

ON A FORMULA OF GAMMELGAARD FOR BEREZIN-TOEPLITZ QUANTIZATION

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ABSTRACT. We give a proof of a slightly refined version of Gammelgaard's graph theoretic formula for Berezin-Toeplitz quantization on (pseudo-)Kähler manifolds. Our proof has the merit of giving an alternative approach to Karabegov-Schlichenmaier's identification theorem. We also identify the dual Karabegov-Bordemann-Waldmann star product.

1. INTRODUCTION

Deformation quantization on a symplectic manifold M was introduced by Bayen et al. [2] as a deformation of the usual pointwise product of $C^\infty(M)$ into a noncommutative associative \star -product of the formal series $C^\infty(M)[[\nu]]$. The celebrated work of Kontsevich [25] completely solved existence and classification of star-products on general Poisson manifolds. See [10] for a comprehensive survey of deformation quantization on symplectic and Poisson manifolds.

This paper will restrict to study differentiable deformation quantization with separation of variables on a Kähler manifold (see below for definitions). Let (M, g) be a Kähler manifold of dimension n . On a coordinate chart Ω , the Kähler form is given by

$$\omega_g = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}.$$

If Ω is contractible, there exists a Kähler potential Φ satisfying

$$\partial\bar{\partial}\Phi = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}.$$

Around any point $x \in M$, there exists a normal coordinate system such that

$$(1) \quad g_{i\bar{j}}(x) = \delta_{ij}, \quad g_{i\bar{j}k_1 \dots k_r}(x) = g_{i\bar{j}\bar{l}_1 \dots \bar{l}_r}(x) = 0$$

for all $r \leq N \in \mathbb{N}$, where N can be chosen arbitrary large and $g_{i\bar{j}k_1 \dots k_r} = \partial_{k_1 \dots k_r} g_{i\bar{j}}$.

The canonical Poisson bracket of two functions $f_1, f_2 \in C^\infty(M)$ is given by

$$(2) \quad \{f_1, f_2\} = ig^{k\bar{l}} \left(\frac{\partial f_1}{\partial z^k} \frac{\partial f_2}{\partial \bar{z}^l} - \frac{\partial f_2}{\partial z^k} \frac{\partial f_1}{\partial \bar{z}^l} \right).$$

Let $C^\infty(M)[[\nu]]$ denote the algebra of formal power series in h over $C^\infty(M)$. A star product is an associative $C[[\nu]]$ -bilinear product \star such that $\forall f_1, f_2 \in C^\infty(M)$,

$$(3) \quad f_1 \star f_2 = \sum_{j=0}^{\infty} \nu^j C_j(f_1, f_2),$$

where the \mathbb{C} -bilinear operators C_j satisfy

$$(4) \quad C_0(f_1, f_2) = f_1 f_2, \quad C_1(f_1, f_2) - C_1(f_2, f_1) = i\{f_1, f_2\}.$$

A star product is called *differentiable*, if each C_j is a bidifferential operator. According to Karabegov [18], a star product has the property of *separation of variables* (*Wick type*), if it satisfies $f \star h = f \cdot h$ and $h \star g = h \cdot g$ for any locally defined antiholomorphic function f , holomorphic function g and an arbitrary function h . If the role of holomorphic and antiholomorphic variables are swapped, we call

Key words and phrases. Berezin-Toeplitz quantization, Berezin transform, Karabegov form.
MSC(2010) 53D55 32J27.

it a star product of *anti-Wick type*. In particular, the Berezin-Toeplitz star product is of Wick type and the Berezin star product is of anti-Wick type. They are dual opposite to each other.

Motivated by the work of Reshetikhin and Takhtajan [31], Gammelgaard [16] obtained a remarkable universal formula for any star product with separation of variables corresponding to a given classifying Karabegov form. Using Karabegov-Schlichenmaier's identification theorem [24] for Berezin-Toeplitz quantization (cf. Theorem 5.1), Gammelgaard's formula specializes to a graph expansion for Berezin-Toeplitz quantization over acyclic graphs, which can be equivalently formulated in the following theorem. See Remark 4.2.

Theorem 1.1. *The Berezin-Toeplitz star product \star_{BT} on a Kähler manifold is given by*

$$(5) \quad f_1 \star_{BT} f_2(x) = \sum_{\Gamma=(V \cup \{f\}, E) \in \mathcal{G}_{BT}^{ss}} \nu^{|E|-|V|} \frac{(-1)^{|E|}}{|\text{Aut}(\Gamma)|} \Gamma(f_1, f_2),$$

where \mathcal{G}_{BT}^{ss} (see (27) for the definition) is a certain subset of strongly connected semistable one-pointed graphs and $\Gamma(f_1, f_2)$ is the partition function of Γ (see Definition 3.8).

In [37], an explicit formula of Berezin transform in terms of strongly connected graphs was obtained, building on the works of Engliš [13], Loi [26], and related results of Charles [8]. At a first look, acyclic graphs and strongly connected graphs are quite different. As noted by Schlichenmaier [35], it shall be interesting to clarify their relations.

We will give a purely graph theoretic proof of Theorem 1.1 in §4 by developing a technique of computing the inverse to the Berezin transform. Our proof also provides some clarifications to Gammelgaard's formula and Karabegov-Schlichenmaier's identification theorem for Berezin and Berezin-Toeplitz star products. In §6, we identify the Karabegov form of the dual Karabegov-Bordemann-Waldmann star product.

All graphs in this paper represent partial derivatives of Kähler metrics and functions. The following two identities can be used to convert covariant derivatives of curvature tensors to partial derivatives of metrics and vice versa.

$$(6) \quad R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}k\bar{l}} + g^{m\bar{p}} g_{m\bar{j}\bar{l}} g_{ik\bar{p}},$$

$$(7) \quad T_{\alpha_1 \dots \alpha_p; \gamma} = \partial_\gamma T_{\alpha_1 \dots \alpha_p} - \sum_{i=1}^p \Gamma_{\gamma \alpha_i}^\delta T_{\alpha_1 \dots \alpha_{i-1} \delta \alpha_{i+1} \dots \alpha_p},$$

where $T_{\alpha_1 \dots \alpha_p}$ is a covariant tensor and $\Gamma_{\beta\gamma}^\alpha = 0$ except for $\Gamma_{jk}^i = g^{i\bar{l}} g_{j\bar{l}k}$, $\Gamma_{\bar{j}\bar{k}}^{\bar{i}} = g^{\bar{l}i} g_{l\bar{j}\bar{k}}$.

We want to point out that for graph expressions in this paper, we can either sum over stable, semistable or all strongly connected graphs. Their difference is briefly indicated here. For a summation over stable graphs, the equation holds only at the center of the normal coordinate system, but it is enough to uniquely recover the curvature tensor expression. For a summation over semistable graphs, the equation holds globally, which is needed when taking derivatives. Finally, we may enlarge the summation over all strongly connected graphs in order to simplify the computation of derivatives (see Remark 3.7).

Acknowledgements The author thanks Professor Kefeng Liu, Hongwei Xu and Shing-Tung Yau for their kind help during his years of graduate studies at Center of Mathematical Sciences (CMS), Zhejiang University. This paper is dedicated to CMS on the occasion of its 10th birthday.

2. BEREZIN TRANSFORM

In this section, we briefly recall the asymptotic expansion of Berezin transform and related constructions of star products. More detailed expositions and historical remarks can be found in recent nice surveys [34, 35].

Let $\Phi(x, y)$ be an almost analytic extension of $\Phi(x)$ to a neighborhood of the diagonal, i.e. $\bar{\partial}_x \Phi$ and $\partial_y \Phi$ vanish to infinite order for $x = y$. We can assume $\overline{\Phi(x, y)} = \Phi(y, x)$.

