Exact scaling exponents in Korn and Korn-type inequalities for cylindrical shells

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Abstract

Understanding asymptotics of gradient components in relation to the symmetrized gradient is important for the analysis of buckling of slender structures. For circular cylindrical shells we obtain the exact scaling exponent of the Korn constant as a function of shell's thickness. Equally sharp results are obtained for individual components of the gradient in cylindrical coordinates. We also derive an analogue of the Kirchhoff ansatz, whose most prominent feature is its singular dependence on the slenderness parameter, in marked contrast with the classical case of plates and rods.

1 Introduction

Korn's inequalities [17, 18] play a central role in the theory of linear and non-linear elasticity, and other areas of physics (see the review [13]). In the study of buckling of slender structures under compression [12, 3, 10] and in the larger study of safe loads [1, 4, 6] of fundamental importance is the dependence of the Korn constant on parameters of the problem. With the application to buckling in mind we study the scaling of the Korn constant as a power of $h = t/R$, where t is the wall thickness and R is the outer radius of the circular cylindrical shell. (See [23, 24] for the application of this theory to rods and plates.) Traditionally, the Korn inequality is proved either for the functions in the orthogonal complement to the space of infinitesimal motions [17, 18, 7] or for functions vanishing on a portion of the boundary [16]. However, for the study of buckling and in other contexts one needs to examine the Korn constant for more general spaces of functions that contain no infinitesimal motions [15, 25, 5, 10, 20].

In order to obtain an asymptotically sharp estimate of the Korn constant one needs to provide an "ansatz": a family of near-minimizers for the variational definition of the Korn constant and then prove an "ansatz-free" inequality establishing the sharpness of the ansatz. This program can be completed for both linear and non-linear versions of the Korn inequality via a compactness theorem for rods and plates [8], justifying the Kirchhoff ansatz [14]. However, the compactness does not hold for cylindrical shells and new approaches, including a new ansatz are needed. The ansatz presented in this paper involves oscillations on a scale $h^{1/4}$, intermediate between the macroscopic and the length scale h of the shell wall. Our method of proof of the ansatz-free bound reduces the first Korn inequality for the circular cylindrical shell to 2D Korn-type inequalities defined on the cylindrical coordinate "plane" cross-sections. These Korn-type inequalities can be regarded to be a cross between the first and second Korn inequalities [22]. The proof uses the method of harmonic projections from [16]. The great flexibility of this method was further explored in [11] with the eventual goal of establishing Korn's inequalities for other shells or for imperfect cylindrical shells, needed for understanding the strong sensitivity to imperfections of the critical buckling load of axially compressed circular cylindrical shells.

Another quantity called for by the buckling theory from [10] is the Korn-like constants in the Korn-like inequalities for gradient components. These inequalities have the form of the first Korn inequality but with a specific component of the gradient matrix in place of the full gradient. We show that, perhaps surprisingly, the Korn-like constants scale in h differently from the Korn constant. This phenomenon is a manifestation of a high degree of symmetry in circular cylindrical shells. With this understanding it is natural that our proof makes full use of that symmetry through the periodicity and the transformation of the problem to the

Fourier space. We conjecture that the imperfections breaking the symmetry will also destroy the distinct power laws in the Korn-like inequalities for gradient components. We believe that this effect of imperfections is related to the large discrepancy between the theoretical buckling load [21, 26] and the experimentally observed values [2, 19]. These ideas are made more precise in our companion paper [9].

This paper is organized as follows. In Section 2 we introduce Korn and Korn-like constants and state our main results for the cylindrical shell. The new oscillatory ansatz is also derived there. We reduce the ansatz-free Korn inequality for the cylindrical shell to the 2D Korn-type inequalities in Section 3. These inequalities are proved by means of the harmonic projection method in Section 4. In Section 5 we prove Korn-like inequalities for gradient components by going to the Fourier space, and using the understanding that simple algebra in Fourier space translates into highly non-trivial statements in the language of differential calculus.

2 Korn and Korn-type inequalities for cylindrical shells

Let $\Omega \subset \mathbb{R}^3$ be an open set. Let V be a subspace of $W^{1,2}(\Omega;\mathbb{R}^3)$ such that $W_0^{1,2}(\Omega;\mathbb{R}^3) \subset V$. We recall that the Korn's constant $K = K(V)$ is defined by

$$K(V) = \sup\{K \geq 0 : \|e(\boldsymbol{\phi})\|^2 \geq K\|\nabla\boldsymbol{\phi}\|^2 \text{ for all } \boldsymbol{\phi} \in V\}, \tag{2.1}$$

where

$$e(\boldsymbol{\phi}) = \frac{1}{2}\left(\nabla\boldsymbol{\phi} + (\nabla\boldsymbol{\phi})^T\right)$$

and $\|\cdot\|$ always denotes the L^2 norm on the domain of definition of the function within the norm symbol. Equivalently,

$$K(V) = \inf_{\boldsymbol{\phi}\in V}\frac{\|e(\boldsymbol{\phi})\|^2}{\|\nabla\boldsymbol{\phi}\|^2}.$$

In this paper we consider a family of circular cylindrical shells given in cylindrical coordinates (r,θ,z) as

$$\mathcal{C}_h = \{(r,\theta,z) : r \in I_h,\ \theta \in \mathbb{T},\ z \in [0,L]\}, \qquad I_h = \left[1-\frac{h}{2}, 1+\frac{h}{2}\right],$$

where $\mathbb{T}$ is a 1-dimensional torus (circle) describing 2π-periodicity in θ.

Our goal is to examine the asymptotics of the Korn constant $K(V_h)$, as $h \to 0$, where V_h is one of the subspaces

$$V_h^1 = \{\boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h;\mathbb{R}^3) : \boldsymbol{\phi}(r,\theta,0) = \mathbf{0},\ \phi_r(r,\theta,L) = \phi_\theta(r,\theta,L) = 0\} \tag{2.2}$$

or

$$V_h^2 = \{\boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h;\mathbb{R}^3) : \phi_\theta(r,\theta,0) = \phi_z(r,\theta,0) = \phi_\theta(r,\theta,L) = \phi_z(r,\theta,L) = 0\}. \tag{2.3}$$

This problem arises in the theory of buckling of slender bodies [10], applied to circular cylindrical shells in our companion paper [9]. In the first case the bottom of the shell is kept fixed, while the top is allowed only vertical displacements under the applied loads, in the second case the loaded shell can "breathe", since the radial displacements are not prescribed at either end. In our notation the dependence on L is suppressed, while the essential dependence on h is emphasized.

