Debt Runs and the Value of Liquidity Reserves*

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Abstract

This article analyzes a firm prone to debt runs, and the effect of its portfolio liquidity composition on the run behavior of its creditors. The firm holds cash and an illiquid cash flow generating asset, and is financed with debt held by a continuum of creditors. At each point in time, a constant fraction of the firm's outstanding liabilities matures, leading the maturing creditors to decide whether to roll-over or ask for their funds back. When the firm's portfolio value deteriorates, creditors are inclined to run, but their propensity to run decreases with the amount of available liquidity resources. The theory has policy implications for micro-prudential bank liquidity regulation: for any leverage ratio, it characterizes the quantity of liquidity reserves a firm should hold in order to deter a run. I solve the model numerically and perform comparative statics, varying the firm's illiquid asset characteristics and the firm's debt maturity profile. I discuss the influence of the firm's portfolio choice and dividend policy on the run behavior of creditors. The model can also be transported into an international macroeconomic context: the firm can be reinterpreted as a central bank/government, having issued foreign-currency denominated sovereign debt that is regularly rolled over. A high debt-to-GDP ratio combined with low levels of foreign currency reserves will prompt foreign creditors to run. The theory can therefore provide guidance on the appropriate sizing of central banks' foreign currency reserves for countries issuing large amounts of short term foreign exchange debt.

Key Words: Bank Runs, Cash Holdings

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1 Introduction

Corporates die of cancer, but financial firms die of heart attacks. This quote from Valukas et al. (2010) emphasizes an important characteristic of firms prone to debt runs: they rarely fail for a lack of equity capital, but rather for a lack of liquidity. The last-minute rescue of Bear Stearns in March 2008 and the bankruptcy of Lehman Brothers in September 2008 illustrate well this phenomenon: days before their collapse, both U.S. broker-dealers were in compliance with the SEC's net capital rules, and had investment grade ratings by all major credit rating agencies. Instead, it is the shortage of available funds and liquid unencumbered assets that precipitated those firms' demise. Of course, liquidity and solvency should be viewed as tightly interconnected concepts: a deteriorating capital situation at a run-prone institution will cause its creditors to be reluctant to roll-over their maturing debt, which in turn leads to increasing liquidity needs for the institution, and potentially to its default. It seems that, at the time, institutions prone to run risk held insufficient liquidity reserves.

The funding stress suffered by financial institutions during the 2008-2009 financial crisis attracted the attention of both academia and policy makers. Brunnermeier, Krishnamurthy, and Gorton (2013) focus on the U.S. banking sector and measure the mismatch between the market liquidity of bank assets and the funding liquidity of their liabilities. Their "liquidity mismatch index" worsens dramatically in the two years leading to the crisis. This liquidity mismatch also prompted the Basel Committee on Banking Supervision to develop several regulations that would force banks to hold minimum amounts of cash and liquid unencumbered assets. The Liquidity Coverage Ratio test, for example, ensures that banks hold sufficient high quality liquid assets to survive a significant stress scenario lasting 30 calendar days, whereas the Net Stable Funding Ratio test requires that at least 100% of a bank's long term asset portfolio is financed with stable funding.

But those rules seem ad-hoc and fail to capture certain key characteristics of runs on run-prone institutions. Why is 30 day the right time horizon for the Liquidity Coverage Ratio test? Should rules driving a bank's liquidity pool also be linked to its solvency – in other words, should a bank with lower capital ratios be forced to hold more liquidity reserves than a healthier bank, since it is potentially more prone to runs? How do these rules capture the strategic behavior of creditors who determine their actions not only by looking at the bank's asset composition, but also by reacting to the assumed behavior of other creditors?

My paper will provide some answers to these micro-prudential regulation design issues. Leveraging the canonical framework of He and Xiong (2012), I develop a model in which a financial firm holds a portfolio consisting of cash and liquid reserves on the one hand, and illiquid assets on the other. Cash and liquid reserves can be sold immediately at their fundamental price, whereas illiquid assets can only be disposed of at a discount to their intrinsic value. The firm finances itself with debt that is purchased by a continuum of creditors. Each year, a constant fraction of the firm's outstanding liabilities matures, a feature first introduced by Leland (1994). This feature guarantees that the firm's average debt maturity is constant. When a creditor's debt claim matures, the creditor decides whether to continue financing the firm, or to stop rolling over its debt claim. The creditor behaves strategically by taking into account the firm's asset composition and the assumed behavior of the firm's other creditors.

I look for a symmetric Markov perfect equilibrium in cutoff strategies for creditors: they run when the firm's illiquid asset value falls below an endogenous threshold that depends on the amount of liquid resources available at the firm. Otherwise they roll-over their maturing claims. The runs in my model are thus directly linked to the solvency and liquidity position of the firm. A deteriorating solvency position combined with a weak liquidity situation leads creditors to start running on the firm.

For the model parameters considered, I find only one equilibrium in cutoff strategies. This result stems from two key ingredients first introduced by Frankel and Pauzner (2000): creditors make asynchronous roll/run decisions, and the state variables (the fundamental value of the illiquid asset and the liquid resources of the firm) are time varying. This proves fundamental in looking at the model from a policy recommendation's perspective. I use this feature to compute probabilities of runs and probabilities of bankruptcy for any balance-sheet composition. This can provide clear quantitative guidance to policy makers interested in designing bank micro-prudential regulations.

In the model, cash is a key stabilizing force against runs. When the firm's portfolio value deteriorates, creditors will be inclined to run, but their propensity to run will decrease with the amount of liquid resources available to the firm. Valukas et al. (2010) mentions this phenomenon in the context of Lehman Brothers' bankruptcy: *[...] the size of Lehman's liquidity pool provided comfort to market participants and observers, including rating agencies. The size of Lehman's liquidity pool encouraged counterparties to continue providing essential short-term financing and intraday credit to Lehman. In addition, the size of Lehman's liquidity pool provided assurance to investors that if certain sources of short-term financing were to disappear, Lehman could still survive.* My model provides a theoretical justification to this assertion by characterizing the set of firm's portfolios that leave creditors indifferent between rolling over their debt claims and running. This boundary can again provide transparent guidance to policy makers designing bank liquidity regulations: for any leverage ratio, it characterizes clearly the quantity of liquidity reserves a firm should hold in order to deter a run.

My paper contributes to a growing literature on dynamic debt runs that includes He and Xiong (2012), Schroth, Suarez, and Taylor (2014) or Cheng and Milbradt (2011). I build on previous work by He and Xiong (2012). They assume that a firm subject to a run can rely on an emergency credit line that might fail. I, on the other hand, make the assumption that the firm maintains a cash buffer that can be used to pay off maturing creditors. This added dimensionality enriches the model in two important dimensions. First, the extended model replicates some of the stylized facts characteristic of debt runs. Second, it also allows me to focus on aspects that have not been studied before, such as quantifying the value of cash as run deterrent for run-prone firms.

On the empirical side, I know from the data that debt runs are not instantaneous events, but rather can be prolonged before an institution runs out of cash and defaults. In my model, when creditors start running, the firm uses its available cash to meet debt redemptions. Only after all liquidity resources have been exhausted does the firm sell its illiquid assets to repay remaining creditors and potentially defaults. Moreover, debt runs do not always lead to the failure of the firm being the target of such run. The experience of Goldman Sachs in the fall of 2008 is well suited. In response to a request from the Financial Crisis Inquiry Committee ("FCIC") related to its liquidity pool at the time, the firm indicated that its cash buffer, which averaged \$113bn in the third quarter 2008, declined to a low of \$66bn on September 18th, following *both anticipated contractual obligations and other flows of cash and collateral that were driven by counterparty confidence and market volatility.* In other words, Goldman Sachs did suffer the equivalent of a debt run in September 2008. However, the firm's liquidity position, at the time, was strong enough to deter the run, as its response to the FCIC highlights: *Our liquidity policies and position gave us enough time to make these tactical and strategic decisions in an appropriate manner that preserved the markets confidence in our institution. This confidence led to the reduction of customer-driven outflows of liquidity and allowed us to return to our pre-crisis liquidity buffer levels, with our buffer at an average of \$111bn in the fourth quarter of 2008.*[1] In my model, during a run, creditors' strategy switches from running to rolling over once the firm's solvency or its liquidity position improves sufficiently.