For $m > 0$, consider the weighted Bergman space of all holomorphic function on Ω square-integrable with respect to the measure $e^{-m\Phi} \frac{w_g^n}{n!}$. We denote by $K_m(x, y)$ the reproducing kernel. Locally

$$(8) \quad K_m(x, y) \sim e^{m\Phi(x, y)} \sum_{k=0}^{\infty} B_k(x, y) m^{n-k},$$

which converges if Ω is a strongly pseudoconvex domain with real analytic boundary (cf. [3, 13]). For discussions on the convergence in the compact Kähler case, see [24, 38].

The asymptotic expansion of the Bergman kernel in the setting when Ω is a compact Kähler manifold was also extensively studied. For recent works, see e.g. [9, 11, 17, 27].

The *Berezin transform* is the integral operator

$$(9) \quad I_m f(x) = \int_{\Omega} f(y) \frac{|K_m(x, y)|^2}{K_m(x, x)} e^{-m\Phi(y)} \frac{w_g^n(y)}{n!}.$$

At any point for which $K_m(x, x)$ invertible, the integral converges for each bounded measurable function f on Ω . Note that (8) implies that for any x , $K_m(x, x) \neq 0$ if m is large enough.

The Berezin transform has an asymptotic expansion for $m \rightarrow \infty$ (cf. [13, 24]),

$$(10) \quad I^{(m)} f(x) = \sum_{k=0}^{\infty} Q_k f(x) m^{-k},$$

where Q_k are linear differential operators.

The Berezin star product was introduced by Berezin [3] through symbol calculus for linear operators on weighted Bergman spaces (cf. [7, 14, 34]). Karabegov [19] noted that any star product with separation of variables can be constructed from a unique formal Berezin transform. In our case, if we write

$$(11) \quad Q_j f = \sum_{\alpha, \beta \text{ multiindices}} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f,$$

then the coefficients of Berezin star product are given by bilinear differential operators

$$(12) \quad C_j(f_1, f_2) := \sum_{\alpha, \beta} c_{j\alpha\beta} (\bar{\partial}^\beta f_1)(\partial^\alpha f_2).$$

The Berezin star product \star_B is equivalent to the Berezin-Toeplitz star product \star_{BT} via the Berezin transform (cf. [24])

$$(13) \quad f_1 \star_{BT} f_2 = I^{-1}(I f_1 \star_B I f_2),$$

where $I := I^{(1/\nu)}$ is obtained from substituting m by $1/\nu$ in $I^{(m)}$.

Recall that the Toeplitz operator $T_f^{(m)}$ for $f \in C^\infty(M)$ is defined by

$$(14) \quad T_f^{(m)} := \Pi^{(m)}(f \cdot) : H^0(M, L^m) \rightarrow H^0(M, L^m),$$

where $\Pi^{(m)} : L^2(M, L^m) \rightarrow H^0(M, L^m)$ is the orthogonal projection. See [5] for a detailed study of their semiclassical properties. The following celebrated theorem of Schlichenmaier [32] shows that Berezin-Toeplitz operator quantization and deformation quantization are closed related.

Theorem 2.1 ([32]). *The Berezin-Toeplitz star product (13) is the unique star product*

$$(15) \quad f_1 \star_{BT} f_2 := \sum_{j=0}^{\infty} \nu^j C_j^{BT}(f_1, f_2),$$

such that the following asymptotic expansion holds

$$(16) \quad T_{f_1}^{(m)} T_{f_2}^{(m)} \sim \sum_{j=0}^{\infty} m^{-j} T_{C_j^{BT}(f_1, f_2)}^{(m)}, \quad m \rightarrow \infty.$$

The Berezin transform was introduced by Berezin [4] for symmetric domains in \mathbb{C}^n and later extended by many authors (see e.g. [12, 14]). Karabegov and Schlichenmaier [24] proved the asymptotic expansion of the Berezin transform for compact Kähler manifolds.

Berezin-Toeplitz quantization for compact Kähler manifolds was extensively studied in the literature and had found many applications. Karabegov [21] constructed an algebra of Toeplitz elements that is isomorphic to the algebra of Berezin-Toeplitz quantization. Ma and Marinescu [29] developed the theory of Toeplitz operators on symplectic manifolds twisted with a vector bundle. Zelditch [39] studied quantizations of symplectic maps on compact Kähler manifolds and uncovered a connection between Berezin-Toeplitz quantization and quantum chaos. Andersen [1] proved asymptotic faithfulness of the mapping class groups action on Verlinde bundles by using Berezin-Toeplitz technique (cf. also [33]).

3. DIFFERENTIAL OPERATORS ENCODED BY GRAPHS

Throughout this paper, a *digraph*, or simply a graph, $G = (V, E)$ is defined to be a directed multi-graph, i.e. it has finite number of vertices and edges with multi-edges and loops allowed.

Recall the definition of stable and semistable graphs in [36, 37]. These graphs were used to represent Weyl invariants, which encode the coefficients of the asymptotic expansion.

Definition 3.1. We call a vertex v of a digraph G *semistable* if we have

$$\deg^-(v) \geq 1, \deg^+(v) \geq 1, \deg^-(v) + \deg^+(v) \geq 3.$$

v is call *stable* if $\deg^-(v) \geq 2, \deg^+(v) \geq 2$.

Definition 3.2. An *m-pointed graph* $\Gamma = (V \cup \{f_1, \dots, f_m\}, E)$ is defined to be a digraph with m distinguished vertex labeled by f_1, \dots, f_m . An automorphism of Γ fixes each of the m distinguished vertices and its automorphism group is denoted by $\text{Aut}(\Gamma)$. Γ is called *semistable* (*stable*) if each ordinary vertex $v \in V$ is semistable (*stable*). We denote by

- $V(\Gamma) = V \cup \{f_1, \dots, f_m\}$;
- Γ_- the subgraph of Γ obtained by removing distinguished vertices;
- $w(\Gamma) = |E| - |V|$ the weight of Γ ;
- $\overline{\mathcal{G}}_m$ the set of all m -pointed graphs;
- $\mathcal{G}_m^{\text{sccon}}$ the set of all strongly connected m -pointed graphs;
- $\mathcal{G}_m^{\text{ss}}$ the set of all strongly connected semistable m -pointed graphs;
- \mathcal{G}_m the set of all strongly connected stable m -pointed graphs.

For an m -pointed graph G , we denote by \dot{G} the one-pointed graph obtained by merging the m distinguished points of G into one point. G is called *strongly connected* if \dot{G} is strongly connected. It is obvious that $\mathcal{G}_m \subset \mathcal{G}_m^{\text{ss}} \subset \mathcal{G}_m^{\text{sccon}} \subset \overline{\mathcal{G}}_m$.

We also denote by

- $\overline{\mathcal{G}}_m(k)$ the set of all m -pointed graphs with weight k ;
- $\overline{\mathcal{G}} = \cup_{m \geq 0} \overline{\mathcal{G}}_m$ the set of all pointed graphs.
- \vdots

The same notation applies to other sets of graphs listed above.

The one-pointed graph in Figure 1 represents an Weyl invariant $g_{i\bar{i}k\bar{l}p}g_{j\bar{j}l\bar{k}q}f_{q\bar{p}}$ in partial derivatives of metrics and functions. Note that $(i, \bar{i}), (j, \bar{j})$ etc. are paired indices to be contracted. The weight of a directed edge is the number of multi-edges. The number attached to a vertex denotes the number of its self-loops. A vertex without loops will be denoted by a small hollow circle \circ . The distinguished vertex is denoted by a solid circle \bullet .

The following theorem on graph theoretic formulae for Bergman kernel and Berezin transform will be used in the sequel.

$$(17) \quad B(x) = \sum_{k \geq 0} \nu^k B_k(x), \quad If(x) = \sum_{k \geq 0} \nu^k Q_k f(x).$$

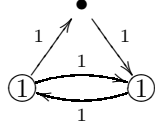


FIGURE 1. A strongly connected one-pointed graph

Theorem 3.3 ([36]). *On a Kähler manifold M , the Bergman kernel has the expansion*

$$(18) \quad B(x) = \exp \left(\sum_{G=(V,E) \in \mathcal{G}_0^{ss}} \nu^{|E|-|V|} \frac{-\det(A(G) - I)}{|\text{Aut}(G)|} G \right),$$

where G runs over all strongly connected semistable graphs.

