

Polychromatic Coloring for Half-Planes.

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Abstract

We prove that for every integer k , every finite set of points in the plane can be k -colored so that every half-plane that contains at least $2k - 1$ points, also contains at least one point from every color class. We also show that the bound $2k - 1$ is best possible. This improves the best previously known lower and upper bounds of $\frac{4}{3}k$ and $4k - 1$ respectively. We also show that every finite set of half-planes can be k colored so that if a point p belongs to a subset H_p of at least $3k - 2$ of the half-planes then H_p contains a half-plane from every color class. This improves the best previously known upper bound of $8k - 3$. Another corollary of our first result is a new proof of the existence of small size ϵ -nets for points in the plane with respect to half-planes.

1 Introduction

In this contribution, we are interested in coloring finite sets of points in \mathbb{R}^2 so that any half-plane that contains at least some fixed number of points, also contains at least one point from each of the color classes.

Before stating our results, we introduce the following definitions:

A *range space* (or *hypergraph*) is a pair (V, \mathcal{E}) where V is a set (called the *ground set*) and \mathcal{E} is a set of subsets of V .

A *coloring* of a hypergraph is an assignment of colors to the elements of the ground set. A k -*coloring* is a function $\chi : V \rightarrow \{1, \dots, k\}$. A hyperedge $S \in \mathcal{E}$ is said to be *polychromatic* with respect to some k -coloring χ if it contains a point from each of the k color classes. That is, for every $i \in \{1, \dots, k\}$ $S \cap \chi^{-1}(i) \neq \emptyset$. We are interested in hypergraphs induced by an infinite family of geometric regions. Let \mathcal{R} be a family of regions in \mathbb{R}^d (such as all balls, all axis-parallel boxes, all half-spaces, etc.)

Consider the following two functions defined for \mathcal{R} (notations are taken from [3]):

1. Let $f = f_{\mathcal{R}}(k)$ denote the minimum number such that any finite point set $P \subset \mathbb{R}^d$ can be k -colored so that every range $R \in \mathcal{R}$ containing at least f points of P is polychromatic.
2. Let $\bar{f} = \bar{f}_{\mathcal{R}}(k)$ denote the minimum number such that any finite sub-family $\mathcal{R}' \subset \mathcal{R}$ can be k -colored so that for every point $p \in \mathbb{R}^d$, for which the subset $\mathcal{R}'_p \subset \mathcal{R}'$ of regions containing p is of size at least \bar{f} , \mathcal{R}'_p is polychromatic.

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We note that the functions $f_{\mathcal{R}}(k)$ and $\bar{f}_{\mathcal{R}}(k)$ might not be bounded even for $k = 2$. Indeed, suppose \mathcal{R} is the family of all convex sets in the plane and P is a set of more than $2f - 2$ points in convex position. Note that any subset of P can be cut-off by some range in \mathcal{R} . By the pigeon-hole principle, any 2 coloring of P contains a monochromatic subset of at least f points, thus illustrating that $f_{\mathcal{R}}(2)$ is not bounded in that case. Also note that $f_{\mathcal{R}}(k)$ and $\bar{f}_{\mathcal{R}}(k)$ are monotone non-decreasing, since any upper bound for $f_{\mathcal{R}}(k)$ would imply an upper-bound for $f_{\mathcal{R}}(k - 1)$ by merging color classes. We sometimes abuse the notation and write $f(k)$ when the family of ranges under consideration is clear from the context.

The functions defined above are related to the so-called *cover-decomposable* problems or the decomposition of *c-fold coverings* in the plane. It is a major open problem to classify, for which families \mathcal{R} those functions are bounded, and in those cases to provide sharp bounds on $f_{\mathcal{R}}(k)$ and $\bar{f}_{\mathcal{R}}(k)$. Pach [11] conjectured that $\bar{f}_{\mathcal{T}}(2)$ exists whenever \mathcal{T} is a family of all translates of some fixed compact convex set. These functions have been the focus of many recent research papers and some special cases are resolved. See, e.g., [1, 2, 6, 12–14, 16–18]. We refer the reader to the introduction of [3] for more details on this and related problems.