In addition, by adding the cash dimension to the firm's problem, I am able to calculate the marginal value of cash for an institution that is subject to run risk. Although there is a large and growing literature discussing the value of cash holdings for firms, nobody has yet studied the value of cash as run deterrent. Décamps et al. (2011) analyze the optimal dividend policy of a firm holding a cash-flow generating asset and facing external financing

[1] source: http://www.goldmansachs.com/media-relations/in-the-news/archive/response-to-fcic-folder/gs-liquidity.pdf

costs. In their model, the firm balances the cost of holding cash (which earns an interest lower than the firm's discount rate due to agency costs) with the savings realized due to less frequent equity issuances upon the occurrence of operating losses. Bolton, Chen, and Wang (2011) and Bolton, Chen, and Wang (2013) enhance the model by studying optimal investment and cash retention under similar capital markets frictions. However, none of the firms studied in those papers issue any debt that might lead to roll-over risk[2]. Hugonnier and Morellec (2014) analyze the effects of liquidity and leverage requirements on banks' solvency risk. The authors build from the model developed by Décamps et al. (2011), but assume an asset dividend process that makes the bank debt claims risky. The bank's motive for holding liquidity reserves is however identical to the previously cited papers: the bank faces flotation costs, and thus stores cash as buffer mechanism to save on future issuance costs. Hugonnier, Malamud, and Morellec (2011) study a firm's investment, payout and financing policies when capital markets are imperfect due to search frictions: the firm has to look for investors when in need of capital, leading to the need to store cash. In my paper, internal cash is valuable to the firm as a run deterrent, since creditors' incentive to pull back their funding will decrease when the cash internal to the firm increases. It is to my knowledge a novel role for cash within a firm.

Finally, the model's added dimensionality opens the number of issues that I hope to study in subsequent research. What is the optimal dynamic portfolio choice (cash and liquid low-yielding securities vs. illiquid higher yielding long term investments) for an institution subject to run risk? How does a run-prone firm's dividend policy influence creditors' run behavior, and what is the firm's optimal dividend policy? I am hoping to shed some light on these questions in future work thanks to the model developed in this paper.

My approach also differs from a separate dynamic bank run literature, which builds upon the canonical model of Qi (1994), mainly due to its mechanism to deter runs and to the uniqueness of its threshold equilibrium. This literature combines elements of Diamond and Dybvig (1983) with an overlapping generation's economy. Martin, Skeie, and Von Thadden (2010) and Martin, Skeie, and von Thadden (2014) for example study financial intermediaries involved in maturity transformation (thus subject to run risk) and derive liquidity and solvency conditions that prevent runs from occurring. In their setup, a bank's liquidity needs are met with under-investment in a given period, which erodes profits in future periods. Thus, future profits are essential to forestall runs and serve as systemic buffer, while in my paper, available liquidity reserves play that role. Their debt runs result from self-fulfilling expectations, they are not related to the fundamentals of the financial intermediary. More-

[2] Bolton, Chen, and Wang (2011) does analyze a firm with a credit line, but there is no strategic interactions amongst creditors and the credit line is perpetual – i.e. does not mature.

over, in this literature, the synchronous run decisions of creditors lead to multiple equilibria, which limits the ability to derive comparative statics or policy recommendations.

Finally, my model can be reinterpreted in an international macroeconomic context and can provide answers to questions related to the optimal sizing of central banks' foreign currency reserves for countries where monetary policy is not totally independent of the government[3]. As an example, the 1997 Asian financial crisis featured countries (Thailand, Indonesia, South-Korea and the Philippines) with large amounts of foreign-currency denominated debt and high debt-to-GDP ratios experiencing sudden withdrawals of dollar funding. Central banks in those crisis countries reacted to the collapse of their currencies by raising interest rates and intervening in the foreign exchange markets. In my model, the analog of the firm is the government/central bank of a country such as Thailand. Its illiquid cash flow generating asset is now the country's fiscal revenues (converted into USD), and its liquid reserves are the USD and other foreign currencies held at its central bank. The country has financed itself with foreign-currency denominated sovereign debt held by a continuum of creditors. Creditors then face a decision to run or roll-over their debt claims, and their decision depends on the debt-to-GDP ratio of the country, as well as the amount of foreign currency reserves held by its central bank.

My paper thus contributes to the literature on currency attacks and international reserve holdings – a large literature that has gone through multiple phases over the past 40 years. The seminal papers of Krugman (1979) and Flood and Garber (1984) focus on currency crisis in a country with a pegged exchange rate regime and domestic credit expansion, which result in a depletion of the central bank's foreign exchange reserves and a currency attack by agents with perfect foresight. Within that class of models, Obstfeld (1986) emphasizes the role of expectations about future government policies and highlights the possibility for self-fulfilling crisis and multiple equilibria in economies where a fixed exchange rate would otherwise be sustainable. In order to move away from equilibrium indeterminacy and self-fullfilling beliefs, Morris and Shin (1998) leverage the global games framework. By assuming heterogeneous information across agents, their model features a unique equilibrium in cutoff strategies, which facilitates comparative statics and policy analysis.

Chang and Velasco (2001) adapts the celebrated framework of Diamond and Dybvig (1983) to an international context, by assuming that domestic banks can borrow from international lenders in addition to obtaining funds from domestic depositors. They illustrate the connection between the behavior of international lenders and domestic creditor runs, and show that the presence of international reserves can prevent equilibria where international creditors refuse to roll-over their debt. Hur and Kondo (2013) extends the concept to a

[3]I want to thank Fernando Alvarez for pointing this out to me.

multi-period setting and analyze a country that is financed by foreign creditors who are hit by exogeneous random liquidity shocks The country has access to an illiquid production technology, giving rise to maturity mismatch and the need to store international reserves. They study the optimal reserve holdings and sudden stop probability as a function of liquidity risk, but assume away the "bad" Diamond and Dybvig (1983) run equilibrium. The authors replicate some stylized facts about sudden stops: when rollover risk increases, sudden stops occur and countries optimally increase their stock of foreign currency reserves. This class of models takes as exogeneous the demand for liquidity by some creditors, whereas my model generates runs purely due to concerns over liquidity and solvency.

Finally, Bianchi, Hatchondo, and Martinez (2012) build upon the model of Eaton and Gersovitz (1981) to study a country that issues long-term defaultable debt to facilitate consumption smoothing, and that is exposed to exogeneous sudden stop shocks. Since a sovereign default leads to financial autarky, the accumulation of international reserves enables the country to survive (at least in the short run) to a sudden stop shock without having to immediately default, which is welfare-improving. Similarly, Jeanne and Ranciere (2011) derive a useful close form expression for the optimal international reserve holding for a country that might be locked out of international capital markets with some exogeneous probability. In both articles, the elasticity of intertemporal substitution crucially influences the demand for international reserves, whereas my model operates in a environment where all agents are risk-neutral. Those last two papers also assume that sudden stop events are entirely exogeneous, while my sudden stops occur following a deterioration of macroeconomic fundamentals [4].

My article is organized as follows. I first discuss some empirical facts that motivate this research. I then describe the basic model. I establish a few properties that will be useful in solving numerically for the threshold equilibrium. I derive the Hamilton-Jacobi-Bellman equations that the key equilibrium value functions of my model satisfy. I solve the model numerically and perform comparative statics, varying the firm's illiquid asset characteristics and the firm's debt maturity profile. Finally, I analyze the influence of the firm's dividend policy and portfolio choice on the run behavior of creditors.

[4]Note that the source of sudden stops is still a hotly debated topic in the empirical literature on international capital flows and the forces driving such flows. While Forbes and Warnock (2012) concludes that global factors are significantly associated with extreme capital flow episodes, Gourinchas and Obstfeld (2011) shows that currency crisis for emerging market economies tend to be preceded by high external debt, a deteriorating current account, and low levels of foreign currency reserves.

2 Empirical Facts

2.1 Bank Runs and Liquidity Reserves

The failure of several banks and broker dealers during the 2008-2009 crisis illustrates the fact that financial firms, while in compliance with their mininimum capital requirements, can collapse due to a shortage of available funds.

In 2008, bank failures occurred in the U.S. even with the ability to access the Federal Reserve discount window. Washington Mutual Bank for example was placed into the receivership of the FDIC on September 25, 2008. Its demise resulted from a deteriorating mortgage portfolio, beginning in 2007, combined with a sudden outflow of deposits[5][6]. In the United Kingdom, Northern Rock received emergency funding from the Bank of England in September 2007, following a deterioration of its mortgage portfolio and an inability to roll over its short and medium term wholesale funding, as documented extensively by Shin (2009).