In particular, there are two strongly connected semistable graphs of weight 1, so we have

$$(19) \quad B(x) = \exp \left(1 + \nu \left(-\frac{1}{2} [\textcircled{2}] + \frac{1}{2} \left[\circ \begin{array}{c} \xrightarrow{2} \\ \xleftarrow{1} \\ \circ \end{array} \right] \right) + O(\nu^2) \right).$$

Theorem 3.4 ([37]). *On a Kähler manifold M , the Berezin transform has the expansion*

$$(20) \quad I(f) = \sum_{\Gamma=(V \cup \{f\}, E) \in \mathcal{G}_1^{scon}} \nu^{|E|-|V|} \frac{\det(A(\Gamma_-) - I)}{|\text{Aut}(\Gamma)|} \Gamma.$$

In particular, coefficients up to ν^3 in (20) (keeping only stable graphs) were obtained in [37].

$$(21) \quad I(f) = f + \nu \left[\bullet \textcircled{1} \right] + \frac{\nu^2}{2} \left[\bullet \textcircled{2} \right] + \nu^3 \left(\frac{1}{6} \left[\bullet \textcircled{3} \right] - \frac{1}{4} \left[\circ \begin{array}{c} \xrightarrow{2} \\ \xleftarrow{2} \\ \circ \end{array} \bullet \right] \right. \\ \left. - \frac{1}{2} \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \xrightarrow{1} \circ \\ \swarrow 1 \quad \searrow 1 \\ \circ \end{array} \right] - \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \xrightarrow{1} \circ \\ \swarrow 1 \quad \searrow 1 \\ \circ \end{array} \right] + \frac{1}{2} \left[\textcircled{2} \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \\ \bullet \end{array} \right] \right) + O(\nu^4).$$

Note that in (20) we sum over all strongly connected one-point graph, no matter the graph is semistable or not. The reason is that it will simplify the computation of the inverse Berezin transform as explained in Remark 3.7.

Fix a normal coordinate system around x on a Kähler manifold M , then (18) and (20) still hold at x if we only sum over those stable graphs.

We now define three special subsets of \mathcal{G}_1 :

$$(22) \quad \mathcal{G}_B = \{\Gamma \in \mathcal{G}_1 \mid 1 \text{ is not an eigenvalue of } A(\Gamma_-)\},$$

$$(23) \quad \mathcal{G}_{BT} = \{\Gamma \in \mathcal{G}_1 \mid \text{each SCC of } \Gamma_- \text{ is either a single vertex or a linear digraph}\},$$

$$(24) \quad \mathcal{G}_S = \{\Gamma \in \mathcal{G}_1 \mid \text{each SCC of } \Gamma_- \text{ is a single vertex without loops}\},$$

where SCC is an abbreviation for *strongly connected component*. More explicitly, $\Gamma \in \mathcal{G}_{BT}$ if each SCC of Γ_- belongs to

$$(25) \quad \circ \textcircled{1} \quad \circ \begin{array}{c} \xrightarrow{1} \\ \xleftarrow{1} \\ \circ \end{array} \quad \begin{array}{c} \circ \\ \swarrow 1 \quad \searrow 1 \\ \circ \xrightarrow{1} \circ \\ \swarrow 1 \quad \searrow 1 \\ \circ \end{array} \quad \dots$$

And $\Gamma \in \mathcal{G}_S$ can also be characterized by saying that Γ_- is a directed acyclic graph (DAG).

Remark 3.5. For later use, we also need to relax the condition “stable” to semistable or just strongly connected in the above definition. The corresponding sets are denoted by $\mathcal{G}_B^{ss}, \mathcal{G}_{BT}^{ss}, \mathcal{G}_S^{ss}$ and $\mathcal{G}_B^{scon}, \mathcal{G}_{BT}^{scon}, \mathcal{G}_S^{scon}$. For example,

$$(26) \quad \mathcal{G}_{BT}^{scon} = \{\Gamma \in \mathcal{G}_1^{scon} \mid \text{each SCC of } \Gamma_- \text{ is either a single vertex} \\ \text{or a linear digraph}\},$$

$$(27) \quad \mathcal{G}_{BT}^{ss} = \{\Gamma \in \mathcal{G}_{BT}^{scon} \mid \Gamma \text{ is semistable}\}.$$

For any $k \geq 0$, the sets $\mathcal{G}_B(k), \mathcal{G}_{BT}(k), \mathcal{G}_S(k)$ are respectively in one-to-one correspondence with weight k terms of Berezin, Berezin-Toeplitz and Karabegov-Bordemann-Waldmann star products.

We have computed the cardinalities of these sets when $k \leq 6$ in Table 1.

TABLE 1. Numbers of strongly connected stable one-pointed graphs

k	0	1	2	3	4	5	6
$ \mathcal{G}_1(k) $	1	1	2	9	61	538	5906
$ \mathcal{G}_B(k) $	1	1	1	5	36	331	3704
$ \mathcal{G}_{BT}(k) $	1	1	2	6	24	112	620
$ \mathcal{G}_S(k) $	1	1	1	2	5	15	54

Definition 3.6. Let $\Gamma \in \overline{\mathcal{G}}_1$ be a one-pointed graph, the *partition function* $D_\Gamma(f_1, f_2)$ is defined to be a Weyl invariant generated from Γ by replacing the vertex f in with two vertices f_1 and f_2 , such that all outward edges of f are connected to f_1 and all inward edges of f are connected to f_2 .

Let $H \in \overline{\mathcal{G}}$ be an arbitrary pointed graph, we define $D_\Gamma(H)$ to be the Weyl invariant generated by replacing f in Γ with H .

For example, if Γ is the graph in Figure 1, then

$$(28) \quad D_\Gamma(f_1, f_2) = g_{i\bar{i}k\bar{l}p} g_{j\bar{j}l\bar{k}\bar{q}} \partial_q f_1 \partial_{\bar{p}} f_2,$$

$$(29) \quad D_\Gamma(H) = g_{i\bar{i}k\bar{l}p} g_{j\bar{j}l\bar{k}\bar{q}} \partial_{q\bar{p}} H.$$

Similarly for two pointed graphs $H_1, H_2 \in \overline{\mathcal{G}}$, we define $D_\Gamma(H_1, H_2)$ to be the Weyl invariant generated by replacing f_1, f_2 with H_1, H_2 in $D_\Gamma(f_1, f_2)$. We may linearly extend $D_\Gamma(\cdot)$ to be defined on linear combination of graphs. We also use the notation $D_\Gamma^{op}(H_1, H_2) = D_\Gamma(H_2, H_1)$.

When expanding $\partial_{q\bar{p}} H$ in (29), we may need to take derivatives of $g^{i\bar{j}}$,

$$(30) \quad \partial_\alpha g^{i\bar{j}} = -g^{p\bar{j}} g^{i\bar{q}} g_{p\bar{q}\alpha},$$

where new vertex appears as shown in Figure 2.

