Inferential or Differential: Privacy Laws Dictate

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ABSTRACT

So far, privacy models follow two paradigms. The first paradigm, termed *inferential privacy* in this paper, focuses on the risk due to statistical inference of sensitive information about a target record from other records in the database. The second paradigm, known as *differential privacy*, focuses on the risk to an individual when included in, versus when not included in, the database. The contribution of this paper consists of two parts. The first part presents a critical analysis on differential privacy with two results: (i) the differential privacy mechanism does not provide inferential privacy, (ii) the impossibility result about achieving Dalenius's privacy goal [5] is based on an adversary simulated by a Turing machine, but a *human* adversary may behave differently; consequently, the practical implication of the impossibility result remains unclear. The second part of this work is devoted to a solution addressing three major drawbacks in previous approaches to inferential privacy: lack of flexibility for handling variable sensitivity, poor utility, and vulnerability to auxiliary information.

1. INTRODUCTION

There has been a significant interest in the analysis of data sets whose individual records are too sensitive to expose directly. Examples include medical records, financial data, insurance data, web query logs, user rating data for recommender systems, personal data from social networks, etc. Data of this kind provide rich information for data analysis in a variety of important applications, but access to such data may pose a significant risk to individual privacy, as illustrated in the following example.

EXAMPLE 1. *A hospital maintains an online database for answering count queries on medical data like the table T in Table 1. T contains three columns, Gender, Zipcode, and Disease, where Disease is a sensitive attribute. Suppose that an adversary tries to infer the disease of an individual Alice, with the background knowledge that Alice, a female living in the area with Zipcode 61434, has a record in T. The adversary issues the following two queries Q_1 and Q_2:*

Q_1: SELECT COUNT(*) FROM T WHERE Gender=F AND Zipcode=61434

Q_2: SELECT COUNT(*) FROM T WHERE Gender=F AND Zipcode=61434 AND Disease=HIV

Each query returns the number of participants (records) who match the description in the WHERE clause. Suppose that the answers for Q_1 and Q_2 are x and y, respectively. The adversary then estimates that Alice has HIV with probability y/x, and if y/x and x are "sufficiently large", there will be a privacy breach.

Gender	Zipcode	Disease
M	54321	Brain Tumor
M	54322	Indigestion
F	61234	Cancer
F	61434	HIV
...	...	...

Table 1: A table T

1.1 Inferential vs Differential

In the above example, the adversary infers that the rule

$$(Gender = F \wedge Zipcode = 61434) \rightarrow (Disease = HIV)$$

holds with the probability y/x and that Alice has HIV with the probability y/x, assuming that the (diseases of) records follow some underlying probability distribution. This type of reasoning, which learns information about *one record* from the statistics of *other records*, is found in many advanced applications such as recommender systems, prediction models, viral marketing, social tagging, and social networks. The same technique could be misused to infer sensitive information about an individual like in the above example. According to the *Privacy Act* of Canada, publishing the above query answers would breach Alice's privacy because they disclose Alice's disease with a high accuracy. In this paper, *inferential privacy* refers to the requirement of limiting the statistical inference of sensitive information about a target record from other records in the database. See [1] for a list of works in this field.

One recent breakthrough in the study of privacy preservation is *differential privacy* [5][7]. In an "impossibility result", the authors of [5][7] showed that it is impossible to achieve Dalenius's absolute privacy goal for statistical databases: any-

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thing that can be learned about a respondent from the statistical database should be learnable without access to the database. Instead of limiting what can be learnt about one record from other records, the differential privacy mechanism hides the presence or absence of a participant in the database,

by producing noisy query answers such that the distribution of query answers changes very little when the database differs in any *single* record. The following definition is from [4].

DEFINITION 1. *A randomized function K gives ε-differential privacy if for all data sets T and T' differing on at most one record, for all queries Q, and for all outputs x, $Pr[K(T,Q) = x] \leq exp(\varepsilon)Pr[K(T',Q) = x]$.*

With a small ε, the presence or absence of an individual is hidden because T and T' are almost equally likely to be the underlying database that produces the final output of the query. Some frequently cited claims of the differential privacy mechanism are that it provides privacy without any assumptions about the data and that it protects against arbitrary background information. But there is no free lunch in data privacy, as pointed out by Kifer and Machanavajjhala recently [14]. Their study shows that assumptions about the data and the adversaries are required if hiding the *evidence* of participation, instead of the presence/absence of records in the database, is the privacy goal, which they argue should be a major privacy definition.

1.2 Contributions

The contribution of this paper consists of two parts. In the first part, we argue that differential privacy is insufficient because it does not provide inferential privacy. We present two specific results:

- (Section 2.1) Using a *differential inference theorem*, we show that the noisy query answers returned by the differential privacy mechanism may derive an inference probability that is arbitrarily close to the inference probability obtained from the noise-free query answers. This study suggests that providing inferential privacy remains a meaningful research problem, despite the protection of differential privacy.
- (Section 2.2) While the impossibility result in [5] is based on an adversary simulated by a Turing machine, a *human* adversary may behave differently when evaluating the sensitivity of information. We use the Terry Gross example, which is a key motivation of differential privacy, to explain this point. This study suggests that the practical implication of the impossibility result remains unclear.

Given that inferential privacy remains relevant, the second part of this work is devoted to stronger solutions for inferential privacy. Previous approaches suffer from three major limitations. Firstly, most solutions are unable to handle sensitive values that have skewed distribution and varied sensitivity. For example, with the Occupation attribute in the Census data (Section 7) having the minimum and maximum frequency of 0.18% and 7.5%, the maximum ℓ-diversity [19] that can be provided is 13-diversity because of the eligibility requirement $1/\ell \geq 7.5\%$ [22]. Therefore, it is impossible to protect the infrequent items at the tail of the distribution or more sensitive items by a larger ℓ-diversity, say 50-diversity, which is more than 10 times the prior 0.18%. Secondly, even if it is possible to achieve such ℓ-diversity, enforcing ℓ-diversity with a large ℓ across *all* sensitive values leads to a large information loss. Finally, previous solutions are vulnerable to additional auxiliary information [21][13][17]. We address these issues in three steps.

- (Section 3) To address the first two limitations in the above, we consider a sensitive attribute with domain values $x_1, \cdots, x_m$ such that each x_i has a different sensitivity, thus, a tolerance f'_i on inference probability. We consider a bucketization problem in which buckets of *different* sizes can be formed to accommodate different requirements f'_i. The goal is to find a collection of buckets for a given set of records so that a notion of information loss related to bucket size is minimized and the privacy constraint f'_i of all x_i's is satisfied.
- (Sections 4, 5, and 7)

  We present an efficient algorithm for the case of two distinct bucket sizes (but many buckets) with *guaranteed optimality*, and a heuristic algorithm for the general case. The empirical study on real life data sets shows that both solutions are good approximations of optimal solutions in the general case and better deal with a sensitive attribute of skewed distribution and varied sensitivity.
- (Section 6) We adapt our solutions to guard against two previously identified strong attacks, corruption attack [21] and negative association attack [13][17] (see more details in Section 6).

1.3 Related Work

Limiting statistical disclosure has been a topic extensively studied in the field of statistical databases, see [1] for a list of works. This problem was recently examined in the context of privacy preserving data publishing and some representative privacy models include ρ_1-ρ_2 privacy [9], ℓ-diversity principle [19], and t-closeness[16]. All of these works assume uniform sensitivity across all sensitive values. One exception is the personalized privacy in [23] where a record owner can specify his/her privacy threshold. Another exception is [18] where each sensitive value may have a different privacy setting. To achieve the privacy goal, these works require a taxonomy of domain values to generalize the attributes, thus, cannot be applied if such taxonomy is not available. The study in [22] shows that generalized attributes are not useful for count queries on raw values. Dealing with auxiliary information is a hard problem in data privacy [21][13][17], and so far there is little satisfactory solution.

There have been a great deal of works in differential privacy since the pioneer work [7][5]. This includes, among others, contingency table releases [2], estimating the degree distribution of social networks [11], histogram queries [12] and the number of permissible queries [24]. These works are concerned with applications of differential privacy in various scenarios. Unlike previous works, the authors of [14] argue that hiding the evidence of participation, instead of the presence/absence of records in the database, should be a major privacy definition, and this privacy goal cannot be achieved with making assumptions about the data and the adversaries.

2. ANALYZING DIFFERENTIAL PRIVACY

This section presents a critical analysis on the differential privacy mechanism. In Section 2.1 we show that the differential privacy mechanism allows violation of inferential privacy. In Section 2.2 we argue that a human adversary may behave differently from some assumptions made in the impossibility result of [5], thus, the practical implication of the impossibility result remains unclear.

2.1 On Violating Inferential Privacy

One popularized claim of the differential privacy mechanism is that it protects an individual's information even if an attacker knows about all other individuals in the data. We quote the original discussion from [3] (pp 3):

> "If there is information about a row that can be learned from other rows, this information is not truly under the control of that row. Even if the row in question were to sequester itself away in a high mountaintop cave, information about the row that can be gained from the analysis of other rows is still available to an adversary. It is for this reason that we focus our attention on those inferences that can be made about rows without the help of others."

