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Saddlepoint methods for option pricing

Peter Carr
Bloomberg LP, 731 Lexington Avenue, New York, NY 10022, USA;
email: pcarr4@bloomberg.com

Dilip Madan
Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, USA;
email: dbm@rhsmith.umd.edu

A single saddlepoint approximation for call prices seen as complementary probabilities that log price exceeds log strike by an independent exponential under the share measure is developed using a non-Gaussian base. The suggested base is that of a Gaussian random variable less an exponential with parameter λ. It is suggested that λ be chosen to match the volatility under the share measure. The method is implemented and observed to be exact for the Black–Scholes model. Six other models with closed forms for the cumulant generating function are also investigated.

1 INTRODUCTION

The Fourier transform of the density for the logarithm of the stock price has seen numerous financial applications. For a theoretical perspective we cite as examples Duffie *et al* (2000) and Bakshi and Madan (2000). This transform has become a standard calibration engine following the methods of Carr and Madan (1999) who invoked the fast Fourier transform for its speed. Direct Fourier inversion has also been used and we cite Heston (1993), Bates (1996) and Scott (1997).

Though adequate for near-money options, and this suffices for most calibration exercises, the method is known to break down for deep out-of-the-money options where it often gives rise to negative prices. Rogers and Zane (1999) suggest the use of classical saddlepoint methods to compute option prices and employ in particular the Lugannini and Rice (1980) approximation as developed in Daniels (1987) and extensively studied in Jensen (1995). They consider mainly a Gaussian base density but suggest that we may follow Wood *et al* (1993) and Butler (2007), for non-Gaussian bases. Working with a Gaussian base these methods are also used by Xiong *et al* (2005) for a variety of models with stochastic rates and volatilities.

These applications of classical saddlepoint methods are used to compute the probability that the stock is in-the-money for the risk-neutral probability and the reweighted probability when the stock is itself taken as a numeraire. Thus two saddlepoint computations are involved in constructing one call option price.

We recognize following Madan *et al* (2008) that the call price is itself quite generally a complementary probability itself. Hence one should be able to apply saddlepoint methods directly in one step to determine the call price. However, even in the classical Black–Scholes case, this density is not Gaussian and the use of a Gaussian base is not exact. It turns out that in the Black–Scholes case the density reflected in call prices as a complementary probability is the density of a Gaussian variable less an independent exponential. This leads us to select for a non-Gaussian base the convolution of a normal random variable with a negative exponential. With this altered base we observe that a single saddlepoint computation does yield exact Black–Scholes prices on adopting the Wood *et al* (1993) generalization of the Lugannini–Rice (denoted as LR in tables and figures) approximation.

We then adopt this base more generally for a host of well known option pricing models. In the present paper we study six models. They are the Carr, Geman, Madan and Yor (CGMY) model, Carr *et al* (2002), its spectrally negative form CGYSN studied in Eberlein and Madan (2009), the VGSSD Sato process of Carr *et al* (2007), the Merton (1976) jump diffusion model with stochastic volatility (SVJ) as studied in Bakshi *et al* (1997), the Heston stochastic volatility (HSV) model, Heston (1993) and its generalization to Lévy processes as developed in the VGSA model of Carr *et al* (2003). For all these models we demonstrate that the single saddlepoint method with the non-Gaussian base given by the density of a Gaussian variate less an independent exponential provides effective call prices that work well for deep out-of-the-money options where the more traditional Fourier methods break down into negative prices.

The outline for the rest of the paper is as follows. Section 2 describes call prices as a complementary probability. Section 3 summarizes the generalized Lugannini–Rice approximation of Wood *et al* (1993). In Section 4 we analyze the Black–Scholes case and show that the base given by the convolution of a Gaussian variate with a negative exponential would be exact in this case. Computations for the Black–Scholes case are conducted in Section 5 and it is shown that the new base is exact. This base is then adopted more generally for the other models and results are presented for the six models discussed above in Section 6. Section 7 concludes.