Application to Battery Consumption in Sensor Networks Let \mathcal{R} be a collection of sensors, each of which monitors the area within a given shape A . Assume further that each sensor has a battery life of one time unit. The goal is to monitor the region A for as long as possible. If we activate all sensors in \mathcal{R} simultaneously, A will be monitored for only one time unit. This can be improved if \mathcal{R} can be partitioned into c pairwise disjoint subsets, each of which covers A . Each subset can be used in turn, allowing us to monitor A for c units of time. Obviously if there is a point in A covered by only c sensors then we cannot partition \mathcal{R} into more than c families. Therefore, it makes sense to ask the following question: what is the minimum number $\bar{f}(k)$ for which we know that, if every point in A is covered by $\bar{f}(k)$ sensors, then we can partition \mathcal{R} into k pairwise disjoint covering subsets? This is exactly the type of problem that we described. For more on the relation between these partitioning problems and sensor networks, see the paper of Buchsbaum *et al.* [4].

Our results For the family \mathcal{H} of all half-planes, Pach and Tóth showed in [14] that $f_{\mathcal{H}}(k) = O(k^2)$. Aloupis *et al.* [3] showed that $\frac{4k}{3} \leq f_{\mathcal{H}}(k) \leq 4k - 1$. In this paper, we settle the case of half-planes by showing that the exact value of $f_{\mathcal{H}}(k)$ is $2k - 1$. Keszegh [8] showed that $\bar{f}_{\mathcal{H}}(2) \leq 4$ and Fulek [5] showed that $\bar{f}_{\mathcal{H}}(2) = 3$. Aloupis *et al.* [3] showed that $\bar{f}_{\mathcal{H}}(k) \leq 8k - 3$. In this paper, we obtain the improved bound of $\bar{f}_{\mathcal{H}}(k) \leq 3k - 2$.

An Application to ϵ -Nets for Half-Planes Let $H = (V, \mathcal{E})$ be a hypergraph where V is a finite set. Let $\epsilon \in [0, 1]$ be a real number. A subset $N \subseteq V$ is called an ϵ -net if for every hyperedge $S \in \mathcal{E}$ such that $|S| \geq \epsilon|V|$, we have also $S \cap N \neq \emptyset$. In other words, N is a hitting set for all “large” hyperedges. Haussler and Welzl [7] proved the following fundamental theorem regarding the existence of small ϵ -nets for hypergraphs with a small VC-dimension.

Theorem 1.1 (ϵ -net theorem [7]). *Let $H = (V, \mathcal{E})$ be a hypergraph with VC-dimension d . For every $\epsilon \in (0, 1]$, there exists an ϵ -net $N \subset V$ with cardinality at most $O\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$*

The notion of ϵ -nets is central in several mathematical fields, such as computational learning theory, computational geometry, discrete geometry and discrepancy theory.

Most hypergraphs studied in discrete and computational geometry have a finite VC-dimension. Thus, by the above-mentioned theorem, these hypergraphs admit small size ϵ -nets. K6mlos *et al.* [9] proved that the bound $O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ on the size of an ϵ -net for hypergraphs with VC-dimension d is best possible. Namely, for a constant d they construct a hypergraph H with VC-dimension d such that any ϵ -net for H must have size of at least $\Omega(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$. However, their construction is random and seems far from being a “nice” geometric hypergraph. It is believed that for most hypergraphs with VC-dimension d that arise in the geometric context, one can improve on the bound $O(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$.

Consider a hypergraph $H = (P, E)$ where P is a finite set of points in the plane and

$$E = \{P \cap h : h \text{ is a half-plane}\}.$$

For this special case, Woeginger [19] showed that for any $\epsilon > 0$ there exists an ϵ -net for H of size at most $\frac{2}{\epsilon} - 1$ (see also, [15]).

As a corollary of our result, we obtain yet another proof for this fact.

2 Coloring Points with Respect to Half-Planes

Let \mathcal{H} denote the family of all half-planes in \mathbb{R}^2 . In this section we prove our main result by finding the exact value of $f_{\mathcal{H}}(k)$, for the family \mathcal{H} of all half-planes.

Theorem 2.1. $f_{\mathcal{H}}(k) = 2k - 1$.