In cylindrical coordinates the gradient of $\boldsymbol{\phi} = \phi_r\boldsymbol{e}_r + \phi_\theta\boldsymbol{e}_\theta + \phi_z\boldsymbol{e}_z$,has the form

$$\nabla\boldsymbol{\phi} = \begin{bmatrix} \phi_{r,r} & \dfrac{\phi_{r,\theta}-\phi_\theta}{r} & \phi_{r,z} \\ \phi_{\theta,r} & \dfrac{\phi_{\theta,\theta}+\phi_r}{r} & \phi_{\theta,z} \\ \phi_{z,r} & \dfrac{\phi_{z,\theta}}{r} & \phi_{z,z} \end{bmatrix}. \tag{2.4}$$

2.1 Ansatz

The famous Kirchhoff ansatz [14] for columns and plates can be derived by substituting the quadratic Taylor expansion of a test function $\boldsymbol{\phi}(x, y, z, h)$ defined on $\omega \times [0, h]$ around $(z, h) = (0, 0)$ into $e(\boldsymbol{\phi})$ and postulating cancellation of zeroth order terms [10]. This simple and natural method for obtaining the ansatz for the Korn inequality does not work in the case of a cylindrical shell, implying that the dependence on (r, h) is not smooth. This may manifest itself in the formation of small scale microstructure as $h \to 0$. We postulate that the dependence on r is, in fact, regular:

$$\boldsymbol{\phi}^h(r, \theta, z) = \boldsymbol{u}^h(\theta, z) + (r - 1)\boldsymbol{v}^h(\theta, z). \tag{2.5}$$

We substitute this ansatz into the the formula for $e(\boldsymbol{\phi}^h)$ in cylindrical coordinates and attempt to eliminate all terms of order zero in $r - 1$. This is possible, except for $\phi^h_{z,z}$:

$$v^h_r = 0, \quad v^h_\theta = -u^h_{r,\theta} + u^h_\theta, \quad v^h_z = -u^h_{r,z}, \quad u^h_r = -u^h_{\theta,\theta}, \quad u^h_{z,\theta} = -u^h_{\theta,z}.$$

The last equation can be replaced with

$$u^h_\theta = w^h_{,\theta}, \qquad u^h_z = -w^h_{,z}.$$

Hence, we obtain the ansatz that depends on a singe function $w^h(\theta, z)$:

$$\phi^h_r = -w^h_{,\theta\theta}, \quad \phi^h_\theta = rw^h_{,\theta} + (r - 1)w^h_{,\theta\theta\theta}, \quad \phi^h_z = -w^h_{,z} + (r - 1)w^h_{,\theta\theta z}.$$

In this case

$$\nabla\boldsymbol{\phi}^h = \begin{bmatrix} 0 & -w^h_{,\theta} - w^h_{,\theta\theta\theta} & -w^h_{,\theta\theta z} \\ w^h_{,\theta} + w^h_{,\theta\theta\theta} & \dfrac{(r-1)(w^h_{,\theta\theta} + w^h_{,\theta\theta\theta\theta})}{r} & rw^h_{,\theta z} + (r-1)w^h_{,\theta\theta\theta z} \\ w^h_{,\theta\theta z} & \dfrac{-w^h_{,\theta z} + (r-1)w^h_{,\theta\theta\theta z}}{r} & -w^h_{,zz} + (r-1)w^h_{,\theta\theta zz} \end{bmatrix}$$

$$e(\boldsymbol{\phi}^h) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{(r-1)(w^h_{,\theta\theta} + w^h_{,\theta\theta\theta\theta})}{r} & \dfrac{(r-1)(w^h_{,\theta z} + w^h_{,\theta\theta\theta z})}{2r} \\ 0 & \dfrac{(r-1)(w^h_{,\theta z} + w^h_{,\theta\theta\theta z})}{2r} & -w^h_{,zz} + (r-1)w^h_{,\theta\theta zz} \end{bmatrix}$$

We now assume that the functions $w^h(\theta, z)$ exhibit a small scale microstructure:

$$w^h(\theta, z) = W\left(\frac{\theta}{a_h}, \frac{z - L/2}{b_h}\right), \quad \theta \in [-\pi, \pi],\ z \in [0, L],$$

where

$$\sqrt{h} < a_h \le 1, \quad h < b_h \le 1, \quad \lim_{h\to 0} a_h b_h = 0.$$

Here the function $W(\eta, \zeta)$ can be any smooth compactly supported function on $(-1, 1)^2$, while the function $w^h(\theta, z)$ is extended as a 2π-periodic function in $\theta \in \mathbb{R}$. Thus, we compute

$$|\nabla\boldsymbol{\phi}^h|^2 = O\left(\max\left\{\frac{1}{a_h^6}, \frac{1}{a_h^4 b_h^2}, \frac{1}{b_h^4}\right\}\right).$$

$$|e(\boldsymbol{\phi}^h)|^2 = O\left(\max\left\{\frac{h^2}{a_h^6 b_h^2}, \frac{h^2}{a_h^8}, \frac{1}{b_h^4}\right\}\right).$$

Thus,

$$K(V_h) \leq C \min_{(a,b)\in[h,1]^2} \frac{\max\left\{\dfrac{h^2}{a^6b^2}, \dfrac{h^2}{a^8}, \dfrac{1}{b^4}\right\}}{\max\left\{\dfrac{1}{a^6}, \dfrac{1}{a^4b^2}, \dfrac{1}{b^4}\right\}}.$$

It is a matter of simple analysis to show that the minimum is achieved at $a = \sqrt[4]{h}$, $b = 1$, giving $K(V_h) \leq Ch\sqrt{h}$. Thus, we have proved the following theorem.

THEOREM **2.1** (Ansatz)**.** *Let*

$$V_h^0 = V_h^1 \cap V_h^2 = \{\boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h;\mathbb{R}^3) : \boldsymbol{\phi}(r,\theta,0) = \boldsymbol{\phi}(r,\theta,L) = \mathbf{0}\}.$$

Then there exist an absolute constant C_0 such that

$$K(V_h^0) \leq C_0 h\sqrt{h}. \tag{2.6}$$

This is established via the ansatz

$$\begin{cases} \phi_r^h(r,\theta,z) = & -W_{,\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) \\ \phi_\theta^h(r,\theta,z) = & r\sqrt[4]{h}W_{,\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) + \frac{r-1}{\sqrt[4]{h}}W_{,\eta\eta\eta}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \\ \phi_z^h(r,\theta,z) = & (r-1)W_{,\eta\eta z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right) - \sqrt{h}W_{,z}\left(\frac{\theta}{\sqrt[4]{h}}, z\right), \end{cases} \tag{2.7}$$

where the function $W(\eta, z)$ is a smooth compactly supported function on $(-1,1)\times(0,L)$, while the function $\boldsymbol{\phi}^h(\theta, z)$ is extended as a 2π-periodic function in $\theta \in \mathbb{R}$.

We remark that inequality (2.6) holds for V_h^1 and V_h^2, given by (2.2) and (2.3), respectively, since $V_h^0 \subset V_h^i$, $i = 1, 2$.

2.2 Ansatz-free lower bounds

THEOREM **2.2** (Ansatz free lower bound)**.** *There exist a constant $C(L)$ depending only on L such that*

$$K(V_h^i) \geq C(L)h^{3/2}, \quad i = 1, 2. \tag{2.8}$$

The proof is conducted in two steps. In Section 3 we reduce inequality (2.8) to the Korn-type inequalities in 2D that can be regarded as refined versions of the 2D Korn inequality. In Section 4 these 2D Korn-type inequalities are proved.

The intended application of these inequalities to buckling of cylindrical shells requires that we also estimate the L^2 norms of the individual components of the gradient matrix (2.4) in terms of $\|e(\boldsymbol{\phi})\|^2$. We first observe that the inequalities

$$\|(\nabla\boldsymbol{\phi})_{rr}\|^2 \leq \|e(\boldsymbol{\phi})\|^2, \qquad \|(\nabla\boldsymbol{\phi})_{\theta\theta}\|^2 \leq \|e(\boldsymbol{\phi})\|^2, \qquad \|(\nabla\boldsymbol{\phi})_{zz}\|^2 \leq \|e(\boldsymbol{\phi})\|^2$$

are obvious, as are the inequalities

$$\|(\nabla\boldsymbol{\phi})_{\theta r}\| = 2\|e(\boldsymbol{\phi})_{r\theta} - \frac{1}{2}(\nabla\boldsymbol{\phi})_{r\theta}\| \leq 2\|e(\boldsymbol{\phi})_{r\theta}\| + \|(\nabla\boldsymbol{\phi})_{r\theta}\| \leq 2\|e(\boldsymbol{\phi})\| + \|(\nabla\boldsymbol{\phi})_{r\theta}\|$$

$$\|(\nabla\boldsymbol{\phi})_{zr}\| = 2\|e(\boldsymbol{\phi})_{rz} - \frac{1}{2}(\nabla\boldsymbol{\phi})_{rz}\| \leq 2\|e(\boldsymbol{\phi})_{rz}\| + \|(\nabla\boldsymbol{\phi})_{rz}\| \leq 2\|e(\boldsymbol{\phi})\| + \|\phi_{r,z}\|$$

$$\|(\nabla\boldsymbol{\phi})_{z\theta}\| = 2\|e(\boldsymbol{\phi})_{\theta z} - \frac{1}{2}(\nabla\boldsymbol{\phi})_{\theta z}\| \leq 2\|e(\boldsymbol{\phi})_{\theta z}\| + \|(\nabla\boldsymbol{\phi})_{\theta z}\| \leq 2\|e(\boldsymbol{\phi})\| + \|\phi_{\theta,z}\|.$$

