MIND SWITCHES IN FUTURAMA AND STARGATE

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Abstract

Let P be a permutation expressed as a product of nontrivial disjoint cycles. When writing P as a product of distinct transpositions none equal to a factor of P, what is the smallest number of transpositions that can be used? We answer this question and give applications to mind-switching problems that have arisen in connection with the popular sci-fi television series *Futurama* and *Stargate SG-1*.

1 Introduction

For a permutation P expressed as a product of $m \geq 1$ nontrivial disjoint cycles, let $n \geq 2$ denote the number of entries. A paper by Mackiw [7] opens with the following question: When writing P as a product of transpositions, what is the smallest number of transpositions that can be used? The well known answer is $n - m$; see [7], [6, Theorem 4].

The purpose of this paper is to answer the following related question: When writing P as a product of distinct transpositions *none equal to a factor of* P, what is the smallest number M of transpositions that can be used? (It can be shown that the distinctness requirement does not affect the value of M except in the case $n = 2$.) In Section 3, we show that M is given by

$$M = \begin{cases} 5, & \text{if } \ n = 2 \\ n - m + r + \epsilon_r, & \text{if } \ n > 2, \end{cases} \tag{1.1}$$

where r is the number of transposition factors in the product P, and

$$\epsilon_r = \begin{cases} 0, & \text{if } \ 2 \mid r \\ 1, & \text{if } \ 2 \nmid r. \end{cases} \tag{1.2}$$

Note that $(12) = Q$, where $Q := (23)(13)(23)$ has fewer than 5 factors, but this does not contradict the case $n = 2$ of (1.1), because the factors of Q are not distinct.

In Section 2, we motivate the problem of determining M by showing that (1.1) can be applied to solve and extend mind-switching problems that have arisen in connection with *Futurama* and *Stargate SG-1*.

In the sequel, we sometimes write $P = P(n, m, r)$ and $M = M(n, m, r)$ in order to emphasize the dependence on n, m, and r.

2 Futurama and Stargate

"The Prisoner of Benda" [8], a 2010 episode of *Futurama*, features a two-body mind-switching machine. Any pair can enter the machine to swap minds, but the machine has the limitation that it will not work more than once on the same pair of bodies. A two-body mind-switching machine with exactly the same limitation is featured in "Holiday" [4], a 1999 episode of *Stargate SG-1*. In both episodes, the participants in the mind swapping eventually wish to return to their original bodies. Brilliant characters (Globetrotters in *Futurama*, Captain Samantha Carter in *Stargate SG-1*) save the day by showing how to reverse the switching. It is natural to ask: What is the smallest number of mind switches needed to bring everyone back to normal?

We first look at a problem from *Futurama*. In "The Prisoner of Benda", nine characters take part in a mind swapping spree involving seven switches; for brevity, we name their bodies $1, 2, \ldots, 9$. The product

$$F_9 := (45)(89)(12)(39)(56)(37)(36) \tag{2.1}$$

represents the sequence of seven switches: first the pair of bodies 3, 6 use the machine to swap minds, then the pair 3, 7, and so on. One can check that (as permutations)

$$F_9 = Q_9 := (23)(19)(18)(17)(16)(15)(14)(13)(29). \tag{2.2}$$

No factor of Q_9 is a factor of F_9. Thus Q_9^{-1} provides a way to "undo" F_9 using nine switches: to restore normalcy, first the pair 2, 3 swap minds, then the pair 1, 9, and so on. Note that due to the limitation of the machine, in order for a product of transpositions to undo F_9, it is necessary that it have distinct factors, none equal to a factor of F_9.

In a 2010 video [5], Cambridge University mathematician James Grime asked if F_9 could be undone with *fewer* than nine switches. We can apply (1.1) to see that the answer is no. For

$$F_9 = P = P(9, 2, 1) = (12)(3456789), \tag{2.3}$$

and (1.1) shows that in order to write P as a product of distinct transpositions none equal to (12), one needs at least $M(9, 2, 1) = 9$ transposition factors.

In place of $P(9, 2, 1)$, consider the more general permutations $P(n, 2, 1)$ for any $n \geq 5$. Let F_n represent a sequence of mind switches involving bodies

$1, 2, \dots, n$ where as in (2.1), the pair 1, 2 swap with each other but not with anyone else. Suppose that, as in (2.3),

$$F_n = P = P(n, 2, 1) = (12)(345 \dots n).$$

As in (2.2), we have

$$P(n, 2, 1) = F_n = Q_n := (23) \cdot (1n) \cdots (15)(14)(13) \cdot (2n), \tag{2.4}$$

and Q_n has no factor in common with F_n. Thus Q_n^{-1} provides a way to undo F_n with n switches.

