}$ which implies there is a nonzero map $f\colon S[1] \to T$. Let $M = \mathrm{Cone}(f)[-1]$. Since T is a simple in $\mathcal{H}_1$, we have $M \in \mathcal{H}_1$. On the other hand, $S[1]$ is a simple in $H[1]$, we have $M \in \mathcal{H}$ contradicting to $M \in \mathcal{H}_1 \subset \mathcal{P}_1 \subset \mathcal{P}[1]$. Hence $T = S[1]$. □

We can now describe how forward tilting the base heart transforms the based exchange graphs. There is an obvious modification for backwards tilting.

Proposition 9.4. *For any heart $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_3 Q)$ and $S \in \mathrm{Sim}\,\mathcal{H}$, the exchange graph $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\sharp_S)$ can be obtained from $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})$ by applying a 'half-twist', that is, applying ϕ_S^{-1} to $\mathrm{EG}(\Gamma_3 Q, \mathcal{H})^-_S$, reversing all edges with label $S[1]$ in $\mathrm{EG}(\Gamma_3 Q, \mathcal{H})$ and relabelling them with $S[2]$.*

Proof. First, by Proposition 9.1, we have $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^+_S = \mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\sharp_S)^-_{S[1]}$ and

$$\begin{aligned} &\phi_S^{-1}\left(\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^-_S\right) = \phi_S^{-1}\left(\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\flat_S)^+_{S[-1]}\right) \\ = \;&\mathrm{EG}^\circ_3(\Gamma_3 Q, \phi_S^{-1}(\mathcal{H}^\flat_S))^+_{\phi_S^{-1}(S[-1])} = \mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\sharp_S)^+_{S[1]}. \end{aligned}$$

Second, by Lemma 9.3, we know that any edge connecting $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^\pm_S$ is labelled by $S[1]$ and any edge connecting $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\sharp_S)^\pm_{S[1]}$ is label-ed by $S[2]$. The result follows because the half twist turns the edge $(\mathcal{H}_1 \xrightarrow{S[1]} \mathcal{H}_2)$ in $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})$ into the edge $\left(\phi_S^{-1}(\mathcal{H}_1) \xleftarrow{S[2]} \mathcal{H}_2\right)$ in $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\sharp_S)$. □

Example 9.5. Let Q be the A_3-type quiver in Example 4.3, $\mathcal{H} = \mathcal{H}_\Gamma$ and $\mathcal{I}(Y) = S$. In Figure 4, the black and blue parts of the top graph are $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H})^\pm_S$; the green and black parts of the bottom graph are $\mathrm{EG}^\circ_3(\Gamma_3 Q, \mathcal{H}^\sharp_S)^\pm_{S[1]}$ respectively. Moreover, the red arrows which connect $\mathrm{EG}^\circ_3(\Gamma_3 Q, ?)^\pm_?$ are $S[1]$-parallel edges in the top graph and $S[2]$-parallel edges in the bottom graph. The vertices $\otimes$ and $\odot$ are the unique source and sink in the graphs, respectively.

Applying Proposition 9.4 inductively, starting from Theorem 8.5 applied to the standard heart $\mathcal{H}_\Gamma$, we obtain the following result.

