

# THE ENTROPY OF RANDOM-FREE GRAPHONS AND PROPERTIES

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**ABSTRACT.** Every graphon defines a random graph on any given number  $n$  of vertices. It was known that the graphon is random-free if and only if the entropy of this random graph is subquadratic. We prove that for random-free graphons, this entropy can grow as fast as any subquadratic function. However, if the graphon belongs to the closure of a random-free graph property, then the entropy is  $O(n \log n)$ . We also give a simple construction of a non-stepfunction random-free graphon for which this entropy is linear, refuting a conjecture of Janson.

## 1. INTRODUCTION

In recent years a theory of convergent sequences of dense graphs has been developed. One can construct a limit object for such a sequence in the form of certain symmetric measurable functions called graphons. Every graphon defines a random graph on any given number of vertices. In [HJS] several facts about the asymptotics of the entropies of these random variables are established. These results provide a good understanding of the situation when the graphon is not “random-free”. However in the case of the random-free graphons they completely trivialize. The purpose of this article is to study these entropies in the case of the random-free graphons.

**1.1. Preliminaries.** For every natural number  $n$ , denote  $[n] := \{1, \dots, n\}$ . In this paper all graphs are simple and finite. For a graph  $G$ , let  $V(G)$  and  $E(G)$ , respectively denote the set of the vertices and the edges of  $G$ . Let  $\mathcal{U}$  denote set of all graphs up to an isomorphism. Moreover, for  $n \geq 0$ , let  $\mathcal{U}_n \subset \mathcal{U}$  denote the set of all graphs in  $\mathcal{U}$  with exactly  $n$  vertices. We will usually work with labeled graphs. For every  $n \geq 1$ , denote by  $\mathcal{L}_n$  the set of all graphs with vertex set  $[n]$ .

The *homomorphism density* of a graph  $H$  in a graph  $G$ , denoted by  $t(H; G)$ , is the probability that a random mapping  $\phi : V(H) \rightarrow V(G)$  preserves adjacencies, i.e.  $uv \in E(H)$  implies  $\phi(u)\phi(v) \in E(G)$ . The *induced density* of a graph  $H$  in a graph  $G$ , denoted by  $p(H; G)$ , is the probability that a random *embedding* of the vertices of  $H$  in the vertices of  $G$  is an embedding of  $H$  in  $G$ .

We call a sequence of finite graphs  $(G_n)_{n=1}^\infty$  *convergent* if for every finite graph  $H$ , the sequence  $\{p(H; G_n)\}_{n=1}^\infty$  converges. It is not difficult to construct convergent sequences  $(G_n)_{n=1}^\infty$  such that their limits cannot be recognized as graphs, i.e. there is no graph  $G$ , with  $\lim_{n \rightarrow \infty} p(H; G_n) = p(H; G)$  for every  $H$ . Thus naturally one considers  $\overline{\mathcal{U}}$ , the completion of  $\mathcal{U}$  under this notion of convergence. It is not hard to see that  $\overline{\mathcal{U}}$  is a compact metrizable space which contains  $\mathcal{U}$  as a dense subset. The elements of the complement  $\mathcal{U}^\infty := \overline{\mathcal{U}} \setminus \mathcal{U}$  are called *graph limits*. Note that a sequence of graphs  $(G_n)_{n=1}^\infty$  converges to a graph limit  $\Gamma$  if and only if  $|V(G_n)| \rightarrow \infty$  and  $p(H; G_n) \rightarrow p(H; \Gamma)$  for every graph  $H$ . Moreover, a graph limit is uniquely determined by the numbers  $p(H; \Gamma)$  for all  $H \in \mathcal{U}$ .