2 CALL PRICES AS PROBABILITIES

Recently Madan *et al* (2008) have shown that quite generally the call price is the probability that the stock price was equal to the strike for the last time before the maturity. Equivalently it is the probability that after the maturity the stock stays below the strike forever. Here we conduct a much simpler analysis at the level of random variables as opposed to processes, and enquire into the nature of the random variable represented by the call price seen as a probability. We begin by writing the call price as a function of the log strike k. That is $c(k)$, which is given by:

$$c(k) = \mathrm{e}^{-rt} \int_k^\infty (\mathrm{e}^x - \mathrm{e}^k) f(x)\, \mathrm{d}x$$

where $f(x)$ is the density for x, the logarithm of the stock price, and r is the risk-free interest rate, assumed to be constant. We divide this call price by the spot price S_0 and note that as:

$$S_0 = \mathrm{e}^{-rt} \int_{-\infty}^{\infty} \mathrm{e}^x f(x)\,\mathrm{d}x$$

we get that:

$$\begin{aligned}\tilde{c}(k) &= \frac{c(k)}{S_0} \\ &= \frac{\int_k^{\infty} (\mathrm{e}^x - \mathrm{e}^k) f(x)\,\mathrm{d}x}{\int_{-\infty}^{\infty} \mathrm{e}^x f(x)\,\mathrm{d}x}\end{aligned} \tag{1}$$

It is clear that this relativized call price (1) is unity at a zero strike and decreases to zero as the log strike tends to infinity and hence this function is a complementary probability. The negative of its derivative is therefore a density and computation yields that:

$$-\tilde{c}'(k) = \frac{\int_k^{\infty} \mathrm{e}^k f(x)\,\mathrm{d}x}{\int_{-\infty}^{\infty} \mathrm{e}^x f(x)\,\mathrm{d}x}$$

Ignoring the normalizing constant we consider the unnormalized function:

$$g(k) = \int_k^{\infty} \mathrm{e}^k f(x)\,\mathrm{d}x$$

that tends to zero as the strike goes to positive or negative infinity. Computing the Fourier transform of g we observe that:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{iuk} g(k)\,\mathrm{d}k &= \int_{-\infty}^{\infty} \mathrm{e}^{iuk}\mathrm{e}^k (1 - F(k))\,\mathrm{d}k \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{e}^{(1+iuk)}}{1+iu} f(k)\,\mathrm{d}k \\ &= \frac{\phi(u-i)}{1+iu}\end{aligned}$$

where $\phi(u)$ is the characteristic function for the logarithm of the stock price. On incorporating the normalization we observe the transform:

$$\gamma(u) = \frac{\phi(u-i)}{\phi(-i)(1+iu)} \tag{2}$$

The transform $\gamma(u)$ is the characteristic function of a density whose complementary distribution function is the normalized call price taken at the log strike. We also observe that (2) is the product of two characteristic functions, the first being $\phi(u-i)/\phi(-i)$ and the second is $(1+iu)^{-1}$ which is the characteristic function of a negative exponential random variable. The first characteristic function is easily seen to be the density of the log of the stock, tilted by the stock price itself, or the exponential of this log stock price. Prices of claims paid out in US dollars can equivalently be seen as prices of claims paid out in shares, but now valued using this share price tilted measure. This measure is also called the share measure as we are transforming payouts from US dollars to shares when employing such a measure. Hence, by virtue of γ being a product of characteristic functions, the derivative of the call price normalized by the forward price is the density of:

$$Z = X - Y$$

where X is the logarithm of the stock under the share measure and Y is an independent exponential. The normalized call price is then the probability that $Z > \log(K)$ or equivalently that:

$$X > \ln K + Y$$

the probability that under the share measure the logarithm of the stock exceeds log strike by an independent exponential variate.

We can obtain this result without transform methods on noting that the normalized call price is:

$$\begin{aligned}\frac{C(K)}{S_0} &= \frac{E[(S-K)^+]}{E[S]} \\ &= \widetilde{E}\left[\left(1 - \frac{K}{S}\right)^+\right]\end{aligned}$$

where $\widetilde{E}$ is the share measure. Now we define:

$$\frac{K}{S} = \mathrm{e}^{-y}$$

and let $f(y)$ be the density for y under the share measure with $F(y)$ the corresponding distribution function then:

$$\begin{aligned}\frac{C(K)}{S_0} &= \int_0^\infty (1 - \mathrm{e}^{-y}) f(y)\,\mathrm{d}y \\ &= \int_0^\infty (1 - F(y))\mathrm{e}^{-y}\,\mathrm{d}y\end{aligned}$$

But as e^{-y} is the density of a positive exponential we have here the probability that:

$$\ln S - \ln K > Y$$

or the probability that the log of the stock exceeds log strike by an independent exponential.