We start by proving the lower bound $f_{\mathcal{H}}(k) \geq 2k - 1$. Our lower bound construction is simple, and is inspired by a lower bound construction for ϵ -nets with respect to half-planes given in [19]. We need to show that there exists a finite set P in \mathbb{R}^2 such that for every k -coloring of P there is a half-plane that contains $2k - 2$ points, and is not polychromatic. In fact, we show a stronger construction. For every $n \geq 2k - 1$ there is such a set P with $|P| = n$. We construct P as follows: We place $2k - 1$ points on a concave curve γ (e.g., the parabola $y = x^2$, $-1 < x < 1$). Let $p_1, p_2, \dots, p_{2k-1}$ be the points ordered from left to right along their x -coordinates. Notice that for every point p_i on γ there is an open positive half-plane h_i that does not contain p_i and contains the rest of the $2k - 2$ points that are on γ . Namely, $h_i \cap \{p_1, \dots, p_{2k-1}\} = \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{2k-1}\}$. We choose $h_1, h_2, \dots, h_{2k-1}$ in such a way that $\cap_{i=1}^{2k-1} \overline{h_i} \neq \emptyset$ where $\overline{h_i}$ is the complement of h_i . We place $n - (2k - 1)$ points in $\cap_{i=1}^{2k-1} \overline{h_i}$. Let $\chi : P \rightarrow \{1, \dots, k\}$ be some k -coloring of P . There exists a color c that appears at most once among the points on γ (for otherwise we would have at least $2k$ points). If no point on γ is colored with c then a (positive) half-plane bounded by a line separating the parabola from the rest of the points is not polychromatic. Let p_j be the point colored with c . As mentioned, the open half-plane h_j contains all the other points on γ (and only them), so h_j contains $2k - 2$ points and misses the color c . Hence, it is not polychromatic. Thus $f_{\mathcal{H}}(k) > 2k - 2$ and this completes the lower bound construction. See Figure 1 for an illustration.

Next, we prove the upper-bound $f_{\mathcal{H}}(k) \leq 2k - 1$. In what follows, we assume without loss of generality that the set of points P under consideration is in general position, namely, that no three points of P lie on a common line. Indeed, we can slightly perturb the point set to obtain a set P' of points in general position. The perturbation is done in such a way that for any subset of the points of the form $h \cap P$ where h is a half-plane, there is another half-plane

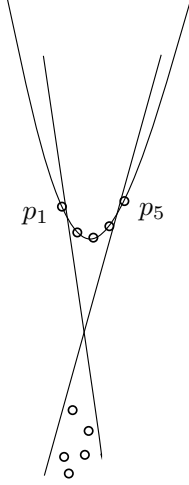


Figure 1: A construction showing that $f(k) > 2k - 2$ for $n = 10$ and $k = 3$

h' such that $h \cap P = h' \cap P'$. Thus any valid polychromatic k -coloring for P' also serves as a valid polychromatic k -coloring for P .

For the proof of the upper bound we need the following lemma:

Lemma 2.2. *Let P be a finite point set in the plane in general position and let $t \geq 3$ be some fixed integer. Let $H' = (P, \mathcal{E}')$ be a hypergraph where $\mathcal{E}' = \{P \cap h : h \in \mathcal{H}, |P \cap h| = t\}$. Let $P' \subseteq P$ be the set of extreme points of P (i.e., the subset of points in P that lie on the boundary of the convex-hull $CH(P)$ of P). Let $N \subseteq P'$ be a (containment) minimal hitting set for H' . Then for every $E \in \mathcal{E}'$ we have $|N \cap E| \leq 2$.*

Proof. First notice that such a hitting set $N \subset P'$ for H' indeed exists since P' is a hitting set.

Assume to the contrary that there exists a hyperedge $E \in \mathcal{E}'$ such that $|N \cap E| \geq 3$. Let h be a half-plane such that $h \cap P = E$ and let l be the line bounding h . Assume, without loss of generality, that l is parallel to the x -axis and that the points of E are below l . If l does not intersect the convex hull $CH(P)$ or is tangent to $CH(P)$ then h contains P and $|P| = t$. Thus any minimal hitting set N contains exactly one point of P , a contradiction. Hence, the line l must intersect the boundary of $CH(P)$ in two points.