The task is, therefore, to estimate the ratios $\|(\nabla\boldsymbol{\phi})_{r\theta}\|/\|e(\boldsymbol{\phi})\|$, $\|\phi_{r,z}\|/\|e(\boldsymbol{\phi})\|$, and $\|\phi_{\theta,z}\|/\|e(\boldsymbol{\phi})\|$.

THEOREM **2.3.** *There exists a constant $C(L)$ depending only on L such that for any $\boldsymbol{\phi}\in V_h^1\cup V_h^2$ we have*

$$\frac{\|(\nabla\boldsymbol{\phi})_{r\theta}\|^2}{\|e(\boldsymbol{\phi})\|^2}\le\frac{C(L)}{h\sqrt{h}}, \tag{2.9}$$

$$\frac{\|\phi_{\theta,z}\|^2}{\|e(\boldsymbol{\phi})\|^2}\le\frac{C(L)}{\sqrt{h}}, \tag{2.10}$$

$$\frac{\|\phi_{r,z}\|^2}{\|e(\boldsymbol{\phi})\|^2}\le\frac{C(L)}{h}. \tag{2.11}$$

We observe that inequality (2.9) is an immediate consequences of the Korn inequality (2.8). The other two inequalities are proved in Section 5. The remarkable feature of inequalities (2.9)–(2.11) is the presence of 3 distinct scaling laws for different components of the gradient. This is a consequence of the high degree of symmetry possessed by the circular cylindrical shell. We conjecture that deviations from the perfect symmetry will "mix" the three cylindrical components producing a single scaling exponent determined by the Korn constant. Another important observation is that ansatz (2.7) exhibits the scaling laws given by the upper bounds for all 3 ratios in Theorem 2.3.

3 Reduction to 2D Korn-type inequalities

In this section we give the proof of Theorem 2.2 modulo 2D Korn-type inequalities, which constitute the technical core of our method. The argument in this section splits naturally into a sequence of successive steps.

Step 1. In this step we prove that one can replace $\nabla\boldsymbol{\phi}$ and $e(\boldsymbol{\phi})$ in Theorem 2.2 by

$$\boldsymbol{A}=\begin{bmatrix}\phi_{r,r} & \phi_{r,\theta}-\phi_\theta & \phi_{r,z}\\ \phi_{\theta,r} & \phi_{\theta,\theta}+\phi_r & \phi_{\theta,z}\\ \phi_{z,r} & \phi_{z,\theta} & \phi_{z,z}\end{bmatrix},\quad\text{and}\quad \boldsymbol{A}_{\rm sym}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T),$$

respectively. The justification is based on a simple observation that

$$\|e(\boldsymbol{\phi})-\boldsymbol{A}_{\rm sym}\|^2\le\|\nabla\boldsymbol{\phi}-\boldsymbol{A}\|^2\le h^2\|\boldsymbol{A}\|^2. \tag{3.1}$$

Indeed, if we can prove that $\|\boldsymbol{A}\|^2\le Ch^{-3/2}\|\boldsymbol{A}_{\rm sym}\|^2$, then

$$\|\boldsymbol{A}\|^2\le Ch^{-3/2}\|\boldsymbol{A}_{\rm sym}\|^2\le Ch^{-3/2}(\|e(\boldsymbol{\phi})\|^2+h^2\|\boldsymbol{A}\|^2),$$

and therefore $(1-C\sqrt{h})\|\boldsymbol{A}\|^2\le Ch^{-3/2}\|e(\boldsymbol{\phi})\|^2$. Thus, for sufficiently small h we also have

$$\|\boldsymbol{A}\|^2\le Ch^{-3/2}\|e(\boldsymbol{\phi})\|^2,$$

concluding that

$$\|\nabla\boldsymbol{\phi}\|^2\le 2\|\nabla\boldsymbol{\phi}-\boldsymbol{A}\|^2+2\|\boldsymbol{A}\|^2\le 2(h^2+1)\|\boldsymbol{A}\|^2\le Ch^{-3/2}\|e(\boldsymbol{\phi})\|^2.$$

Step 2. In order to prove the Korn inequality for $\boldsymbol{A}$ we need to estimate the quantities

$$G_{12}^2=\|\phi_{\theta,r}\|^2+\|\phi_{r,\theta}-\phi_\theta\|^2,\qquad G_{13}^2=\|\phi_{r,z}\|^2+\|\phi_{z,r}\|^2,\qquad G_{23}^2=\|\phi_{z,\theta}\|^2+\|\phi_{\theta,z}\|^2$$

in terms of

$$E_{12}^2=\|\phi_{\theta,r}+\phi_{r,\theta}-\phi_\theta\|^2,\qquad E_{13}^2=\|\phi_{r,z}+\phi_{z,r}\|^2,\qquad E_{23}^2=\|\phi_{z,\theta}+\phi_{\theta,z}\|^2,$$

$$E_{11}^2=G_{11}^2=\|\phi_{r,r}\|^2,\qquad E_{22}^2=G_{22}^2=\|\phi_{\theta,\theta}+\phi_r\|^2,\qquad E_{33}^2=G_{33}^2=\|\phi_{z,z}\|^2.$$

Estimate for G_{23}. This estimate is the simplest to make. Integration by parts, using the boundary conditions $\phi_\theta = 0$ at $z = 0$ and $z = L$, common to the spaces V_h^1 and V_h^2, and the periodicity in θ, gives

$$|(\phi_{z,\theta}, \phi_{\theta,z})| = |(\phi_{z,z}, \phi_{\theta,\theta})| \le \|\phi_{z,z}\|\|\phi_{\theta,\theta}\| \le E_{33}(E_{22} + \|\phi_r\|),$$

where (f, g) denotes the inner product of f and g in $L^2(\mathcal{C}_h)$. It follows that

$$G_{23}^2 = E_{23}^2 - 2(\phi_{z,\theta}, \phi_{\theta,z}) \le E_{23}^2 + E_{22}^2 + E_{33}^2 + 2E_{33}\|\phi_r\| \le 2\|\boldsymbol{A}_{\text{sym}}\|(\|\boldsymbol{A}_{\text{sym}}\| + \|\phi_r\|). \tag{3.2}$$

Estimate for G_{13}. It would seem that the most natural way to estimate G_{13} is by the Korn inequality on the rectangle $I_h \times [0, L]$ [15]:

$$\|e(\boldsymbol{\Phi})\|^2 \ge Ch^2\|\nabla\boldsymbol{\Phi}\|^2, \tag{3.3}$$

where $\boldsymbol{\Phi}(r, z) = (\phi_r(r, \theta_0, z), \phi_z(r, \theta_0, z))$ for each fixed θ_0, since $\phi_z(r, \theta_0, 0) = 0$. However, inequality (3.3) is incapable of delivering the correct scaling law $h^{3/2}$ of the 3D Korn constant, and, hence, a more delicate estimate is required.