While non-bank U.S. financial firms did not have access to the discount window, the Federal Reserve had created, in March 2008, two facilities to enable primary dealers to borrow funds against eligible collateral either overnight (through the Primary Dealer Credit Facility) or for a period of 28 days (for the Term Securities Lending Facility). This did not prevent Lehman Brothers to file for bankruptcy in September 2008[7]. Valukas et al. (2010) discusses extensively the liquidity position of the firm in the months leading to its collapse. According to the authors, *by the second week of September 2008, Lehman found itself in a liquidity crisis; it no longer had sufficient liquidity to fund its survival.* The report describes a variety of collateral and other margin calls during August and early September 2008 that depleted Lehman Brothers' liquidity resources. Goldman Sachs also suffered a run in September 2008, but it managed to survive, highlighting the fact that runs are not always fatal to a financial institution[8].

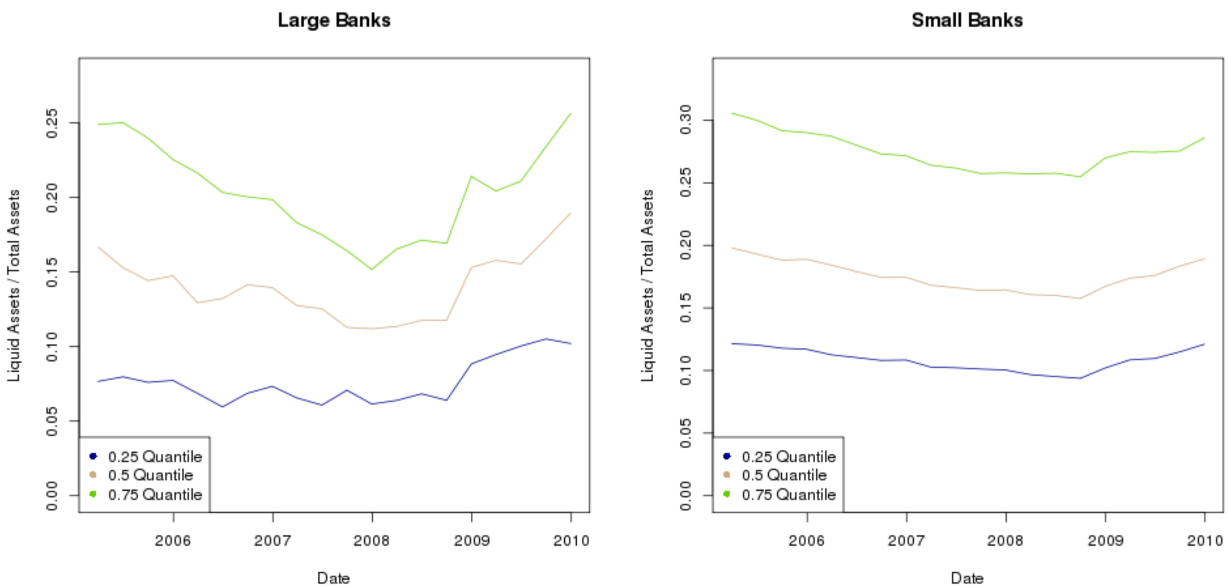
[5]John Reich, director of the federal Office of Thrift Supervision, is quoted by the Huffington Post on September 25, 2008, as saying the following: "Pressure on WaMu intensified in the last three months as market conditions worsened. An outflow of deposits, that began on Sept. 15, reached $16.7 billion, and without sufficient cash to meet its obligations, WaMu was in an unsafe and unsound condition to transact business".

[6]It is worthwhile noting that Washington Mutual Bank had lost primary-credit access to the Federal Reserve discount window in September 2008 due to its CAMELS rating downgrade by the OTS – an institution with a CAMELS rating of 4 or below is not eligible to receive primary credit at the Federal Reserve discount window. At the time of its bankruptcy filing however, Washington Mutual Bank still had secondary-credit discount window access, with the Federal Reserve Bank of San Francisco.

[7]Note that Bear Stearns, at the time of its collapse in March 2008, had access neither to the PDCF, nor to the TSLF.

[8]The financial section of Goldman Sachs' 2008 annual report mentions the following: *In the latter half of 2008, we were unable to raise significant amounts of long-term unsecured debt in the public markets, other*

Figure 1: Liquid Assets - Total Assets Ratio



(a) Banks with Total Assets > $10bn

(b) Banks with Total Assets < $10bn

Figure 1 gives a picture of aggregate liquidity of the U.S. banking sector between 2005 and 2010. For that time period, I plot the ratio of (a) cash, U.S. treasury securities and U.S. agency mortgage backed securities divided by (b) total assets for different quantiles of the distribution of banks regulated by the FDIC[9]. The figure also separates large banks (defined as banks with assets over $10bn) from small banks, as those two groups differ by several important factors[10]. Figure 1 indicates that U.S. banks' liquidity reserves (as a percentage of total assets) trended downwards in the 3 years preceding the financial crisis. Those liquidity reserves then rebounded rapidly starting in the fourth quarter 2008, most likely due to (a) a more conservative portfolio liquidity choice by banks' management in a period of funding stress, as well as (b) anticipation of future liquidity requirements enacted by regulators worldwide[11].

than as a result of the issuance of securities guaranteed by the FDIC under the TLGP. It is unclear when we will regain access to the public long-term unsecured debt markets on customary terms or whether any similar program will be available after the TLGPs scheduled June 2009 expiration. However, we continue to have access to short-term funding and to a number of sources of secured funding, both in the private markets and through various government and central bank sponsored initiatives.

[9]Source: FDIC call reports.

[10]As Figure 15 and Figure 16 in the appendix show, large U.S. banks have (i) tier 1 risk based capital ratios and (ii) retail deposits to total asset ratios significantly lower than small banks.

[11]It is worth noting that the multiple rounds of quantitative easing implemented by the Federal Reserve, starting in December 2008, did not impact the liquidity reserves of U.S. banks, since these operations merely transformed U.S. treasuries and agency MBS into reserves, therefore not impacting banks' total liquidity pools.

The 2008-2009 financial crisis prompted the Basel Committee on Banking Supervision to develop new liquidity regulations. The most important of these rules, to be implemented in 2015, constrains banks to maintain their Liquidity Coverage Ratio ("LCR") at a level greater than 100%[12]. The LCR is defined as (i) High Quality Liquid Assets ("HQLA") divided by (ii) total 30-days Net Cash Outflows ("NCO"):

$$LCR = \frac{HQLA}{NCO}$$

Eligible HQLA are defined as unencumbered assets that can be readily converted to cash at little or no loss of value. Some of these assets are only contributing a fraction of their market value, due to regulatory haircuts used to account for a potentially lower market liquidity in times of stress. Table 1 lists eligible HQLA and their related regulatory haircuts.

Asset Category	Asset Classification	Regulatory Haircut
Cash and Bank Notes	Level 1	0%
Sovereign Bonds with 0% Basel II risk-weights	Level 1	0%
Domestic Currency Sovereign Bonds	Level 1	0%
Sovereign Bonds with 20% Basel II risk-weights	Level 2A	15%
AA Non-Financial Corporate Bonds	Level 2A	15%
AA Non-Financial Covered Bonds	Level 2A	15%
AA RMBS	Level 2B	25%
A or BBB Non-Financial Corporate Bonds	Level 2B	50%
Common Stocks	Level 2B	50%

Table 1: Basel III HQLA: Regulatory Haircuts

The NCO is calculated as the difference between (i) total expected cash outflows and (ii) total expected cash inflows over the next 30 calendar days during a period of market stress. Each category of liability that is projected to come due within 30 days, weighted by its probability of not being rolled over, contributes to the total expected cash outflows. Similarly, cash inflows projected to be received within the next 30 days are discounted at various haircut rates driven by the probability of those cash inflows being actually received in a period of market stress. Table 2 shows the different liability runoff rates assumed by the Basel Committee on Banking Supervision.

While international bank regulators are just starting to discuss implementation of minimum liquidity requiments, certain countries have been imposing such requirements for some time now. Regulators in the Netherlands for example impose a floor on domestic banks' "Liquidity Balance" – a ratio that resembles the LCR ratio designed by the Basel Committee on

[12]In 2015, banks only need to maintain their LCR at a level above 60%, and this minimum level increases each year, reaching 100% in 2019.