It is shown in [LS06] that every graph limit  $\Gamma$  can be represented by a *graphon*, which is a symmetric measurable function  $W : [0, 1]^2 \rightarrow [0, 1]$ . The set of all graphons are denoted by  $\mathcal{W}_0$ . Given a graph  $G$  with vertex set  $[n]$ , we define the corresponding graphon  $W_G : [0, 1]^2 \rightarrow [0, 1]$  as follows. Let  $W_G(x, y) := A_G(\lceil xn \rceil, \lceil yn \rceil)$  if  $x, y \in (0, 1]$ , and if  $x = 0$  or  $y = 0$ , set  $W_G$  to 0. It is easy to see that if  $(G_n)_{n=1}^\infty$  is a graph sequence that converges to a graph limit  $\Gamma$ , then for every graph  $H$ ,

$$p(H; \Gamma) = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \prod_{uv \in E(H)} W_{G_n}(x_u, x_v) \prod_{uv \in E(H)^c} (1 - W_{G_n}(x_u, x_v)) \right],$$

where  $\{x_u\}_{u \in V(H)}$  are independent random variables taking values in  $[0, 1]$  uniformly, and  $E(H)^c = \{uv : u \neq v, uv \notin E(H)\}$ . Lovász and Szegedy [LS06] showed that for every graph limit  $\Gamma$ , there exists a graphon

$W$  such that for every graph  $H$ , we have  $p(H; \Gamma) = p(H; W)$  where

$$p(H; W) := \mathbb{E} \left[ \prod_{uv \in E(H)} W(x_u, x_v) \prod_{uv \in E(H)^c} (1 - W(x_u, x_v)) \right].$$

Furthermore, this graphon is unique in the following sense: If  $W_1$  and  $W_2$  are two different graphons representing the same graph limit, then there exists a measure-preserving map  $\sigma : [0, 1] \rightarrow [0, 1]$  such that

$$W_1(x, y) = W_2(\sigma(x), \sigma(y)), \tag{1.1}$$

almost everywhere [BCL10]. With these considerations, sometimes we shall not distinguish between the graph limits and their corresponding graphons. We define the  $\delta_1$  distance of two graphons  $W_1$  and  $W_2$  as

$$\delta_1(W_1, W_2) = \inf \|W_1 - W_2 \circ \sigma\|_1,$$

where the infimum is over all measure-preserving maps  $\sigma : [0, 1] \rightarrow [0, 1]$ .

A graphon  $W$  is called a *stepfunction*, if there is a partition of  $[0, 1]$  into a finite number of measurable sets  $S_1, \dots, S_n$  so that  $W$  is constant on every  $S_i \times S_j$ . The partition classes will be called the *steps* of  $W$ .

Let  $W$  be a graphon and  $x_1, \dots, x_n \in [0, 1]$ . The random graph  $G(x_1, \dots, x_n, W) \in \mathcal{L}_n$  is obtained by including the edge  $ij$  with probability  $W(x_i, x_j)$ , independently for all pairs  $(i, j)$  with  $1 \leq i < j \leq n$ . By picking  $x_1, \dots, x_n$  independently and uniformly at random from  $[0, 1]$ , we obtain the random graph  $G(n, W) \in \mathcal{L}_n$ . Note that that for every  $H \in \mathcal{L}_n$ ,

$$\Pr[G(n, W) = H] = p(H; W).$$

**1.2. Graph properties and Entropy.** A subset of the set  $\mathcal{U}$  is called a *graph class*. Similarly a *graph property* is a property of graphs that is invariant under graph isomorphisms. There is an obvious one-to-one correspondence between graph classes and graph properties and we will not distinguish between a graph property and the corresponding class. Let  $\mathcal{Q} \subseteq \mathcal{U}$  be a graph class. For every  $n > 1$ , we denote by  $\mathcal{Q}_n$  the set of graphs in  $\mathcal{Q}$  with exactly  $n$  vertices. We let  $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{U}}$  be the closure of  $\mathcal{Q}$  in  $\overline{\mathcal{U}}$ .

Define the *binary entropy* function  $h : [0, 1] \mapsto \mathbb{R}_+$  as  $h(x) = -x \log(x) - (1 - x) \log(1 - x)$  for  $x \in (0, 1)$  and  $h(0) = h(1) = 0$  so that  $h$  is continuous on  $[0, 1]$  where here and throughout the paper  $\log(\cdot)$  denotes the logarithm to the base 2. The *entropy* of a graphon  $W$  is defined as

$$\text{Ent}(W) := \int_0^1 \int_0^1 h(W(x, y)) dx dy.$$

Note that it follows from the uniqueness result (1.1) that entropy is a function of the underlying graph limit, and it does not depend on the choice of the graphon representing it. It is shown in [Ald85] and [Jan, Theorem D.5] that

$$\lim_{n \rightarrow \infty} \frac{\text{Ent}(G(n, W))}{\binom{n}{2}} = \text{Ent}(W). \tag{1.2}$$