In other words, the differential privacy framework does not consider violation to inferential privacy and the reason is that it is not under the control of the target row. Two points need clarification. Firstly, a user submits her sensitive data to an organization because she trusts that the organization will do everything possible to protect her sensitive information; indeed, the data publisher has full control in how to release the data or query answers in order to protect individual privacy. Secondly, learning information about one record from other records could pose a risk to an individual if the learnt information is accurate about the individual. This type of learning assumes that records follow some underlying probability distribution, which is widely adapted by prediction models in many real applications. Under this assumption, suppose Q_1 and Q_2 in Example 1 have the answers $x = 100$ and $y = 99$, even if Alice's record is removed from the database, it is still valid to infer that Alice has HIV with a high probability.

Next, we show that even if the differential privacy mechanism adds noises to the answers for queries Q_1 and Q_2, Alice's disease can still be inferred using the noisy answers.

Let x and y be the true answers to Q_1 and Q_2. We assume that x and y are non-zero. The differential privacy mechanism will return the noisy answers $X = x+\xi_1$ and $Y = y+\xi_2$ for Q_1 and Q_2, after adding noises ξ_1 and ξ_2. Consider the most used Laplace distribution $Lap(b) = \frac{1}{2b}exp(-|\xi|/b)$ for the noise ξ, where b is the scale factor. The mean $E[\xi]$ is zero and the variance $var[\xi]$ is $2b^2$. The next theorem is due to [5].

THEOREM 1. *[5] For a count query Q, the mechanism K that adds independently generated noise ξ with distribution $Lap(1/\varepsilon)$ to the output enjoys ε-differential privacy.*

The next theorem shows that Y/X is a good approximation of y/x.

THEOREM 2 (DIFFERENTIAL INFERENCE THEOREM). *Given two queries Q_1 and Q_2 as above, let x and y be the true answers and let X and Y be the answers returned by the ε-differential privacy mechanism. $E[\frac{Y}{X}] = \frac{y}{x}(1 + \frac{2b^2}{x^2})$ and $var[\frac{Y}{X}] = \frac{2b^2}{x^2}(1 + (\frac{y}{x})^2)$, where $b = 1/\varepsilon$.*

PROOF. Using the Taylor expansion technique [8] [20], the mean $E[\frac{Y}{X}]$ and variance $var[\frac{Y}{X}]$ of Y/X can be approximated as follows:

$$E[\frac{Y}{X}] \simeq \frac{E[Y]}{E[X]} + \frac{cov[X,Y]}{E[X]^2} + \frac{var[X]E[Y]}{E[X]^3}$$

$$var[\frac{Y}{X}] \simeq \frac{var[Y]}{E[X]^2} - \frac{2E[Y]}{E[X]^3}cov[X,Y] + \frac{E[Y]^2}{E[X]^4}var[X]$$

$E[X]$ and $E[Y]$ are equal to the true answers x and y of Q_1 and Q_2. $var[X]$ and $var[Y]$ are $2b^2$ for $Lap(b)$. $cov[X,Y] = cov[x+\xi_1, y+\xi_2] = cov[\xi_1, \xi_2]$. Since ξ_1 and ξ_2 are unrelated, $cov[\xi_1, \xi_2] = 0$. Simplifying the above equations, we get $E[\frac{Y}{X}]$ and $var[\frac{Y}{X}]$ as required.

□

The next corollary follows from the fact that $\frac{y}{x} \leq 1$ and b is a constant for a given ε-differential privacy mechanism K.

COROLLARY 1. *Let X, Y be defined as in Theorem 2. As the query size x for Q_1 increases, $E[\frac{Y}{X}]$ gets arbitrarily close to $\frac{y}{x}$ and $var[\frac{Y}{X}]$ gets arbitrarily close to zero.*

Corollary 1 suggests that Y/X, where Y and X are the noisy query answers returned by the differential privacy mechanism, can be a good estimate of the inference probability y/x for a large query answer x. For example, for $\varepsilon = 0.1$ and $x = 100$, $\frac{2b^2}{x^2} = 0.02$, and following Theorem 2, $E[\frac{Y}{X}]$ is 1.02 times $\frac{y}{x}$; if $x = 1000$, $E[\frac{Y}{X}]$ is 1.0002 times $\frac{y}{x}$. If y/x is high, inferential privacy is violated. Note that $var[\frac{Y}{X}]$ is small in these cases.

2.2 On The Impossibility Results

A key motivation behind differential privacy is the impossibility result about the Dalenius's privacy goal [5]. Intuitively, it says that for any privacy mechanism and any distribution satisfying certain conditions, there is always some particular piece of auxiliary information, z, so that z alone is useless to an adversary who tries to win, while z in combination with access to the data through the privacy mechanism permits the adversary to win with probability arbitrarily close to 1. The proof assumes an adversary simulated by a Turing machine. We argue that a *human* adversary, who also considers the "semantics" when evaluating the usefulness of information, may behave differently. Let us explain this point by the Terry Gross example that was originally used to capture the intuition of the impossibility result in [6].

In the Terry Gross example, the exact height is considered private, thus, useful to an adversary, whereas the auxiliary information of being two inches shorter than an unknown average is considered not private, thus, not useful. Under this assumption, accessing the statistical database, which returns the average height, is to blame for disclosing Terry Gross's privacy. Mathematically, knowing the exact height is a remarkable progress from knowing two inches shorter than an unknown average. However, to a *human* adversary, the information about how an individual *deviates from the statistics* already discloses the sensitive information, regardless of what the statistics is. For example, once knowing that someone took the HIV check-up ten times more frequently than an unknown average, his/her privacy is already leaked. Here, a human adversary is able to interpret "deviation" as a sensitive notion based on "life experiences", even though mathematically deviation does not derive the exact height. It is unclear whether such a human adversary can be simulated by a Turing machine.

In practice, a realistic privacy definition does allow disclosure of sensitive information in a controlled manner and there are scenarios where it is possible to protect inferential privacy while retaining a reasonable level of data utility. For example, the study in [10] shows that the anonymized data is useful for training a classifier because the training does not depend on detailed personal information. Another scenario is when the utility metric is different from the adversary's target. Suppose that the attribute *Disease* is sensitive and the response attribute R (to a medicine) is not. Learning the following rules does not violate privacy

$$(Disease = x_1) \rightarrow (R = Positive)$$
$$(Disease = x_2) \rightarrow (R = Positive)$$

in that a positive response does not indicate a specific disease with certainty. However, these rules are useful for a researcher to exclude the diseases x_1 and x_2 in the absence of a positive response. Even for a sensitive attribute like *Disease*, the varied sensitivity of domain values (such as Flu and HIV) could be leveraged to retain more utility for less sensitive values while ensuring strong protection for highly sensitive items. In the rest of the paper, we present an approach of leveraging such varied sensitivity to address some drawbacks in previous approaches to inferential privacy.

3. PROBLEM STATEMENT

This section defines the problem we will study. First, we present our model of adversaries, privacy, and data utility.

3.1 Preliminaries

The database is a microdata table $T(QI, SA)$ with each record corresponding to a participant. QI is a set of non-sensitive attributes $\{A_1, \cdots, A_d\}$. SA is a sensitive attribute and has the domain $\{x_1, \cdots, x_m\}$. m is the domain size of SA, also written $|SA|$. Each x_i is called a sensitive value or a SA value. o_i denotes the number of records for x_i in T and f_i denotes the frequency $o_i/|T|$, where $|T|$ is the cardinality of T. For a record r in T, $t[QI]$ and $r[SA]$ denote the values of r on QI and SA. Table 3 lists some of the notations used in this paper.

An adversary wants to infer the SA value of a target individual t. The adversary has access to a published version of T, denoted by T^*. For each SA value x_i, $Pr(x_i|t, T^*)$ denotes the probability that t is inferred to have x_i. For now, we consider an adversary with the following auxiliary information: a t's record is contained in T, t's values on QI, i.e., $t[QI]$, and the algorithm used to produce T^*. Additional auxiliary information will be considered in Section 6.

One approach for limiting $Pr(x_i|t, T^*)$ is *bucketization* [22]. In this approach, the records in T are grouped into small-size buckets and each bucket is identified by a unique bucket ID,

Gender	Zipcode	BID
M	54321	1
M	54322	1
F	61234	2
F	61434	2
...	...	...

(a) QIT

BID	Disease
1	Brain Tumor
1	Indigestion
2	Cancer
2	HIV
...	...

(b) ST

Table 2: An anonymized table T^*

T, $\|T\|$	the raw data and its cardinality
m	domain size of SA
x_i	a sensitive value
o_i	number of occurrence of x_i in T
f_i	$o_i/\|T\|$
f'_i	privacy threshold for x_i
F'-privacy	a collection of f'_i for x_i
$B_j(S_j, b_j)$	b_j buckets of size S_j
$s(B_j)$	total size of buckets in B_j

Table 3: Notations

BID. We use g to refer to both a bucket and the bucket ID of a bucket, depending on the context. T^* is published in two tables, $QIT(QI, BID)$ and $ST(BID, SA)$. For each record r in T that is grouped into a bucket g, QIT contains a record $(r[QI], g)$ and ST contains a record $(g, r[SA])$ (with duplicates preserved). For a target individual t with $t[QI]$ contained in a bucket g, the probability of inferring a SA value x_i using g, $Pr(x_i|t, g)$, is equal to $|g, x_i|/|g|$, where $|g, x_i|$ denotes the number of occurrence of (g, x_i) in ST and $|g|$ denotes the size of g. $Pr(x_i|t, T^*)$ is defined to be the maximum $Pr(x_i|t, g)$ for any bucket g containing $t[QI]$ [22].