Again we can apply (1.1) to see that n is the smallest possible number of switches that can undo F_n. For in order to write $P(n, 2, 1)$ as a product of distinct transpositions none equal to (12), one needs at least $M(n, 2, 1) = n$ transposition factors.

For solutions to other mind-switching problems arising from *Futurama*, see [3].

We next turn to a problem from *Stargate SG-1*. In "Holiday", a crisis is created when Ma'chello tricks Daniel into swapping minds with him. In an attempt to help their colleague, Jack and Teal'c then blunder into swapping themselves. This sequence of two mind swaps can be represented by the product

$$P = P(4, 2, 2) = (12)(34),$$

where the bodies 1,2,3,4 stand for Teal'c, Jack, Ma'chello, and Daniel, respectively. We have

$$P = (12)(34) = Q := (24)(13)(23)(14), \tag{2.5}$$

where P and Q have no factor in common. Thus Q^{-1} serves to undo P with four switches. It is easy to check directly that $P(4, 2, 2)$ cannot be undone with fewer than four switches.

In place of $P(4, 2, 2)$, consider the more general permutations $P(2r, r, r)$ for any $r \geq 2$. First suppose that $r = 3$, and write

$$P = P(6, 3, 3) = (12)(34)(56).$$

We have

$$P = (12)(34)(56) = Q_7 := (15)(25)(35)(46)(45)(16)(13). \tag{2.6}$$

The product Q_7 has no factor in common with (12)(34)(56), so Q_7^{-1} serves to undo P with seven switches.

For general $r > 3$, we can undo $P = P(2r, r, r)$ as follows. If r is even, undo the first two factors of P using four transpositions (cf. (2.5)), then repeat the process for the next two factors of P, etc. In this way, we undo P with $2r$ transpositions. If r is odd, undo the first three factors of P using seven transpositions (cf. (2.6)), then undo consecutive pairs of factors as in the case where r is even. In this way, we undo P with $2r + 1$ transpositions. In summary, for any $r \geq 2$, the product $P(2r, r, r)$ can be undone with $2r + \epsilon_r$ switches.

We can apply (1.1) to see that $2r + \epsilon_r$ is the smallest possible number of switches that can undo $P = P(2r, r, r)$. For in order to write P as a product of distinct transpositions none equal to a factor of P, one needs at least $M(2r, r, r) = 2r + \epsilon_r$ transposition factors.

3 Determination of $M(n, m, r)$

The aim of this section is to prove the formula for $M(n, m, r)$ given in (1.1). The proof depends on an elementary result in graph theory: a connected graph on N vertices has at least $N - 1$ edges [2, Theorem 11.2.1, p. 163].

First consider the case $n = 2$, and suppose $P = (12)$. We can write P as a product of five transpositions, none equal to (12), as follows:

$$P = (12) = (34)(23)(14)(13)(24).$$

Suppose for the purpose of contradiction that $P = (12)$ could be written as a product of u distinct transpositions, none equal to (12), with $u < 5$. Then by the Parity Theorem [1, pp. 82, 149], we would have $u = 3$, so that some product of four distinct transpositions would equal the identity. This is easily seen to be impossible, so the proof is complete in the case $n = 2$.

From now on, let $n > 2$. Our object is to prove that $M(n, m, r) = n - m + r + \epsilon_r$. We begin by showing that

$$M(n, m, r) \leq n - m + r + \epsilon_r. \tag{3.1}$$

Write

$$P(n, m, r) = (12)(34) \cdots (2r - 1, 2r) C_1 \cdots C_{m-r}, \tag{3.2}$$

where the C_i are disjoint cycles of length $\ell_i > 2$. Observe that

$$\sum_{i=1}^{m-r} \ell_i \;=\; n - 2r. \tag{3.3}$$

Note also that each C_i equals a product of $\ell_i - 1$ transpositions (for example, $(abcdef) = (ab)(bc)(cd)(de)(ef)$). Consequently, $P(n, m, r)$ has parity $n - m$.

First consider the case $r = 1$. By (2.4), the permutation $(12)C_1$ equals a product of $2 + \ell_1$ distinct transpositions, none equal to (12). Thus $P(n, m, 1) = (12)C_1 \cdots C_{m-1}$ equals a product of

$$2 + \ell_1 + \sum_{i=2}^{m-1} (\ell_i - 1) = n - m + 2$$

distinct transpositions, none equal to (12). This proves (3.1) in the case $r = 1$.