Liability Type	Runoff Probability
Insured/“Stable” Retail Deposits	5%
“Less Stable” Retail Deposits	10%
Non-Finance Wholesale Funding	25%
Financial Unsecured Wholesale Funding	100%
Secured Wholesale Funding vs. Level 1 Assets	0%
Secured Wholesale Funding vs. Level 2A Assets	15%
Secured Wholesale Funding vs. Level 2B Assets	50%
Secured Wholesale Funding vs. other Assets	100%
Derivatives Liabilities Funding Needs	100%
Collateral Posting Upon Downgrade	100%
ABCP Liquidity Facilities	100%

Table 2: Basel III NCO: Liability Run-off Rates

Banking Supervision. The availability of high-frequency data on banks' liquid asset holdings in the Netherlands enables De Haan and van den End (2013) to analyze the determinants of such liquidity holdings for Dutch banks from 2004 to 2010. By regressing banks' liquid asset holdings vs. measures of short term liabilities and short term cash inflows and outflows, the authors find that Dutch banks store liquid assets as a buffer against short term liabilities, as well as longer term liabilities and other projected net cash outflows, up to one year ahead. By including measures of solvency on the right-handside of their regressions, the authors also find that better capitalized banks hold less liquid assets against their stock of short term liabilities, an empirical result that will resonate with the model developed in this paper.

At this point, it should be clear to the reader that regulators' approach to liquidity risk completely abstracts from any solvency considerations. In other words, two banks that have the same short term debt and short term net cash outflows will have the same minimum liquidity holdings requirements, irrespective of the fact that one bank might be significantly better capitalized than the other. Unfortunately, liquidity and solvency are highly interconnected concepts. A bank whose solvency deteriorates will find it more difficult to find creditors willing to purchase its debt claims. A reduction of the bank's ability to refinance itself will force the bank to use its available liquid resources to repay maturing creditors that are not rolling over. This downward pressure on the liquidity resources of the bank might eventually deplete the cash buffer of such bank, forcing the bank to sell less liquid assets to service its debt. Eventually, such bank might end up totally illiquid and having to default. This reasoning motivates the model presented in the next section: a regulatory framework for bank minimum liquidity holdings must take into account the bank's solvency level.

2.2 Sudden Stops and Currency Runs

As discussed in the introduction, the model developed in this paper can be reinterpreted in an international macroeconomic context. In that reinterpretation, the consolidated government includes its central bank, which holds a stock of foreign currency reserves. The government finances some of its activities through the issuance of foreign currency denominated liabilities, and relies on fiscal receipts to repay its creditors. The fiscal receipts can be viewed as an "illiquid asset" on the government's balancesheet. Foreign creditors financing this government will focus on both the fiscal revenues of the government, as well as its stock of foreign currency reserves when making roll-over decisions. A "sudden stop" happens when both the fiscal revenues and the foreign currency reserves of the country have declined below certain threshold levels. It is thus informative to look empirically at the occurrence of sudden stops and understand the extent to which those sudden stops are related to a country's external debt-to-GDP ratio and its foreign-currency reserves-to-GDP ratio.

Gourinchas and Obstfeld (2011) studies exactly this question, using data on 57 emerging market economies over the period 1973 – 2010. For different types of crisis episodes (sovereign, banking, and currency crisis), using a fixed-effect panel specification, the authors analyze the evolution of different macroeconomic variables pre- and post- crisis, relative to "tranquil times". For emerging market countries, in the years preceding either a currency crisis or a sovereign default, the authors find that (a) external leverage is high relative to "tranquil times", (b) the current account is low relative to "tranquil times", (c) foreign exchange reserves are below their value in "tranquil times", and (d) short term external debt is high relative to "tranquil times". As will become clear in the next few sections, some of these empirical facts will be featured in the model developed in this article.

3 The Model

3.1 Agents

Time is continuous, indexed by t, and the horizon is infinite. I consider a firm that owns two types of assets: illiquid assets that can only be sold at a discount to fundamental value, and liquid reserves (which I will refer to as cash). The firm is controlled by risk-neutral shareholders, who discount their flow profits at the rate ρ. The firm has financed itself by issuing debt that was sold to a continuum of risk-neutral creditors, with initial measure 1. A given creditor will be indexed by i, and creditors' discount rate will be ρ. I am assuming that the firm cannot raise additional outside equity or debt financing: it has to finance its operations and costs using cash generated from its asset portfolio.

3.2 The Firm's Assets

The firm's portfolio consists of two different assets. First, the firm holds cash and short term liquid assets, which can be sold at any point in time at a price of 1, and which pay a flow interest at rate r_c, where $0 < r_c < \rho$. The assumption that the cash yield is lower than the time preference rate of agents creates a friction that can be motivated as follows. First, consistent with Jensen (1986), the firm's managers could be engaging in wasteful activities – such as expensing the available cash to derive private benefits. Those wasteful activities lead to agency costs that can be modeled in a reduced form fashion by assuming that cash yields a rate r_c that is lower than the time preference rate ρ. Second, cash in my model is valuable as it can be liquidated immediately without incurring transaction costs. But this benefit comes at a shadow cost, in the form of a lower cash yield. I will note $C(t)$ the cash value of those short term liquid assets at the firm at time t.

Second, the firm holds $N(t)$ units of a long term illiquid investment, which generates cash-flows $Y(t)$ per unit of time and per unit of investment. $Y(t)$ follows the stochastic differential equation $\frac{dY(t)}{Y(t)} = \mu dt + \sigma dB(t)$, where $B(t)$ is a Brownian motion and $\mu < \rho$. The fundamental value $Q(t)$ of one unit of the illiquid investment satisfies:

$$Q(t) = \mathbb{E}^Y \left[\int_t^{+\infty} e^{-\rho(s-t)} Y(s) ds \right] = \frac{Y(t)}{\rho - \mu}$$

The notation $\mathbb{E}^Y$ represents the expectation operator conditioned on $Y(t) = Y$. I will note $P(t) = N(t)Q(t)$ the aggregate fundamental value of the illiquid investments held by the firm. The firm's long term illiquid investment "matures" according to a Poisson process with arrival intensity ϕ. At such maturity date, the illiquid investment is sold for its fundamental value, and the proceeds, in addition to any cash available at the firm, are distributed to creditors and shareholders according to a standard priority of payments. Outside the maturity date, the long term illiquid investments can only be sold at a discount α to the fundamental value $P(t)$.

In the first part of my analysis, firm's portfolio cash flows received in excess of funding costs and potential debt redemptions are entirely reinvested into the cash reserve, meaning that $N(t) = N$ is constant (normalized to 1). I then analyze a modified environment where the firm reinvests a portion of these net cash inflows into the illiquid asset by solving a portfolio choice problem. I do not allow fractional sales of the illiquid investment: if sold before its maturity date, the illiquid investment is sold in its entirety. I then discuss a modified environment where the firm has the option to liquidate fractions of the illiquid asset. Finally, I assume that the illiquid asset cannot be pledged for collateralized funding purposes, in contrast with Martin, Skeie, and von Thadden (2014), which assumes that the firm's assets can be pledged under repo contracts.

3.3 The Firm's Liability Structure

I assume that each creditor has invested 1 unit of cash into the firm, in the form of short term debt. This short term debt matures according to a Poisson arrival process with intensity λ. Thus, at each point in time, a constant fraction of the firm's oustanding liabilities matures, guaranteeing that the firm's debt average life remains constant. At the maturity of a given debt instrument, the relevant creditor decides whether or not to roll-over into a new short term debt instrument. Creditors receive a flow interest rate r_d (with $r_d > \rho$) on their debt claims. The parameter restriction $r_d > \rho$ ensures that creditors have some incentive to be invested in the debt issued by the firm. Since the firm's creditors have an initial measure equal to 1, the aggregate short term debt at time $t = 0$ is equal to $D(0) = 1$, and generally, the aggregate short term debt at the firm at time t is equal to $D(t)$. In my model, $D(t)$ is a weakly decreasing function of time: it is constant when creditors continuously roll over their maturing claims, but it decreases when maturing creditors refuse to roll and receive their principal balance back. If the firm needs to repay a creditor before the maturity date of the firm's illiquid investment, the firm will first use its cash reserves before selling a single unit of the illiquid investment, since such illiquid investment can only be sold at a discount to fundamental value. Before the maturity date of the illiquid asset, if the firm's cash reserves are depleted, the firm has to sell its illiquid assets and distribute the proceeds to creditors. If any money is left once creditors have been repaid, the balance is distributed to the firm's shareholders. Going forward, I will assume the following parameter restriction.