EXAMPLE 2. *For the microdata T in Table 1, Gender and Zipcode are the QI attributes and Disease is SA. Table 2 shows the QIT and ST for one bucketization. To infer the SA value of Alice with $QI = \langle F, 61434\rangle$, the adversary first locates the bucket that contains $\langle F, 61434\rangle$, i.e., $BID = 2$. There are two diseases in this bucekt, Cancer and HIV, each occurring once. So $Pr(x_i|Alice, 2) = 50\%$, where x_i is either Cancer or HIV.*

3.2 Privacy Specification

We consider the following privacy specification.

DEFINITION 2 (F'-PRIVACY). *For each SA value x_i, f'_i-*privacy *specifies the requirement that $Pr(x_i|t, T^*) \leq f'_i$, where f'_i is a real number in the range (0,1]. F'-*privacy *is a collection of f'_i-privacy for all SA values x_i.*

For example, the publisher may set $f'_i = 1$ for some x_i's that are not sensitive at all, set f'_i manually to a small value for a few highly sensitive values x_i, and set $f'_i = min\{1, a \times f_i + b\}$ for the rest of SA values whose sensitivity grows linearly with their frequency, where a and b are constants. Our approach assumes that f'_i is specified but does not depend on how f'_i is specified. The next lemma follows easily and the proof is omitted.

LEMMA 1. *A bucketization T^* satisfying F'-privacy exists if and only if $f'_i \geq f_i$ for all x_i.*

Remark 1. *To model a given F'-privacy specification by ℓ-diversity [19], the smallest ℓ required is set by $\ell = \lceil 1/min_i f'_i \rceil$. If some x_i is highly sensitive, i.e., has a small f'_i, this ℓ will be too large for less sensitive x_i's. This leads to poor utility for two reasons. First, the previous bucketization [22] produces buckets of the sizes ℓ or $\ell + 1$. Thus, a large ℓ leads to large buckets and a large information loss. Second, a large ℓ implies that the* eligibility requirement *[22] for having a ℓ-diversity T^*, i.e., $1/\ell \geq max_i f_i$, is more difficult to satisfy. In contrast, the corresponding eligibility requirement for having F'-privacy T^* is $f'_i \geq f_i$ for all x_i's (Lemma 1), which is much easier to satisfy. In Section 3.4, we will address the large bucket size issue by allowing buckets of different sizes to be formed to accommodate different requirements f'_i.*

3.3 Utility Metrics

Within each bucket g, the QI value of every record is equally likely associated with the SA value of every record through the common BID. Therefore, the bucket size $|g|$ serves as a measure of the "disorder" of such association. This observation motivates the following notion of information loss.

Definition 3. *Let T^* consist of a set of buckets $\{g_1, \cdots, g_b\}$. The Mean Squared Error (MSE) of T^* is defined by*

$$MSE(T^*) = \frac{\sum_{i=1}^{b}(|g_i| - 1)^2}{|T| - 1} \qquad (1)$$

Any bucketization T^* has a MSE in the range $[0, |T| - 1]$. The raw data T is one extreme where each record itself is a bucket, so $MSE = 0$. The single bucket containing all records is the other extreme where $MSE = |T| - 1$. With $|T|$ being fixed, to minimize MSE, we shall minimize the following loss metric:

$$Loss(T^*) = \sum_{i=1}^{b}(|g_i| - 1)^2 \qquad (2)$$

Note that $Loss$ has the additivity property: if $T^* = T_1^* \cup T_2^*$, then $Loss(T^*) = Loss(T_1^*) + Loss(T_2^*)$.

3.4 Problem Description

To minimize $Loss$, we consider a general form of bucketization in which buckets of different sizes can be formed so that a large bucket size is used for records having a more sensitive x_i (i.e., a small f'_i) and a small bucket size is used for records having less sensitive x_i (i.e., a larger f'_i). A collection of buckets can be specified by a *bucket setting* of the form $\langle B_1(S_1, b_1), \cdots, B_q(S_q, b_q)\rangle$, where b_j is the number of buckets of the size S_j, $j = 1, \cdots, q$, and $S_1 < \cdots < S_q$. We also denote a bucket setting simply by $\cup B_j$. $s(B_j) = b_j S_j$ denotes the total size of the buckets in B_j. Following Definition 2, the collection of buckets specified by $\cup B_j$ has the loss $\sum_{j=1}^{q} b_j \times (S_j - 1)^2$. We denote this loss by $Loss(\cup B_j)$.

A bucket setting $\cup B_j$ is *feasible* wrt T if $\sum_j s(B_j) = |T|$. A feasible bucket setting is *valid* wrt F'-privacy if there is an assignment of the records in T to the buckets in $\cup B_j$ such that no SA value x_i has a frequency more than f'_i in any bucket g, i.e., $Pr(x_i|t, g) \leq f'_i$. Such assignment is called a *valid* record assignment.

Definition 4 (Optimal multi-size bucket setting). *Given T and F'-privacy, we want to find a valid bucket setting $\langle B_1(S_1, b_1), \cdots, B_q(S_q, b_q)\rangle$ that has the minimum $Loss(\cup B_j)$ among all valid bucket settings.*

This problem must determine the number q of distinct bucket sizes, each bucket size S_j and the number b_j of buckets for the size S_j, $1 \leq j \leq q$. The following special case is a building block of our solution.

Definition 5 (Optimal two-size bucket setting). *Given T and F'-privacy, we want to find a valid two-size bucket setting $\langle B_1(S_1, b_1), B_2(S_2, b_2)\rangle$ that has the minimum loss among all valid two-size bucket settings.*

Remark 2. *The bucket setting problem is challenging for several reasons. Firstly, allowing varied sensitivity f'_i and buckets of different sizes S_j introduces the new challenge of finding the best bucket setting that can fulfil the requirement f'_i for all x_i's. Even for a given bucket setting, it is non-trivial to validate whether there is a valid record assignment to the buckets. Secondly, the number of feasible bucket settings of the form $\langle (S_1, b_1), \cdots, (S_q, b_q)\rangle$ is huge, rendering it prohibitive to enumerate all bucket settings. For example, suppose that S_1 and S_2 are chosen from the range of $[3, 20]$, and $|T| = 1,000,000$, there are a total of 2,077,869 feasible bucket settings of the form (S_1, b_1) and (S_2, b_2). This number will be much larger if $q > 2$. Finally, the number of distinct bucket sizes q is unknown in advance and must be searched.*

Section 4 presents an algorithm for validating a two-size bucket setting. Section 5 presents an efficient algorithm for the optimal two-size bucket setting problem with *guaranteed* optimality, and a heuristic algorithm for the multi-size bucket setting problem.

4. VALIDATING TWO-SIZE BUCKET SETTING

Let $Valid(B, T, F')$ denote a function that tests if a bucket setting B is valid. We assume that the number of occurrence o_i for x_i in T has been collected, $1 \leq i \leq m$. In Section 4.1, we consider buckets having the same size and we give an $O(m)$ time and space algorithm for evaluating $Valid(B, T, F')$. In Section 4.2, we consider buckets having two different sizes and give an $O(m)$ time and space algorithm for $Valid(B, T, F')$. In both cases, we give a linear time algorithm for finding a valid record assignment for a valid bucket setting.

4.1 One-Size Bucket Setting

Let $B = \{g_0, \cdots, g_{b-1}\}$ be a set of b buckets of the *same* size S. To validate this bucket setting, we introduce a round-robin assignment of records to buckets.

Round-Robin Assignment (RRA): For each value x_i, $1 \leq i \leq m$, we assign the t-th record of x_i to the bucket g_s, where $s = (o_1 + \cdots + o_{i-1} + t) \mod b$, where o_i is the number of occurrence of x_i in T. In other words, the records for x_i are assigned to the buckets in a round-robin manner; the order in which x_i is considered by RRA is not important. It is easy to see that the number of records for x_i assigned to a bucket is either $\lfloor |o_i|/b \rfloor$ or $\lceil |o_i|/b \rceil$. The next lemma gives a sufficient and necessary condition for $Valid(B, T, F') = true$.

LEMMA 2 (VALIDATING ONE-SIZE BUCKET SETTING). *Let B be a set of b buckets of size S such that $|T| = s(B)$. The following are equivalent: (1) $Valid(B, T, F') = true$. (2) There is a valid RRA from T to B wrt F'. (3) For each SA value x_i, $\frac{\lceil o_i/b \rceil}{S} \leq f'_i$. (4) For each SA value x_i, $o_i \leq \lfloor f'_i S \rfloor b$.*

PROOF. We show $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4$. Observe that if r is a real number and i is an integer, $r \leq i$ if and only if $\lceil r \rceil \leq i$, and $i \leq r$ if and only if $i \leq \lfloor r \rfloor$. Then the following rewriting holds.

$\frac{\lceil o_i/b \rceil}{S} \leq f'_i \Leftrightarrow \lceil o_i/b \rceil \leq f'_i S \Leftrightarrow \lceil o_i/b \rceil \leq \lfloor f'_i S \rfloor \Leftrightarrow o_i/b \leq \lfloor f'_i S \rfloor \Leftrightarrow o_i \leq \lfloor f'_i S \rfloor b$. This shows the equivalence of 4 and 3.