Theorem **3.1** ("First-and-a-half Korn inequality")**.** *There exists a constant $C_0(L) > 0$ depending only on L, such that, if the vector field $\boldsymbol{\phi} = (u, v) \in H^1(I_h \times [0, L]; \mathbb{R}^2)$ satisfies $v(x, 0) = 0$, $x \in I_h$ in the sense of traces, then for any $h \in (0, 1)$ and any $L > 0$*

$$\|\nabla\boldsymbol{\phi}\|^2 \le C_0(L)\|e(\boldsymbol{\phi})\| \left(\frac{\|u\|}{h} + \|e(\boldsymbol{\phi})\|\right). \tag{3.4}$$

The theorem is proved in Section 4. We emphasize that there are no boundary conditions imposed on $u(x, y)$. Applying Theorem 3.1 to the vector field $\boldsymbol{\Phi}(r, z)$ for every θ_0 and integrating the resulting inequality in θ_0 over $[0, 2\pi]$ we obtain, via the Cauchy-Schwartz inequality for the product term

$$G_{13}^2 \le C(L)\left(E_{11}^2 + E_{13}^2 + E_{33}^2 + \frac{\|\phi_r\|}{h}(E_{11} + E_{13} + E_{33})\right) \le C(L)\|\boldsymbol{A}_{\text{sym}}\|\left(\|\boldsymbol{A}_{\text{sym}}\| + \frac{\|\phi_r\|}{h}\right). \tag{3.5}$$

Estimate for G_{12}. The estimate for G_{12} requires an even more delicate Korn-type inequality for rectangles than the estimate for G_{13}.

Theorem **3.2.** *Suppose that the vector field $\boldsymbol{\phi} = (u, v) \in H^1(I_h \times [0, 2\pi]; \mathbb{R}^2)$ satisfies $\boldsymbol{\phi}(x, 0) = \boldsymbol{\phi}(x, 2\pi)$ in the sense of traces. Then*

$$\|u\|^2 \le \|\boldsymbol{e}_*\|^2 + 2\|\boldsymbol{G}_*\|\|v\| + 2\|v\|^2, \tag{3.6}$$

where

$$\boldsymbol{G}_* = \begin{bmatrix} u_x & u_y - v \\ v_x & v_y + u \end{bmatrix}, \qquad \boldsymbol{e}_* = \frac{1}{2}(\boldsymbol{G}_* + \boldsymbol{G}_*^T).$$

In addition, there exist absolute numerical constants $\sigma > 0$ and $C_0 > 0$, such that for any $h \in (0, \sigma)$

$$\|\boldsymbol{G}_*\|^2 \le C_0\left(\|\boldsymbol{e}_*\|^2 + \|\boldsymbol{e}_*\|\frac{\|u\|}{h} + \|v\|^2\right). \tag{3.7}$$

The theorem is proved in Section 4.

We apply inequality (3.7) to the vector field

$$\boldsymbol{\Phi}(r, \theta) = (\phi_r(r, \theta, z_0), \phi_\theta(r, \theta, z_0)) \tag{3.8}$$

for every $z_0 \in [0, L]$. Integrating the resulting inequality in z_0 and using the Cauchy-Schwarz inequality for the product term, we obtain

$$G_{12}^2 \le C_0\left(\|\boldsymbol{A}_{\text{sym}}\|^2 + \|\boldsymbol{A}_{\text{sym}}\|\frac{\|\phi_r\|}{h} + \|\phi_\theta\|^2\right).$$

We estimate via the 1D Poincaré inequality and (3.2)

$$\|\phi_\theta\|^2 \le \frac{L^2}{\pi^2}\|\phi_{\theta,z}\|^2 \le \frac{L^2}{\pi^2}G_{23}^2 \le \frac{2L^2}{\pi^2}(\|\boldsymbol{A}_{\rm sym}\|^2 + \|\boldsymbol{A}_{\rm sym}\|\|\phi_r\|). \tag{3.9}$$

Thus, there exists a constant $C(L) \le C_0(L^2(\sigma+1)+1)$ such that

$$G_{12}^2 \le C(L)\|\boldsymbol{A}_{\rm sym}\|\left(\|\boldsymbol{A}_{\rm sym}\| + \frac{\|\phi_r\|}{h}\right). \tag{3.10}$$

Combining inequalities (3.2), (3.5) and (3.10) we obtain the 3D Korn-type inequality

$$\|\boldsymbol{A}\|^2 \le C_1(L)\|\boldsymbol{A}_{\rm sym}\|\left(\frac{\|\phi_r\|}{h} + \|\boldsymbol{A}_{\rm sym}\|\right). \tag{3.11}$$

It is now clear that in order to prove the Korn inequality (2.8) we need to estimate $\|\phi_r\|$.
Estimate for $\|\phi_r\|$. The estimate for $\|\phi_r\|$ is based on inequality (3.6) in Theorem 3.2 applied to the vector field $\boldsymbol{\Phi}$, given by (3.8). Integrating the resulting inequality in z_0, and using the Cauchy-Schwarz inequality for the product term we obtain

$$\|\phi_r\|^2 \le \|\boldsymbol{A}_{\rm sym}\|^2 + 2\|\boldsymbol{A}\|\|\phi_\theta\| + 2\|\phi_\theta\|^2 \le \|\boldsymbol{A}_{\rm sym}\|^2 + \epsilon^2\|\boldsymbol{A}\|^2 + \left(2+\frac{1}{\epsilon^2}\right)\|\phi_\theta\|^2$$

for any $\epsilon > 0$. The small parameter $\epsilon \in (0,1)$ will be chosen later to optimize the resulting inequality. By the "Poincaré inequality" (3.9) we obtain for sufficiently small ϵ

$$\|\phi_r\|^2 \le \left(\frac{L^2}{\epsilon^2}+1\right)\|\boldsymbol{A}_{\rm sym}\|^2 + \epsilon^2\|\boldsymbol{A}\|^2 + \frac{L^2}{\epsilon^2}\|\boldsymbol{A}_{\rm sym}\|\|\phi_r\|.$$

Therefore,

$$\|\phi_r\|^2 \le 2\left(\frac{L^2}{\epsilon^2}+1\right)^2\|\boldsymbol{A}_{\rm sym}\|^2 + 2\epsilon^2\|\boldsymbol{A}\|^2.$$

Thus,

$$\|\phi_r\| \le \sqrt{2}\left(\left(\frac{L^2}{\epsilon^2}+1\right)\|\boldsymbol{A}_{\rm sym}\| + \epsilon\|\boldsymbol{A}\|\right). \tag{3.12}$$

Substituting this inequality into (3.11), we conclude that there is a constant $C(L)$, depending only on L, such that

$$\|\boldsymbol{A}\|^2 \le C(L)\left(\frac{1}{h\epsilon^2} + \frac{\epsilon^2}{h^2}\right)\|\boldsymbol{A}_{\rm sym}\|^2.$$

We now choose $\epsilon = h^{1/4}$ to minimize the upper bound:

$$\|\boldsymbol{A}\|^2 \le \frac{C(L)}{h\sqrt{h}}\|\boldsymbol{A}_{\rm sym}\|^2, \tag{3.13}$$

which, due to Step 1 completes the proof of Theorem 2.2, modulo Theorems 3.1 and 3.2.