Assumption 1. The liquidation fraction α is such that $1 < \frac{1}{\alpha} < \frac{r_d+\lambda}{\rho-\mu}$.

In the first part of the paper, I assume that no dividends are paid to shareholders. The only cash-flows received by shareholders come either (a) at the maturity date of the illiquid asset, when the firm's cash and illiquid assets are liquidated at their fundamental value, or (b) when the firm's cash reserves are entirely depleted and the firm's illiquid asset is sold for a fraction α of its fundamental value. In each of these events, shareholders are entitled to the residual proceeds (if any) after creditors have been repaid the funds they are owed. I will then analyze a modified environment where the firm has the ability to pay dividends to shareholders. I will do so by first assuming an exogeneous dividend payment rule, and will then discuss the firm's optimal dividend payment policy. In each of these environments, I need to assume that the firm can credibly commit to a dividend payment policy. Indeed, consider the situation where the firm's illiquid asset suffers a sequence of bad shocks: in that case, the balance-sheet of the firm deteriorates, prompting creditors to run. In that situation, the cash available at the firm starts being depleted, putting the firm closer to bankruptcy and making future cash payments to shareholders less likely. In such situation, the firm's management, maximizing shareholder's value, would have an incentive to instead pay the available cash out to shareholders. To prevent such a scenario from occurring, I thus need to assume that the firm's management can credibly commit to a dividend payment rule. I will note by $U(t)$ the cumulative payouts made to shareholders.

4 Model Solution

4.1 Strategy Space

In the next few sections and unless otherwise specified, I assume that no dividend is payable to shareholders before the maturity date of the illiquid asset, and that all the firm's net cash-flows above debt service and potential debt redemptions are reinvested in cash – in other words, $N(t) = N = 1$ at all times. I will study the problem faced by a specific creditor i, and will focus on symmetric Markov perfect equilibria, as defined in more details below. A given creditor only makes decisions when his debt claim matures. The state variables that are payoff-relevant for creditor i's decision at such time t are (i) the value of the illiquid portfolio $P(t)$ (since the cash flows related to such portfolio are simply proportional to $P(t)$), (ii) the amount of cash $C(t)$ in the liquidity reserve, and (iii) the amount of debt outstanding $D(t)$. A pure Markov strategy for a given creditor can then be defined as a mapping $s : \mathbb{R}_+^3 \to \{0, 1\}$, where action 0 corresponds to the decision to roll over, and action 1 corresponds to the decision to run. Under strategy s, at each time t at which creditor i's debt claim matures, creditor i will roll into a new debt claim if $s\left(P(t), C(t), D(t)\right) = 0$, and creditor i will instead request his principal balance back if $s\left(P(t), C(t), D(t)\right) = 1$. In what follows, I will adopt the following convention: s will denote the strategy assumed to be followed by a given creditor i, while S is the strategy assumed to be followed by all other creditors. Note that a pure Markov strategy for creditors can be also characterized by the subset of $\mathbb{R}_+^3$ for which creditors elect to run. In other words, for any stragegy S, the set $\mathcal{R}_S \equiv \{(P, C, D) : S\left(P, C, D\right) = 1\}$ (or "run region") uniquely characterizes the creditors' strategy. I will use similarly the creditors' roll region $\mathcal{NR}_S \equiv \{(P, C, D) : S\left(P, C, D\right) = 1\}$.

4.2 State Space

Since I am first assuming that the firm's illiquid asset holding $N(t)$ is constant and normalized to 1, the aggregate value of the firm's illiquid investments $P(t) = N(t)Q(t)$ follows the same dynamics as the price of one unit of the illiquid investment:

$$dP(t) = \mu P(t)dt + \sigma P(t)dB(t) \quad (1)$$

For a given arbitrary strategy $S : \mathbb{R}^3_+ \to \{0, 1\}$ followed by the firm's creditors, the endogenous state variable C follows the dynamics:

$$dC(t) = (\rho - \mu)P(t)dt + r_c C(t)dt - r_d D(t)dt - 1_{\{S(P(t),C(t),D(t))=1\}} \lambda D(t)dt \quad (2)$$

$C(t)$ increases with cash flows $Y(t)dt = (\rho - \mu)P(t)dt$ received on the illiquid investment and with interest $r_c C(t)dt$ collected on the liquid assets kept by the firm, and $C(t)$ decreases with interest $r_d D(t)dt$ paid to creditors and with redeeming creditors $\lambda D(t)dt$, whenever the state $(P(t), C(t), D(t))$ is in the "run" region. Similarly, the endogenous state variable $D(t)$ follows the dynamics:

$$dD(t) = -1_{\{S(P(t),C(t),D(t))=1\}} \lambda D(t)dt \quad (3)$$

Thus, the outstanding debt balance of the firm is constant in the roll region, but declines exponentially in the run region.

4.3 Payoff Functions

Let $\tau_b = \inf\{t : C(t) = 0, (\rho - \mu)P(t) < (r_d + \lambda 1_{\{S(P(t),C(t),D(t))=1\}})D(t)\}$ be the firm's default time. For the firm to default, its cash reserve needs to hit zero, and the drift of its cash reserve needs to be strictly negative. I focus on a specific creditor, following a strategy s. Let $\{\tau^n_\lambda\}_{n\geq 1}$ be a sequence of independent exponentially distributed stopping times, with arrival intensity λ. Let τ_r be defined as follows:

$$\tau_r = \inf_k \{\sum_{i=1}^k \tau^i_\lambda \text{ s.t. } s\left(P(\sum_{i=1}^k \tau^i_\lambda), C(\sum_{i=1}^k \tau^i_\lambda), D(\sum_{i=1}^k \tau^i_\lambda)\right) = 1\}$$

τ^n_λ are times at which creditor i has the opportunity to roll over its debt claim, and τ_r is the first time at which creditor i elects to run. For a given strategy s followed by creditor i and a given strategy S followed by all other creditors, the payoff function V of creditor i is equal to the following:

$$V(P,C,D;s,S) = \mathbb{E}^{P,C,D}\left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_r\}} + e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \min\left(1, \frac{P(\tau)+C(\tau)}{D(\tau)}\right) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \min\left(1, \alpha\frac{P(\tau)}{D(\tau)}\right)\right] \quad (4)$$

In the above, $\tau \equiv \tau_r \wedge \tau_b \wedge \tau_\phi$ is the earliest of (a) the creditor running, (b) the firm filing for bankruptcy, and (c) the illiquid asset maturing. τ is potentially infinite. Equation (4) says that creditor i derives value (a) from flow interest payments at the rate r_d per unit of time, (b) at the first creditor's debt maturity τ_r for which the creditor elects not to roll (if such stopping time comes first) from the collection of his principal balance, (c) at the illiquid asset maturity date τ_ϕ (if such stopping time comes first) from the value $\min(1, \frac{P(\tau)+C(\tau)}{D(\tau)})$ of collecting his principal balance to the extent of funds available at the firm, and (d) at the firm's bankruptcy date τ_b (if such stopping time comes first) from the value $\min(1, \frac{\alpha P(\tau)}{D(\tau)})$ of collecting his principal balance to the extent of funds available at the firm. The notation above emphasizes that the payoff function V depends on creditor i's strategy s, as well as the strategy of all other creditors S (via the dynamics of the state variables). Note that I can similarly define shareholders' payoff function E as follows:

$$E(P,D,C;S) = \mathbb{E}^{P,D,C}\left[e^{-\rho(\tau_\phi\wedge\tau_b)} 1_{\{\tau_\phi<\tau_b\}} \max\left(0, P(\tau_\phi)+C(\tau_\phi)-D(\tau_\phi)\right)\right] + \mathbb{E}^{P,D,C}\left[e^{-\rho(\tau_\phi\wedge\tau_b)} 1_{\{\tau_b<\tau_\phi\}} \max\left(0, \alpha P(\tau_b)-D(\tau_b)\right)\right] \quad (5)$$

Shareholders derive value (a) at the illiquid asset maturity date τ_ϕ (if such stopping time comes first) from the excess value $\max(0, P(\tau_\phi)+C(\tau_\phi)-D(\tau_\phi))$ of the firm's asset portfolio over the aggregate debt oustanding, and (b) at the firm's bankruptcy date τ_b (if such stopping time comes first) from the excess value $\max(0, \alpha P(\tau_b) - D(\tau_b))$ of the firm's illiquid asset liquidation proceeds over the aggregate debt oustanding. I will be studying strategies that have a particular homogeneity property.