To see $3 \Rightarrow 2$, observe that $\frac{\lceil o_i/b \rceil}{S}$ is the maximum frequency of x_i in a bucket generated by RRA. Condition 3 implies that this assignment is valid. $2 \Rightarrow 1$ follows because every valid RRA is a valid assignment. To see $1 \Rightarrow 4$, observe that F'-privacy implies that the number of occurrence of x_i in a bucket of size S is at most $\lfloor f'_i S \rfloor$. Thus for any valid assignment, the total number of occurrence o_i in the b buckets of size S is no more than $\lfloor f'_i S \rfloor b$. □

4.2 Two-Size Bucket Setting

Now we consider a two-size bucket setting of the form $\langle B_1(S_1, b_1), B_2(S_2, b_2) \rangle$. The next lemma follows trivially.

LEMMA 3. *$Valid(B_1 \cup B_2, T, F') = true$ if and only if there is a partition of T, $\{T_1, T_2\}$, such that $Valid(B_1, T_1, F') = true$ and $Valid(B_2, T_2, F') = true$.*

DEFINITION 6. *Given F'-privacy, for each x_i and for $j = 1, 2$, we define $u_{ij} = \lfloor f'_i S_j \rfloor b_j$ and $a_{ij} = min\{u_{ij}, o_i\}$.*

From Lemma 2(4), u_{ij} is the upper bound on the number of records for x_i that can be allocated to B_j without violating f'_i-privacy, assuming unlimited supply of x_i records. a_{ij} is the upper bound, assuming the *actual* supply of x_i records, i.e., o_i. The next theorem gives the condition for $Valid(B_1 \cup B_2, T, F') = true$.

THEOREM 3 (VALIDATING TWO-SIZE BUCKET SETTING). *$Valid(B_1 \cup B_2, T, F') = true$ if and only if all of the following conditions hold:*

$$\forall i : a_{i1} + a_{i2} \geq o_i \quad (Privacy\ Constraint(PC)) \quad (3)$$

$$j = 1, 2 : \sum_i a_{ij} \geq s(B_j) \quad (Fill\ Constraint(FC))) \quad (4)$$

$$|T| = s(B_1) + s(B_2) \quad (Capacity\ Constraint(CC))) \quad (5)$$

PROOF. Intuitively, Equation (3) says that the number of occurrence of x_i does not exceed the upper bound $a_{i1}+a_{i2}$ imposed by F'-privacy on all buckets collectively, thus, the name Privacy Constraint. Equation (4) says that under this upper bound constraint it is possible to fill up the buckets in B_j without leaving unused slots, thus, the name Fill Constraint. Equation (5) says that the total bucket capacity matches the data cardinality, thus the name Capacity Constraint. Clearly, all these conditions are necessary for a valid assignment. The sufficiency proof is given by the algorithm in the next subsection that finds a valid assignment of the records in T to the buckets in B_1 and B_2, assuming that the above conditions hold. □

In the rest of the paper, PC, FC, and CC denote Privacy Constraint, Fill Constraint, and Capacity Constraint in Theorem 3.

COROLLARY 2. *For a set buckets B with at most two bucket sizes, $Valid(B, T, F') = true$ can be tested in $O(m)$ time and $O(m)$ space.*

4.3 Record Partitioning

Suppose that PC, FC and CC in Theorem 3 hold. We show how to find a partition $\{T_1, T_2\}$ of T such that $Valid(B_1, T_1, F') = true$ and $Valid(B_2, T_2, F') = true$. This provides the sufficiency proof for Theorem 3 because Lemma 3 implies $Valid(B_1 \cup B_2, T, F') = true$. By finding the partition $\{T_1, T_2\}$, we also provide an algorithm for assigning records from T to the buckets in $B_1 \cup B_2$, that is, simply applying RRA to each of (T_j, B_j), $j = 1, 2$.

The partition $\{T_1, T_2\}$ can be created as follows. For each SA value x_i, initially T_1 contains any a_{i1} records and T_2 contains the remaining $o_i - a_{i1}$ records for x_i. Since $a_{i1} \leq u_{i1}$, Lemma 2(4) holds on (T_1, B_1). (Note that in this case, o_i in Lemma 2 is the number of occurrence of x_i in T_1.) PC implies that the number of occurrence of x_i in T_2, i.e., $o_i - a_{i1}$, is no more than a_{i2}, therefore, Lemma 2(4) also holds on (T_2, B_2). FC implies $|T_1| \geq s(B_1)$. If $|T_1| = s(B_1)$, $|T_2| = s(B_2)$ (i.e., CC), from the above discussion and Lemma 2, $Valid(B_1, T_1, F') = true$ and $Valid(B_2, T_2, F') = true$. We are done.

We now assume $|T_1| > s(B_1)$, thus $|T_2| < s(B_2)$. We need to move $|T_1| - s(B_1)$ records from T_1 to T_2 without exceeding the upper bound a_{i2} for T_2. FC implies that such moves are possible because there must be some x_i for which less than a_{i2} records are found in T_2. For such x_i, we move records of x_i from T_1 to T_2 until the number of records for x_i in T_2 reaches a_{i2} or until $|T_2| = s(B_2)$, whichever comes first. Since we move a record for x_i to T_2 only when there are less than a_{i2} records for x_i in T_2, the condition of Lemma 2(4) is preserved on (T_2, B_2). Clearly, moving a record out of T_1 always preserves the condition of Lemma 2(4) on (T_1, B_1). As long as $|T_2| < s(B_2)$, the above argument can be repeated to move more records from T_1 to T_2.

Eventually, we have $|T_2| = s(B_2)$, so $Valid(B_1, T_1, F') = true$ and $Valid(B_2, T_2, F') = true$. The $\{T_1, T_2\}$ is the partition required.

1	2	3	4	5	6	7	8	9	10	11	12	13	14

(a) The bucket for B_1

9	10	12	13
9	11	12	14
9	11	13	14
9	11	13	14
9	11	13	14
10	11	13	14
10	12	13	14
10	12	13	14
10	12	13	14

(b) The buckets for B_2

Figure 1: The record assignment for Example 3

EXAMPLE 3. *Suppose $f'_i = 2 \times f_i + 0.05$. Consider a table T containing 50 records with o_i for x_i as follows:*

x_1-x_8: $o_i = 1$, $f_i = 0.02$ *and* $f'_i = 0.09$.

x_9-x_{12}: $o_i = 6$, $f_i = 0.12$ and $f'_i = 0.29$.
x_{13}-x_{14}: $o_i = 9$, $f_i = 0.18$ and $f'_i = 0.41$.

Consider the bucket setting $B_1(S_1 = 4, b_1 = 9), B_2(S_2 = 14, b_2 = 1)$. Note CC in Theorem 3 holds. Let us compute a_{i1} and a_{i2}.

$a_{i1} = min\{u_{i1}, o_i\}$: For x_1-x_8, $u_{i1} = \lfloor f'_i S_1 \rfloor b_1 = \lfloor 0.09 \times 4 \rfloor \times 9 = 0$, so $a_{i1} = 0$. For x_9-x_{12}, $u_{i1} = \lfloor 0.29 \times 4 \rfloor \times 9 = 9$, $a_{i1} = 6$. For x_{13}-x_{14}, $u_{i1} = \lfloor 0.41 \times 4 \rfloor \times 9 = 9$, $a_{i1} = 9$.

$a_{i2} = min\{u_{i2}, o_i\}$: For x_1-x_8, $u_{i2} = \lfloor f'_i S_2 \rfloor b_2 = \lfloor 0.09 \times 14 \rfloor \times 1 = 1$, $a_{i2} = 1$. For x_9-x_{12}, $u_{i2} = \lfloor 0.29 \times 14 \rfloor \times 1 = 4$, $a_{i2} = 4$. For x_{13}-x_{14}, $u_{i2} = \lfloor 0.41 \times 14 \rfloor \times 1 = 5$, $a_{i2} = 5$.

It can be verified that PC and FC in Theorem 3 hold. To find the partitioning $\{T_1, T_2\}$, initially T_1 contains $a_{i1} = 0$ record for each of x_1-x_8, $a_{i1} = 6$ records for each of x_9-x_{12}, and $a_{i1} = 9$ records for each of x_{13}-x_{14}. T_2 contains the remaining records in T. Since T_1 contains 42 records, but $s(B_1) = 36$, we need to move 6 records from T_1 to T_2 without exceeding the upper bound a_{i2} for T_2. This can be done by moving one record for each of $x_9 - x_{14}$ from T_1 to T_2. Figure 1 shows a record assignment generated by RRA for (B_1, T_1) and (B_2, T_2).

5. FINDING OPTIMAL BUCKET SETTINGS

We now present an efficient algorithm for finding the optimal bucket setting. Section 5.1 presents an exact solution for the two-size bucket setting problem. Section 5.2 presents a heuristic solution for the multi-size bucket setting problem.