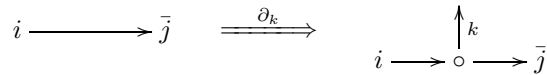


FIGURE 2. Illustration of $\partial_k g^{i\bar{j}} = -g^{p\bar{j}} g^{i\bar{q}} g_{p\bar{q}k}$

Remark 3.7. In the recursive computation of the inverse Berezin transform I^{-1} from $I^{-1}I = id$ or $I \cdot I^{-1} = id$, we need to handle terms like (29). The possible action on the edge (cf. Figure 2) made the computation much more complicated. However, thanks to the special structure of I and I^{-1} , we may extend their summations over all strongly connected graphs (cf. (20), (34)) such that when taking derivatives on a graph, the derivatives only go to vertices. This follows from Lemma 3.9 (note the minus sign in (30)) and the fact that any strongly connected graph can be obtained by adding finite number of vertices to edges of a semistable graph. Adding a vertex to an edge may change the automorphism group of a graph, but this will be compensated when taking derivatives by Leibniz rule. This assertion can be made rigorous by following the argument of Lemma 3.10. For example,

the automorphism group of the left-hand side (LHS) graph in Figure 3 has order 2. It becomes rigid (i.e. have no nontrivial automorphism) after adding a vertex on an edge. But there are exactly two ways of adding a vertex to the LHS graph to turn it into the RHS graph. In fact, throughout the

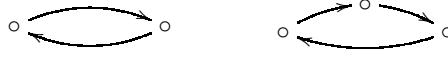


FIGURE 3. Adding a vertex to an edge

paper, when dealing with $D_\Gamma(H)$ and $D_\Gamma(H_1, H_2)$, it is sufficient to stipulate that no derivatives will be taken on edges. Thus we will be able to use Lemma 3.10.

For convenience, we introduce the following notation.

Definition 3.8. Let $\Gamma \in \overline{\mathcal{G}}_1$ and $H, H_1, H_2 \in \overline{\mathcal{G}}$. Define $\Gamma(H)$ and $\Gamma(H_1, H_2)$ respectively to be the summation of those graph terms in the expansion of $D_\Gamma(H)$ and $D_\Gamma(H_1, H_2)$ with derivatives only acting on the vertices of H, H_1, H_2 . Namely we discard all graphs in $D_\Gamma(H)$ and $D_\Gamma(H_1, H_2)$ if some vertex of it was created through taking derivative on an edge. We also use the notation $\Gamma^{op}(H_1, H_2) = \Gamma(H_2, H_1)$.

Lemma 3.9. Let $\Gamma \in \overline{\mathcal{G}}_1$ be a one-pointed graph and Γ' be the graph obtained by adding a vertex to an edge $e = (v_1, v_2)$ of Γ . Then $\det(A(\Gamma'_-) - I) = -\det(A(\Gamma_-) - I)$.

Proof. First we assume that both v_1, v_2 are not the distinguished vertex of Γ . Let v_1, \dots, v_n be the vertices of Γ_- and a_{ij} be the number of directed edges from v_i to v_j . Then

$$A(\Gamma'_-) - I = \begin{bmatrix} -1 & 0 & 1 & \dots & 0 \\ 1 & a_{11}-1 & a_{12}-1 & & \\ 0 & a_{21} & a_{22}-1 & & \\ \vdots & & & \ddots & \vdots \\ 0 & & \dots & & a_{nn}-1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 \\ 1 & a_{11}-1 & a_{12} & & \\ 0 & a_{21} & a_{22}-1 & & \\ \vdots & & & \ddots & \vdots \\ 0 & & \dots & & a_{nn}-1 \end{bmatrix},$$

which implies that $\det(A(\Gamma'_-) - I) = -\det(A(\Gamma_-) - I)$.

If at least one of v_1, v_2 is the distinguished vertex, the lemma is obvious. \square

Let $G, H \in \overline{\mathcal{G}}_m$ be m -pointed graphs and H be a subgraph of G requiring that the m distinguished points of H are exactly the m distinguished points of G . We define G/H to be a one-pointed graph obtained from G by contracting H to a point, which will be the distinguished vertex of G/H . The ordinary vertices of G/H are just the vertices not in H and edges of G/H are just edges not in H .

Let $\Gamma \in \overline{\mathcal{G}}_1$ and $G, H \in \overline{\mathcal{G}}_m$. We define a natural number

$$(31) \quad \alpha(\Gamma, H; G) := \#\{\text{subgraphs } H' \text{ of } G \mid H' \cong H, G/H' \cong \Gamma\}$$

Lemma 3.10. Let $\Gamma \in \overline{\mathcal{G}}_1$ and $H \in \overline{\mathcal{G}}$. Then

$$(32) \quad \frac{1}{|\text{Aut}(\Gamma)||\text{Aut}(H)|} \Gamma(H) = \sum_G \frac{\alpha(\Gamma, H; G)}{|\text{Aut}(G)|} G,$$

where G in the summation runs over isomorphism classes of graphs appearing in the expansion of $\Gamma(H)$ by Leibniz rule.

Proof. Note that the group $\text{Aut}(\Gamma) \times \text{Aut}(H)$ has a natural action on the multiset of all graphs in the expansion of $\Gamma(H)$ by Leibniz rule. Then it is not difficult to see that the set of orbits corresponds to isomorphism classes of graphs and the isotropy group at a graph G is $\text{Aut}(G)_{H'}$, the subgroup of $\text{Aut}(G)$ that leaves invariant a subgraph H' in the set at the right-hand side of (31). So we have

$$\Gamma(H) = \sum_G \frac{|\text{Aut}(\Gamma)||\text{Aut}(H)|}{|\text{Aut}(G)_{H'}|} G = \sum_G \frac{\alpha(\Gamma, H; G)|\text{Aut}(\Gamma)||\text{Aut}(H)|}{|\text{Aut}(G)|} G,$$

as claimed. \square

Definition 3.11. Given a digraph G , we call a subgraph L of G a *generalized linear subgraph* if each connected component of L is either a vertex without loops or a linear graph (i.e. belongs to the graphs in (25)). We denote by $\mathcal{L}(G)$ the set of all spanning generalized linear subgraphs L of G .

The following theorem is a corollary of the so-called coefficient theorem in spectral graph theory.

Theorem 3.12. *For a digraph G with n vertices, we have*

$$(33) \quad \det(A(G) - I) = \sum_{L \in \mathcal{L}(G)} (-1)^{n+p(L)} = \sum_{L \in \mathcal{L}(G)} \prod_{C \in L} (-1)^{\ell(C)+1},$$

where $p(L)$ denotes the number of components of L and C runs over components of L and $\ell(C)$ denotes the length of C . We regard a single vertex as a 0-cycle and a loop as a 1-cycle.

4. BEREZIN-TOEPLITZ QUANTIZATION

The first few terms of Berezin-Toeplitz quantization have been computed in [14, 24, 28, 37]. Since Berezin-Toeplitz quantization corresponds to the inverse Berezin transform, the following theorem completely determines the structure of Berezin-Toeplitz quantization and implies (5) in Theorem 1.1.

Theorem 4.1. *On a Kähler manifold, the inverse Berezin transform is given by*

$$(34) \quad I^{-1}(f) = \sum_{\Gamma=(V \cup \{f\}, E) \in \mathcal{G}_{BT}^{scn}} \nu^{|E|-|V|} \frac{(-1)^{|E|}}{|\text{Aut}(\Gamma)|} \Gamma.$$

In particular, coefficients up to ν^3 in (34) (keeping only stable graphs) were obtained in [37].

$$(35) \quad I^{-1}(f) = f - \nu \left[\bullet \circlearrowleft_1 \right] + \nu^2 \left(\frac{1}{2} \left[\bullet \circlearrowleft_2 \right] - \left[\textcircled{1} \right] \right) \\ + \nu^3 \left(-\frac{1}{6} \left[\bullet \circlearrowleft_3 \right] + \left[\textcircled{1} \right] + \frac{1}{4} \left[\circ \right] + \frac{1}{2} \left[\textcircled{2} \right] \right. \\ \left. + \frac{1}{2} \left[\textcircled{1} \right] - \left[\textcircled{1} \right] \right) + O(\nu^4).$$

The 24 strongly connected stable graphs in $\mathcal{G}_{BT}(4)$ are listed in Table 2.