Corollary 3.3. *Inequality (3.11) remains valid if A is replaced by $\nabla\boldsymbol{\phi}$, i.e.,*

$$\|\nabla\boldsymbol{\phi}\|^2 \le C(L)\|e(\boldsymbol{\phi})\|\left(\frac{\|\phi_r\|}{h} + \|e(\boldsymbol{\phi})\|\right). \tag{3.14}$$

Proof. Combining inequalities (2.8) and (3.1) we get

$$(1 - C(L)h^{1/4})\|\boldsymbol{A}_{\rm sym}\| \le \|e(\boldsymbol{\phi})\| \le (1 + C(L)h^{1/4})\|\boldsymbol{A}_{\rm sym}\|, \tag{3.15}$$

which together with (3.11) implies (3.14). □

4 Korn and Korn-type inequalities in two dimensions

In this section our goal is to prove Theorems 3.1 and 3.2. We begin with an auxiliary lemma that will be essential in the proof of both theorems.

LEMMA **4.1.** *Suppose that the vector field $\boldsymbol{\phi}(x,y) = (u(x,y), v(x,y)) \in H^1(I_h \times [0,p];\mathbb{R}^2)$ satisfies $u(x,0) = u(x,p)$ in the sense of traces. Then there exists a constant $C_0(p)$ depending only on p such that for any $\alpha \in [-1,1]$, any $h \in (0,1)$ and any $p > 0$*

$$\|\boldsymbol{G}_\alpha\|^2 \leq C_0(L)\|\boldsymbol{e}_\alpha\| \left(\frac{\|u\|}{h} + \|\boldsymbol{e}_\alpha\| \right), \tag{4.1}$$

where

$$\boldsymbol{G}_\alpha = \boldsymbol{G}_\alpha(\boldsymbol{\phi}) = \begin{bmatrix} u_x & u_y \\ v_x & v_y + \alpha u \end{bmatrix}, \qquad \boldsymbol{e}_\alpha = \boldsymbol{e}_\alpha(\boldsymbol{\phi}) = \frac{1}{2}(\boldsymbol{G}_\alpha(\boldsymbol{\phi}) + (\boldsymbol{G}_\alpha(\boldsymbol{\phi}))^T).$$

We emphasize that there are no boundary conditions imposed on $v(x,y)$. If $\alpha = 0$, and $p = L$ then inequality (4.1) reduces to (3.4). However, the assumed boundary conditions in Lemma 4.1 and Theorem 3.1 do not match. If $\alpha = 1$, $p = \pi$, then the boundary conditions in Lemma 4.1 and Theorem 3.2 are the same and inequalities (4.1) and (3.7) are similar, but not identical. These small discrepancies will be rectified in the proof of the lemma.

Proof. Following the argument of Kondratiev and Oleinik in [16], one can assume, without loss of generality, that u is harmonic. Indeed, suppose $w(x,y)$ solves

$$\begin{cases} \Delta w(x,y) = 0, & (x,y) \in \Omega \\ w(x,y) = u(x,y), & (x,y) \in \partial\Omega, \end{cases} \tag{4.2}$$

where $\Omega = I_h \times [0,p]$. Then ∇w is the Helmholtz projection of ∇u onto the space of divergence-free fields in $L^2(\Omega;\mathbb{R}^2)$, and the following bounds hold:

LEMMA **4.2.** *Let $\Omega = I_h \times [0,p]$ and $\boldsymbol{\phi} = (u,v) \in H^1(\Omega;\mathbb{R}^2)$. If $w(x,y)$ is defined by (4.2), then for any $\alpha \in [-1,1]$, any $h \in (0,1)$, and any $p > 0$*

$$\|\nabla u - \nabla w\| \leq \pi K_0 \|\boldsymbol{e}_\alpha\|, \qquad \|u - w\| \leq K_0 h \|\boldsymbol{e}_\alpha\|, \qquad K_0 = \frac{1}{\pi}\left(\sqrt{2} + \frac{1}{\pi}\right). \tag{4.3}$$

Proof. Using the idea that the Laplacian can be expressed in terms of partial derivatives of components of the symmetrized gradient [16] we compute, using the fact that w is harmonic,

$$\Delta(u-w) = \Delta u = (e_{11}^\alpha - e_{22}^\alpha)_x + 2(e_{12}^\alpha)_y + \alpha e_{11}^\alpha,$$

in the sense of distributions. Here e_{ij}^α denote the components of the matrix $\boldsymbol{e}_\alpha$. Then, since $u - w \in H_0^1(\Omega)$, we have

$$\|\nabla(u-w)\|^2 = \int_\Omega \{(e_{11}^\alpha - e_{22}^\alpha)(u-w)_x + 2e_{12}^\alpha(u-w)_y + \alpha e_{11}^\alpha(w-u)\} dxdy.$$

By the Cauchy-Schwarz inequality we get

$$\|\nabla(u-w)\|^2 \leq \|\boldsymbol{e}_\alpha\|(\sqrt{2}\|\nabla(u-w)\| + |\alpha|\|u-w\|).$$

By the Poincaré inequality

$$\int_{I_h} |u-w|^2 dx \leq \frac{h^2}{\pi^2} \int_{I_h} |(u-w)_y|^2 dx.$$

Hence,

$$\|u-w\| \leq \frac{h}{\pi}\|\nabla(u-w)\|,$$

and (4.3) follows. □

Next we prove a Korn-like inequality for harmonic functions.

LEMMA **4.3.** *Suppose $w \in H^1(I_h \times [0,p])$ is harmonic and satisfies $w(x,0) = w(x,p)$ in the sense of traces. Then*

$$\|w_y\|^2 \le \frac{2\sqrt{3}}{h}\|w\|\|w_x\| + \|w_x\|^2. \tag{4.4}$$

Proof. By the method of separation of variables

$$w(x,y) = \sum_{n\in\mathbb{Z}} (A_n e^{\frac{2\pi nx}{p}} + B_n e^{-\frac{2\pi nx}{p}}) e^{\frac{2\pi nyi}{p}}$$

in $H^1(I_h \times [0,p])$. Therefore,

$$\|w\|^2 = ph \sum_{n\in\mathbb{Z}} \left\{ \psi\left(\frac{2\pi nh}{p}\right)\left(|A_n|^2 e^{\frac{2\pi nh}{p}} + |B_n|^2 e^{\frac{-2\pi nh}{p}}\right) + 2\mathfrak{Re}(A_n\overline{B_n})\right\}, \quad \psi(x) = \frac{\sinh(x)}{x}.$$

In the expansions of w_x and w_y we simply replace A_n and B_n with $2\pi nA_n/p$, $-2\pi nB_n/p$ and $2\pi inA_n/p$, $2\pi inB_n/p$, respectively:

$$\|w_x\|^2 = 4ph \sum_{n\in\mathbb{Z}} \frac{\pi^2 n^2}{p^2} \left\{ \psi\left(\frac{2\pi nh}{p}\right)\left(|A_n|^2 e^{\frac{2\pi nh}{p}} + |B_n|^2 e^{\frac{-2\pi nh}{p}}\right) - 2\mathfrak{Re}(A_n\overline{B_n})\right\},$$

$$\|w_y\|^2 = 4ph \sum_{n\in\mathbb{Z}} \frac{\pi^2 n^2}{p^2} \left\{ \psi\left(\frac{2\pi nh}{p}\right)\left(|A_n|^2 e^{\frac{2\pi nh}{p}} + |B_n|^2 e^{\frac{-2\pi nh}{p}}\right) + 2\mathfrak{Re}(A_n\overline{B_n})\right\},$$