Assumption 2. Creditors' strategies S are homogeneous of degree zero in (P,C,D).

This assumption makes intuitive sense: since creditors are infinitesimally small, a given creditor should be indifferent between a credit exposure to the risk of a firm with illiquid assets worth P, cash worth C, and debt worth D, or a credit exposure to the risk of a firm with illiquid assets worth aP, cash worth aC, and debt worth aD, for any $a > 0$. One question that arises naturally is whether a creditor's best response $s^*(S)$ to a common homogeneous strategy S followed by all other creditors is also homogeneous. The following lemma helps shed light on this issue.

Lemma 1. For any homogeneous of degree zero strategy $S : \mathbb{R}^3_+ \to \{0,1\}$ followed by all other firm's creditors, creditor i's best response $s^*(S) \equiv \arg\max_s V(\cdot,\cdot,\cdot;s,S)$ is homogeneous of degree zero.

Thus, homogeneous strategies are "stable", in the sense that a creditor, taking into account the fact that all other creditors follow a common homogeneous strategy, will respond by also following a homogeneous strategy. All the results that follow are thus derived under Assumption 2, satisfied for (i) all other creditors (strategy S), and (ii) for the particular creditor of interest (strategy s). The homogeneity property discussed above simplifies considerably the problem to be studied, as Lemma 2 demonstrates.

Lemma 2. For any strategy $S : \mathbb{R}^3_+ \to \{0,1\}$ followed by the firm's creditors and strategy $s : \mathbb{R}^3_+ \to \{0,1\}$ followed by creditor i, the payoff function V is homogeneous of degree zero, and the value function E is homogeneous of degree one.

Lemma 2 enables me to summarize the state of the system by two state variable only:

$$c(t) \equiv \frac{C(t)}{D(t)}$$
$$p(t) \equiv \frac{P(t)}{D(t)}$$

Note that p and c are appropriately defined: since creditors' claims mature with Poisson arrival rate λ, I know that $D(t) > 0$ for all t almost surely, irrespective of the decisions made by creditors or the firm. With a slight abuse of notation, a strategy will now be a mapping $S : \mathbb{R}^2_+ \to \{0,1\}$. I now focus on the dynamics of the state variables (p,c), for a given arbitrary strategy $S : \mathbb{R}^2_+ \to \{0,1\}$ followed by creditors. Using Ito's lemma, I have;

$$dp(t) = \left(\mu + \lambda 1_{\{S(p(t),c(t))=1\}}\right) p(t)dt + \sigma p(t) dB(t) \tag{6}$$
$$dc(t) = \left((\rho-\mu)p(t) + (r_c + \lambda 1_{\{S(p(t),c(t))=1\}})c(t) - (r_d + \lambda 1_{\{S(p(t),c(t))=1\}})\right) dt \tag{7}$$

The drift of the illiquid asset price (per unit of outstanding debt) $p(t)$ is greater (by the term $+\lambda p(t)$) when a run is occuring, since the outstanding debt of the firm decreases, meaning that the remaining creditors can rely on a greater "share of the pie". Similarly, when a run occurs, there are two effects on the drift of the cash per unit of outstanding debt $c(t)$: cash is used to pay down maturing creditors (the drift term $-\lambda dt$), but since the number of remaining creditors is decreasing, those remaining creditors are entitled to a greater "share of the pie" (the drift term $+\lambda c(t)dt$). Note that in the region of the state space where the firm holds less that 1 unit of cash per unit of debt outstanding (in other words, where $c < 1$ – arguably the empirically relevant region), a run leads to a decrease of the cash drift. Given the homogeneity properties described above, I can write $V(P, D, C; s, S) = v(p, c; s, S)$, and $E(P, D, C; S) = De(p, c; S)$. Using the state variables $(p(t), c(t))$, the creditor's payoff function V can be re-written as follows:

$$v(p, c; s, S) = \mathbb{E}^{p,c}\Big[\int_0^{\tau} e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_r\}} + e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \min(1, p(\tau) + c(\tau)) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \min(1, \alpha p(\tau))\Big] \quad (8)$$

In the above, $\tau = \tau_r \wedge \tau_\phi \wedge \tau_b$. Similarly, the shareholders' payoff function, per unit of debt outstanding, can be re-written:

$$e(p, c; S) = \mathbb{E}^{p,c}\left[e^{-\rho(\tau_b\wedge\tau_\phi)}\left[1_{\{\tau_\phi<\tau_b\}} \max(0, p(\tau_\phi) + c(\tau_\phi) - 1) + 1_{\{\tau_b<\tau_\phi\}} \max(0, \alpha p(\tau_b) - 1)\right]\right] \quad (9)$$

4.4 Symmetric Markov Perfect Equilibrium

For a given strategy $S : \mathbb{R}_+^2 \to \{0, 1\}$, creditor i finds the strategy $s : \mathbb{R}_+^2 \to \{0, 1\}$ that maximizes his payoff function. In other words, creditor i solves for any (p, c):

$$v^*(p, c; S) = \sup_s v(p, c; s, S)$$

Definition 1. A symmetric Markov perfect equilibrium of the game is a mapping $S : \mathbb{R}_+^2 \to \{0, 1\}$ such that for any $(p, c) \in \mathbb{R}^2$:

$$v(p, c; S, S) = \sup_s v(p, c; s, S)$$

4.5 Strategic Complementarity?

One question that naturally arises is whether creditors' payoff functions exhibit strategic complementarity, or more generally supermodularity. The answer to this question will guide the solution method adopted. Indeed, if the game that creditors play is supermodular, I can rely on lattice theory and iterated deletions of interim dominated strategies, as illustrated by

Vives (1990) or Milgrom and Roberts (1990), in order to establish the existence of equilibria of the game described in this paper. Such a strategy has been used in the bank run literature, for example by Rochet and Vives (2004) or more recently Vives (2014).

Intuitively, one might want to postulate that the game's payoffs indeed exhibit strategic complementarity. When creditors run, they reduce the cash balance available for the firm to service the remaining creditors' debt, making the length of time needed to hit the zero-cash boundary shorter and prompting the remaining creditors to be tempted to run at the first opportunity. Proving that the game considered in this paper is supermodular would require showing that for an arbitrary strategy S_1 followed by all creditors, and an arbitrary strategy $S_2 \geq S_1$ (i.e. creditors run in more states under S_2 than under S_1), a specific creditor's incentive to run is higher when responding to S_2 than when responding to S_1:

$$1 - v(p, c; s, S_2) \geq 1 - v(p, c; s, S_1) \quad \text{for any strategy } s \text{ and } (p, c) \in \mathbb{R}_+^2$$

Instead, I will show that this condition does not hold for the game considered in this paper.

Proposition 1. The payoff function $v(\cdot, \cdot; s, S)$ does not exhibit supermodularity.

I develop the proof in the appendix, by focusing on the region of the state space $p = 0$. The failure of the game to be supermodular stems from the fact that conflicting forces are at play upon the occurrence of a run. On one side, the firm's cash balance decreases due to debt redemptions, putting the firm closer to an illiquid situation, at which point the firm might have to sell its illiquid asset and default on its debts. This contributes to increasing the incentives for a given creditor to run. However, a run has several beneficial effects for the remaining creditors. As equations (6) and (7) indicate, upon the occurrence of a run, the drift of the illiquid asset price (per unit of debt outstanding) is higher (than in the no-run regime) by the term $\lambda p(t)$. As discussed previously, this means that the remaining creditors can claim a bigger "share of the pie" upon a firm's default. In addition, upon the occurrence of a run, the firm's expensive debt (which yields r_d) is being paid down, helping the firm reduce its future debt interest expense. This is another source of "good news" for a remaining creditor, leading to a potentially counterintuitive result that the run incentive might decrease when the run region expands. The lack of strategic complementarity of the creditor payoff function is one of the striking differences between this paper and He and Xiong (2012), whose payoff function does exhibit strategic complementarity in terminal payoffs and could therefore be solved using iterated deletions of interim dominated strategies[13].

[13]Note that He and Xiong (2012) establish the existence and uniqueness of the equilibrium of their game by leveraging closed-form solutions for the payoff function of creditors; assuming the constant

This result reminds of a similar observation made by Goldstein and Pauzner (2005) for their model: their bank run payoffs do not exhibit global, but rather local strategic complementarity. This prompts the authors to use a solution technique that differs from the traditional tools available in the presence of supermodularity.