Remark 4.2. Gammelgaard's universal formula [16] was expressed as a summation over acyclic graphs with two external vertices (one sink and one source). All internal vertices are weighted. We refer the reader to [35] for a brief summary of Gammelgaard's work. In fact, if we identify the two external vertices, then we get a strongly connected one-pointed graph Γ such that Γ_- is acyclic. In the case of Berezin-Toeplitz quantization, by Karabegov-Schlichenmaier's identification theorem, a vertex of weight -1 is just a single vertex without loops and a vertex of weight 0 comes from the Ricci curvature, which may be regarded as a vertex with exactly one loop. There are no vertices of weight greater than 0. If we repeatedly take derivatives on a vertex of weight 0 by Leibniz rule, we will get a cycle (see (30) and Figure 2). Finally we get a graph in \mathcal{G}_{BT} . It is not difficult to prove that Gammelgaard's formula for Berezin-Toeplitz quantization is equivalent to (5) in Theorem 1.1 by using Lemma 3.10 and Remark 3.7. For general quantizations, we may roughly say that the cycles in the graphs were absorbed into vertices with weight ≥ 0 in Gammelgaard's formula.

By Lemma 3.10 and (20), in order to prove (34), we need only prove the following purely graph theoretic theorem.

TABLE 2. The 24 graphs in $\mathcal{G}_{BT}(4)$

$\bullet 4$					
1/24	-1/4	-1/12	-1/12	-1/2	-1/2
-1/6	-1/2	-1/4	-1/6	1/2	1/8
1/4	1/2	1/2	1/4	1/2	1
1/2	1	1	1/2	1/2	-1

Theorem 4.3. Let $G \in \mathcal{G}_1^{scon}$ be a nontrivial strongly connected one-pointed graph. Then

$$(36) \quad \sum_{H \in \mathcal{B}(G)} (-1)^{|E(H)|} \det(A((G/H)_-) - I) = 0,$$

where $\mathcal{B}(G)$ consists of all strongly connected one-pointed subgraphs H of G with $H \in \mathcal{G}_{BT}^{scon}$.

Proof. First we consider a specific graph G of the form as depicted in Figure 4. Namely there exists

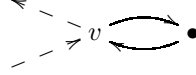


FIGURE 4. A one-pointed graph

an ordinary vertex v of G such that there is no other edge connected to \bullet besides an edge from v to \bullet and an edge from \bullet to v . Then the graph in $\mathcal{B}(G)$ is either the trivial graph \bullet or the graph with a cycle of length $\ell \geq 0$ attached at v . By Theorem 3.12, the left-hand side of (36) is equal to

$$\det(A(G_-) - I) + \sum_{L \in \mathcal{L}(G_-)} (-1)^{\ell(C_v)} \prod_{\substack{C \in L \\ v \notin C}} (-1)^{\ell(C)+1} = 0,$$

where C_v is the cycle in L containing v .

For general graph $G \in \mathcal{G}_1^{scon}$ and a fixed subgraph $H \in \mathcal{B}(G)$, the strongly connected components of H and a spanning generalized linear subgraph of $(G/H)_-$ together constitute a spanning generalized linear subgraph of G_- . On the other hand, for a fixed spanning generalized linear subgraph $L \in \mathcal{L}(G_-)$, we may compute the contributions of the summation in (36) to L . In fact, we may contract each cycle in L to a vertex, then it is not difficult to see from the following Lemma 4.4 that the contributions add up to zero for each $L \in \mathcal{L}(G_-)$. So we proved (36). \square

Lemma 4.4. Let $G \in \mathcal{G}_1^{scon}$ be a nontrivial strongly connected one-pointed graph. Then

$$(37) \quad \sum_{H=(V \cup \{f\}, E) \in \mathcal{S}(G)} (-1)^{w(H)} = 0,$$

where $\mathcal{S}(G)$ consists of all strongly connected one-pointed subgraphs H of G with $H \in \mathcal{G}_S^{scon}$ and $w(H) = |E| - |V|$ is the weight of H .

Proof. First we note that the lemma still holds for any one-pointed graph $G \in \overline{\mathcal{G}}_1$ such that the strongly connected component containing the distinguished vertex \bullet has at least one edge.

Next we prove that if the distinguished vertex of G has $k > 0$ loops, denoted by $\{e_1, \dots, e_k\}$, then (37) holds. Denote by $\mathcal{S}'(G)$ the graphs H in $\mathcal{S}(G)$ such that H has no loops at the distinguished vertex. Then each graph in $\mathcal{S}(G)$ is obtained by attaching some loops in $\{e_1, \dots, e_k\}$ to a graph in $\mathcal{S}'(G)$. So the left-hand side of (37) is equal to

$$\sum_{H \in \mathcal{S}'(G)} \sum_{i=0}^k \binom{k}{i} (-1)^{w(H)+i},$$

which is zero if $k > 0$.

In order to prove (37), we may assume that G has no loops or multi-edges. In fact, if there are $k > 1$ edges from v_1 to another vertex v_2 , denote by G' the graph obtained from G by merging the k edges a single edge e , we have

$$\begin{aligned} \sum_{H \in \mathcal{S}(G)} (-1)^{w(H)} &= \sum_{\substack{H \in \mathcal{S}(G') \\ e \notin H}} (-1)^{w(H)} + \sum_{\substack{H \in \mathcal{S}(G') \\ e \in H}} \sum_{i=1}^k \binom{k}{i} (-1)^{w(H)+i-1} \\ &= \sum_{\substack{H \in \mathcal{S}(G') \\ e \notin H}} (-1)^{w(H)} + \sum_{\substack{H \in \mathcal{S}(G') \\ e \in H}} (-1)^{w(H)} \\ &= \sum_{H \in \mathcal{S}(G')} (-1)^{w(H)}. \end{aligned}$$

We may also assume that G_- is acyclic. This is because G can be written as a finite union of maximal strongly connected subgraphs G' with G'_- acyclic and each $H \in \mathcal{B}(G)$ must lie in at least one of these G' , so we can apply the inclusion-exclusion principle and use induction on the number of edges of G .

If G_- is acyclic, then there exists an ordinary vertex v which is a source of G_- , i.e. all edges entering v must come from the distinguished vertex \bullet . Since G is strongly connected, so there exists an edge e from \bullet to v . If we contract e and absorb v into \bullet in G , we get a new strongly connected graph G' with less number of vertices. It is not difficult to see that $H \in \mathcal{S}(G)$ are in one-to-one correspondence with $H' \in \mathcal{S}(G')$ given by

$$H' = \begin{cases} H & \text{if } e \notin H; \\ H/\{e\} & \text{if } e \in H, \end{cases}$$

where $H/\{e\}$ is obtained by contracting e in H . Moreover $w(H) = w(H')$. Namely we have

$$\sum_{H \in \mathcal{S}(G)} (-1)^{w(H)} = \sum_{H' \in \mathcal{S}(G')} (-1)^{w(H')}.$$

So the lemma follows by induction. \square

Now we verify the associativity of Berezin-Toeplitz star product directly from (5).

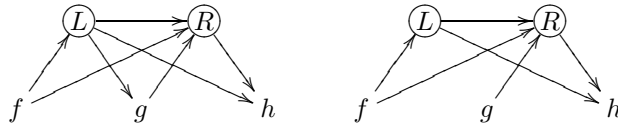


FIGURE 5. Two graphs with 3 distinguished vertices

Proposition 4.5. *For any three functions f, g, h on a Kähler manifold, we have*

$$(38) \quad (f \star_{BT} g) \star_{BT} h = f \star_{BT} (g \star_{BT} h).$$

Proof. By (5),

$$(39) \quad f \star_{BT} g = \sum_{\Gamma \in \mathcal{G}_1^{scn}} \nu^{w(h)} \frac{(-1)^{|E|}}{\text{Aut}(\Gamma)} \Gamma(f, g).$$

We need to prove that for any strongly connected three-pointed graph $G = (V \cup \{f, g, h\}, E)$, the coefficients of G at both sides of (38) are equal. All graphs G that may appear in (38) has two typical forms as depicted in in Figure 5 (each arrow may represent multi-edges).

We denote by \dot{G} the one-pointed graph obtained by merging the 3 distinguished vertices of G into one vertex (see Definition 3.2).