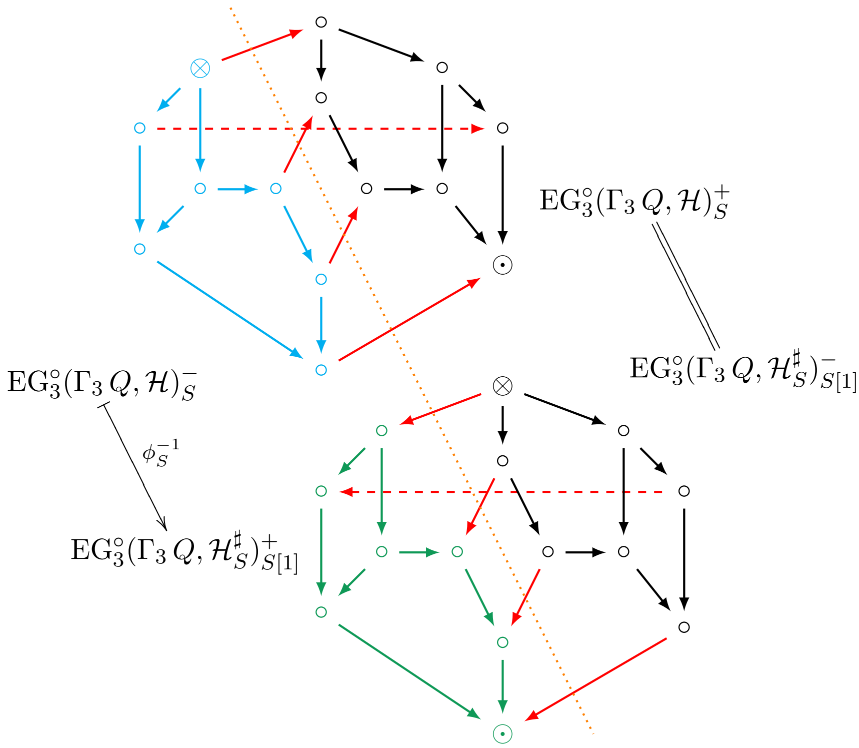


FIGURE 4. Half twist of $\mathrm{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$ for an A_3-type quiver Q

Theorem 9.6. *For any heart* $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_3 Q)$*, we have* $\overline{\mathrm{EG}_3^\circ}(\Gamma_3 Q, \mathcal{H}) \cong \mathrm{CEG}_2(Q)$*, or equivalently,* $\mathrm{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$ *induces an orientation of the (unoriented) cluster exchange graph* $\mathrm{CEG}(Q)^*$.

In particular, Theorem 9.6 says that $\mathrm{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$ is a fundamental domain for $\mathrm{EG}^\circ(\Gamma_3 Q)/\mathrm{Br}_3$, for every $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_3 Q)$, while in Calabi-Yau-$N$ case, this is only proved (and most likely only true) for the standard heart $\mathcal{H}_\Gamma$ as base (cf. Theorem 8.5).

Remark 9.7. Proposition 9.4 describes precisely how the subgraphs $\mathrm{EG}_3^\circ(\Gamma_3 Q, \mathcal{H})$, each of which is an orientation of the cluster exchange graph $\mathrm{CEG}(Q)^*$ by Theorem 9.6, glue together to form $\mathrm{EG}^\circ(\Gamma_3 Q)$. Several of the underlying ideas in this section, including (9.1), already appear in the work of Plamondon [18], Nagao [17] and Keller [13].

10. CONSTRUCTION OF A2-TYPE EXCHANGE GRAPH VIA THE FAREY GRAPH

In this section, we consider a quiver Q of type A_2 and, by abuse of notation, write $\Gamma_N A_2$ for $\Gamma_N Q$. We will demonstrate a roughly dual relationship between the quotient graph $\mathrm{EG}^\circ(\Gamma_3 Q)/[1]$ and the Farey graph FG, in its natural embedding in the hyperbolic disc. This graph has vertices the rational points on the boundary of the disc

$$\mathrm{FG}_0 = \mathbb{Q} \cup \{\infty\},$$

with an edge from p/q to r/s if and only if $|ps - rq| = 1$. By convention here $\infty = 1/0$. These edges, as hyperbolic geodesics, give a triangulation of the disc.

The Farey graph arises in a variety of contexts; for example, it is the curve complex of a (once-punctured) torus, whose vertices are homotopy classes of simple closed curves and whose edges join curves with intersection number one. More directly relevant here, FG_0 can be identified with a conjugacy class of parabolic elements in $\mathrm{PSL}_2(\mathbb{Z})$ of the form

$$\Psi_{p/q} = \begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix}.$$

Note that the natural action of $\mathrm{PSL}_2(\mathbb{Z})$ on the hyperbolic disc preserves FG and furthermore $\Psi_{p/q}$ fixes p/q.