4.6 Dominance Regions

In what follows, I will want to restrict the equilibria of the game to cutoff Markov perfect equilibria.

Definition 2. A strategy $S : \mathbb{R}^2_+ \to \{0, 1\}$ followed by the firm's creditors is a cutoff strategy if the sets $\mathcal{R}_S \equiv \{(p, c) \in \mathbb{R}^2_+ : S(p, c) = 1\}$ and $\mathcal{NR}_S \equiv \{(p, c) \in \mathbb{R}^2_+ : S(p, c) = 0\}$ are disjoint connected sets that form a partition of $\mathbb{R}^2_+$.

In words, under a cutoff strategy S, the northeast quadrant is divided into two disjoint areas: one area in which creditors run, and one area in which creditors roll over their debt contracts. I will show that there exists a Markov perfect equilibrium in cutoff strategies, as defined above. In order to do this, I first establish some preliminary results.

Proposition 2. For any strategy $S : \mathbb{R}^2_+ \to \{0, 1\}$ followed by the firm's creditors, the payoff function v is non-negative and bounded above by $\frac{r_d+\phi}{\rho+\phi}$.

The upper bound established in Proposition 2 corresponds to the value for a creditor rolling his debt claim into a new debt claim forever while facing no credit risk. Such creditor earns an interest rate r_d greater than his discount rate ρ until the time the illiquid asset matures, yielding a present value $\frac{r_d+\phi}{\rho+\phi} > 1$.

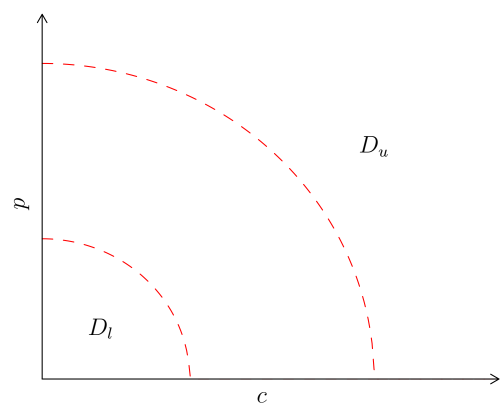
Proposition 3. There exists non-empty lower and upper dominance regions, in other words there exists $\mathcal{D}_l \subset \mathbb{R}^2_+$ and $\mathcal{D}_u \subset \mathbb{R}^2_+$ such that for any strategy $S : \mathbb{R}^2_+ \to \{0, 1\}$ followed by creditors, any strategy $s : \mathbb{R}^2_+ \to \{0, 1\}$ followed by creditor i, I have:

$$(p, c) \in \mathcal{D}_l \Rightarrow v(p, c; s, S) < 1$$
$$(p, c) \in \mathcal{D}_u \Rightarrow v(p, c; s, S) > 1$$

value of debt is $D = 1$, the authors could instead have analyzed the properties of the value function $V(P; s, S) = \mathbb{E}^P \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_r\}} + e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \max(1, P(\tau)) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \max(1, \alpha P(\tau))\right]$. τ_r is as usual the first (stopping) time at which the state is such that $s(P_{\tau_r}) = 1$, and τ_b is the default time, which is the stopping time at which $S(P_{\tau_b}) = 1$ and the credit line fails. The value function V as defined is supermodular, as Doh (2015) shows: $1 - V(P, s, S) \leq 1 - V(P, s, S')$, whenever $S < S'$, for any strategy s and $P > 0$.

The proof of the propositions above can be found in the appendix. The lower dominance region is a region of the state space near the origin $(p, c) = (0, 0)$: for small values of p and c, it is a dominant strategy for a creditor to run when he has the opportunity to do so, irrespective of the strategy S employed by all other creditors. The upper dominance region is a region of the state space characterized by high values of p, c, or both. In that region, it is a dominant strategy for a creditor to roll over his debt claim upon the maturity of his existing debt claim, irrespective of the strategy S employed by all other creditors. The lower and upper dominance regions are illustrated in Figure 2. Given that these regions are non-empty, I can define $\mathcal{D}_l$ as the largest connected set containing $(0, 0)$ such that running is a dominant strategy, and similarly I can define $\mathcal{D}_u$ as the largest connected set containing $(+\infty, +\infty)$ such that rolling is a dominant strategy. The existence of dominance regions suggests that I should be looking for symmetric Markov perfect Equilibria in cutoff strategies.

Figure 2: Dominance Regions



4.7 Creditor's Problem

For a given strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by all other creditors, I now study the optimization problem that creditor i solves:

$$v^* (p, c; S) = \sup_s v (p, c; s, S)$$

The following proposition establishes existence and uniqueness of v^* as the solution to a standard functional equation.

Proposition 4. For any cutoff strategy $S : \mathbb{R}_+^2 \rightarrow \{0, 1\}$ followed by the firm's creditors, creditor i's optimal value function is the unique continuous bounded function that is solution to the following fixed point problem:

$$\begin{aligned} v^* (p, c; S) = \mathbb{E}^{p,c} & \left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} \max (1, v^* (p(\tau), c(\tau); S)) \right] \\ & + \mathbb{E}^{p,c} \left[e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \min (1, p(\tau) + c(\tau)) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \min (1, \alpha p(\tau)) \right] \quad (10) \end{aligned}$$

Where τ_λ is exponentially distributed (with parameter λ), and $\tau = \tau_\lambda \wedge \tau_\phi \wedge \tau_b$.

The proof of Proposition 4 relies on an appropriately constructed contraction map. The functional equation for $v^* (\cdot, \cdot; S)$ reflects the fact that for $\tau = \tau_\lambda$ (in other words, when the first stopping time to occur is the maturity date of creditor i's debt claim), the creditor has the option to either roll over into a new debt claim (with payoff $v^* (p(\tau_\lambda), c(\tau_\lambda); S)$), or to take his money out (with payoff 1).

4.8 Cutoff Markov Perfect Equilibrium when $p = 0$

In order to make progress on the characterization of any cutoff Markov perfect equilibrium, I now derive the value function v^* in the special case where the illiquid asset cash flow – and thus the illiquid asset fundamental value – is zero. When p approaches zero, the firm can only rely on its cash resources to pay creditors' interest and principal (for those creditors who are seeking repayment). At the extreme, when $p = 0$, only the cash balance c is relevant for creditors' roll-over decisions. Note that this is a deterministic problem with perfect foresight for creditors: creditors can predict perfectly the evolution of the state variable $c(t)$. Since c is the only state variable, I can postulate a threshold c^* above which it will be optimal for creditors to continue rolling their debt claims, and below which it will be optimal for creditors to run when the opportunity arises. In what follows, I prove that the threshold c^*

is unique. I will note $v_0(c) \equiv v^*(0,c)$ the *equilibrium* value function of a creditor of a game with no illiquid asset – in other words, the value function of a given creditor following the cutoff strategy c^*, when all other creditors follow the same strategy. Similarly, I will use $e_0(c) \equiv e^*(0,c)$ for the shareholder value.

Proposition 5. In an economy without illiquid asset, there exists a unique equilibrium in cutoff strategies, characterized by the cutoff $c^* \in \left(1, \frac{r_d+\lambda}{r_c+\lambda}\right)$. The equilibrium value function v_0 of creditors is equal to:

$$v_0(c) = \begin{cases} \left(\frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)}\right)\left[1-\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}\right]+\frac{\phi}{\phi+\rho-r_c}c & 0 \le c < 1 \\ H_1\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}+\frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} & 1 \le c < c^* \\ H_2\left(1-\frac{r_c}{r_d}c\right)^{\frac{\rho+\phi}{r_c}}+\frac{r_d+\phi}{\rho+\phi} & c^* \le c < \frac{r_d}{r_c} \\ \frac{r_d+\phi}{\rho+\phi} & c \ge \frac{r_d}{r_c} \end{cases} \quad (11)$$