A graphon is called *random-free* if it is  $\{0, 1\}$ -valued almost everywhere. Note that a graphon  $W$  is random-free if and only if  $\text{Ent}(W) = 0$ , which by (1.2) is equivalent to  $\text{Ent}(G(n, W)) = o(n^2)$ . Our first theorem shows that this is sharp in the sense that the growth of  $\text{Ent}(G(n, W))$  for random-free graphons  $W$  can be arbitrarily close to quadratic.

**Theorem 1.1.** *Let  $\alpha : \mathbb{N} \rightarrow \mathbb{R}_+$  be a function with  $\lim_{n \rightarrow \infty} \alpha(n) = 0$ . Then there exists a random-free graphon  $W$  such that  $\text{Ent}(G(n, W)) = \Omega(\alpha(n)n^2)$ .*

A graph property  $\mathcal{Q}$  is called *random-free* if every  $W \in \overline{\mathcal{Q}}$  is random-free. Our next theorem shows that in contrast to Theorem 1.1, when a graphon  $W$  is the limit of a sequence of graphs with a random-free property, then  $\text{Ent}(G(n, W))$  cannot grow faster than  $O(n \log n)$ .

**Theorem 1.2.** *Let  $\mathcal{Q}$  be a random-free property, and let  $W$  be the limit of a sequence of graphs in  $\mathcal{Q}$ . Then  $\text{Ent}(G(n, W)) = O(n \log n)$ .*

*Remark 1.3.* We defined  $G(n, W)$  as a *labeled* graph in  $\mathcal{L}_n$ . Both Theorems 1.1 and 1.2 remain valid if we consider the random variable  $G_u(n, W)$  taking values in  $\mathcal{U}_n$  obtained from  $G(n, W)$  by forgetting the labels. Indeed,  $\text{Ent}(G_u(n, W)) = \text{Ent}(G(n, W)) - \text{Ent}(G(n, W) \mid G_u(n, W))$  and  $\text{Ent}(G(n, W) \mid G_u(n, W) = H) = O(n \log n)$  for every  $H \in \mathcal{U}_n$ . It follows that

$$\text{Ent}(G(n, W)) - O(n \log n) \leq \text{Ent}(G_u(n, W)) \leq \text{Ent}(G(n, W)).$$

## 2. PROOF OF THEOREM 1.1

For every positive integer  $m$ , let  $F_m$  denote the unique bigraph  $([m], [2^m], E)$  with the property that the vertices in  $[2^m]$  all have different sets of neighbors. The *transversal-uniform* graph is the unique graph (up to an isomorphism) with vertex set  $\mathbb{N}$  which satisfies the following property. The vertices are partitioned into sets  $\{A_i\}_{i=1}^\infty$  with  $\log |A_i| = \sum_{j=1}^{i-1} |A_{j-1}|$ . There are no edges inside  $A_i$ 's, and for every  $i$ , the bigraph induced by  $(\cup_{j=1}^{i-1} A_j, A_i)$  is isomorphic to  $F_{\sum_{j=1}^{i-1} |A_j|}$ .

Let  $\mathcal{I} = \{I_i\}_{i \in \mathbb{N}}$  be a partition of  $[0, 1]$  into intervals. We define its corresponding *transversal-uniform* graphon  $W_{\mathcal{I}}$  by assigning weights  $|I_i|/|A_i|$  to all the vertices in  $A_i$  in the transversal-uniform graph  $G_U$  described above. More precisely, we partition each  $I_i$  into  $|A_i|$  equal size intervals (corresponded with elements in  $A_i$ ), and mapping all the points in each of these subintervals to its corresponding vertex in  $A_i$ . This measurable surjection  $\pi_{\mathcal{I}} : [0, 1] \rightarrow \mathbb{N}$ , together with the transversal-uniform graph described above defines the transversal-uniform graphon  $W_{\mathcal{I}}$  by setting

$$W_{\mathcal{I}}(x, y) = \begin{cases} 1 & \text{if } \pi(x)\pi(y) \in E(G_U), \\ 0 & \text{if } \pi(x)\pi(y) \notin E(G_U). \end{cases}$$