10.1. **Spherical twists and vertices in FG.** Denote by $\mathrm{Sph}(\Gamma_N A_2)$ the set of all spherical objects which are simples in some hearts in $\mathrm{EG}^\circ(\Gamma_N A_2)$ and

$$\mathrm{Tw}(\Gamma_N A_2) = \{\phi_S \mid S \in \mathrm{Sph}(\Gamma_N A_2)\} \subset \mathrm{Br}_3\,.$$

Since $\phi_S = \phi_{S[1]}$, there is a surjective map

$$\Phi : \mathrm{Sph}(\Gamma_N A_2)/[1] \to \mathrm{Tw}(\Gamma_N A_2). \tag{10.1}$$

Moreover, suppose $\phi_S = \phi_T$ for some $S, T \in \mathrm{Sph}(\Gamma_N A_2)$. Then $\phi_T(S) = \phi_S(S) = S[1-N]$. Since there is no non-zero map from S to $S[1-N]$, we must have

$$T \otimes \mathrm{Hom}^\bullet(T, S) = S \oplus S[-N],$$

which implies $T = S[m]$ for some integer m. Thus Φ in (10.1) is in fact an bijection.

Let $\mathrm{Sim}\,\mathcal{H}_\Gamma = \{X_0, X_\infty\}$ with $\mathrm{Ext}^1(X_0, X_\infty) \neq 0$. Then $\{\phi_{X_0}, \phi_{X_\infty}\}$ is a generating set of Br_3. By (7.4), we inductively deduce that

$$\mathrm{Tw}(\Gamma_N A_2) = \{\phi \circ \phi_{X_0} \circ \phi^{-1} \mid \phi \in \mathrm{Br}_3\},$$

that is, $\mathrm{Tw}(\Gamma_N A_2)$ is the conjugacy class of one of the generators of the braid group. It is well-known that Br_3 is a central extension

$$0 \to \mathbb{Z} \to \mathrm{Br}_3 \xrightarrow{p} \mathrm{PSL}_2(\mathbb{Z}) \to 0,$$

where the central generator is a braid of non-zero 'length' (i.e. number of positive minus number of negative crossings). Hence the map p is injective restricted to the conjugacy class $\mathrm{Tw}(\Gamma_N A_2)$ and we can identify $\mathrm{Tw}(\Gamma_N A_2)$ with its image, which is $\{\Psi_a \mid a \in \mathrm{FG}_0\}$. Indeed we can arrange that $p(\phi_{X_a}) = \Psi_a$ for $a = 0, 1, \infty$, where $X_1 = \phi_{X_0}(X_\infty)$, and thereby obtain a bijection

$$\chi = (p \circ \Phi)^{-1} \circ \Psi : \mathrm{FG}_0 \xrightarrow{\sim} \mathrm{Sph}(\Gamma_N A_2)/[1],$$

which satisfies

$$\chi(\Psi_a(b)) = \phi_{\chi(a)}(\chi(b)),$$

for any $a, b \in \mathrm{FG}_0$. In other words, the spherical twists act on $\mathrm{Sph}(\Gamma_N A_2)/[1]$ in the same way that the parabolic elements acting on FG_0.

10.2. **L-immersions and triangles in FG.** By the action of the $\mathrm{PSL}_2(\mathbb{Z})$ symmetry, the properties of the triangle $\triangle = (\infty, 1, 0)$ in FG can be extended to any clockwise triangle $\triangle = (a, b, c)$ in FG. In particular, for each such triangle, we have the following:

1°. There is a triangle
$$X_a \to X_b \to X_c \to X_a[1] \tag{10.2}$$
in $\mathcal{D}(\Gamma_N A_2)$ such that X_j is in the shift orbit $\chi(j)$ for $j = a, b, c$ satisfying
$$X_b = \phi_{X_c}(X_a), \quad X_c = \phi_{X_a}(X_b), \quad X_a[1] = \phi_{X_b}(X_c).$$

2°. There is an L-immersion $\mathcal{L}_\triangle \colon \mathcal{D}(A_2) \to \mathcal{D}(\Gamma_N A_2)$, unique up to the action of $\operatorname{Aut}\mathcal{D}(A_2)$, determined by
$$\mathcal{L}_\triangle(\operatorname{Ind}\mathcal{D}(A_2)) = X_a[\mathbb{Z}] \cup X_b[\mathbb{Z}] \cup X_c[\mathbb{Z}],$$
where $X_j[\mathbb{Z}]$ means $\{X_j[m]\}_{m\in\mathbb{Z}}$.