If G is the first graph, then the contributions of the two sides of (38) come from $\dot{R}(\dot{L}(f, g), h)$ and $\dot{L}(f, \dot{R}(g, h))$ respectively, the choices of subgraphs L and R in G are unique by their strongly connectedness. So they are equal by Lemma 3.10 and (39).

If G is the second graph, then the contributions of the two sides of (38) come from $\dot{G}(\dot{f}, h)$ and $\dot{L}(f, \dot{R}(g, h))$ respectively, which are again equal by Lemma 3.10 and (39). \square

5. KARABEGOV FORM OF A STAR PRODUCT

Let (M, ω_{-1}) be a pseudo-Kähler manifold, i.e. we only assume ω_{-1} to be a nondegenerate closed $(1, 1)$ -form. A formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal $(1, 1)$ -form,

$$(40) \quad \hat{\omega} = \frac{1}{\nu}\omega_{-1} + \omega_0 + \nu\omega_1 + \nu^2\omega_2 + \dots,$$

where each ω_k is a closed $(1, 1)$ -form. Karabegov [18] has shown that deformation quantizations with separation of variables on the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parametrized by such formal deformations.

Given a star product \star of anti-Wick type, its Karabegov form is computed as following: Let z^1, \dots, z^n be local holomorphic coordinates on an open subset U of M . Karabegov proved that there exists a set of formal functions on U , denoted by u^1, \dots, u^n ,

$$u^k = \frac{1}{\nu}u_{-1}^k + u_0^k + \nu u_1^k + \nu^2 u_2^k + \dots,$$

satisfying $u^k \star z^l - z^l \star u^k = \delta^{kl}$. Then the classifying Karabegov form of \star , which is a global form on M , is given by $\hat{\omega}|_U = -\sqrt{-1} \bar{\partial}(\sum_k u^k dz^k)$ on the coordinate neighborhood U .

In a remarkable paper [24], Karabegov and Schlichenmaier identified the corresponding Karabegov forms for Berezin and Berezin-Toeplitz star products on compact Kähler manifolds. Their original proof uses microlocal analysis to obtain off-diagonal expansion of Bergman kernel, then they derived integral representations of Berezin transform and a newly introduced twisted product in terms of off-diagonal Bergman kernels. The integral representations were used to prove the existence of their asymptotic expansions. Finally the Berezin-Toeplitz star product was identified by applying Schlichenmaier's Theorem 2.1, and the Berezin star product was identified using Zelditch's theorem [38].

We hope our approach through the graph theoretic formulae of Berezin transform and Bergman kernel could provide further insights to Karabegov-Schlichenmaier's identification theorem.

Theorem 5.1 ([24]). (i) *The Karabegov form of Berezin-Toeplitz star product is*

$$-\frac{1}{\nu}\omega_{-1} + Ric,$$

where $Ric = \sqrt{-1} \partial \bar{\partial} \log \det g$ is the Ricci curvature.

(ii) *The Karabegov form of Berezin star product is*

$$\frac{1}{\nu}\omega_{-1} + \sqrt{-1} \partial \bar{\partial} \log B(x),$$

where $B(x)$ is the Bergman kernel given in (18).

Proof. Let $\omega_{-1} = \sqrt{-1}\partial\bar{\partial}\Phi$, i.e. Φ is the potential function for the pseudo-Kähler metric ω_{-1} .

First we prove Theorem 5.1 (i). Since $\hat{\omega}|_U = -\sqrt{-1}\partial(\sum_k u^k dz^k)$, it is sufficient to prove that

$$(41) \quad u^k = -\frac{1}{\nu} \frac{\partial\Phi}{\partial z^k} + \frac{\partial \log \det g}{\partial z^k} = -\frac{1}{\nu} \frac{\partial\Phi}{\partial z^k} + g_{\bar{i}k}$$

satisfy the equation

$$(42) \quad \delta^{kl} = u^k \star_{BT}^{op} z^l - z^l u^k = \sum_{\substack{\Gamma \in \mathcal{G}_{BT} \\ w(\Gamma) > 0}} \nu^{w(\Gamma)} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op}(u^k, z^l) \\ = \sum_{d=0}^{\infty} \nu^d \sum_{\substack{\Gamma \in \mathcal{G}_{BT}(d+1) \\ \text{deg}^+(\bullet) = \text{deg}^-(\bullet) = 1}} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op} \left(-\frac{\partial\Phi}{\partial z^k}, z^l \right) \\ + \sum_{d=1}^{\infty} \nu^d \sum_{\substack{\Gamma \in \mathcal{G}_{BT}(d) \\ \text{deg}^-(\bullet) = 1}} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op}(g_{\bar{i}k}, z^l)$$

for $1 \leq k, l \leq n$, where \star_{BT}^{op} is the opposite Berezin-Toeplitz star product of anti-Wick type, i.e. the holomorphic and anti-holomorphic variables are swapped.

Let us compute the first few coefficients of $u^k \star_{BT}^{op} z^l - z^l u^k$ by using (35). It is easy to see that the constant term (the coefficient of ν^0) is equal to

$$-\left[\bullet \circlearrowleft_1 \right]^{op} \left(-\frac{\partial\Phi}{\partial z^k}, z^l \right) = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l} = \delta^{kl}.$$

The coefficient of ν (keeping only stable graphs) is equal to

$$-\left[\text{graph with two nodes and two edges} \right]^{op} \left(-\frac{\partial\Phi}{\partial z^k}, z^l \right) - \left[\bullet \circlearrowleft_1 \right]^{op} (g_{\bar{i}k}, z^l) \\ = \left[\text{graph with two nodes and two edges} \right] - \left[\text{graph with two nodes and two edges} \right] = 0.$$

The coefficient of ν^2 (keeping only stable graphs) is equal to

$$-\left[\text{graph with two nodes and three edges} \right]^{op} \left(-\frac{\partial\Phi}{\partial z^k}, z^l \right) - \left[\text{graph with two nodes and two edges} \right]^{op} (g_{\bar{i}k}, z^l) \\ = \left[\text{graph with two nodes and three edges} \right] - \left[\text{graph with two nodes and three edges} \right] = 0.$$

In general, the contribution of $u^k \star_{BT}^{op} z^l - z^l u^k$ to the graph H in Figure 7 comes from $\dot{H}^{op}(-\frac{\partial\Phi}{\partial z^k}, z^l)$, where $\dot{H} \in \mathcal{G}_{BT}$ is obtained from H by gluing the head of k and the tail of \bar{l} , and $\Gamma^{op}(g_{\bar{i}k}, z^l)$, where Γ is a one-pointed graph from Berezin-Toeplitz star product. So by (20) and Lemma 3.10, the coefficient of H is equal to

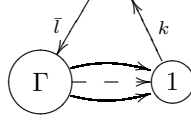
$$-1 \cdot (-1)^{|E(\dot{H})|} + (-1)^{|E(\Gamma)|} = 0,$$

since $|E(\dot{H})| = |E(\Gamma)| + 2$. Note that all H that may appear in $u^k \star_{BT}^{op} z^l - z^l u^k$ is of the form in Figure 6. So we proved $u^k \star_{BT}^{op} z^l - z^l u^k = \delta^{kl}$.

Now we prove Theorem 5.1 (ii). By (18), we need to check that

$$(43) \quad u^k = \frac{1}{\nu} \frac{\partial\Phi}{\partial z^k} + \sum_{G=(V,E) \in \mathcal{G}_0^{ss}} \nu^{|E|-|V|} \frac{-\det(A(G)-I)}{|\text{Aut}(G)|} \frac{\partial G}{\partial z^k}$$

satisfy $u^k \star_B z^l - z^l u^k = \delta^{kl}$ for $1 \leq k, l \leq n$.