Denoting

$$a_n = A_n e^{\frac{\pi nh}{p}}, \qquad b_n = B_n e^{-\frac{\pi nh}{p}}, \qquad \tau_n = \frac{2\pi nh}{p}$$

we simplify the above expressions:

$$\frac{\|w\|^2}{h^2} = 4ph \sum_{n\in\mathbb{Z}} \frac{\pi^2 n^2}{\tau_n^2 p^2} \{(\psi(\tau_n) - 1)(|a_n|^2 + |b_n|^2) + |a_n + b_n|^2\},$$

$$\|w_y\|^2 = 4ph \sum_{n\in\mathbb{Z}} \frac{\pi^2 n^2}{p^2} \{(\psi(\tau_n) - 1)(|a_n|^2 + |b_n^2|) + |a_n + b_n|^2\},$$

$$\|w_x\|^2 = 4ph \sum_{n\in\mathbb{Z}} \frac{\pi^2 n^2}{p^2} \{(\psi(\tau_n) - 1)(|a_n|^2 + |b_n|^2) + |a_n - b_n|^2\},$$

Obviously,

$$\|w_y\|^2 - \|w_x\|^2 = 16ph \sum_{n\in\mathbb{Z}} \frac{\pi^2 n^2}{p^2} \mathfrak{Re}(a_n\overline{b_n}) \le 16ph \sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{p^2} \mathfrak{Re}(a_n\overline{b_n}),$$

where $\mathcal{P} = \{n \in \mathbb{Z} : \mathfrak{Re}(a_n\overline{b_n}) > 0\}$. Next we estimate

$$\begin{aligned} \frac{\|w\|^2}{h^2} &\ge 4ph \sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{\tau_n^2 p^2} \{(\psi(\tau_n) - 1)(|a_n|^2 + |b_n|^2) + |a_n + b_n|^2\} \\ &\ge 8ph \sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{\tau_n^2 p^2} (\psi(\tau_n) + 1)\mathfrak{Re}(a_n\overline{b_n}). \end{aligned}$$

Similarly,

$$\|w_x\|^2 \ge 8ph \sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{p^2} (\psi(\tau_n) - 1)\mathfrak{Re}(a_n\overline{b_n}).$$

Now we have

$$\sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{p^2}\mathfrak{Re}(a_n\overline{b_n}) = \sum_{n\in\mathcal{P}} \left(\frac{\pi n}{p}\sqrt{(\psi(\tau_n)-1)\mathfrak{Re}(a_n\overline{b_n})}\right)\left(\frac{\pi n}{p}\sqrt{\frac{\mathfrak{Re}(a_n\overline{b_n})}{\psi(\tau_n)-1}}\right).$$

Applying the Cauchy-Schwarz inequality we obtain

$$\sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{p^2}\mathfrak{Re}(a_n\overline{b_n}) \le \sqrt{\sum_{n\in\mathcal{P}} \frac{\pi^2 n^2}{p^2}(\psi(\tau_n)-1)\mathfrak{Re}(a_n\overline{b_n})}\sqrt{\sum_{n\in\mathcal{P}} \Psi(\tau_n)\frac{\pi^2 n^2}{\tau_n^2 p^2}(\psi(\tau_n)+1)\mathfrak{Re}(a_n\overline{b_n})},$$

where

$$\Psi(\tau) = \frac{\tau^2}{\psi(\tau)^2-1} = \frac{\tau^4}{\sinh^2(\tau)-\tau^2}.$$

The function $\Psi(\tau)$ is monotone decreasing on $(0,+\infty)$, and hence,

$$\Psi(\tau_n) \le \Psi(\tau_1) \le \Psi(0) = 3.$$

Therefore,

$$\|w_y\|^2 - \|w_x\|^2 \le \frac{2\sqrt{\Psi(\tau_1)}}{h}\|w\|\|w_x\|, \tag{4.5}$$

and inequality (4.4) follows. □

We remark that inequality (4.4) is sharp, since

$$w(x,y) = \cosh\left(\frac{\pi}{p}\left(x-\frac{h}{2}\right)\right)\sin\left(\frac{\pi y}{p}\right)$$

turns the inequality (4.5) into equality.

We can now finish the proof of Lemma 4.1. By the triangle inequality and Lemma 4.2 we get

$$\|\boldsymbol{G}_\alpha\|^2 = \|\boldsymbol{e}_\alpha\|^2 + \frac{1}{2}\|v_x-\phi_y\|^2 = \|\boldsymbol{e}_\alpha\|^2 + \frac{1}{2}\|(v_x+\phi_y)-2(\phi_y-w_y)-2w_y\|^2 \le$$
$$\|\boldsymbol{e}_\alpha\|^2 + \frac{3}{2}\|\phi_y+v_x\|^2 + 6\|\phi_y-w_y\|^2 + 6\|w_y\|^2 \le (4+6\pi^2K_0^2)\|\boldsymbol{e}_\alpha\|^2 + 6\|w_y\|^2.$$

We estimate $\|w_y\|$ by means of Lemma 4.3. By the triangle inequality and Lemma 4.2

$$\|w\| \le \|u\| + \|u-w\| \le \|u\| + K_0 h\|\boldsymbol{e}_\alpha\|,$$
$$\|w_x\| \le \|\phi_x\| + \|w_x-\phi_x\| \le (1+\pi K_0)\|\boldsymbol{e}_\alpha\|.$$

Therefore,

$$\|w_y\|^2 \le \frac{2\sqrt{3}(1+\pi K_0)}{h}\|u\|\|\boldsymbol{e}_\alpha\| + (1+\pi K_0)(1+(2\sqrt{3}+\pi)K_0)\|\boldsymbol{e}_\alpha\|^2.$$

Thus, using somewhat arbitrary integer overestimation, we obtain

$$\|\boldsymbol{G}_\alpha\|^2 \le 100\|\boldsymbol{e}_\alpha\|\left(\frac{\|u\|}{h}+\|\boldsymbol{e}_\alpha\|\right). \tag{4.6}$$

The proof of Lemma 4.1 is complete. □

Proof of Theorem 3.1. Theorem 3.1 follows from Lemma 4.1 via the even-odd extension method, whereby we define the new displacement $\widetilde{\boldsymbol{\phi}} = (\tilde{u},\tilde{v},)$ on the rectangle $I_h\times[-p,p]$, where $\tilde{u}$ and $\tilde{v}$ are extensions of u and v such that

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{if } y\in[0,p] \\ u(x,-y) & \text{if } y\in[-p,0] \end{cases} \qquad \tilde{v}(x,y) = \begin{cases} v(x,y) & \text{if } y\in[0,p] \\ -v(x,-y) & \text{if } y\in[-p,0] \end{cases}.$$

We observe that due to the boundary condition $v(x,0)=0$, the extension $\widetilde{\boldsymbol{\phi}}$ is an $H^1(I_h\times[-p,p])$ vector field, while $\tilde{u}(x,-p)=\tilde{u}(x,p)$. Moreover,

$$\nabla\widetilde{\boldsymbol{\phi}}(x,y)=\begin{cases}\begin{bmatrix}u_x(x,y) & u_y(x,y)\\ v_x(x,y) & v_y(x,y)\end{bmatrix} & \text{if } \ y\in[0,p],\\ \begin{bmatrix}u_x(x,-y) & -u_y(x,-y)\\ -v_x(x,-y) & v_y(x,-y)\end{bmatrix} & \text{if } \ y\in[-p,0].\end{cases}$$

Therefore, setting $\widetilde{\Omega}=I_h\times[-p,p]$ we get,

$$\|\nabla\widetilde{\boldsymbol{\phi}}\|^2_{L^2(\widetilde{\Omega})}=2\|\nabla\boldsymbol{\phi}\|^2_{L^2(\Omega)},\qquad \|e(\widetilde{\boldsymbol{\phi}})\|^2_{L^2(\widetilde{\Omega})}=2\|e(\boldsymbol{\phi})\|^2_{L^2(\Omega)}.$$

It is also clear that $\|\widetilde{u}\|^2_{L^2(\widetilde{\Omega})}=2\|u\|^2_{L^2(\Omega)}$. An application of Lemma 4.1 to the vector field $\widetilde{\boldsymbol{\phi}}$ in the domain $\widetilde{\Omega}$ completes the proof.