Note that by construction  $W_{\mathcal{I}}$  has the following property. Let  $s < k$  be positive integers, and  $x_1, \dots, x_s \in \cup_{i < k} I_i$  belong to pairwise distinct intervals in  $\mathcal{I}$ . For every  $f : [s] \rightarrow \{0, 1\}$ , we have

$$\Pr[\forall i, W_{\mathcal{I}}(x_i, y) = f(i) \mid y \in I_k] = \frac{1}{2^s},$$

where  $y$  is a random variable taking values uniformly in  $[0, 1]$ . It follows that for every graph  $H$  on  $s$  vertices,

$$\Pr[G(x_1, \dots, x_s, W_{\mathcal{I}}) = H \mid \forall i, x_i \in I_{k_i}] = \frac{1}{2^{\binom{s}{2}}}, \tag{2.1}$$

where  $x_1, \dots, x_s$  are now i.i.d. random variables taking values uniformly in  $[0, 1]$ , and  $k_1, k_2, \dots, k_s$  are distinct natural numbers.

We translate (2.1) into a lower bound on (conditional) entropy of transversal-uniform graphons. First we need a simple lemma.

**Lemma 2.1.** *Let  $W_{\mathcal{I}}$  be a transversal-uniform graphon, and  $\phi : [n] \rightarrow [0, 1]$  be a uniformly random map. For every  $\rho : [n] \rightarrow \mathbb{N}$ , we have*

$$\text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}}) \mid \pi_{\mathcal{I}} \circ \phi = \rho) \geq \binom{|\text{Im}(\rho)|}{2}.$$

*Proof.* Pick a set of representatives  $K \subseteq [n]$  so that  $\rho|_K : K \rightarrow \text{Im}(\rho)$  is a bijection. Equation (2.1) implies that for every graph  $H$  with  $V(H) = K$ ,

$$\Pr[G(\phi(1), \dots, \phi(n), W_{\mathcal{I}})[K] = H \mid \pi_{\mathcal{I}} \circ \phi = \rho] = \frac{1}{2^{\binom{|\text{Im}(\rho)|}{2}}}.$$

Therefore,

$$\text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}}) \mid \pi_{\mathcal{I}} \circ \phi = \rho) \geq \text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}})[K] \mid \pi_{\mathcal{I}} \circ \phi = \rho) = \binom{|\text{Im}(\rho)|}{2}.$$

□

In the proof of Theorem 1.1 below we will make use of the following well-known inequality about conditional entropy. For discrete random variables  $X$  and  $Y$ ,

$$\text{Ent}(X | Y) := \sum_{y \in \text{supp}(Y)} \Pr[Y = y] \text{Ent}(X | Y = y) \leq \text{Ent}(X). \quad (2.2)$$

*Proof of Theorem 1.1.* For every positive integer  $k$ , define

$$g_k := \max \{ \{2^{k+5}\} \cup \{n \mid \alpha(n) > 2^{-2k-9}\} \}.$$

The numbers  $g_k$  are well-defined, as the condition  $\lim_{n \rightarrow \infty} \alpha(n) = 0$  implies that the set  $\{n \mid \alpha(n) > 2^{-2k-9}\}$  is finite. Define the sums  $G_k := \sum_{i=1}^k g_k$ , and set  $\beta_i = \frac{1}{g_k 2^k}$  for all the  $g_k$  indices  $i \in (G_{k-1}, G_k]$ . Let  $\mathcal{I} = \{I_i\}_{i \in \mathbb{N}}$  be a partition of  $[0, 1]$  into intervals with  $|I_i| = \beta_i$ , and let  $W_{\mathcal{I}}$  be the corresponding transversal-uniform graphon.