3 GENERALIZED LUGANNINI–RICE APPROXIMATIONS

Wood *et al* (1993) generalized the Lugannini–Rice approximation to an arbitrary base density. For the implementation of the approximation to a particular base one needs access to the cumulant generating function (CGF) of the base random variable Z:

$$G(w) = \log(E[\exp(wZ)])$$

the cumulative distribution function (cdf), $\Gamma(w)$, and the probability density function (pdf), $\gamma(w)$. For a more detailed presentation of the method we refer the reader to Butler (2007, Theorem 16.1.1, pp. 528–531).

Suppose we wish to evaluate the probability that $X > y$ for a random variable with cumulant function:

$$K(t) = \log(E[\exp(tX)])$$

The Lugannini–Rice approximation requires that we first evaluate the Fenchel transform of the base CGF as:

$$\begin{aligned} H(\xi) &= G(w(\xi)) - \xi w(\xi) \\ G'(w(\xi)) &= \xi \end{aligned} \tag{3}$$

We then define $\hat{t}$, by the saddlepoint equation:

$$K'(\hat{t}) = y$$

The dominant term in the representation of the target density at y in terms of its cumulant is then equated to the corresponding dominant term with respect to the base by:

$$H(\hat{\xi}) = K(\hat{t}) - \hat{t}y$$

There are two solutions for $\hat{\xi}$ as H is concave and we take $\hat{\xi} < G'(0)$ if $y < K'(0)$ and $\hat{\xi} > G'(0)$ if $y > K'(0)$ otherwise.

We further define the required second order terms by:

$$\begin{aligned} \hat{u} &= \hat{t}(K''(\hat{t}))^{1/2} \\ \hat{u}_{\xi} &= \hat{u}(G''(w(\hat{\xi})))^{-1/2} \end{aligned}$$

The complementary probability is then estimated by:

$$P(X > y) = 1 - \Gamma(\hat{\xi}) + \gamma(\hat{\xi})\left(\frac{1}{\hat{u}_{\xi}} - \frac{1}{w(\hat{\xi})}\right)$$

For a Gaussian base we have the classical Lugannini–Rice approximation. We also know that base density will be exact for all models that are a shift and scale transform of the base. The base is therefore determined up to shift and scale.

4 THE BLACK–SCHOLES CASE

We would like to choose a base that is exact for the Black–Scholes case. This leads us to ask what the base is in this case. We first verify independently that the Black–Scholes price of a call option relative to the initial spot price is the probability that the log of the stock under the share measure exceeds log strike by an independent exponential.

Under the share measure the logarithm of the final stock price is given by:

$$X = \ln(S_0) + \left(r - q + \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z$$

where Z is a standard normal variate. We therefore seek the probability that:

$$X > \ln K + Y$$

for an independent exponential Y.

Conditional on the exponential variate this is given by the probability that:

$$Z > \frac{\ln K/S_0}{\sigma\sqrt{t}} - \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} + \frac{Y}{\sigma\sqrt{t}}$$

or:

$$N\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{Y}{\sigma\sqrt{t}}\right)$$

where $N(x)$ represents the cumulative distribution function of a standard normal distribution and $n(x)$ is its corresponding density function.

The unconditional probability is then:

$$\begin{aligned}
&\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-y} N\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{y}{\sigma\sqrt{t}}\right) \\
&\quad = -\mathrm{e}^{-y} N\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{y}{\sigma\sqrt{t}}\right)\Bigg|_0^\infty \\
&\qquad - \int_0^\infty \mathrm{e}^{-y} n\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{y}{\sigma\sqrt{t}}\right)\frac{1}{\sigma\sqrt{t}}\,\mathrm{d}y \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(on integration by parts)} \\
&\quad = N\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t}\right) \\
&\qquad - \int_0^\infty \mathrm{e}^{-y} n\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{y}{\sigma\sqrt{t}}\right)\frac{1}{\sigma\sqrt{t}}\,\mathrm{d}y \\
&\quad = N\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t}\right) \\
&\qquad - \int_0^\infty \mathrm{e}^{-y} \frac{1}{\sqrt{2\pi}\sigma\sqrt{t}} \exp\left(-\frac{1}{2}\left(\frac{y}{\sigma\sqrt{t}} - \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{\ln S_0/K}{\sigma\sqrt{t}}\right)^2\right)\mathrm{d}y
\end{aligned}$$