Let q, q' be the left and right points of $l \cap \partial CH(P)$ respectively. Let p, r, u be three points in $N \cap E$ ordered according to their counter-clockwise order on $\partial CH(P)$. By the minimality property, there is a half-plane h_r such that $h_r \cap P \in \mathcal{E}'$ and such that $N \cap h_r = \{r\}$, for otherwise, $N \setminus \{r\}$ is also a hitting-set for \mathcal{E}' contradicting the minimality of N . See Figure 2 for an illustration.

Denote the line bounding h_r by l_r and denote by \bar{h}_r the complement half-plane of h_r . Notice that l_r can not intersect the line l in the interior of the segment qq' . Indeed assume to the contrary that l_r intersects the segment qq' in some point x . Then, by convexity, the open segment rx lies in h_r . However, the segment rx must intersect the segment pu . This is impossible since both p and u lie in \bar{h}_r and therefore, by convexity also the segment pu lies in \bar{h}_r . Thus the segment pu and the segment rx are disjoint.

Next, suppose without loss of generality that the line l_r intersects l to the right of the segment qq' . Let q'' denote the point $l \cap l_r$. We have that $|h_r \cap P| = t$ and also $|E| = |h \cap P| = t$,

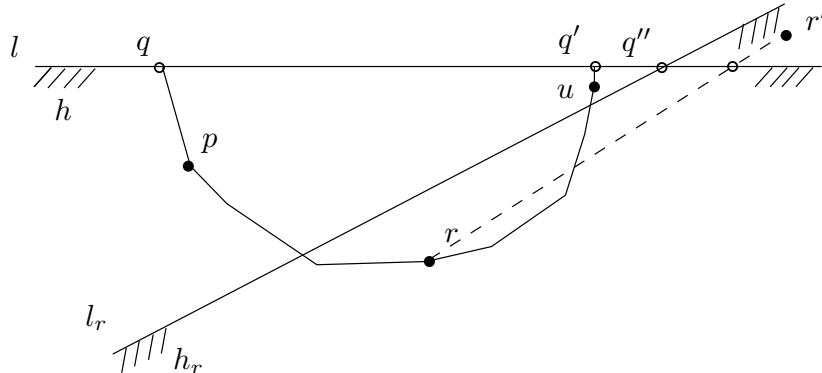


Figure 2: The line l intersects the boundary of $CH(P)$ in two points.

therefore there is at least one point r' that is contained in $h_r \cap P$ and is not contained in h , hence it lies above the line l . The segment rr' must intersect the line l to the right of the point q'' . Also, by convexity, the segment rr' is contained in $CH(P)$. This implies that the line l must intersect $\partial CH(P)$ to the right of q' , i.e intersects $\partial CH(P)$ in three points, a contradiction.

□

We are ready to prove the second part of Theorem 2.1: Recall that for a given finite planar set $P \subset \mathbb{R}^2$ and an integer k , we need to show that there is a k -coloring for P such that every half-plane that contains at least $2k - 1$ points is polychromatic.

For $k = 1$ the theorem is obvious. For $k = 2$, put $t = 3$ and let N be a hitting set as in lemma 2.2. We assign the points of N the color 2 and assign the points of $P \setminus N$ the color 1. Let h be a half-plane such that $|h \cap P| \geq 3$. Assume without loss of generality that h is a negative half-plane. Let l denote the line bounding h . Translate l downwards to obtain a line l' , such that for the negative half-plane h' bounded by l' , we have $h' \cap P \subseteq h \cap P$ and $|h' \cap P| = 3$. We can assume without loss of generality that no line parallel to l passes through two points of P . Indeed, this can be achieved by rotating l slightly. Obviously $h' \cap N \neq \emptyset$. Moreover, by lemma 2.2 we have that $h' \cap (P \setminus N) \neq \emptyset$. Hence, h' contains both a point colored with 1 and a point colored with 2, i.e., h' is polychromatic. Thus h is also polychromatic.