3°. Up to shift, there are $3(N-1)$ hearts induced by $\mathcal{L}_\triangle$, given as follows
$$\mathcal{H}_j^{ca} = \langle X_a, X_c[j-1]\rangle, \quad \mathcal{H}_j^{ab} = \langle X_b, X_a[j]\rangle, \quad \mathcal{H}_j^{bc} = \langle X_c, X_b[j]\rangle. \tag{10.3}$$
where $j = 1, \dots, N-1$.

4°. In (10.3), only the three hearts $\mathcal{H}_1^*$ are induced by $\mathcal{L}_\triangle$ from standard hearts in $\mathcal{D}(A_2)$. Their images in $\mathrm{EG}(\Gamma_N A_2)/[1]$ form a three-cycle $T_\triangle$. Moreover, the images of hearts $\mathcal{H}_{j-1}^*$ and $\mathcal{H}_j^*$ in $\mathrm{EG}(\Gamma_N A_2)/[1]$ form a two-cycle $C_{*,j}$, for $j = 2, \dots, N-1$.

5°. If two triangles share an edge (c, a), then the corresponding induced hearts $\mathcal{H}_j^{ca}$ and $\mathcal{H}_{N-j}^{ac}$ coincide (up to shift). N.B. the objects X_a and X_c for the two triangles will also differ by shifts.

Moreover, by iterated tilting from $\mathcal{H}_\Gamma$, we can obtain every heart $\mathcal{H} \in \mathrm{EG}^\circ(\Gamma_N A_2)$ as (a shift of) one of the hearts in (10.3). In particular, if $\operatorname{Sim}\mathcal{H} = \{A, C\}$, then $(A[\mathbb{Z}], C[\mathbb{Z}])$ corresponds an edge in FG.

Thus we can naturally draw the (oriented) quotient graph
$$\mathcal{G}_N = \mathrm{EG}^\circ(\Gamma_N A_2)/[1]$$
on top of the Farey graph, as illustrated in Figure 5 in the case $N = 3$ and in Figure 6 in the cases $N = 2, 4$. It consists of the three-cycles $T_\triangle$, for each triangle $\triangle$ of FG, joined by a chain of $N-2$ two-cycles $C_{\Lambda,j}$, for each edge Λ of FG, and thus $\mathcal{G}_N$ is 'roughly' dual to FG.

10.3. **Lifting to the exchange graph.** The exchange graph $\mathrm{EG}^\circ(\Gamma_N A_2)$ may be recovered from $\mathcal{G}_N$ as a $\mathbb{Z}$-cover $\Pi : \widetilde{\mathcal{G}_N} \to \mathcal{G}_N$ sitting inside $\mathcal{G}_N \times \frac{1}{6}\mathbb{Z}$ and determined by grading each edge e by
$$\mathrm{gr}(e) = \begin{cases} \frac{1}{3}, & \text{if } e \in T_\triangle, \\ \frac{1}{2}, & \text{if } e \in C_{\Lambda,j}. \end{cases}$$
More precisely we require that, for each vertex v of $\mathcal{G}_N$, we have $\Pi^{-1}(v) = \{v\} \times \mathbf{N}_v$, where $\mathbf{N}_v$ is a coset of $\mathbb{Z}$ in $\frac{1}{6}\mathbb{Z}$. Furthermore, each edge e of $\mathcal{G}_N$ starting at v, lifts to edges $\widetilde{e}$, starting at each point of $\{v\} \times \mathbf{N}_v$, with an additional vertical shift by $\mathrm{gr}(e)$.

Notice, in particular, that $\widetilde{\mathcal{G}_3}$ is covered by the countably many oriented pentagons which sit between vertices (v, t) and $(v, t+1)$, as described in Remark 9.7. More precisely,