We analyze the final integral as follows:

$$\int_0^\infty e^{-y} \frac{1}{\sqrt{2\pi}\sigma\sqrt{t}} \exp\left(-\frac{1}{2}\left(\frac{y}{\sigma\sqrt{t}} - \left(\frac{r-q}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} - \frac{\ln S_0/K}{\sigma\sqrt{t}}\right)^2\right) dy$$

$$= \int_{-((r-q)/\sigma+\sigma/2)\sqrt{t}+(\ln S_0 K)/\sigma\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} - \left[\sigma\sqrt{t}z + (r-q)t + \frac{\sigma^2 t}{2} + \ln S_0/K\right]\right) dz$$

by a change of variable:

$$= \frac{K}{S_0} e^{-rt+qt-\sigma^2 t/2} \int_{-((r-q)/\sigma+\sigma/2)\sqrt{t}-(\ln S_0/K)/\sigma\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} - \sigma\sqrt{t}z\right) dz$$

$$= \frac{K}{S_0} e^{-rt+qt} \int_{-((r-q)/\sigma-\sigma/2)\sqrt{t}-(\ln S_0/K)/\sigma\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

by another change of variable:

$$= K e^{-rt+qt} N\left(\frac{\ln S_0/K}{\sigma\sqrt{t}} + \left(\frac{r-q}{\sigma} - \frac{\sigma}{2}\right)\sqrt{t}\right)$$

Hence on multiplication by $S_0\, e^{-qt}$ we get the traditional Black–Scholes formula.

5 THE CHOICE OF A NON-GAUSSIAN BASE

We observe from the analysis of the previous section that the density reflected in call prices for the Black–Scholes model is, up to a shift and change of scale, a Gaussian density less an independent exponential. This leads us to consider for a base model the one parameter family of a zero mean variable given by:

$$Z + \frac{1}{\lambda} - Y$$

where Z is a standard Gaussian variate and Y is a positive exponential with parameter λ. To employ such a base in a saddlepoint approximation requires a knowledge of its CGF, cdf, pdf and the Fenchel transform of the CGF.

The CGF of our suggested base is:

$$G(w) = \frac{w^2}{2} + \frac{w}{\lambda} - \ln(\lambda + w) + \ln(\lambda)$$

and its first and second derivatives are:

$$G'(w) = w + \frac{1}{\lambda} - \frac{1}{\lambda + w}$$

$$G''(w) = 1 + \left(\frac{1}{\lambda + w}\right)^2$$

For the complementary cdf we evaluate directly:

$$\begin{aligned}\Gamma(y) &= \Pr\left(Z + \frac{1}{\lambda} - Y > a\right)\\ &= \int_{a-1/\lambda}^{\infty} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}\left(1 - \exp\left(-\lambda\left(z - \left(a - \frac{1}{\lambda}\right)\right)\right)\right)\mathrm{d}z\\ &= N\left(\frac{1}{\lambda} - a\right) - \exp\left(\lambda\left(a - \frac{1}{\lambda}\right)\right)\int_{a-1/\lambda}^{\infty}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2-\lambda z}\,\mathrm{d}z\\ &= N\left(\frac{1}{\lambda} - a\right) - \exp\left(\lambda a - 1 + \frac{\lambda^2}{2}\right)N\left(\frac{1}{\lambda} - a - \lambda\right)\end{aligned}$$

The pdf follows on differentiation as:

$$\begin{aligned}\gamma(y) = n\left(\frac{1}{\lambda} - a\right) &+ \lambda\exp\left(\lambda a - 1 + \frac{\lambda^2}{2}\right)N\left(\frac{1}{\lambda} - a - \lambda\right)\\ &- \exp\left(\lambda a - 1 + \frac{\lambda^2}{2}\right)n\left(\frac{1}{\lambda} - a - \lambda\right)\end{aligned}$$

We also need the Gauss–Fenchel transform of the CGF and for this we solve for:

$$w(\xi) = -\lambda + \frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}$$

$$c = \xi - \frac{1}{\lambda} + \lambda$$

The Gauss–Fenchel transform may then be directly determined from Equation (3).