We prove the theorem by induction on the number of colors k . The induction hypothesis is that the theorem holds for all values $i < k$. Let $k > 2$ be an integer. Put $t = 2k - 1$ and let N be a minimal hitting set as in Lemma 2.2. We assign all points in N the color k . Put $P' = P \setminus N$. By the induction hypothesis, we can color the points of P' with $k - 1$ colors, such that for every half-plane h with $|h \cap P'| \geq 2k - 3$, h is polychromatic, i.e., h contains representative points from all the $k - 1$ color classes. We claim that this coloring together with the color class N forms a valid k -coloring for P . Consider a half-plane h such that $|h \cap P| \geq 2k - 1$. As before, let h' be a half-plane such that $h' \cap P \subseteq h \cap P$ and $|h' \cap P| = 2k - 1$. It is enough to show that h' is polychromatic. By lemma 2.2 we know that $1 \leq |h' \cap N| \leq 2$, therefore we can find a half-plane h'' such that $h'' \cap P \subseteq h' \cap P$ and $|h'' \cap (P \setminus N)| = 2k - 3$. By the induction hypothesis, h'' contain representative points from all the initial $k - 1$ colors. Thus h' contain a point from N (i.e., colored with k) and a point from each of the initial $k - 1$ colors. Hence h' is polychromatic and so is h . This completes the proof of the theorem.

Remark: The above theorem also provides a recursive algorithm to obtain a valid k -coloring for a given finite set P of points. See Algorithm 1. Here, we do not care about the running time of the algorithm. Assume that we have a “black-box” that finds a hitting set N as in lemma 2.2 in time bounded by some function $f(n, t)$.