5.1 Algorithms for Two-Size Bucket Settings

Given T and F'-privacy, we want to find the valid bucket setting of the form $\langle B_1(S_1, b_1), B_2(S_2, b_2)\rangle$, where $b_j \geq 0$ and $S_1 < S_2$, such that the following loss is minimized

$$Loss(B_1 \cup B_2) = b_1(S_1 - 1)^2 + b_2(S_2 - 1)^2 \tag{6}$$

One approach is applying Theorem 3 to validate each feasible bucket setting (B_1, B_2), but this is inefficient because the number of such bucket settings can be huge (Remark 2). We present a more efficient algorithm that prunes the bucket settings that are not valid or do not have the minimum loss. Observe that f'_i-privacy implies that a record for x_i must be placed in a bucket of size at least $\lceil 1/f'_i \rceil$; therefore, the minimum size for S_1 and S_2 is $M = min_i\{\lceil 1/f'_i \rceil\}$. The maximum bucket size M' for S_1 and S_2 is constrained by the maximum loss allowed. We assume that M' is given, where $M' > M$. We consider only (S_1, S_2) such that $M \leq S_1 < S_2 \leq M'$. Note that a valid bucket setting may not exist in this range of size.

5.1.1 Indexing Bucket Settings

We first present an "indexing" structure for feasible bucket settings to allow a direct access to any feasible bucket setting. We say that a pair (b_1, b_2) is *feasible* (resp. *valid*) wrt (S_1, S_2) if the bucket setting $\langle B_1(S_1, b_1), B_2(S_2, b_2)\rangle$ is feasible (resp. valid). A valid pair (b_1, b_2) is *optimal* wrt (S_1, S_2) if $Loss(B_1 \cup B_2)$ is minimum among all valid pairs wrt (S_1, S_2). We define $\Gamma(S_1, S_2)$ to be the list of all feasible (b_1, b_2) in the descending order of b_1, thus, in the ascending order of b_2. Intuitively, an earlier bucket setting has more smaller buckets, thus, a smaller *Loss*, than a later bucket setting. Below, we show that the i-th pair in $\Gamma(S_1, S_2)$ can be generated *directly* using the position i without scanning the list. We will use this property to locate all valid pairs by a binary search on $\Gamma(S_1, S_2)$ without storing the list. To this end, it suffices to identify the first and last pairs in $\Gamma(S_1, S_2)$, and the increments of b_1 and b_2 between two consecutive pairs.

The first pair in $\Gamma(S_1, S_2)$, denoted (b_1^0, b_2^0), has the largest possible b_1 such that $S_1 b_1 + S_2 b_2 = |T|$. So (b_1^0, b_2^0) is the solution to the following integer linear program:

$$min\{b_2 \mid S_1 b_1 + S_2 b_2 = |T|\} \tag{7}$$

b_1 and b_2 are variables of non-negative integers and $S_1, S_2, |T|$ are constants.

Next, consider two consecutive pairs (b_1, b_2) and $(b_1 - \Delta_1, b_2 + \Delta_2)$ in $\Gamma(S_1, S_2)$. Since $S_1 b_1 + S_2 b_2 = |T|$ and $S_1(b_1 - \Delta_1) + S_2(b_2 + \Delta_2) = |T|$, $S_1 \Delta_1 = S_2 \Delta_2$. Since Δ_1 and Δ_2 are the smallest positive integers such that this equality holds, $S_2 \Delta_2$ must be the least common multiple of S_1 and S_2, denoted by $LCM(S_1, S_2)$. Δ_2 and Δ_1 are then given by

$$\Delta_2 = LCM(S_1, S_2)/S_2, \ \Delta_1 = LCM(S_1, S_2)/S_1 \tag{8}$$

Therefore, the ith pair in $\Gamma(S_1, S_2)$ has the form $(b_1^0 - i * \Delta_1, b_2^0 + i * \Delta_2)$, where $i \geq 0$. The last pair has the maximum i such that $0 \leq b_1^0 - i * \Delta_1 < \Delta_1$, or $b_1^0/\Delta_1 - 1 < i \leq b_1^0/\Delta_1$. The only integer i satisfying this condition is given by

$$k = \lfloor b_1^0/\Delta_1 \rfloor \tag{9}$$

LEMMA 4. *$\Gamma(S_1, S_2)$ has the form*

$$(b_1^0, b_2^0), (b_1^0 - \Delta_1, b_2^0 + \Delta_2), \cdots, (b_1^0 - k * \Delta_1, b_2^0 + k * \Delta_2) \tag{10}$$

where $b_1^0, b_2^0, \Delta_1, \Delta_2, k$ are defined in Equations (7-9).

REMARK 3. *$\Gamma(S_1, S_2)$ in Lemma 4 has several important properties for dealing with a large data set. Firstly, we can access the i-th element of $\Gamma(S_1, S_2)$ without storing or scanning the list. Secondly, we can represent any sublist of $\Gamma(S_1, S_2)$ by a bounding interval $[i, j]$ where i is the starting position and j is the ending position of the sublist. Thirdly, the common sublist of two sublists L and L' of $\Gamma(S_1, S_2)$, denoted by $L \cap L'$, is given by the intersection of the bounding intervals of L and L'.*

EXAMPLE 4. *Let $|T| = 28, S_1 = 2, S_2 = 4$. $LCM(S_1, S_2) = 4$. $\Delta_2 = 4/4 = 1$ and $\Delta_1 = 4/2 = 2$. $b_1^0 = 14, b_2^0 = 0$. $k = \lfloor 14/2 \rfloor = 7$. $\Gamma(S_1, S_2)$ is (14,0), (12,1), (10,2), (8,3), (6,4), (4,5), (2,6), (0,7).*

The length k of $\Gamma(S_1, S_2)$, given by Equation (9), is proportional to the cardinality $|T|$. b_1^0 is as large as $|T|/S_1$ (when $b_2^0 = 0$) and Δ_1 is no more than S_2. Thus k is as large as $|T|/(S_1 S_2)$. With S_1 and S_2 being small, k is proportional to $|T|$. Therefore, examining all pairs in $\Gamma(S_1, S_2)$ is not scalable. In the rest of this section, we explore two pruning strategies to prune unpromising pairs (b_1, b_2) in $\Gamma(S_1, S_2)$, one based on loss minimization and one based on privacy requirement.

5.1.2 Loss-Based Pruning

Our first strategy is pruning the pairs in $\Gamma(S_1, S_2)$ that do not have the minimum loss wrt (S_1, S_2), by exploiting the following monotonicity of *Loss*, which follows from the descending order of b_1, $S_1 < S_2$, and Equation (6).

LEMMA 5 (MONOTONICITY OF LOSS). *If (b_1, b_2) precedes (b'_1, b'_2) in $\Gamma(S_1, S_2)$. $Loss(B_1 \cup B_2) < Loss(B'_1 \cup B'_2)$, where B_j contains b_j buckets of size S_j, and B'_j contains b'_j buckets of size S_j, $j = 1, 2$.*

Thus the first valid pair in $\Gamma(S_1, S_2)$ is the optimal pair wrt (S_1, S_2). Lemma 5 can also be exploited to prune pairs across different (S_1, S_2). Let $Best_{loss}$ be the minimum loss found so far and (S_1, S_2) be the next pair of sizes to be considered. From Lemma 5, all the pairs in $\Gamma(S_1, S_2)$ that have a loss less than $Best_{loss}$ must form a prefix of $\Gamma(S_1, S_2)$. Let (b_1^*, b_2^*) be the cutoff point of this prefix, where $b_1^* = b_1^0 - k^* * \Delta_1$ and $b_2^* = b_2^0 + k^* * \Delta_2$. k^* is the maximum integer satisfying $b_1^*(S_1 - 1)^2 + b_2^*(S_2 - 1)^2 < Best_{loss}$. k^* is given by

$$k^* = max\{0, \lfloor \frac{Best_{loss} - b_1^0(S_1 - 1)^2 - b_2^0(S_2 - 1)^2}{\Delta_2(S_2 - 1)^2 - \Delta_1(S_1 - 1)^2} \rfloor\} \quad (11)$$

The next lemma revises $\Gamma(S_1, S_2)$ by the cutoff point based on $Best_{loss}$.

LEMMA 6 (LOSS-BASED PRUNING). *Let $Best_{loss}$ be the minimum loss found so far and let (S_1, S_2) be the next pair of sizes to consider. Let $k' = min\{k, k^*\}$, where k is given by Equation (9) and k^* is given by Equation (11). Let $\Gamma'(S_1, S_2)$ denote the prefix of $\Gamma(S_1, S_2)$ that contains the first $k' + 1$ pairs. It suffices to consider $\Gamma'(S_1, S_2)$.*

In the rest of this section, Γ' denotes $\Gamma'(S_1, S_2)$ when S_1 and S_2 are clear from context.

5.1.3 Privacy-Based Pruning

From Lemma 5, the optimal pair wrt (S_1, S_2) is the first valid pair in Γ'. Our second strategy is to locate the first valid pair in Γ' *directly* by exploiting a certain monotonicity property of the condition for a valid pair. First, we introduce some terminology. Consider any sublist L of Γ' and any boolean condition C on a pair. $H(C, L)$ denotes the set of all pairs in L on which C holds, and $F(C, L)$ denotes the set of all pairs in L on which C fails. C is *monotone* in L if whenever C holds on a pair in L, it holds on all later pairs in L, and *anti-monotone* in L if whenever C fails on a pair in L, it fails on all later pairs in L. A monotone C splits L into two sublists $F(C, L)$ and $H(C, L)$ in that order, and an anti-monotone C splits L into two sublists $H(C, L)$ and $F(C, L)$ in that order. Therefore, if we can show that FC and PC in Theorem 3 are monotone or anti-monotone, we can locate all valid pairs in Γ', i.e., those satisfying both FC and PC, by a binary search over Γ'. We consider FC and PC separately.