We now develop the saddlepoint method for the Black–Scholes model using this Gauss minus exponential (GME) base. The exact CGF for the random variable associated with the call price is the CGF for the log of the stock under the share measure less an independent exponential and this is:

$$K(x) = \frac{\sigma^2 t x^2}{2} + \frac{\sigma^2 t}{2}x - \ln(1 + x)$$

To observe that the suggested base is exact for this CGF we must observe that we may recover K by a shift and change of scale of G. Consider an arbitrary shift of a and scale of b to get that:

$$ax + G(bx) = ax + \frac{\sigma^2 t x^2}{2} + \frac{\sigma\sqrt{t}}{\lambda}x - \ln\left(\lambda + \sigma\sqrt{t}x\right) + \ln(\lambda)$$

Now choose $\lambda = \sigma\sqrt{t}$ to get:

$$ax + G(bx) = ax + \frac{\sigma^2 t x^2}{2} + x - \ln(1 + x)$$

TABLE 1 Lugannini–Rice saddlepoint prices for Black–Scholes.

Strike	Maturity	LR GME	LR Gauss	Black–Scholes
36.78	0.5	63.7597	62.9690	63.7597
60.65	0.5	40.2569	39.8812	40.2569
81.87	0.5	20.2050	20.1427	20.2050
90.48	0.5	13.4486	13.3367	13.4486
110.52	1	7.08205	6.7907	7.08205
122.14	1	4.00475	3.8300	4.00475
174.87	1	0.365935	0.3524	0.365935
271.82	1	$0.0^3 48714$	$0.0^3 4722$	$0.0^3 48714$

Finally we choose:

$$a = \frac{\sigma^2 t}{2} - 1$$

to get that $ax + G(bx) = K(x)$.

We computed the saddlepoint Lugannini–Rice approximation for the Black–Scholes model using this GME base for a volatility of 25%, and a sample of strikes and maturities with the initial spot at 100, an interest rate of 3% and a zero dividend yield with the choice of λ at $\sigma\sqrt{t}$. We also computed the Lugannini–Rice approximation using a Gaussian base for the same options. Finally we presented the Black–Scholes price computed from the usual Black–Scholes formula. We know that the Lugannini–Rice approximation with the GME base is exact and the fact that they match up serves as a check on our new program for the saddlepoint method, where the classical Black–Scholes formula serves as a benchmark. The results are presented in Table 1.

6 A GENERAL PURPOSE SADDLEPOINT PRICER

We develop a general purpose saddlepoint pricer using the GME base with the function call:

$$ww = optionpriceGMESP(pp, kk, rr, qq, xx, tt, uu, model)$$

that takes as inputs the vectors *pp* for the initial spot, *kk* for the strikes, *rr* for the interest rates, *qq* for the dividend yields, *xx* the parameter values of a prospective model, *tt* for the option maturities, *uu* a vector denoting *iscall*, and *model* the name of the model. The pricer requires that the following three functions are available for an option price to be computable. These are the CGF, and its first and second derivatives. Given that a Gaussian base is not exact for the classical Black–Scholes formula as observed in Section 5 above, we did not write a general purpose pricer for this base. In fact we first applied the Gaussian base to the Black–Scholes model and observed the difference from the classical price; this then led to a need to alter the base. In this section we present results for just the GME base.