Consider a sufficiently large  $n \in \mathbb{N}$ , and let  $k \in \mathbb{N}$  be chosen to be maximum so that  $2^{k+4} \leq n$  and  $\alpha(n) \leq 2^{-2k-7}$ . We have  $n < 2^{k+5}$  or  $\alpha(n) > 2^{-2k-9}$ . Therefore  $n \leq g_k$  by the definition of  $g_k$ . Let  $\phi : [n] \rightarrow [0, 1]$  be random and uniform. By Lemma 2.1, for any fixed  $\rho : [n] \rightarrow \mathbb{N}$ , we have

$$\text{Ent}(G(\phi(1), \dots, \phi(n), W_{\mathcal{I}}) | \pi_{\mathcal{I}} \circ \phi = \rho) \geq \binom{|\text{Im}(\rho)|}{2}.$$

Thus

$$\text{Ent}(G(n, W_{\mathcal{I}})) \geq \text{Ent}(G(n, W_{\mathcal{I}}) | \pi_{\mathcal{I}} \circ \phi) \geq \Pr[|\text{Im}(\pi_{\mathcal{I}} \circ \phi)| \geq n2^{-k-2}] \binom{n2^{-k-2}}{2}. \quad (2.3)$$

Define the random variable  $X := |\text{Im}(\pi_{\mathcal{I}} \circ \phi) \cap (G_{k-1}, G_k]| \leq |\text{Im}(\pi_{\mathcal{I}} \circ \phi)|$ . We have

$$\mathbb{E}[X] = \sum_{i \in (G_{k-1}, G_k]} \Pr[\phi^{-1}(I_i) \neq \emptyset] = \sum_{i \in (G_{k-1}, G_k]} (1 - (1 - \beta_i)^n) = g_k \left( 1 - \left( 1 - \frac{1}{g_k 2^k} \right)^n \right) \geq n2^{-k-1},$$

where we used the fact that  $g_k 2^k \geq 2n$  and that  $(1 - x)^n \leq 1 - nx + n^2 x^2 \leq 1 - nx/2$  for  $x \in [0, 1/2n]$ . As the events  $\phi^{-1}(I_i) \neq \emptyset$  and  $\phi^{-1}(I_j) \neq \emptyset$  are negatively correlated for  $i \neq j$ , we have  $\text{Var}[X] \leq \mathbb{E}[X]$ . Hence by Chebyshev's inequality

$$\begin{aligned} \Pr[|\text{Im}(\pi_{\mathcal{I}} \circ \phi)| \geq n2^{-k-2}] &\geq \Pr[X \geq n2^{-k-2}] \geq 1 - \Pr\left[|X - \mathbb{E}[X]| \geq \frac{\mathbb{E}[X]}{2}\right] \\ &\geq 1 - \frac{4\text{Var}[X]}{\mathbb{E}[X]^2} \geq 1 - \frac{4}{n2^{-k-2}} \geq \frac{1}{2}. \end{aligned}$$

Substituting in (2.3) we obtain

$$\text{Ent}(G(n, W_{\mathcal{I}})) \geq \frac{1}{2} \binom{n2^{-k-2}}{2} \geq n^2 2^{-2k-7} \geq \alpha(n) n^2,$$

as desired.  $\square$

### 3. PROOF OF THEOREM 1.2

In [LS] Lovász and Szegedy obtained a combinatorial characterization of random-free graph properties. To state this result it is convenient to distinguish between bipartite graphs and bigraphs. A *bipartite* graph is a graph  $(V, E)$  whose node set has a partition into two classes such that all edges connect nodes in different classes. A *bigraph* is a triple  $(U_1, U_2, E)$  where  $U_1$  and  $U_2$  are finite sets and  $E \subseteq U_1 \times U_2$ . So a bipartite graph becomes a bigraph if we fix a bipartition and specify which bipartition class is first and second. On the other hand, if  $F = (V, E)$  is a graph, then  $(V, V, E')$  is an associated bigraph, where  $E' = \{(x, y) : xy \in E\}$ .

If  $G = (V, E)$  is a graph, then an induced sub-bigraph of  $G$  is determined by two (not necessarily disjoint) subsets  $S, T \subseteq V$ , and its edge set consists of those pairs  $(x, y) \in S \times T$  for which  $xy \in E$  (so this is an induced subgraph of the bigraph associated with  $G$ ).