Monotonicity of FC. Let $FC(S_1)$ denote FC for $j = 1$, and $FC(S_2)$ denote FC for $j = 2$. Note that $H(FC, \Gamma')$ is given by $H(FC(S_1), \Gamma') \cap H(FC(S_2), \Gamma')$.

LEMMA 7 (MONOTONICITY OF FC). *$FC(S_1)$ is monotone in Γ' and $FC(S_2)$ is anti-monotone in Γ'.*

PROOF. We rewrite FC as

$$\sum_i min_i\{\lfloor f'_i S_1 \rfloor b_1, o_i\} \geq S_1 b_1 \quad (12)$$

$$\sum_i min_i\{\lfloor f'_i S_2 \rfloor b_2, o_i\} \geq S_2 b_2 \quad (13)$$

Assume that (b_1, b_2) precedes (b'_1, b'_2) in Γ'. Then $b_1 > b'_1$ and $b_2 < b'_2$. As b_1 decreases to b'_1, both $\lfloor f'_i S_1 \rfloor b_1$ and $S_1 b_1$ decreases by a factor by b'_1/b_1, but o_i remains unchanged. Therefore, if Equation (12) holds for (b_1, b_2), it holds for (b'_1, b'_2) as well; so Equation (12) is monotone on Γ'. For a similar reason, if Equation (13) fails on (b_1, b_2), it remains to fail on (b'_1, b'_2) as well; thus Equation (13) is anti-monotone on Γ'. □

Monotonicity of PC. Let $PC(x_i)$ denote PC for x_i. $H(PC, \Gamma')$ is given by $\cap_i H(PC(x_i), \Gamma')$. To compute $H(PC(x_i), \Gamma')$, we rewrite $PC(x_i)$ as

$$min\{\lfloor f'_i S_1 \rfloor b_1, o_i\} + min\{\lfloor f'_i S_2 \rfloor b_2, o_i\} \geq o_i \quad (14)$$

Since b_1 is decreasing and b_2 is increasing in Γ', $\lfloor f'_i S_1 \rfloor b_1 \geq o_i$ is anti-monotone and $\lfloor f'_i S_2 \rfloor b_2 \geq o_i$ is monotone in Γ'. Note Equation (14) holds in $H(\lfloor f'_i S_1 \rfloor b_1 \geq o_i, \Gamma')$ and $H(\lfloor f'_i S_2 \rfloor b_2 \geq o_i, \Gamma')$.

Let us consider the remaining part of Γ', denoted by $\Gamma'(x_i)$:

$$F(\lfloor f'_i S_1 \rfloor b_1 \geq o_i, \Gamma') \cap F(\lfloor f'_i S_2 \rfloor b_2 \geq o_i, \Gamma').$$

In this part, Equation (14), thus $PC(x_i)$, degenerates into

$$\lfloor f'_i S_1 \rfloor b_1 + \lfloor f'_i S_2 \rfloor b_2 \geq o_i \quad (15)$$

Consider

$$\lfloor f'_i S_2 \rfloor \Delta_2 \geq \lfloor f'_i S_1 \rfloor \Delta_1 \quad (16)$$

and any two consecutive pairs (b_1, b_2) and $(b_1 - \Delta_1, b_2 + \Delta_2)$ in $\Gamma'(x_i)$. If Equation (16) holds, Equation (15) holding on (b_1, b_2) implies that it holds on $(b_1 - \Delta_1, b_2 + \Delta_2)$, thus, Equation (15) is monotone; if Equation (16) fails, Equation (15) failing on (b_1, b_2) implies that it fails on $(b_1 - \Delta_1, b_2 + \Delta_2)$, thus, Equation (15) is anti-monotone. Recall that in $\Gamma'(x_i)$, $PC(x_i)$ degenerates into Equation (15). The next lemma summarizes the above discussion.

LEMMA 8 (MONOTONICITY OF PC). *(i) $\lfloor f'_i S_1 \rfloor b_1 \geq o_i$ is anti-monotone in Γ' and $\lfloor f'_i S_2 \rfloor b_2 \geq o_i$ is monotone in Γ'. (ii) If Equation (16) holds, $PC(x_i)$ is monotone in $\Gamma'(x_i)$, and if Equation (16) fails, $PC(x_i)$ is anti-monotone in $\Gamma'(x_i)$.*

COROLLARY 3. *$H(PC(x_i), \Gamma')$ consists of $H(\lfloor f'_i S_1 \rfloor b_1 \geq o_i, \Gamma')$, $H(PC(x_i), \Gamma'(x_i))$, and $H(\lfloor f'_i S_2 \rfloor b_2 \geq o_i, \Gamma')$.*

5.1.4 Algorithms

The next theorem gives a computation of all pairs in Γ' satisfying both PC and FC, i.e., all valid pairs in Γ'.

THEOREM 4 (COMPUTING ALL VALID PAIRS IN Γ'). *Let Γ^* be the intersection of $H(FC(S_1), \Gamma')$, $H(FC(S_2), \Gamma')$, and $\cap_i H(PC(x_i), \Gamma')$. (i) Γ^* contains exactly the valid pairs in Γ'. (ii) The first pair in Γ^* (if any) is the optimal pair wrt (S_1, S_2). (iii) Γ^* can be computed in $O(m \log |T|)$ time and $O(m)$ space.*

PROOF. (i) follows from Theorem 3. From Lemma 5, the first pair in Γ^* has the minimum loss wrt (S_1, S_2). To see (iii), the monotonicities in Lemma 7 and Lemma 8, and Corollary 3, imply that each sublist involved in computing Γ^* can be found by a binary search over Γ', which takes $O(m \log |T|)$ time (note that the length k' of Γ' is no more than $|T|$). Note that intersecting two sublists takes $O(1)$. The $O(m)$ space follows from the fact that each sublist is represented by its bounding interval and any element of Γ' examined by a binary search can be generated based on its position without storing the list. □

Algorithm 1 Optimal Two-Size Bucketing

TwoSizeBucketing(T, F', M, M')
Input: T, $1 \le i \le m$, F', M, M'
Output: the optimal bucket setting $\langle (S_1, b_1), (S_2, b_2) \rangle$

1: compute o_i, $1 \le i \le m$
2: $Best_{loss} \leftarrow \infty$
3: $Best_{setting} \leftarrow NULL$
4: **for all** $\{S_1 = M;\ S_1 \le M' - 1;\ S_1 + +\}$ **do**
5:     **for all** $\{S_2 = S_1 + 1;\ S_2 \le M';\ S_2 + +\}$ **do**
6:         compute Γ^* using Theorem 4
7:         **if** Γ^* is not empty **then**
8:             let (b_1, b_2) be the first pair in Γ^*
9:             let B_j be the set of b_j buckets of size S_j, $j = 1, 2$
10:             **if** $Best_{loss} > Loss(B_1 \cup B_2)$ **then**
11:                 $Best_{setting} \leftarrow \langle (S_1, b_1), (S_2, b_2) \rangle$
12:                 $Best_{loss} \leftarrow Loss(B_1 \cup B_2)$
13: return $Best_{setting}$

Algorithm 1 presents the algorithm for finding the optimal two-size bucket setting based on Theorem 4, *TwoSizeBucketing*. The input consists of a table T, a privacy parameter F', and the minimum and maximum bucket sizes M and M'. Line 1 computes o_i in one scan of T. Lines 2 and 3 initialize $Best_{loss}$ and $Best_{setting}$. Lines 4 and 5 iterate through all pairs (S_1, S_2) with $M \le S_1 < S_2 \le M'$. For each pair (S_1, S_2), Line 6 computes the list Γ^* using Theorem 4. Lines 8-12 compute *Loss* of the first pair in Γ^* and update $Best_{loss}$ and $Best_{setting}$ if necessary. Line 13 returns $Best_{setting}$. The algorithm uses both loss-based pruning and privacy-based pruning. The former is through the prefix Γ' obtained by the upper bound $Best_{loss}$ as computed in Lemma 6, and the latter is through the binary search of valid pairs implicit in the computation of Γ^*. To tighten up $Best_{loss}$, Lines 4 and 5 examine smaller sizes (S_1, S_2) before larger ones.

5.2 Algorithms for Multi-Size Bucket Settings

A natural next step is to extend the solution for the two-size problem to the multi-size problem. To do so, we must extend Theorem 3 to validate a three-size bucket setting. The next example shows that this does not work.