 FIGURE 6. A graph H for Berezin-Toeplitz quantization

By (21), the constant term (the coefficient of ν^0) in $u^k \star_B z^l - z^l u^k$ is equal to

$$\left[\bullet \circlearrowleft_1 \right]^{op} \left(\frac{\partial \Phi}{\partial z^k}, z^l \right) = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l} = \delta^{kl}.$$

The coefficient of ν is vacuously zero. The coefficient of ν^2 (keeping only stable graphs) is equal to

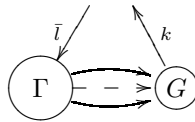
$$\begin{aligned} & \left(-\frac{1}{2} \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \quad \circ \\ \leftarrow 2 \end{array} \right] - \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \quad \circ \\ \leftarrow 1 \end{array} \right] + \frac{1}{2} \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \quad \circ \\ \leftarrow 1 \end{array} \right] \right)^{op} \left(\frac{\partial \Phi}{\partial z^k}, z^l \right) \\ & + \left[\bullet \circlearrowleft_1 \right]^{op} \left(-\frac{1}{2} \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \quad \circ \\ \leftarrow k \end{array} \right] + \left[\begin{array}{c} \bullet \\ \swarrow 1 \quad \searrow 1 \\ \circ \quad \circ \\ \leftarrow k \end{array} \right] + \frac{1}{2} \left[\begin{array}{c} \bullet \\ \swarrow 2 \quad \searrow 2 \\ \circ \quad \circ \\ \leftarrow k \end{array} \right] \right), \end{aligned}$$

which is easily seen to be zero.

In general, the contribution of $u^k \star_B z^l - z^l u^k$ to the graph H in Figure 7 comes from $\dot{H}^{op}(\frac{\partial \Phi}{\partial z^k}, z^l)$, where \dot{H} is obtained from H by gluing the head of k and the tail of \bar{l} , and $\Gamma^{op}(\frac{\partial G}{\partial z^k}, z^l)$, where G is a graph from Bergman kernel and Γ is a one-pointed graph from Berezin star product. We may assume that $\frac{\partial}{\partial z^k}$ may act only on vertices of G , since we can extend (18) to be a summation over strongly connected graphs (not necessarily semistable). See Remark 3.7. So by (20) and Lemma 3.10, the coefficient of H is equal to

$$\det(A(\dot{H}_-) - I) + \det(A(\Gamma_-) - I) \left(-\det(A(G) - I) \right) = 0.$$

So we proved $u^k \star_B z^l - z^l u^k = \delta^{kl}$. □


 FIGURE 7. A graph H for Berezin star product

Remark 5.2. The Karabegov form of an anti-Wick star product has leading term $\frac{1}{\nu} \omega_{-1}$ or $-\frac{1}{\nu} \omega_{-1}$ depending on whether its coefficient of $\left[\bullet \circlearrowleft_k \right]$ is $\frac{1}{k!}$ or $\frac{(-1)^k}{k!}$, in order for (3) to hold. In other words, given an anti-Wick star product on a pseudo-Kähler (M, ω_{-1}) with Karabegov form (40), we can multiply $(-1)^k$ on its weight k coefficients to get another anti-Wick star product on $(M, -\omega_{-1})$ with Karabegov form

$$(44) \quad \hat{\omega} = -\frac{1}{\nu} \omega_{-1} + \omega_0 - \nu \omega_1 + \nu^2 \omega_2 - \nu^3 \omega_3 + \dots$$

6. KARABEGOV-BORDEMANN-WALDMANN QUANTIZATION

Karabegov-Bordemann-Waldmann (KBW) quantization, denoted by \star_S , is also called the standard deformation quantization on Kähler manifolds, i.e. it corresponds to the trivial Karabegov form $\frac{1}{\nu}\omega_{-1}$ in Karabegov's classification [18]. It was also independently constructed by Bordemann and Waldmann [6] by modifying Fedosov's geometric construction of star products on symplectic manifolds [15]. The identification of Bordemann and Waldmann's star product was due to Karabegov [20]. Neumaier [30] showed that each star product of separation of variables type can be obtained following Bordemann-Waldmann's construction.

The following theorem is due to Gammelgaard [16]. It can also be deduced from recent works of Karabegov [22, 23].

Theorem 6.1 ([16]). *On a Kähler manifold, the KBW star product \star_S is given by*

$$(45) \quad f_1 \star_S f_2(x) = \sum_{\Gamma=(V \cup \{f\}, E) \in \mathcal{G}_S} \nu^{|E|-|V|} \frac{(-1)^{|V|}}{|\text{Aut}(\Gamma)|} \Gamma(f_1, f_2) \Big|_x,$$

where $\mathcal{G}_S^{\text{con}}$ is the set of strongly connected one-pointed graphs Γ such that each strongly connected component of Γ_- is a single vertex without loops, i.e. Γ_- is acyclic.

In other words, the formal Berezin transform of KBW quantization is

$$(46) \quad I_S(f) = \sum_{\Gamma=(V \cup \{f\}, E) \in \mathcal{G}_S^{\text{con}}} \nu^{|E|-|V|} \frac{(-1)^{|V|}}{|\text{Aut}(\Gamma)|} \Gamma,$$

We write out the terms in (46) up to ν^4 (keeping only stable graphs),

$$\begin{aligned} I_S(f) = & f + \nu \left[\bullet \circlearrowleft_1 \right] + \frac{\nu^2}{2} \left[\bullet \circlearrowleft_2 \right] + \nu^3 \left(\frac{1}{6} \left[\bullet \circlearrowleft_3 \right] - \frac{1}{4} \left[\circ \overset{2}{\underset{2}{\rightleftarrows}} \bullet \right] \right) \\ & + \nu^4 \left(\frac{1}{24} \left[\bullet \circlearrowleft_4 \right] - \frac{1}{4} \left[\circ \overset{2}{\underset{2}{\rightleftarrows}} \bullet \right] - \frac{1}{12} \left[\circ \overset{2}{\underset{3}{\rightleftarrows}} \bullet \right] \right. \\ & \left. - \frac{1}{12} \left[\circ \overset{3}{\underset{2}{\rightleftarrows}} \bullet \right] + \frac{1}{8} \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \leftarrow \quad \rightarrow \\ \circ \end{array} \right] \right) + O(\nu^5), \end{aligned}$$

which agrees with the computations by Karabegov [23].

The coefficient of ν^5 in (46) has 15 terms corresponding to graphs in Table 3.

TABLE 3. The 15 graphs in $\mathcal{G}_S(5)$

1/120	-1/8	-1/12	-1/48	-1/12
-1/36	-1/48	1/4	1/8	1/24
1/8	1/8	1/24	1/24	-1/16

The dual Karabegov-Bordemann-Waldmann star product \star_{DS} is defined by

$$(47) \quad f_1 \star_{DS} f_2 = I_S^{-1}(I_S f_2 \star_S I_S f_1),$$

for any functions f_1, f_2 .