Algorithm 1: Algorithm for polychromatic k -coloring

Input: A finite set $P \subset \mathbb{R}^2$ and an integer $k \geq 1$
Output: A polychromatic k -coloring $\chi : P \rightarrow \{1, \dots, k\}$

```

begin
  if  $k=1$  then
    | Color all points of  $P$  with color 1.
  end
  else
    | Find a minimal hitting set  $N$  as in lemma 2.2 for all the half-planes of size
    |  $2k - 1$ .
    | Color the points in  $N$  with color  $k$ .
    | Set  $P = P \setminus N$  and  $k = k - 1$ . Recursively color  $P$  with  $k$  colors.
  end
end

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Note that a trivial bound on the total running time of the algorithm is $\sum_{i=1}^k f(n, 2i - 1)$.

3 Coloring Half-Planes with Respect to Points

Keszegh [8] investigated the value of $\bar{f}_{\mathcal{H}}(2)$ and proved that $\bar{f}_{\mathcal{H}}(2) \leq 4$. Recently Fulek [5] showed that in fact $\bar{f}_{\mathcal{H}}(2) = 3$. For the general case, Aloupis *et al.* proved in [3] that $\bar{f}_{\mathcal{H}}(k) \leq 8k - 3$. We obtain an improved bound of $\bar{f}_{\mathcal{H}}(k) \leq 3k - 2$. Before proving this bound, let us show a simple proof of the weaker bound $\bar{f}_{\mathcal{H}}(k) \leq 4k - 3$.

Theorem 3.1. $\bar{f}_{\mathcal{H}}(k) \leq 4k - 3$

Theorem 3.1 is a direct corollary of Theorem 2.1 and uses a reduction to coloring points in the plane. This reduction was also used in [3].

Proof. Let $H \subseteq \mathcal{H}$ be a finite set of half-planes. We partition H into two disjoint sets H^+ and H^- where $H^+ \subset H$ (respectively $H^- \subset H$) is the set of all positive half-planes (respectively negative half-planes). It is no loss of generality to assume that all lines bounding the half-planes in H are distinct. Indeed, by a slight perturbation of the lines, one can only obtain a superset of hyperedges in the corresponding hypergraph (i.e, a superset of cells in the arrangement of the bounding lines). Let L^+ (respectively L^-) be the sets of lines bounding the half-planes in H^+ (respectively H^-). Next, we use a standard (incidence-preserving) dualization to transform the set of lines L^+ (respectively L^-) to a set of points L^{+*} (respectively L^{-*}). It has the property that a point p is above (respectively incident or below) a line l if and only if the dual line p^* is above (respectively incident or below) the point l^* . See Figure 3 for an illustration.

We then color the sets L^{+*} and L^{-*} independently. We color each of them with k -colors so that every half-plane containing $2k - 1$ points of a given set, is also polychromatic.

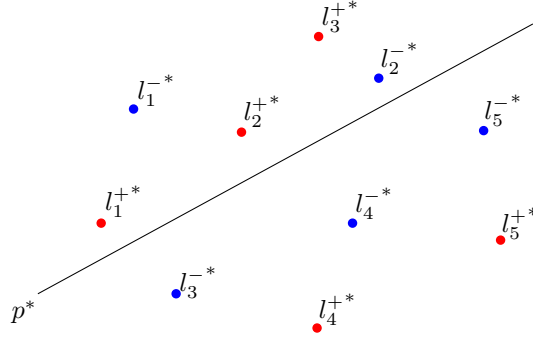


Figure 3: An illustration of the dualization. In the primal, the point p is contained in the half-planes bounded by the lines l_1^-, l_2^-, l_4^+ and l_5^+

Obviously, by Theorem 2.1, such a coloring can be found. This coloring induces the final coloring for the set $H = H^+ \cup H^-$. To prove that this coloring is indeed valid, consider a point p in the plane. Let $H' \subseteq H$ be the set of half-planes containing p . We claim that if $|H'| \geq 4k - 3$ then H' is polychromatic. Indeed, if $|H'| \geq 4k - 3$ then, by the pigeon-hole principle, either $|H' \cap H^+| \geq 2k - 1$ or $|H' \cap H^-| \geq 2k - 1$. Suppose without loss of generality that $|H' \cap H^+| \geq 2k - 1$. Let $L_{H'}^+ \subseteq L^+$ be the set of lines bounding the half-planes in $H' \cap H^+$. In the dual, the points in $L_{H'}^{+,*}$ are in the half-plane below the line p^* . Since $|L_{H'}^{+,*}| \geq 2k - 1$, we also have that $L_{H'}^{+,*}$ is polychromatic, thus the set of half-planes $H' \cap H^+$ is polychromatic and so is the set H' . This completes the proof of the theorem. \square

As promised, we further improve the bound on $\bar{f}_{\mathcal{H}}(k)$.

Theorem 3.2. $\bar{f}_{\mathcal{H}}(k) \leq 3k - 2$

In the proof of Theorem 3.2 we use polar point-line duality (which was also used in [5]). However, this duality applies only if there exists a point $p \in \mathbb{R}^2$ that is not covered by any half-plane in H . If this is not the case, we use the following known fact.