Suppose we have the CGF of the log of the stock under some risk-neutral model given by the function $K(x)$. The CGF of the density embodied in the call price relativized by the spot price is the function:

$$K(x+1) - \ln(1+x)$$

Our base is of the form:

$$\frac{x^2}{2} + \frac{x}{\lambda} - \ln(\lambda + x) + \ln(\lambda)$$

We evaluate at a scale b to write:

$$\frac{b^2x^2}{2} + \frac{bx}{\lambda} - \ln(\lambda + bx) + \ln(\lambda)$$

If we take $\lambda = b$ we get:

$$\frac{b^2x^2}{2} + x - \ln(1+x)$$

This choice forces a matching of the cumulant of the exponential in the rescaled base to the cumulant of the exponential in the target. The means or drifts can always be matched by shifts and so we focus on the second order terms of $K(x+1)$ and $b^2x^2/2$. A matching of the volatility is attained by choosing:

$$\lambda = b = [K''(\hat{t}+1)]^{1/2}$$

We observe that for Black–Scholes this gives the correct solution for $\lambda = \sigma\sqrt{t}$. We could have followed the suggestion of Wood *et al* (1993) of matching standardized skewness but then we would not be exact for Black–Scholes.

7 RESULTS FOR SOME ASSORTED MODELS

Apart from the Gaussian model reported on in Table 1 we applied our general purpose GME saddlepoint pricer to models CGMY of Carr *et al* (2002), the model VGSSD for the Sato process of Carr *et al* (2007), the spectrally negative model CGYSN studied in Eberlein and Madan (2009), and three stochastic volatility models, HSV model of Heston (1993), the Merton (1976) SVJ as described in Bakshi *et al* (1997), and the Lévy process model with stochastic volatility VGSA of Carr *et al* (2003).

The parameter values used for the six models were as follows. For CGMY we used $C = 2$, $G = 5$, $M = 10$ and $Y = 0.5$. For VGSSD we used $\sigma = 0.2$, $\nu = 0.5$, $\theta = -0.15$ and $\gamma = 0.5$. For CGYSN we took $\sigma = 0.1$, $C = 0.5$, $G = 5$ and $Y = 0.5$. The Heston parameters were initial volatility 0.2, long-term volatility 0.2, mean reversion 2, volatility of volatility 0.5 and correlation −0.7. The parameters for SVJ were initial variance 0.02, jump arrival rate 2.5, mean jump size −0.001, standard deviation of jump 0.0155, mean reversion 3, long-term variance 0.3, volatility of volatility 0.5, and correlation −0.7. The parameters for VGSA were initial speed 10, $G = 20$, $M = 40$, mean reversion 3, long-term speed 6, volatility of speed 7.

For all models we used the initial spot at 100, the interest rate at 0.03, a zero dividend yield, and a half-year maturity. The strikes ranged from 10 dollars to 200 dollars in steps of 10 dollars. Two computations were performed, the first is the price using the fast Fourier transform method of Carr and Madan (1999) that is known to break down for deep out-of-the-money options, often returning negative prices in such cases, and the motivation for developing the saddlepoint method developed here. The second is the GME base saddlepoint method. The results are reported in Table 2 (see page 60). We observe from this table that the saddlepoint pricer matches the fast Fourier transform for near-money options and it returns positive prices in the range of deep out-of-the-money options where the fast Fourier transform has broken down.

8 CONCLUSION

Call prices are observed to be the complementary probabilities that log price exceeds log strike by an independent exponential, under the share measure obtained on tilting the log price by its exponential or the normalized final stock price. It is therefore appropriate to approximate deep out-of-the-money call prices using a single saddlepoint approximation for the appropriate complementary probability. The issue arises with respect to the base distribution to be used in the saddlepoint approximation. Typically in a standard application one employs a Gaussian base. For the Black–Scholes model of geometric Brownian motion the exact result is that of the density and cumulant generating function for the random variable formed by a Gaussian variable less an independent exponential and this density is not a scale shift transform of a Gaussian base. Hence the use of a Gaussian base will not be exact for the Black–Scholes model. In order to be exact for the Black–Scholes model the suggested base is that of a Gaussian variable less an independent exponential with parameter λ. The method is then to employ saddlepoint approximations with such a base distribution. The parameter λ of the exponential distribution in the base, that matches, on rescaling, the cumulant of the exponential in the target cumulant, is seen to be the volatility of the risk-neutral distribution under the share measure as given by the square root of second derivative of the unit shifted cumulant taken at the solution of the saddlepoint equation. The methods are implemented and observed to be exact for the Black–Scholes model. Six other models with closed forms for the cumulant generating function are also investigated. These are the CGMY model of Carr *et al* (2002), the Sato process model constructed from the variance gamma model as described in Carr *et al* (2007), the spectrally negative form of CGMY with a diffusion component studied in Eberlein and Madan (2009), the HSV model of Heston (1993), the Merton (1976) jump diffusion model and the stochastic volatility Lévy process, VGSA of Carr *et al* (2003).