EXAMPLE 5. *Let $|B_1| = |B_2| = 20$, $|B_3| = 30$, and $|T| = 70$. There are 11 values $x_1, \cdots, x_{11}$: $o_i = 5$ for $1 \le i \le 10$, and $o_{11} = 20$. Suppose that for $1 \le i \le 10$, $a_{i1} = a_{i2} = 0$, $a_{i3} = 5$, and $a_{11,1} = a_{11,2} = a_{11,3} = 20$. The following extended version of PC, FC and CC in Theorem 3: $\forall i : a_{i1} + a_{i2} + a_{i3} \ge o_i$; for $j = 1, 2, 3$, $\sum_i a_{ij} \ge |B_j|$; $|T| = |B_1| + |B_2| + |B_3|$. However, there is no valid record assignment to these buckets. Note that, for $1 \le i \le 10$, $a_{i1} = a_{i2} = 0$, none of the records for x_i can be assigned to the buckets for B_1 or B_2. So the 50 records for x_i, $1 \le i \le 10$, must be assigned to the buckets for B_3, but B_3 has a capacity of 30.*

Our solution is recursively applying *TwoSizeBucketing* to reduce *Loss*. This algorithm, *MultiSizeBucketing*, is given in Algorithm 2. The input consists of T, a set of records, B, a set of buckets of the same size, and F', M, M' as usual, where $|T| = s(B)$. The algorithm applies *TwoSizeBucketing* to find the optimal two-size bucket setting (B_1, B_2) for T (Line 1). If $Loss(B_1 \cup B_2) < Loss(B)$, Line 3 partitions the records of T into T_1 and T_2 between B_1 and B_2.

Algorithm 2 Heuristic Multi-Size Bucketing

MultiSizeBucketing(T, B, F', M, M')
Input: T, B, F', M, M'
Output: a bucket setting $\langle B_1, \cdots, B_q \rangle$ and $T_1, \cdots, T_q$, where T_j is a set of records for B_j, $1 \le j \le q$

1: $\langle B_1, B_2 \rangle \leftarrow TwoSizeBucketing(T, F', M, M')$
2: **if** $Loss(B_1 \cup B_2) < Loss(B)$ **then**
3:     $(T_1, T_2) \leftarrow RecordPartition(T, B_1, B_2)$ (Section 4.3)
4:     $MultiSizeBucketing(T_1, B_1, F', M, M')$
5:     $MultiSizeBucketing(T_2, B_2, F', M, M')$
6: **else**
7:     return(T, B)

$RecordPartition(T, B_1, B_2)$ is the record partition procedure discussed in Section 4.3. Lines 4 and 5 recur on each of (T_1, B_1) and (T_2, B_2). If $Loss(B_1 \cup B_2) \ge Loss(B)$, Line 7 returns the current bucket setting B for T.

6. ADDITIONAL AUXILIARY INFORMATION

Dealing with an adversary armed with additional auxiliary information is one of the hardest problems in data privacy. As pointed out by [14], there is no free lunch in data privacy. Thus, instead of dealing with all types of auxiliary information, we consider two previously identified attacks, namely, corruption attack [21] and negative association attack [13][17]. To focus on the main idea, we consider F'-privacy such that f_i' is the same for all x_i's. In this case, F'-privacy degenerates into ℓ-diversity with $\ell = \lceil 1/f_i' \rceil$ and the solution in Section 5.1 returns buckets of size $S_1 = \ell$ or $S_2 = \ell + 1$, and each record in a bucket has a distinct SA value.

In the *corruption attack*, an adversary has acquired from an *external source* the SA value x_i of some record r in the data. r is called a *corrupted record*. Armed with this knowledge, the adversary will boost the accuracy of inference by excluding one occurrence of x_i when inferring the sensitive value of the remaining records that share the same bucket with r. To combat the accuracy boosting, we propose to inject some small number σ of *fake SA* values into each bucket g, where a fake value does not actually belong to any record in the bucket. To ensure that the adversary cannot distinguish a fake value from a real value, a fake value must be from the domain of SA and must be distinct in the bucket. Now, for each bucket g, the table QIT contains $|g|$ records and the table ST contains $|g| + \sigma$ distinct SA values, in a random order. The adversary knows σ of these SA values are fake but does not know which ones.

Suppose now that in a corruption attack, the adversary is able to corrupt q records in a bucket g, where $q \le |g|$, so $|g| - q + \sigma$ values remain in g, σ of which are fake. Note that $|g|$ and σ are constants. Therefore, the more records the adversary is able to corrupt (i.e., a larger q), the larger the proportion of fake values among the remaining records in the bucket (i.e., $\frac{\sigma}{|g|-q+\sigma}$) and the more uncertain the adversary is about whether a remaining value in g is a real value or a fake value. Even if all but one record in a bucket are corrupted, the adversary has only $1/(1+\sigma)$ certainty that a remaining value is a real value. The price to pay for this additional protection is the distortion by the σ fake values added to each bucket.

The study in [13][17] shows that under unusual circumstances a *negative association* between a non-sensitive value z and a SA value x may be learnt from the published data T^*, which states that a record having z is less likely to have x. Using such negative association, an adversary could exclude unlikely choices x when inferring the sensitive value for an individual having the non-sensitive value z. Since this attack shares the same mechanism as the corruption attack, i.e., by excluding unlikely values, the above solution proposed for corruption attack can be applied to deter the negative association attack, with one difference: a fake value should not be easily excluded for any record using the negative association knowledge. To ensure this, the publisher can first learn the negative association from T^* and inject only those fake values into a bucket that cannot be removed using the learnt negative association.

Parameters	**Settings**
Cardinality $\|T\|$	100k, 200k, **300k**, 400k, 500k
f'_i-privacy for x_i	$f'_i = min\{1, \theta \times f_i + 0.02\}$
Privacy coefficient θ	2, 4, **8**, 16, 32
M	$min_i\{\lceil 1/f'_i \rceil\}$
M'	50

Table 4: Parameter settings

7. EMPIRICAL STUDIES

We evaluate the effectiveness and efficiency of the algorithms proposed in Section 5. For this purpose, we utilized the real data set CENSUS containing personal information of 500K American adults. This data set was previously used in [22], [15] and [19]. Table 5 shows the eight discrete attributes of the data. Two base tables were generated from CENSUS. The first table *OCC* has *Occupation* as SA and the 7 remaining attributes as the QI-attributes. The second table *EDU* has *Education* as SA and the 7 remaining attributes as the QI-attributes. OCC-n and EDU-n denote the data sets of OCC and EDU of the cardinality n. Figure 2 shows the frequency distribution of SA. The parameters and settings are summarized in Table 4 with the default setting in bold face.

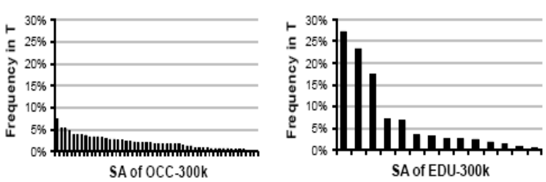


Figure 2: Frequency distribution of SA

We evaluate our algorithms by three criteria: suitability for handling varied sensitivity, data utility, and scalability.

7.1 Criterion 1: Handling Variable Sensitivity

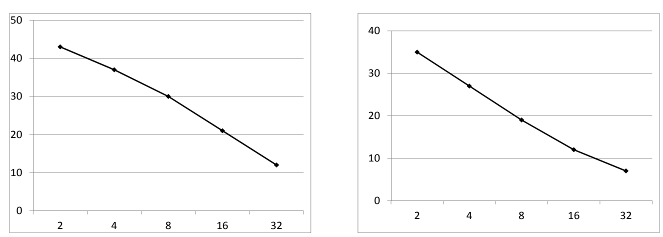
Our first objective is to study the suitability of F'-privacy for handling variable sensitivity and skewed distribution of sensitive values. For concreteness, we specify F'-privacy by $f'_i = min\{1, \theta \times f_i + 0.02\}$, where θ is the *privacy coefficient* chosen from $\{2, 4, 8, 16, 32\}$. This specification models a linear relationship between the sensitivity f'_i and the frequency

Attribute	Domain Size
Age	76
Gender	2
Education	14
Marital	6
Race	9
Work-Class	10
Country	83
Occupation	50

Table 5: Statistics of CENSUS

f_i for x_i. Since $f'_i \geq f_i$ for all x_i's, a solution satisfying F'-privacy always exists (Lemma 1). In fact, a solution exists even with the maximum bucket size constraint $M' = 50$.

For comparison purposes, we apply ℓ-diversity to model the above F'-privacy, where ℓ is set to $\lceil 1/min_i f'_i \rceil$ (Remark 1). For the OCC-300K and EDU-300K data sets, which have the minimum f_i of 0.18% and 0.44%, respectively, Figure 3 plots the relationship between θ and ℓ. Except for $\theta = 32$, a rather large ℓ is required to enforce F'-privacy. As such, the buckets produced by Anatomy [22] have a large size ℓ or $\ell+1$, thus, a large *Loss*. A large ℓ also renders ℓ-diversity too restrictive. As discussed in Remark 1, $1/\ell \geq max_i f_i$ is necessary for having a ℓ-diversity solution. With OCC-300K's maximum f_i being 7.5% and EDU-300K's maximum being 27.3%, this condition is violated for all $\ell \geq 14$ in the case of OCC-300K and all $\ell \geq 4$ in the case of EDU-300K, thus, for most F'-privacy considered. This study suggests that ℓ-diversity is not suitable for handling sensitive values of varied sensitivity and skewed distribution.