Proposition 3.3. *Let H be a set of half-spaces in \mathbb{R}^d ($|H| \geq d + 1$) which covers \mathbb{R}^d . Then there is a subset $H' \subset H$ such that $|H'| = d + 1$ and H' covers \mathbb{R}^d .*

Proof. Assume to the contrary that any subset of H which covers \mathbb{R}^d is of size at least $d + 2$. Let $H' = \{h_1, \dots, h_n\} \subset H$ ($n \geq d + 2$) be a (containment) minimal subset which covers \mathbb{R}^d , i.e., a subset of half-spaces such that for any half-space $h \in H'$, the set $H' \setminus \{h\}$ does not cover \mathbb{R}^d . Since H' is minimal, there exists a point set $P = \{p_1, \dots, p_n\}$ such that every p_i is contained in a half-space $h_i \in H'$ and is not contained in any other half-space in H' , i.e., $p_i \in h_i \setminus \bigcup_{j \neq i} h_j$, for every $1 \leq i \leq n$. It is easy to see that the set P is in convex position. Since $|P| \geq d + 2$ by Radon's lemma (see, e.g., [10]), there exists a partition $P = P_1 \cup P_2$ such that $P_1 \cap P_2 = \emptyset$ and $CH(P_1) \cap CH(P_2) \neq \emptyset$. Let $r \in CH(P_1) \cap CH(P_2)$. Notice that, since P is in convex position, $r \notin P$. Also note that for every half-space h containing r we have $h \cap P_1 \neq \emptyset$ and $h \cap P_2 \neq \emptyset$, and since $r \notin P$, $|h \cap P| \geq 2$. We assumed that H' covers \mathbb{R}^2 . Let $h_i \in H'$ be a half-space that contains r . We have $|h_i \cap P| \geq 2$, a contradiction to the fact that $h_i \cap P = \{p_i\}$. \square

As mentioned, we need the notion of point-line polar duality. A point $p = (a, b)$ (such that $(a, b) \neq (0, 0)$) is transformed to a line p^* with parameterization $ax + by = 1$, and a line

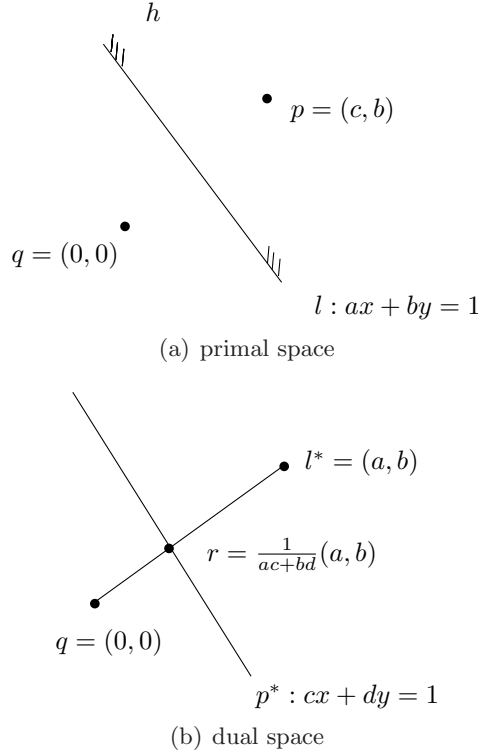


Figure 4: point-line polar duality example

$l : ax + by = 1$ is transformed to a point $l^* = (a, b)$. For the proof of Theorem 3.2, we need the following proposition regarding point-line polar duality.

Proposition 3.4. *Let H be a set of half-planes such that H does not cover \mathbb{R}^2 . Let $q \in \mathbb{R}^2$ be a point not covered by H . Assume without loss of generality that q is the origin, $q = (0, 0)$. Let p be a point contained in a half-plane $h \in H$, and let l_h be the boundary line of h . Then the dual line p^* intersects the segment $\overline{ql_h^*}$, where l_h^* is the point dual to l_h . See Figure 4 for an illustration.*

Proof. Assume that l_h has parameterization $ax + by = 1$ and $p = (c, d)$. We know that h is a half-plane that does not contain the origin, i.e. $h = \{(x, y) | ax + by \geq 1\}$, therefore $ac + bd \geq 1$. The line that passes through the points $q = (0, 0)$ and $l_h^* = (a, b)$ and the line p^* intersect in the point $r = (\frac{a}{ac+bd}, \frac{b}{ac+bd}) = \frac{1}{ac+bd}(a, b)$. Since $ac + bd \geq 1$, r must lie on the segment $\overline{ql_h^*}$. Hence the line p^* intersects the segment $\overline{ql_h^*}$ as claimed. \square

We are ready to prove Theorem 3.2.

Proof. Let H be a finite set of half-planes. If H covers \mathbb{R}^2 , by Proposition 3.3, we know that there is a subset $H_1 \subset H$ of 3 half-planes that covers \mathbb{R}^2 . If $H \setminus H_1$ covers \mathbb{R}^2 , again by Proposition 3.3 there is a subset $H_2 \subset H \setminus H_1$ of 3 half-planes that covers \mathbb{R}^2 . Let i be the number of subsets of half-planes that we find in such a way until we obtain a subset $H' = H \setminus \bigcup_{j=1}^i H_j$ such that H' does not cover \mathbb{R}^2 .

For every $1 \leq j \leq i$, we assign to the 3 half-planes in H_j the color j . If $i < k$ we color the half-planes in H' with $k - i$ colors $\{i + 1, i + 2, \dots, k\}$ in the following way. Since H' does