TABLE 2 Call prices for different models.

Strike	FFT CGMY	GME LR CGMY	FFT VGSSD	GME LR VGSSD	FFT CGYSN	GME LR CGYSN
10	88.2148	90.1571	87.7794	90.1508	88.3875	90.1517
20	79.4972	80.3279	79.1716	80.3051	79.4701	80.3086
30	70.2051	70.5178	70.0198	70.4653	70.1370	70.4738
40	60.6231	60.7397	60.4879	60.6352	60.5219	60.6530
50	51.0359	51.0482	50.7400	50.8222	50.7584	50.8565
60	41.7280	41.6104	40.9403	41.0401	40.9802	41.1042
70	32.9847	32.7303	31.2560	31.3247	31.3234	31.4395
80	25.0952	24.7688	21.9359	21.8432	21.9862	22.0009
90	18.3244	18.0184	13.4748	13.1831	13.3894	13.2631
100	12.8429	12.6191	6.6766	6.4127	6.3690	6.2481
110	8.6739	8.5428	2.4189	2.3804	2.0135	1.9884
120	5.6892	5.6278	0.6988	0.7205	0.3593	0.3611
130	3.6601	3.6387	0.1951	0.2057	0.0314	0.0342
140	2.3324	2.3295	0.0551	6.0095E−02	−9.8120E−04	1.7169E−03
150	1.4840	1.4879	0.0151	1.8456E−02	−2.6410E−03	4.8645E−05
160	0.9482	0.9537	0.0031	5.9918E−03	−2.6894E−03	8.4152E−07
170	0.6012	0.6159	−0.0040	2.0547E−03	−2.8654E−03	9.5923E−09
180	0.3278	0.4018	−0.0236	7.4200E−04	−3.6956E−03	7.7157E−11
190	0.0129	0.2652	−0.0732	2.8116E−04	−5.7047E−03	4.6470E−13
200	−0.4588	0.1772	−0.1700	1.1139E−04	−9.4178E−03	2.2040E−15
	HSV	HSV	SVJ	SVJ	VGSA	VGSA
10	88.2213	90.1511	88.9673	90.1609	88.4818	90.1504
20	79.3860	80.3064	79.7445	80.3240	79.4958	80.3036
30	70.1029	70.4684	70.2563	70.4907	70.1374	70.4614
40	60.5096	60.6416	60.5799	60.6637	60.5149	60.6272
50	50.7435	50.8370	50.7925	50.8491	50.7371	50.8088
60	40.9422	41.0710	40.9711	41.0574	40.9123	41.0192
70	31.2459	31.3837	31.1949	31.3139	31.1521	31.2899
80	21.8595	21.8939	21.6007	21.6865	21.6401	21.7127
90	13.1996	13.0323	12.5598	12.4728	12.8156	12.6732
100	6.0528	5.8409	5.0195	4.8485	5.5627	5.3009
110	1.6344	1.5848	0.8444	0.8202	1.2106	1.1812
120	0.2320	0.2385	0.0521	0.0564	0.1390	0.1503
130	2.48E−02	2.87E−02	5.67E−04	3.46E−03	1.44E−02	1.81E−02
140	6.99E−04	3.56E−03	−2.46E−03	2.44E−04	−4.04E−04	2.41E−03
150	−2.23E−03	4.82E−04	−2.67E−03	2.02E−05	−2.35E−03	3.57E−04
160	−2.62E−03	7.19E−05	−2.69E−03	1.93E−06	−2.63E−03	5.90E−05
170	−3.06E−03	1.18E−05	−2.72E−03	2.11E−07	−2.92E−03	1.07E−05
180	−5.01E−03	2.10E−06	−2.85E−03	2.59E−08	−4.21E−03	2.13E−06
190	−9.97E−03	4.08E−07	−3.18E−03	3.53E−09	−7.46E−03	4.58E−07
200	−1.94E−02	8.51E−08	−3.81E−03	5.31E−10	−1.37E−02	1.06E−07

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