Theorem 6.2. *The formal Berezin transform of the dual KBW quantization is*

$$(48) \quad I_S^{-1}(f) = \sum_{\Gamma=(V \cup \{f\}, E) \in \mathcal{G}_1^{scon}} \nu^{|E|-|V|} \frac{(-1)^{|E|}}{|\text{Aut}(\Gamma)|} \Gamma.$$

Proof. By Lemma 4.4, for any nontrivial strongly connected one-pointed graph $G \in \mathcal{G}_1^{scon}$, we have

$$\sum_{H=(V \cup \{f\}, E) \in \mathcal{S}(G)} (-1)^{|E(G/H)|} (-1)^{|V|} = (-1)^{|E(G)|} \sum_{H \in \mathcal{S}(G)} (-1)^{w(H)} = 0.$$

So (48) follows from (46). \square

We write out terms of (48) up to ν^3 (keeping only stable graphs)

$$\begin{aligned} I_S^{-1}(f) = & f - \nu \left[\bullet \circlearrowleft_1 \right] + \nu^2 \left(\frac{1}{2} \left[\bullet \circlearrowleft_2 \right] - \left[\textcircled{1} \overset{1}{\curvearrowright} \bullet \right] \right) \\ & \nu^3 \left(-\frac{1}{6} \left[\bullet \circlearrowleft_3 \right] + \left[\textcircled{1} \overset{1}{\curvearrowright} \bullet \right] + \frac{1}{4} \left[\circ \overset{2}{\curvearrowright} \bullet \right] \right. \\ & + \frac{1}{2} \left[\textcircled{1} \overset{1}{\curvearrowright}_2 \bullet \right] + \frac{1}{2} \left[\textcircled{1} \overset{2}{\curvearrowright}_1 \bullet \right] + \frac{1}{2} \left[\textcircled{2} \overset{1}{\curvearrowright}_1 \bullet \right] \\ & \left. - \frac{1}{2} \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \textcircled{1} \\ \swarrow \quad \searrow \\ \circ \quad \textcircled{1} \end{array} \right] - \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \textcircled{1} \\ \swarrow \quad \searrow \\ \circ \quad \textcircled{1} \end{array} \right] - \left[\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \textcircled{1} \quad \textcircled{1} \\ \swarrow \quad \searrow \\ \textcircled{1} \quad \textcircled{1} \end{array} \right] \right) + O(\nu^4). \end{aligned}$$

Corollary 6.3. *The Karabegov form of \star_{DS} is*

$$(49) \quad -\frac{1}{\nu} \omega_{-1} + Ric + \sqrt{-1} \partial \bar{\partial} \sum_{G \in \mathcal{G}_0^{ss}} \nu^{w(G)} \frac{(-1)^{|E(G)|+1}}{|\text{Aut}(G)|} G,$$

where G runs over all strongly connected semistable graphs.

Proof. By (48), it is sufficient to prove that

$$(50) \quad u_k = -\frac{1}{\nu} \frac{\partial \Phi}{\partial z^k} + \frac{\partial \log \det g}{\partial z^k} + \sum_{G \in \mathcal{G}_0^{ss}} \nu^{w(G)} \frac{(-1)^{|E(G)|+1}}{|\text{Aut}(G)|} \frac{\partial G}{\partial z^k}$$

$$(51) \quad = -\frac{1}{\nu} \frac{\partial \Phi}{\partial z^k} + g_{i\bar{i}k} + \sum_{G \in \mathcal{G}_0^{ss}} \nu^{w(G)} \frac{(-1)^{|E(G)|+1}}{|\text{Aut}(G)|} \frac{\partial G}{\partial z^k}$$

satisfy the equation

$$\begin{aligned} (52) \quad \delta^{kl} = u^k \star_{DS} z^l - z^l u^k &= \sum_{\substack{\Gamma \in \mathcal{G}_1 \\ w(\Gamma) > 0}} \nu^{w(\Gamma)} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op}(u^k, z^l) \\ &= \sum_{d=0}^{\infty} \nu^d \sum_{\substack{\Gamma \in \mathcal{G}_1(d+1) \\ \deg^+(\bullet) = \deg^-(\bullet) = 1}} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op} \left(-\frac{\partial \Phi}{\partial z^k}, z^l \right) \\ &\quad + \sum_{d=1}^{\infty} \nu^d \sum_{\substack{\Gamma \in \mathcal{G}_1(d) \\ \deg^-(\bullet) = 1}} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op}(g_{i\bar{i}k}, z^l) \end{aligned}$$

$$+ \sum_{d=2}^{\infty} \nu^d \sum_{t=1}^{d-1} \sum_{\substack{\Gamma \in \mathcal{G}_1(t) \\ \deg^-(\bullet)=1}} \sum_{G \in \mathcal{G}_0^{ss}(d-t)} \frac{(-1)^{|E(\Gamma)|}}{|\text{Aut}(\Gamma)|} \Gamma^{op} \left(\frac{(-1)^{E(G)+1}}{|\text{Aut}(G)|} \frac{\partial G}{\partial z^k}, z^l \right).$$

Note that we have

$$(53) \quad \sum_{G \in \mathcal{G}_0^{ss}} \nu^{w(G)} \frac{(-1)^{|E(G)|+1}}{|\text{Aut}(G)|} G = 1 + \nu \left(-\frac{1}{2} [\textcircled{2}] + \frac{1}{2} \left[\circ \begin{array}{c} \textcircled{2} \\ \textcircled{1} \end{array} \circ \right] \right) + O(\nu^2).$$

It is easy to see that the constant term (the coefficient of ν^0) is equal to

$$-\left[\bullet \textcircled{1} \right]^{op} \left(-\frac{\partial \Phi}{\partial z^k}, z^l \right) = \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l} = \delta^{kl}.$$

The coefficient of ν (keeping only stable graphs) is equal to

$$\begin{aligned} & -\left[\textcircled{1} \begin{array}{c} \textcircled{1} \\ \textcircled{1} \end{array} \bullet \right]^{op} \left(-\frac{\partial \Phi}{\partial z^k}, z^l \right) - \left[\bullet \textcircled{1} \right]^{op} (g_{\bar{i}i k}, z^l) \\ & = \left[\bar{l} \begin{array}{c} \textcircled{1} \\ \textcircled{1} \end{array} k \right] - \left[\bar{l} \begin{array}{c} \textcircled{1} \\ \textcircled{1} \end{array} k \right] = 0. \end{aligned}$$

The coefficient of ν^2 (keeping only stable graphs) is equal to

$$\begin{aligned} & \left(\frac{1}{2} \left[\begin{array}{c} \bullet \\ \textcircled{1} \textcircled{1} \end{array} \right] + \left[\begin{array}{c} \bullet \\ \textcircled{1} \textcircled{1} \end{array} \right] - \frac{1}{2} \left[\textcircled{2} \bullet \right] + \left[\begin{array}{c} \bullet \\ \textcircled{1} \textcircled{1} \end{array} \right] \right)^{op} \left(\frac{\partial \Phi}{\partial z^k}, z^l \right) \\ & - \left[\textcircled{1} \begin{array}{c} \textcircled{1} \\ \textcircled{1} \end{array} \bullet \right]^{op} \left(\left[\textcircled{1} \xrightarrow{k} \right], z^l \right) \\ & + \left[\bullet \textcircled{1} \right]^{op} \left(\frac{1}{2} \left[\textcircled{2} \xrightarrow{k} \right] - \left[\textcircled{1} \begin{array}{c} \textcircled{1} \\ \textcircled{1} \end{array} \circ \xrightarrow{k} \right] - \frac{1}{2} \left[\circ \begin{array}{c} \textcircled{2} \\ \textcircled{1} \end{array} \circ \xrightarrow{k} \right], z^l \right), \end{aligned}$$

which is easily seen to be zero.

In general, the contribution of $u^k \star_{DS} z^l - z^l u^k$ to the graph H in Figure 8 comes from $\dot{H}^{op}(\frac{\partial \Phi}{\partial z^k}, z^l)$, where \dot{H} is obtained from H by gluing the head of k and the tail of \bar{l} , and $\Gamma^{op}(\frac{\partial G}{\partial z^k}, z^l)$, where G is the unique sink of \dot{H}_- containing the tail of k and Γ is a one-pointed graph. We may assume that $\frac{\partial}{\partial z^k}$ may act only on vertices of G , since we can extend (53) to be a summation over strongly connected graphs (not necessarily semistable). See Remark 3.7. So by (48) and Lemma 3.10, the coefficient of H is equal to

$$(-1) \cdot (-1)^{|E(\dot{H})|} + (-1)^{|E(\Gamma)|} \cdot (-1)^{|E(G)|+1} = 0,$$

since $|E(\dot{H})| = |E(\Gamma)| + |E(G)| + 1$. So we proved $u^k \star_{DS} z^l - z^l u^k = \delta^{kl}$. \square

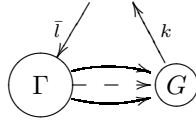


FIGURE 8. A graph H for dual KBW star product

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