For a bigraph  $H = (U_1, U_2, E)$  and a graphon  $W$ , analogous to the definition of the induced density of a graph in a graphon, we define

$$p^b(H; W) = \mathbb{E} \left[ \prod_{\substack{u \in U_1, v \in U_2 \\ uv \in E}} W(x_u, y_v) \prod_{\substack{u \in U_1, v \in U_2 \\ uv \in (U_1 \times U_2) \setminus E}} (1 - W(x_u, y_v)) \right],$$

where  $\{x_u\}_{u \in U_1}, \{y_v\}_{v \in U_2}$  are independent random variables taking values in  $[0, 1]$  uniformly. Now we are ready to state Lovász and Szegedy's characterization of random-free graph properties.

**Theorem 3.1.** [LS] *A graph property  $\mathcal{Q}$  is random-free if and only if there exists a bigraph  $H$  such that  $p^b(H; W) = 0$  for all  $W \in \overline{\mathcal{Q}}$ .*

The following lemma is due to Alon, Fischer, and Newman (See [AFN07, Lemma 1.6]).

**Lemma 3.2.** [AFN07] *Let  $k$  be a fixed integer and let  $\delta > 0$  be a small real. For every graph  $G$ , either there exists stepfunction graphon  $W'$  with  $r \leq \left(\frac{k}{\delta}\right)^{O(k)}$  steps such that  $\delta_1(W_G, W') \leq \delta$ , or for every bigraph  $H$  on  $k$  vertices  $p^b(H; G) \geq \left(\frac{\delta}{k}\right)^{O(k^2)}$ .*

Every random-free graphon  $W$  can be approximated arbitrarily well in the  $\delta_1$  distance with  $W_G$  for some graph  $G$ , and furthermore, for every fixed  $H$ , the function  $p^b(H, \cdot)$  is continuous in the  $\delta_1$  distance. Thus Lemma 3.2 can be generalized to random-free graphons.

**Corollary 3.3.** *Let  $k$  be a fixed integer and let  $\delta > 0$  be a small real. For every random-free graphon  $W$ , either there exists a stepfunction graphon  $W'$  with  $r \leq \left(\frac{k}{\delta}\right)^{O(k)}$  steps such that  $\delta_1(W, W') \leq \delta$ , or for every bigraph  $H$  on  $k$  vertices  $p^b(H; G) \geq \left(\frac{\delta}{k}\right)^{O(k^2)}$ .*

Next we will prove two simple lemmas about entropy.

**Lemma 3.4.** *Let  $\mu_1$  and  $\mu_2$  be two discrete probabilistic distributions on a finite set  $\Omega$ . Then*

$$|\text{Ent}(\mu_1) - \text{Ent}(\mu_2)| \leq |\Omega| h \left( \frac{\|\mu_1 - \mu_2\|_1}{|\Omega|} \right).$$

*Proof.* Define  $0 \log 0 := \lim_{x \rightarrow 0} x \log x = 0$ . By taking the derivative with respect to  $x$ , for fixed  $d$  we see that  $(x + d) \log(x + d) - x \log x$  is monotone for  $0 \leq x \leq 1 - d$ . Therefore, for  $x_1, x_2 \in [0, 1]$  we have

$$|x_2 \log x_2 - x_1 \log x_1| \leq \max\{-|x_2 - x_1| \log |x_2 - x_1|, -(1 - |x_2 - x_1|) \log(1 - |x_2 - x_1|)\} \leq h(|x_2 - x_1|).$$

Thus

$$\begin{aligned} |\text{Ent}(\mu_1) - \text{Ent}(\mu_2)| &= \left| \sum_{x \in \Omega} \mu_1(x) \log \mu_1(x) - \mu_2(x) \log \mu_2(x) \right| \\ &\leq \sum_{x \in \Omega} h(|\mu_1(x) - \mu_2(x)|) \leq |\Omega| h \left( \frac{\|\mu_1 - \mu_2\|_1}{|\Omega|} \right), \end{aligned}$$

where the last inequality is by concavity of the binary entropy function  $h$ . □