Next let $r > 1$. The penultimate paragraph of Section 2 shows that when $r > 1$, $Y := (12)(34) \cdots (2r - 1, 2r)$ can be written as a product of $2r + \epsilon_r$ distinct transpositions, none equal to a factor of Y. Thus $P(n, m, r)$ equals a product of

$$2r + \epsilon_r + \sum_{i=1}^{m-r} (\ell_i - 1) = n - m + r + \epsilon_r$$

distinct transpositions, none equal to a factor of $P(n, m, r)$. This completes the proof of (3.1).

It remains to prove the reverse inequality

$$M(n, m, r) \geq n - m + r + \epsilon_r. \tag{3.4}$$

We need only prove

$$M(n, m, r) \geq n - m + r; \tag{3.5}$$

indeed, (3.4) and (3.5) are equivalent by the Parity Theorem, since $P(n, m, r)$ has parity $n - m$. Hence, supposing that $P = P(n, m, r)$ equals a product Q of w distinct transpositions, none equal to a factor of P, our aim is to show that $w \geq n - m + r$.

Let G be a graph whose vertex set $V(G)$ is the set of entries in Q, and whose edges $[ij]$ correspond to the w factors (ij) of Q. Since $P = Q$, the set $V(G)$ contains all the entries in P, and $V(G)$ may contain other "outside" entries as well. For each i with $1 \leq i \leq m-r$, G has a connected component G_i such that $V(G_i)$ contains all of the entries in the cycle C_i. Let H denote the union of the components G_i. (View H as empty if $m = r$.) The set $\{1, 2, \ldots, 2r\} \subset V(G)$ can be written as a disjoint union

$$\{1, 2, \ldots, 2r\} = A \cup B, \tag{3.6}$$

where $A \subset V(H)$ and B is disjoint from $V(H)$. For the cardinalities, write $\alpha := |A|$, $\beta := |B| = 2r - \alpha$.

Since $V(H)$ contains the $n-2r$ entries in the cycles C_i, we have $|V(H)| \geq n - 2r + \alpha$. By [2, Theorem 11.2.1, p. 163], each G_i contains at least $|G_i| - 1$ edges. Therefore, since the graph H is the disjoint union of at most $m - r$ distinct components G_i, the number of edges in H is at least

$$(n - 2r + \alpha) - (m - r) = n - m + \alpha - r.$$

It follows that Q has at least $n - m + \alpha - r$ distinct transposition factors whose entries are all in $V(H)$.

If each $b \in B$ occurs at least twice as an entry in Q, then we obtain the desired lower bound

$$w \geq (n - m + \alpha - r) + \beta = n - m + r.$$

Thus suppose that B contains "singletons", i.e., elements that occur only once as entries in Q. (For example, in (2.6), Q_7 has the singleton 2.)

A singleton $b \in B$ is paired in Q with some $c \in V(G)$, that is, Q has a factor (cb). Since $b \notin V(H)$, we have $c \notin V(H)$. Let $S(c) := \{b_1, \ldots, b_k\}$ be the set of all singletons in B that are paired with c, and let

$$(cb_1),\ (cb_2),\ \ldots,\ (cb_k)$$

be transpositions appearing in that order in the factorization of Q. Since P and Q have no factor in common, the permutation P cannot map the singleton b_1 to c. Thus c occurs as an entry in Q to the left of the factor (cb_1). Similarly, P cannot map c to the singleton b_k, so c occurs as an entry in Q to the right of (cb_k). Therefore c occurs at least $k+2$ times as an entry, so that all together, the elements in $\{b_1, \ldots, b_k, c\}$ fill at least $2(k+1)$ slots in Q.

If B contains another singleton $b' \notin S(c)$, then repeat this procedure with an element $c' \in V(G)$ that is paired with b'. Note that $S(c')$ is disjoint from

$S(c)$, since an element common to both sets would have to occur at least twice as an entry in Q. Repeat the procedure again and again until all the singletons in B have been exhausted. The non-singletons in B each fill at least two slots in Q. We've thus produced a set $E = B \cup \{c, c', \dots\} \subset V(G)$, disjoint from $V(H)$, whose elements fill at least $2|E|$ slots in Q. Since $|E| \geq \beta$, we have the desired bound

$$w \geq (n - m + \alpha - r) + |E| \geq n - m + r. \quad \square$$

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