Proof of Theorem 3.2. Let $\widetilde{\boldsymbol{\phi}}(x,y)=(u(x,y),(1-x)v(x,y))$, and let

$$\widetilde{\boldsymbol{G}}=\boldsymbol{G}_\alpha(\widetilde{\boldsymbol{\phi}})|_{\alpha=1},\qquad \widetilde{\boldsymbol{e}}=\frac{1}{2}(\widetilde{\boldsymbol{G}}+\widetilde{\boldsymbol{G}}^T).$$

We compute

$$\boldsymbol{G}_*=\widetilde{\boldsymbol{G}}+\begin{bmatrix}0 & -v\\ v+xv_x & xv_y\end{bmatrix},\qquad \widetilde{\boldsymbol{e}}=\boldsymbol{e}_*+\begin{bmatrix}0 & -\dfrac{x}{2}v_x\\ -\dfrac{x}{2}v_x & -xv_y\end{bmatrix}.$$

Thus we immediately obtain that

$$\|\boldsymbol{G}_*\|^2\le 6(\|\widetilde{\boldsymbol{G}}\|^2+\|v\|^2+h^2(\|v_x\|^2+\|v_y\|^2)). \tag{4.7}$$

and

$$\|\widetilde{\boldsymbol{e}}\|\le\|\boldsymbol{e}_*\|+h(\|v_x\|+\|v_y\|). \tag{4.8}$$

We also estimate

$$\|v_x\|\le\|\boldsymbol{G}_*\|,\qquad \|v_y\|\le\|v_y+u\|+\|u\|\le\|\boldsymbol{e}_*\|+\|u\|. \tag{4.9}$$

Now we apply Lemma 4.1 to the vector field $\widetilde{\boldsymbol{\phi}}$ and $\alpha=1$, and obtain

$$\|\widetilde{\boldsymbol{G}}\|^2\le C_0\|\widetilde{\boldsymbol{e}}\|\left(\frac{\|u\|}{h}+\|\widetilde{\boldsymbol{e}}\|\right).$$

Therefore, by (4.7) and (4.8) we obtain

$$\|\boldsymbol{G}_*\|^2\le C_0\left(\|\boldsymbol{e}_*\|^2+\|\boldsymbol{e}_*\|\frac{\|u\|}{h}+\|u\|(\|v_x\|+\|v_y\|)+\|v\|^2+h^2(\|v_x\|^2+\|v_y\|^2)\right).$$

Applying inequalities (4.9) to the terms containing $\|v_x\|$ and $\|v_y\|$ we obtain

$$\|\boldsymbol{G}_*\|^2\le C_0\left(\|\boldsymbol{e}_*\|^2+\|\boldsymbol{e}_*\|\frac{\|u\|}{h}+\|u\|\|\boldsymbol{G}_*\|+\|u\|^2+\|v\|^2+h^2\|\boldsymbol{G}_*\|^2\right).$$

When $h^2<1/(2C_0)$ we get the inequality

$$\|\boldsymbol{G}_*\|^2\le C_0\left(\|\boldsymbol{e}_*\|^2+\|\boldsymbol{e}_*\|\frac{\|u\|}{h}+\|u\|\|\boldsymbol{G}_*\|+\|u\|^2+\|v\|^2\right).$$

We also have

$$C_0\|u\|\|\boldsymbol{G}_*\|\le\frac{1}{2}\|\boldsymbol{G}_*\|^2+\frac{C_0^2}{2}\|u\|^2.$$

Thus, we obtain

$$\|\boldsymbol{G}_*\|^2 \leq C_0 \left(\|\boldsymbol{e}_*\|^2 + \|\boldsymbol{e}_*\| \frac{\|u\|}{h} + \|u\|^2 + \|v\|^2 \right). \tag{4.10}$$

To finish the proof of the theorem we write $\|u\|^2$ using integration by parts and periodic boundary conditions:

$$\|u\|^2 = (u, u + v_y) + (u_y - v, v) + \|v\|^2.$$

Thus,

$$\|u\|^2 \leq \|u\|\|\boldsymbol{e}_*\| + \|\boldsymbol{G}_*\|\|v\| + \|v\|^2,$$

and using $2\|u\|\|\boldsymbol{e}_*\| \leq \|u\|^2 + \|\boldsymbol{e}_*\|^2$ we obtain (3.6). Applying this inequality to the $\|u\|^2$ term in (4.10) we obtain

$$\|\boldsymbol{G}_*\|^2 \leq C_0 \left(\|\boldsymbol{e}_*\|^2 + \|\boldsymbol{e}_*\| \frac{\|u\|}{h} + \|\boldsymbol{G}_*\|\|v\| + \|v\|^2 \right).$$

from which Theorem 3.2 follows.

Remark 4.4. *In the proofs of all of the Korn and Korn-like inequalities, the vanishing of $\phi_z(r, \theta, L)$ was never used. Hence,*

$$c(L)h^{3/2} \leq K(V_h^*) \leq C(L)h^{3/2}, \tag{4.11}$$

where

$$V_h^* = \{\boldsymbol{\phi} \in W^{1,2}(\mathcal{C}_h; \mathbb{R}^3) : \phi_\theta(r, \theta, 0) = \phi_z(r, \theta, 0) = \phi_\theta(r, \theta, L) = 0\}.$$

5 Korn inequality for gradient components

The goal in this section is to prove Korn-like inequalities (2.10) and (2.11) for gradient components. While inequalities (2.9)–(2.11) bear a formal resemblance to the Korn inequality (2.8), the distinct scaling exponents in (2.10)–(2.11) are a consequence of the high degree of metric symmetry in the structure. By contrast, our methods in Sections 3 and 4 exploited only the topological and smooth structures of the cylindrical shell. Not surprisingly, then, the proof of (2.10) and (2.11) is based on exact calculations in Fourier space, rather than on various integral inequalities, as in the proof of (2.8). In fact, the natural periodicity in θ is not sufficient, and we need the periodicity in z variable as well. The boundary conditions in V_h^1 and V_h^2 permit us to achieve this goal in the same way as was done in proof of Theorem 3.1 in Section 4. For V_h^1 we extend ϕ_r and ϕ_θ as odd and ϕ_z as an even function in $z \in [-L, L]$, while for V_h^2 we extend ϕ_r and ϕ_θ as even functions as ϕ_z as odd. We remark that the periodic extension method cannot be applied to the boundary conditions in the definition of space V_h^*. To fix ideas we conduct the proof for the space V_h^1. The proof for V_h^2 is obtained by switching the sine and cosine series in the z variable. Denoting the periodic extensions without relabeling, we expand the vector field $\boldsymbol{\phi}(r, \theta, z)$ in Fourier series in (θ, z):

$$\boldsymbol{\phi}(r, \theta, z) = \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}} \boldsymbol{\phi}^{(m,n)}(r, \theta, z), \tag{5.1}$$

where

$$\begin{cases} \phi_r^{(m,n)} = \widehat{\phi}_r(r; m, n) \sin\left(\dfrac{\pi m z}{L}\right) e^{in\theta}, & \widehat{\phi}_r(r; m, n) = \dfrac{1}{\pi L} \displaystyle\int_0^{2\pi} \int_0^L \phi_r \sin\left(\dfrac{\pi m z}{L}\right) e^{in\theta} dz d\theta \\ \phi_\theta^{(m,n)} = \widehat{\phi}_\theta(r; m, n) \sin\left(\dfrac{\pi m z}{L}\right) e^{in\theta}, & \widehat{\phi}_\theta(r; m, n) = \dfrac{1}{\pi L} \displaystyle\int_0^{2\pi} \int_0^L \phi_\theta \sin\left(\dfrac{\pi m z}{L}\right) e^{in\theta} dz d\theta \\ \phi_z^{(m,n)} = \widehat{\phi}_z(r; m, n) \cos\left(\dfrac{\pi m z}{L}\right) e^{in\theta}, & \widehat{\phi}_z(r; m, n) = \dfrac{1}{\pi L} \displaystyle\int_0^{2\pi} \int_0^L \phi_z \cos\left(\dfrac{\pi m z}{L}\right) e^{in\theta} dz d\theta. \end{cases}$$