**Lemma 3.5.** *Let  $W_1$  and  $W_2$  be two graphons, and let  $\mu_1$  and  $\mu_2$  be the probability distributions on  $\mathcal{L}_n$  induced by  $G(n, W_1)$  and  $G(n, W_2)$ , respectively. Then*

$$\|\mu_1 - \mu_2\|_1 \leq n^2 \delta_1(W_1, W_2).$$

*Proof.* Let  $x_1, \dots, x_n$  be i.i.d. uniform random variables with values in  $[0, 1]$ . Note

$$\begin{aligned} \|\mu_1 - \mu_2\|_1 &\leq \Pr[G(x_1, \dots, x_n, W_1) \neq G(x_1, \dots, x_n, W_2)] \\ &\leq \mathbb{E} \left[ \sum_{i \neq j} |W_1(x_i, x_j) - W_2(x_i, x_j)| \right] \leq n^2 \|W_1 - W_2\|_1. \end{aligned}$$

□

*Proof of Theorem 1.2.* Since  $\mathcal{Q}$  is random-free, by Theorem 3.1, there exists a bigraph  $H$  such that  $p^b(H; W) = 0$  for all  $W \in \overline{\mathcal{Q}}$ . Applying Corollary 3.3 with  $\delta = 1/n^5$  shows that there exists a stepfunction graphon  $W'$  with  $n^{O(1)}$  steps satisfying  $\|W - W'\|_1 \leq \delta$ . Then since  $|\mathcal{L}_n| \leq 2^{n^2}$ , Lemmas 3.4 and 3.5 imply

$$\begin{aligned} |\text{Ent}(G(n, W')) - \text{Ent}(G(n, W))| &\leq 2^{n^2} h\left(\frac{n^2\delta}{2^{n^2}}\right) \\ &= -2^{n^2} \left( \frac{n^2\delta}{2^{n^2}} \log\left(\frac{n^2\delta}{2^{n^2}}\right) + \left(1 - \frac{n^2\delta}{2^{n^2}}\right) \log\left(1 - \frac{n^2\delta}{2^{n^2}}\right) \right) \\ &\leq n^4\delta + n^2\delta(-2\log n - \log \delta) + 2^{n^2} \cdot 2 \frac{n^2\delta}{2^{n^2}} = o(1). \end{aligned}$$

Since  $W'$  is random-free and it has  $n^{O(1)}$  steps,  $|\text{supp}(G(n, W'))| = n^{O(n)}$ . Consequently  $\text{Ent}(G(n, W')) = O(n \log n)$ .  $\square$

#### 4. CONCLUDING REMARKS

**1.** Note that if  $W$  is a random-free stepfunction, then  $\text{Ent}(G(n, W)) = O(n)$ . In [Jan] it is conjectured that the converse is also true. That is  $\text{Ent}(G(n, W)) = O(n)$  if and only if  $W$  is equivalent to a random-free stepfunction. The following simple example disproves this conjecture.

Let  $\mu$  be the probability distribution on  $\mathbb{N}$  defined by  $\mu(\{i\}) = 2^{-i}$ . Consider the random variable  $X = (X_1, \dots, X_n) \in \mathbb{N}^n$  where  $X_i$  are i.i.d. random variables with distribution  $\mu$ . We have  $\text{Ent}(X_i) = \sum_{i=1}^{\infty} 2^{-i} i = 2$ . Hence  $\text{Ent}(X) \leq \sum \text{Ent}(X_i) = 2n$ .

Partition  $[0, 1]$  into intervals  $\{I_i\}_{i=1}^{\infty}$  where  $|I_i| = 2^{-i}$ . Let  $W$  be the graphon that is constant 1 on  $\cup_{i=1}^{\infty} I_i \times I_i$  and 0 everywhere else. Note that

$$\text{Ent}(G(n, W)) \leq \text{Ent}(X) \leq 2n.$$

Therefore  $G(n, W)$  has linear entropy.