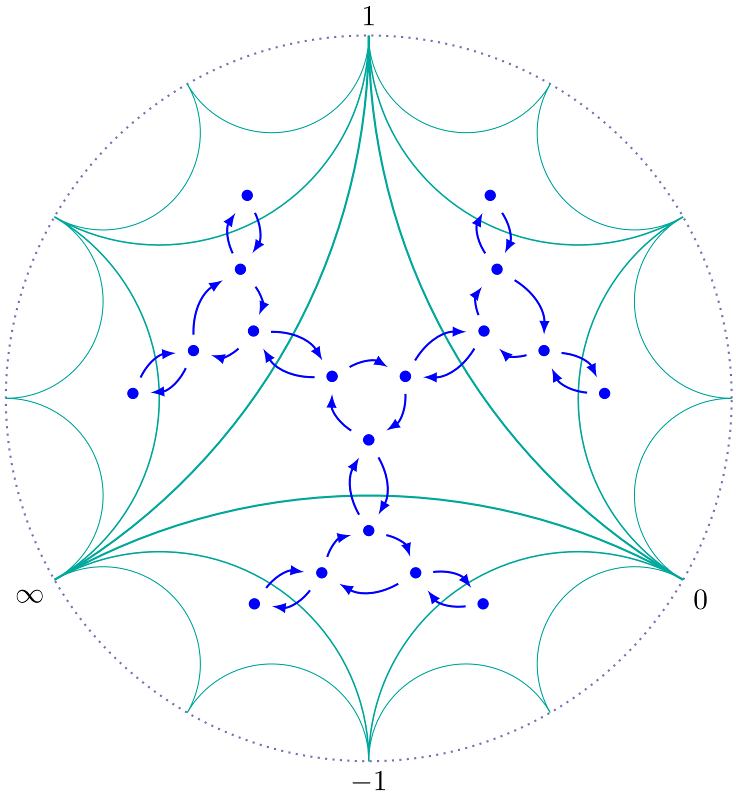


FIGURE 5. The Farey graph FG with 'dual' quotient graph $\mathcal{G}_3$.

such a pentagon consists of the lift of some three-cycle $T_\triangle$ together with the lift of some adjacent two-cycle C_Λ, as illustrated in Figure 7.

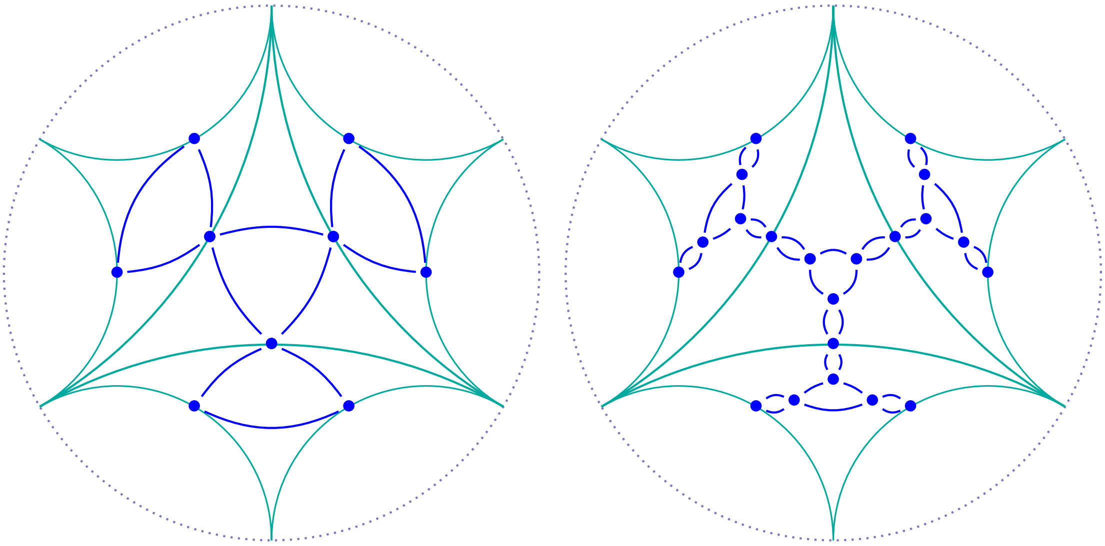


FIGURE 6. The quotient graphs $\mathcal{G}_2$ and $\mathcal{G}_4$ (orientations omitted).