We observe that in cylindrical coordinates

$$\nabla\boldsymbol{\phi}(r,\theta,-z) = -\left[\begin{array}{ccc} -\phi_{r,r}(r,\theta,z) & -\frac{\phi_{r,\theta}(r,\theta,z)-\phi_\theta(r,\theta,z)}{r} & \phi_{r,z}(r,\theta,z) \\ -\phi_{\theta,r}(r,\theta,z) & -\frac{\phi_{\theta,\theta}(r,\theta,z)+\phi_r(r,\theta,z)}{r} & \phi_{\theta,z}(r,\theta,z) \\ \phi_{z,r}(r,\theta,z) & \frac{\phi_{z,\theta}(r,\theta,z)}{r} & -\phi_{z,z}(r,\theta,z) \end{array}\right]$$

Therefore, it is sufficient to prove inequalities (2.10) and (2.11) for functions of the form

$$\boldsymbol{\phi}^{(m,n)}(r,\theta,z) = \left(f_r(r)\sin\left(\frac{\pi m z}{L}\right), f_\theta(r)\sin\left(\frac{\pi m z}{L}\right), f_z(r)\cos\left(\frac{\pi m z}{L}\right)\right)e^{in\theta}.$$

Indeed,

$$\|\phi_{r,z}\|^2 = \pi L\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\|\phi_{r,z}^{(m,n)}\|^2 \le \pi L\sum_{m=0}^{\infty}\sum_{n\in\mathbb{Z}}\frac{C(L)}{h}\|e(\boldsymbol{\phi}^{(m,n)})\|^2 = \frac{C(L)}{h}\|e(\boldsymbol{\phi})\|^2,$$

with the similar bound for $\|\phi_{\theta,z}\|$. Observe that $\boldsymbol{\phi}^{(m,n)} \in V_h^1$ or V_h^2, provided $\boldsymbol{\phi} \in V_h^1$ or V_h^2, respectively. Therefore, Theorem 2.2 and Corollary 3.3 are applicable to such functions. We now fix $m \ge 1$ and $n \in \mathbb{Z}$, and for simplicity of notation we use $\boldsymbol{\phi} = (\phi_r, \phi_\theta, \phi_z)$ instead of $\boldsymbol{\phi}^{(m,n)} = (\phi_r^{(m,n)}, \phi_\theta^{(m,n)}, \phi_z^{(m,n)})$. Notice that if $\|\phi_r\| \le 3\|e(\boldsymbol{\phi})\|$, then Corollary 3.3 implies that

$$\|\phi_{r,z}\|^2 \le \|\nabla\boldsymbol{\phi}\|^2 \le \frac{C(L)}{h}\|e(\boldsymbol{\phi})\|^2,$$

and (2.11) is proved. Let us now prove inequality (2.11) under the assumption that $\|\phi_r\| > 3\|e(\boldsymbol{\phi})\|$. In that case inequalities (3.2) and (3.14) become

$$\|\phi_{z,\theta}\|^2 + \|\phi_{\theta,z}\|^2 \le \frac{8}{3}\|e(\boldsymbol{\phi})\|\|\phi_r\| \tag{5.2}$$

and

$$\|\nabla\boldsymbol{\phi}\|^2 \le \frac{C(L)}{h}\|e(\boldsymbol{\phi})\|\|\phi_r\|, \tag{5.3}$$

respectively. We estimate

$$n^2\|\phi_r\|^2 = \|\phi_{r,\theta}\|^2 \le 2\|\phi_{r,\theta} - \phi_\theta\|^2 + 2\|\phi_\theta\|^2 \le 2\|\nabla\boldsymbol{\phi}\|^2 + \frac{2L^2}{\pi^2}\|\phi_{\theta,z}\|^2 \le C(L)\|\nabla\boldsymbol{\phi}\|^2,$$

where the Poincaré inequality has been used for ϕ_θ. Applying inequality (5.3) we obtain

$$n^2\|\phi_r\|^2 \le C(L)\|\nabla\boldsymbol{\phi}\|^2 \le \frac{C(L)}{h}\|e(\boldsymbol{\phi})\|\|\phi_r\|.$$

Thus,

$$n^2\|\phi_r\| \le \frac{C(L)}{h}\|e(\boldsymbol{\phi})\|. \tag{5.4}$$

We next estimate

$$\|\phi_r\|^2 \le 2\|\phi_r + \phi_{\theta,\theta}\|^2 + 2\|\phi_{\theta,\theta}\|^2 \le 2\|e(\boldsymbol{\phi})\|^2 + 2n^2\|\phi_\theta\|^2,$$

and

$$\frac{m^2\pi^2}{L^2}\|\phi_\theta\|^2 = \|\phi_{\theta,z}\|^2 \le \frac{8}{3}\|e(\boldsymbol{\phi})\|\|\phi_r\|,$$

due to (5.2). Combining the last two inequalities we obtain

$$\|\phi_r\|^2 \le 2\|e(\boldsymbol{\phi})\|^2 + \frac{16L^2n^2}{3m^2\pi^2}\|e(\boldsymbol{\phi})\|\|\phi_r\|. \tag{5.5}$$

By our assumption $\|e(\boldsymbol{\phi})\|^2 < \|\phi_r\|^2/9$. We use this inequality to estimate the first term on the right-hand side of (5.5) and obtain

$$\|\phi_r\| \le \frac{12L^2n^2}{m^2\pi^2}\|e(\boldsymbol{\phi})\|. \tag{5.6}$$

Finally, multiplying (5.4) and (5.6) we get

$$m^2\|\phi_r\|^2 \leq \frac{C(L)}{h}\|e(\boldsymbol{\phi})\|^2,$$

and (2.11) is proved. To prove (2.10) we utilize (3.2) to get,

$$\|\phi_{\theta,z}\|^2 \leq G_{12}^2 \leq 2\|\boldsymbol{A}_{\text{sym}}\|(\|\boldsymbol{A}_{\text{sym}}\| + \|\phi_r\|). \tag{5.7}$$

Choosing $\epsilon = \sqrt[4]{h}$ in (3.12) and applying (3.13) to the resulting inequality, we obtain

$$\|\phi_r\| \leq \frac{C(L)\|\boldsymbol{A}_{\text{sym}}\|}{\sqrt{h}}.$$

Substituting now the last inequality into (5.7) we get

$$\|\phi_{\theta,z}\|^{\leq}\frac{C(L)}{\sqrt{h}}\|\boldsymbol{A}_{\text{sym}}\|^2.$$

Invoking inequality (3.15), gives

$$\|\phi_{\theta,z}\|^2 \leq \frac{C(L)}{\sqrt{h}}\|e(\boldsymbol{\phi})\|^2,$$

for sufficiently small h. This completes the proof for the case $\boldsymbol{\phi} \in V_h^1$. If $\boldsymbol{\phi} \in V_h^2$ we repeat the same proof changing sines to cosines in the expansion (5.1).

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