(a) OCC (b) EDU

Figure 3: The relationship between ℓ (y-axis) and privacy coefficient θ (x-axis)

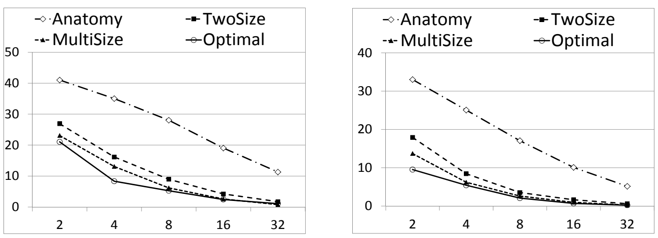
7.2 Criterion 2: Data Utility

Our second objective is to evaluate the utility of T^*. We consider two utility metrics, *Mean Squared Error (MSE)* (Definition 3) and *Relative Error (RE)* for count queries previously used in [22]. We compare *TwoSizeBucketing*, denoted by "TwoSize", and *MultiSizeBucketing*, denoted by "MultiSize", against two other methods. (i) *Optimal multi-size bucketing*, denoted by "Optimal", is the exact solution to the optimal multi-size bucket setting problem, solved by an integer linear program. "Optimal" provides the theoretical lower bound on *Loss*, but it is feasible only for a small domain size $|SA|$. (ii) *Anatomy* [22] with ℓ-diversity being set to $\ell = \lceil 1/min_i f'_i \rceil$. Except for "Anatomy", the minimum bucket size M is set to $min\{\lceil 1/f'_i \rceil\}$ and the maximum bucket size

M' is set to 50.

7.2.1 Mean Squared Error (MSE)

Figure 4 shows MSE vs the privacy coefficient θ on the default OCC-300K and EDU-300K. The study in Section 7.1 shows that for most F'-privacy considered the corresponding ℓ-diversity cannot be achieved on the OCC and EDU data sets. For comparison purposes, we compute the MSE for "Anatomy" based on the bucket size of ℓ or $\ell+1$ while ignoring the privacy constraint. "Anatomy" has a significantly higher MSE than all other methods across all settings of θ because the bucket sizes ℓ and $\ell+1$ are large. "TwoSize" has only a slightly higher MSE than "MultiSize", which has only a slightly higher MSE than "Optimal". This study suggests that the restriction to the two-size bucketing problem causes only a small loss of optimality and that the heuristic solution is a good approximation to the optimal solution of the multi-size bucket setting problem.



(a) OCC (b) EDU

Figure 4: MSE (y-axis) vs privacy coefficient θ (x-axis)

7.2.2 Relative Error (RE)

We adapt *count queries* Q of the form from [22]:

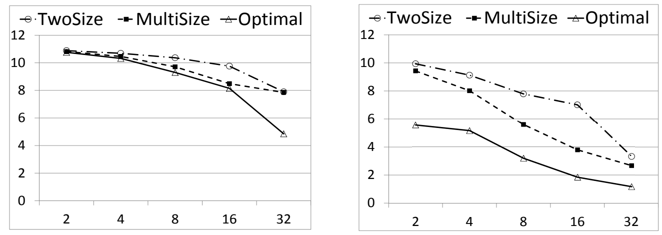
SELECT COUNT(*) FROM T
WHERE $pred(A_1)$ AND ... AND $pred(A_{q_d})$ AND $pred(SA)$

$A_1, \cdots, A_{q_d}$ are randomly selected QI-attributes. q_d is the query dimensionality and is randomly selected from $\{1, \cdots, 7\}$ with equal probability, where 7 is the total number of QI attributes. For any attribute A, $pred(A)$ has the form

$$A = a_1 \text{ OR ... OR } A = a_b,$$

where a_i is a random value from the domain of A. As in [22], the value of b depends on the *expected query selectivity*, which was set to 1% here. The details can be found in [22]. The answer *act* to Q using T is the number of records in T that satisfy the condition in the WHERE clause. We created a pool of 5,000 count queries of the above form. For each query Q in the pool, we compute the estimated answer *est* using T^* in the same way as in [22]. The *relative error (RE)* on Q is defined to be $RE = |act - est|/act$. We report the average RE over all queries in the pool.

Figure 5 shows RE vs the privacy coefficient θ on the default OCC-300K and EDU-300K. For the OCC data set, the maximum RE is slightly over 10%. The RE's for "TwoSize", "MultiSize", and "Optimal" are relatively close to each other, which is consistent with the earlier finding on similar MSE for these algorithms. For the EDU data set, all RE's are no more than 10%. "MultiSize" improves upon "TwoSize" by about 2%, and "Optimal" improves upon "MultiSize" by about 2%. This study suggests that the solutions of the optimal two-size bucketing and the heuristic multi-size bucketing are highly accurate for answering count queries, with the RE below 10% for most F'-privacy considered. "Anatomy" was not included since there is no corresponding ℓ-diversity solution for most F'-privacy considered (see Section 7.1).



(a) OCC (b) EDU

Figure 5: Relative Error (%) (y-axis) vs privacy coefficient θ (x-axis)

7.3 Criterion 3: Scalability

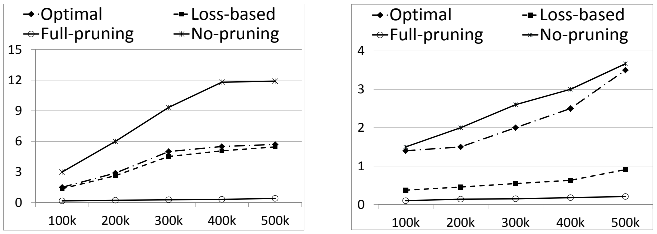
Lastly, we evaluate the scalability for handling large data sets. We focus on *TwoSizeBucketing* because it is a key component of *MultiSizeBucketing*. "No-pruning" refers to the sequential search of the full list Γ without any pruning; "Loss-pruning" refers to the loss-based pruning in Section 5.1.2; "Full-pruning" refers to *TwoSizeBucketing* in Section 5.1.3, which exploits both loss-based pruning and privacy-based pruning. "Optimal" refers to the integer linear program solution to the two-size bucketing problem. We study the *Runtime* with respect to the cardinality $|T|$ and the domain size $|SA|$. The default privacy coefficient setting $\theta = 8$ is used. All algorithms were implemented in C++ and run on a Windows 64 bits Platform with CPU of 2.53 GHz and memory size of 12GB. Each algorithm was run 100 times and the average time is reported here.

7.3.1 Scalability with $|T|$

Figure 6 shows *Runtime* vs the cardinality $|T|$. "Full-pruning" takes the least time and "No-pruning" takes the most time. "Loss-pruning" significantly reduces the time compared to "No-pruning", but has an increasing trend in *Runtime* as $|T|$ increases because of the sequential search of the first valid pair in the list Γ'. In contrast, a larger $|T|$ does not affect "Full-pruning" much because ""Full-pruning" locates the first valid pair by a binary search over Γ'. "Optimal" takes less time than "No-pruning" because the domain size $|SA|$ is relatively small. The next experiment shows that the comparison is reversed for a large domain size $|SA|$.

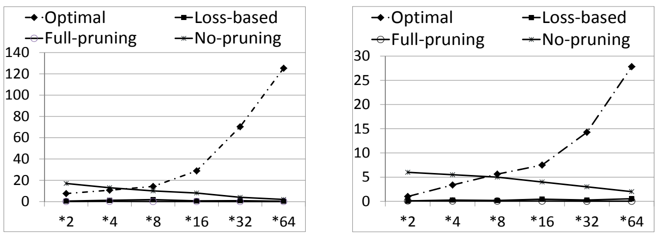
7.3.2 Scalability with $|SA|$

We scale up $|SA|$ for OCC-500K and EDU-500K by a factor γ, where γ is ranged over 2, 4, 8, 16, 32 and 64. Assume that the domain of SA has the form $\{0, 1, \cdots, m-1\}$. For each record t in T, we replace $t[SA]$ in t with the value $\gamma \times t[SA] + r$, where r is an integer selected randomly from the range $[0, \gamma - 1]$ with equal probability. Thus the new do-



(a) OCC (b) EDU

Figure 6: Runtime (seconds) (y-axis) vs cardinality $|T|$ (x-axis)



(a) OCC (b) EDU

Figure 7: Runtime (seconds) (y-axis) vs scale-up factor γ for $|SA|$ (x-axis)

main of SA has the size $m \times \gamma$. Figure 7 shows *Runtime* vs the scale-up factor γ. As γ increases, *Runtime* of "Optimal" increases quickly because the integer linear programming is exponential in the domain size $|SA|$. *Runtime* of the other algorithms increases little because the complexity of these algorithms is linear in the domain size $|SA|$. Interestingly, as $|SA|$ increases, *Runtime* of "No-pruning" decreases. A close look reveals that when there are more SA values, f_i and f'_i become smaller and the minimum bucket size M becomes larger, which leads to a short Γ list. A shorter Γ list benefits most the sequential search based "No-pruning".

In summary, we showed that the proposed methods can better handle sensitive values of varied sensitivity and skewed distribution, therefore, retain more information in the data, and the solution is scalable for large data sets.

8. CONCLUSION

Although differential privacy has many nice properties, it does not address the concern of inferential privacy, which arises due to the wide use of statistical inferences in advanced applications. On the other hand, previous approaches to inferential privacy suffered from major limitations, namely, lack of flexibility in handling varied sensitivity, poor utility, and vulnerability to auxiliary information. This paper developed a novel solution to overcome these limitations. Extensive experimental results confirmed the suitability of the proposed solution for handling sensitive values of varied sensitivity and skewed distribution.

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