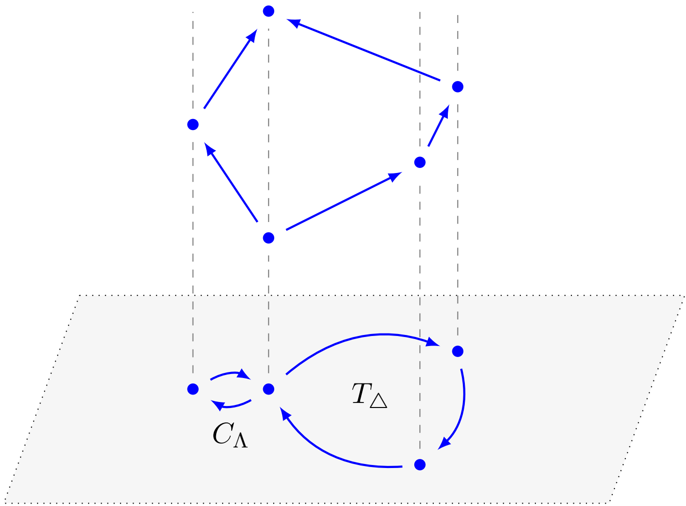


FIGURE 7. A lifting pentagon

References

[1] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, *Ann. Inst. Fourier* 59 (2009), 2525–2590.

[2] I. Assem, D. Simson and A. Skowroski, Elements of the Representation Theory of Associative Algebras 1, *C.U.P.*, 2006.

[3] T. Bridgeland, Stability conditions on triangulated categories, *Ann. Math.* 166 (2007), 317–345. (arXiv:math/0212237v3)

[4] A. Buan, R.J. Marsh, M. Reineke, I. Reiten, and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* 204 (2006), 572–618. (arXiv:math/0402054v1)

[5] A. Buan, I. Reiten and H. Thomas, From m-clusters to m-noncrossing partitions via exceptional sequences, arXiv:1007.0928v1.

[6] A. Buan and H. Thomas, Coloured quiver mutation for higher cluster categories, *Adv. Math.* 222 (2009), 971–995, (arXiv:0809.0691v3).

[7] V. Ginzburg, Calabi-Yau algebras, arXiv:math/0612139v3.

[8] D. Happel, On the derived category of a finite-dimensional algebra, *Comm. Math. Helv.* 62 (1987), 339–389.

[9] D. Happel, I. Reiten, and S.Smalø, Tilting in abelian categories and quasitilted algebras, *Mem. Amer. Math. Soc.* 120 (1996), no. 575, viii+ 88pp.

[10] O. Iyama, and Y. Yoshino, Mutation in triangulated categories and rigid Cohen-Macaulay modules, *Inv. Math.* 172 (2008), 117–168. (arXiv:math/0607736v3)

[11] B. Keller, Simple dg modules for positive dg algebras. arXiv:1009.5904v2.

[12] B. Keller, On Triangulated Orbit Categories. arXiv:math/0503240v4.

[13] B. Keller, On cluster theory and quantum dilogarithm, arXiv:1102.4148v4.

[14] B. Keller, Deformed Calabi-Yau completions, arXiv:0908.3499v5.

[15] B. Keller and D. Vossieck, Aisles in derived categories, *Bull. Soc. Math. Belg.* 40 (1988), 239–253.

[16] M. Khovanov and P. Seidel, Quivers, floer cohomology and braid group actions, *J. Amer. Math. Soc.* 15 (2002), 203–271. (arXiv:math/0006056v2)

[17] K. Nagao, Donaldson-Thomas theory and cluster algebras, arXiv:1002.4884v2.

[18] P-G. Plamondon, Cluster characters for Hom-infinite cluster categories, arXiv:1002.4956v2.

[19] Y. Qiu, Exchange graphs and stability conditions for quivers, Bath Ph.D thesis, 2011.

[20] Y. Qiu, Stability conditions and quantum dilogarithm identities for Dynkin quivers, arXiv:1111.1010v2.

[21] P. Seidel and R. Thomas, Braid group actions on derived categories of coherent sheaves, *Duke Math. J.*, 108 (2001) 37–108. (arXiv:math/0001043v2)

[22] T. Sutherland, The modular curve as the space of stability conditions of a CY3 algebra, arXiv:1111.4184v1.

[23] A. Wraalsen, Rigid objects in higher cluster categories, arXiv:0712.2970v1.

[24] Y Zhou and B Zhu, Cluster combinatorics of d-cluster categories, arXiv:0712.1381v2.

[25] B Zhu, Generalized Cluster Complexes via Quiver Representations, arXiv:math/0607155v4.

Mathematical Sciences, University of Bath, Bath, BA2 7AY, U.K.
E-mail address: A.D.King@bath.ac.uk, Yu.Qiu@bath.edu