not cover \mathbb{R}^2 , there is a point $q \in \mathbb{R}^2$ that is not covered by any half-plane in H' . Without loss of generality, assume that q is the origin. Let $L_{H'}$ be the set of the bounding lines of the half-planes in H' . We transform the set of lines $L_{H'}$ to a set of points $L_{H'}^*$ using point-line polar duality. We color the points in $L_{H'}^*$ with $k - i$ colors using the colors $\{i + 1, i + 2, \dots, k\}$ such that every half-plane that contains at least $2(k - i) - 1$ points is polychromatic. Such a coloring exists by Theorem 2.1.

Next, we show that the above coloring ensures that every point covered by at least $3k - 2$ half-planes is also covered by some half-plane from every color class. If $i \geq k$ then, by our construction every point in the plane is covered by a half-plane from every color class. If $i < k$, let $p \in \mathbb{R}^2$ be some point that is covered by at least $3k - 2$ half-planes, p is covered by at most $3i$ half-planes from the set $\bigcup_{j=1}^i H_j$, therefore p is covered by at least $3k - 2 - 3i = 3(k - i) - 2$ half-planes from H' . Since $k - i \geq 1$ we have that p is covered by at least $2(k - i) - 1$ half-planes from H' . Denote those covering half-planes by H'' . By the property of our coloring of H' we have that H'' contains a half-plane from every color class $c \in \{i + 1, i + 2, \dots, k\}$. Hence the point p is polychromatic, i.e., covered by a half-plane from every color class $\{1, \dots, k\}$. This completes the proof. \square

Remark: Obviously, the result of Fulek [5] implies that the bound $3k - 2$ is not tight already for $k = 2$. It would be interesting to find the exact value of $\bar{f}_{\mathcal{H}}(k)$ for every integer k .

4 Small Epsilon-Nets for Half-Planes

Consider a hypergraph $H = (P, E)$ where P is a set of n points in the plane and $E = \{P \cap h : h \in \mathcal{H}\}$. As mentioned in the introduction, Woeginger [19] showed that for any $1 \geq \epsilon > 0$ there exists an ϵ -net for H of size at most $\frac{2}{\epsilon} - 1$.

As a corollary of Theorem 2.1, we obtain yet another proof for this fact. Recall that for any integer $k \geq 1$ we have $f_{\mathcal{H}}(k) \leq 2k - 1$. Let $\epsilon > 0$ be a fixed real number. Put $k = \lceil \frac{\epsilon n + 1}{2} \rceil$. Let χ be a k -coloring as in Theorem 2.1. Notice that every half-plane containing at least ϵn points contains at least $2k - 1$ points of P . Indeed such a half-plane must contain at least $\lceil \epsilon n \rceil = \lceil 2(\frac{\epsilon n + 1}{2}) - 1 \rceil = 2k - 1$. Such a half-plane is polychromatic with respect to χ . Thus, every color class of χ is an ϵ -net for H . Moreover, by the pigeon-hole principle one of the color classes has size at most $\frac{n}{k} \leq \frac{2n}{\epsilon n + 1} < \frac{2}{\epsilon}$. Thus such a set contains at most $\frac{2}{\epsilon} - 1$ points as asserted.

The arguments above are general and, in fact, we have the following theorem:

Theorem 4.1. *Let \mathcal{R} be a family of regions such that $f_{\mathcal{R}}(k) \leq ck$ for some absolute constant c and every integer k . Then for any ϵ and any finite set P there exists an ϵ -net for P with respect to \mathcal{R} of size at most $\frac{c}{\epsilon} - 1$.*

Applying the above theorem for the dual range space defined by a set of n half-planes with respect to points and plugging Theorem 3.2 we conclude that there exists an ϵ -net for such a range-space of size at most $\frac{3}{\epsilon} - 1$. However, using a more clever analysis one can, in fact, show that for every $\epsilon \leq \frac{2}{3}$ there is an ϵ -net of size at most $\frac{2}{\epsilon}$ for such a range-space. Indeed, let H be a set of half-planes. If H does not cover \mathbb{R}^2 , by using the polar point-line transformation we can obtain a coloring of the half-planes H such that every point that is covered by at least $2k - 1$ half-planes is polychromatic. Hence there is an ϵ -net of size $\frac{2}{\epsilon} - 1$.

If H covers \mathbb{R}^2 then by Proposition 3.3 we can find a set of 3 half-planes $G \subset H$ such that G covers \mathbb{R}^2 . The set G is an ϵ -net, i.e. for $\epsilon \leq \frac{2}{3}$ we have an ϵ -net of size $\frac{2}{\epsilon}$.

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