It remains to verify that  $W$  is not equivalent to a stepfunction. This follows immediately from the fact that  $W$  has infinite rank as a kernel. It can also be verified in a more combinatorial way: A *homogenous set* of vertices in a graph  $H$  is a set of vertices which are either all pairwise adjacent to each other, or all pairwise non-adjacent. If  $W$  is equivalent to a step-function with  $k$  steps, then every  $H \in \text{supp}(G(n, W))$  clearly contains a homogenous set of size at least  $n/k$ . On the other hand, if  $H \in \mathcal{L}_{n^2}$  is a disjoint union of  $n$  complete graphs on  $n$  vertices, then the largest homogenous set in  $H$  has size  $n$ , but  $H \in \text{supp}(G(n^2, W))$  by construction.

**2.** Theorem 1.2 shows that when  $W$  is a limit of a random-free property, then the entropy of  $G(n, W)$  is small. However, the support of  $G(n, W)$  can be comparatively large. For every  $\epsilon > 0$ , we construct examples for which  $\log(|\text{supp}(G(n, W))|) = \Omega(n^{2-\epsilon})$ . Note that Theorem 1.2 implies that  $G(n, W)$  is far from being uniform on the support in these examples, as the entropy of a uniform random variable with support of size  $2^{\Omega(n^{2-\epsilon})}$  is  $\Omega(n^{2-\epsilon})$ .

Let us now describe the construction. Let  $\mathcal{Q}$  be the set of graphs that do not contain  $K_{t,t}$  as a subgraph. Partition  $[0, 1]$  into intervals  $\{S_i\}_{i=1}^{\infty}$  with non-zero lengths, and let  $\{H_i\}_{i=1}^{\infty}$  be an enumeration of graphs in  $\mathcal{Q}$ . Define  $W$  to be the graphon that is 0 on  $S_i \times S_j$  for  $i \neq j$ , and is equivalent to  $W_{H_i}$  (scaled properly) on  $S_i \times S_i$ . By construction  $p(H; W) > 0$  if  $H \in \mathcal{Q}$ . Thus  $|\text{supp}(G(n, W))| \geq |\mathcal{Q}_n|$ . Since there exists  $K_{t,t}$ -free graphs with  $n^{2-2/t}$  edges (See e.g. [Bol78, p. 316, Thm. VI.2.10]), we have  $|\mathcal{Q}_n| \geq 2^{n^{2-2/t}}$ .

It remains to show that  $W$  is a limit of graphs in some random-free property. Unfortunately,  $W \notin \overline{\mathcal{Q}}$ . We construct a larger random-free property  $\mathcal{Q}'$  so that  $W \in \overline{\mathcal{Q}'}$  as follows.

Fix a bigraph  $B$ , so that the corresponding graph contains  $K_{t,t}$  as a subgraph and is connected. Suppose further that no two vertices of  $B$  have the same neighborhood. Note that such a bigraph trivially exists. For example, one can take  $B = (V_1 \cup U_1, V_2 \cup U_2, E)$  so that  $V_1, U_1, V_2, U_2$  are disjoint sets of size  $t$ , every vertex of  $V_1$  is joined to every vertex of  $V_2$ , and the edges between  $V_1$  and  $U_2$ , as well as the edges between  $U_1$  and  $V_2$ , form a matching of size  $t$ . Let  $\mathcal{Q}' \supseteq \mathcal{Q}$  be the set of graphs not containing  $B$  as an induced sub-bigraph. Then  $\mathcal{Q}'$  is random-free by Theorem 3.1, as  $p^b(B, W') = 0$  for every  $W' \in \overline{\mathcal{Q}'}$ .

Let  $r = |V(B)|$  and suppose that  $G = G(x_1, x_2, \dots, x_r, W)$  contains  $B$  as an induced sub-bigraph. Then there exists  $i$  so that  $x_1, x_2, \dots, x_r \in S_i$ , as  $G$  is connected. It follows further that  $G$  is an induced subgraph of  $H_i$ , as no two vertices of  $G$  have the same neighborhood. Thus  $G$  contains no  $K_{t,t}$  subgraph, contradicting our assumption that  $G$  contains  $B$ . We conclude that  $\text{supp}(G(n, W)) \subseteq \mathcal{Q}'$  for every positive integer  $n$ . By [LS06, Lemma 2.6] the sequence  $\{G(n, W)\}_{n=1}^\infty$  converges to  $W$  with probability one. Thus  $W \in \overline{\mathcal{Q}'}$ , as desired.

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