The equilibrium cutoff c^* satisfies $v_0(c^*) = 1$. The formula for c^* is derived in the appendix. The value function v_0 is strictly increasing for $c < \frac{r_d}{r_c}$, and constant for $c > \frac{r_d}{r_c}$. The equilibrium value function for shareholders is equal to:

$$e_0(c) = \begin{cases} 0 & 0 \le c < 1 \\ K_1\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\phi}{r_c+\lambda}}+\frac{\phi}{\phi+\rho-(r_c+\lambda)}c-\frac{\phi}{\rho+\phi}\left(1+\frac{r_d+\lambda}{\phi+\rho-(r_c+\lambda)}\right) & 1 \le c < c^* \\ K_2\left(1-\frac{r_c}{r_d}c\right)^{\frac{\rho+\phi}{r_c}}+\frac{\phi}{\phi+\rho-r_c}c-\frac{\phi}{\rho+\phi}\left(1+\frac{r_d}{\phi+\rho-r_c}\right) & c^* \le c < \frac{r_d}{r_c} \\ \frac{\phi}{\phi+\rho-r_c}c-\frac{\phi}{\rho+\phi}\left(1+\frac{r_d}{\phi+\rho-r_c}\right) & c \ge \frac{r_d}{r_c} \end{cases} \quad (12)$$

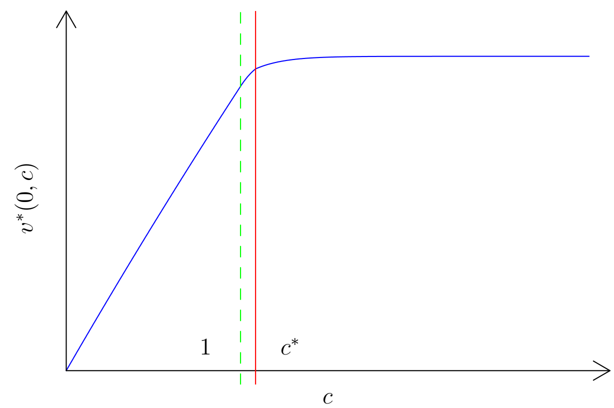
H_1, H_2, K_1 and K_2 are constants described in the appendix.

Proposition 5 is useful. Indeed, when looking for a symmetric cutoff Markov perfect equilibrium strategy, I now know that the cutoff boundary must intersect the axis $p = 0$ at $c = c^*$. Figure 3 illustrates the shape of the value function when the firm does not hold any illiquid asset[14]. Notice that the cutoff c^* is always strictly greater than 1. This means that when the firm's cash reserve is exactly equal to the aggregate outstanding debt, creditors are already running. This makes sense – when $C(t) = D(t)$, the firm owns an asset yielding r_c per unit of time, while it has outstanding debt yielding $r_d > r_c$. This is an irreversible situation for such firm, whose cash reserves will be depleted in finite time, and creditors

[14]The value function is plotted using the following model parameters: $\rho = 0.04$, $\lambda = 0.2$, $\phi = 0.2$, $r_d = 0.05$, $r_c = 0.01$.

choose to run before the asset to debt ratio of the firm is unity. My next proposition looks at the comparative statics for the cutoff c^*.

Figure 3: Value function $v_0(\cdot)$



PROPOSITION 6. In an economy without illiquid asset, the unique equilibrium cutoff strategy c^* is decreasing in λ.

Proposition 6 might seem surprising at first. It says that the longer the average firm's debt maturity $1/\lambda$, the more conservative the equilibrium strategie of creditors, in other words the "earlier" they run. This result is due to the lack of strategic complementarity highlighted by Proposition 1 in the neighborhood of the equilibrium. Specifically, for the game with no illiquid asset, the region of the state space where the equilibrium cutoff c^* is located exhibits strategic substitutability: the earlier other creditors run, the better off a given creditor i is. As the proof of Proposition 6 (in the appendix) shows, this substitutability leads to the seemingly strange result that a longer debt average maturity leads to a more run-prone firm. I will show in the numerical section of the paper that this result also obtains in the presence of illiquid assets.

4.9 Boundary $c = 0$

I now discuss the boundary of the state space $c = 0$. When this boundary is reached (or when the system is started at $c = 0$), two events can occur. Either the drift of the cash reserve is positive, in which case the game continues, or the drift is negative, in which case the firm is forced to sell its illiquid asset and distribute the proceeds to its creditors and its shareholders. The following proposition characterizes the run and roll regions on the subset $\{(p,c) : c = 0\}$, for any symmetric cutoff Markov perfect equilibrium.

Proposition 7. Given any symmetric cutoff Markov perfect equilibrium:

i. If $\frac{1}{\alpha} < \frac{r_d+\lambda}{\rho-\mu}$, the set $\{(p,0) : p < \frac{1}{\alpha}\}$ is in the run region and the set $\{(p,0) : p \geq \frac{1}{\alpha}\}$ is in the roll region;

ii. If $\frac{1}{\alpha} \geq \frac{r_d+\lambda}{\rho-\mu}$, there exists a positive value $\bar{p} \geq \frac{r_d+\lambda}{\rho-\mu}$ such that the set $\{(p,0) : p < \bar{p}\}$ is in the run region and the set $\{(p,0) : p \geq \bar{p}\}$ is in the roll region.

The proof of Proposition 7 is developed in the appendix. It shows that when the recovery rate is above a certain threshold level $\frac{\rho-\mu}{r_d+\lambda}$, I know exactly where any symmetric cutoff Markov perfect equilibrium boundary will be anchored on the axis $c = 0$, which will prove useful for my numerical implementation of this model.

4.10 Existence of Symmetric Cutoff Markov Perfect Equilibrium

The previous sections facilitated a better understanding of the economics of the bank run model developed in this paper in different regions of the state space. The following theorem establishes the existence of a symmetric cutoff Markov perfect equilibrium of the game.

Proposition 8. There exists a symmetric cutoff Markov perfect equilibrium of the game.

5 Hamilton-Jacobi-Bellman Equations

In the section below, I derive HJB equations that the creditors' optimal value function v^* satisfies in different regions of the state space, under the assumption that S is a symmetric Markov perfect equilibrium strategy. The value function v^* implicitly depends on the strategy S followed by all other creditors, but I omit this dependence in this section in order to simplify notation. Finally, I assume that the value function v^* is twice differentiable except at specific boundaries of the state space.

5.1 Domain $\mathcal{NR}$

This is the domain where creditors are rolling over their debt claims. On the interior of the domain $\mathcal{NR}$, the value function v^* satisfies the following HJB equation:

$$\rho v^*(p,c) = r_d + \mu p \partial_p v^*(p,c) + \frac{1}{2}\sigma^2 p^2 \partial_{pp} v^*(p,c) + ((\rho-\mu)p + r_c c - r_d)\, \partial_c v^*(p,c) \\ + \lambda \max(0, 1 - v^*(p,c)) + \phi\,[\min(1, p+c) - v^*(p,c)]$$

On the interior of the domain $\mathcal{NR}$, the value function e^* satisfies the following HJB equation:

$$\rho e^*(p,c) = \mu p \partial_p e^*(p,c) + \frac{1}{2}\sigma^2 p^2 \partial_{pp} e^*(p,c) + ((\rho-\mu)p + r_c c - r_d)\, \partial_c e^*(p,c) \\ + \phi\,[\max(0, p+c-1) - e^*(p,c)]$$

Since the firm cannot pay dividends before the maturity date of the illiquid asset, if $c > \frac{r_d}{r_c}$, the cash available (per unit of outstanding debt) is strictly increasing with time, irrespective of the value of the illiquid asset p and irrespective of creditor's decisions. Thus, a dominant strategy for creditors in this case is to roll over their maturing debt. Creditor's debt is thus risk-free, with a value equal to the discounted stream of interest payments at rate r_d, up to the stopping time τ_ϕ. In other words, I must have for any (p,c) such that $c > \frac{r_d}{r_c}$:

$$v^*(p,c) = \frac{r_d + \phi}{\rho + \phi}$$

Additionally, for p large, the illiquid asset cash flows per unit of time are large as well. For p large enough, the available cash per unit of debt outstanding exceeds the bound $\frac{r_d}{r_c}$ in a short amount of time, at which point the debt is risk-free, as established above. This means that I have:

$$\lim_{p\to+\infty} v^*(p,c) = \frac{r_d + \phi}{\rho + \phi}$$

Those considerations also indicate that:

$$\lim_{p\to+\infty} \partial_p v^*(p,c) = \lim_{c\to+\infty} \partial_c v^*(p,c) = 0$$

Finally, I establish the following lemma for shareholder value when p or c are large.

LEMMA 3. When $p \to +\infty$ or $c \to +\infty$, and when no dividends are payable to shareholders, the function e^* verifies:

$$e^*(p,c) = \frac{\phi}{\rho+\phi-r_c}c + \frac{\phi}{\rho+\phi-\mu}\left(1+\frac{\rho-\mu}{\rho+\phi-r_c}\right)p - \frac{\phi}{\rho+\phi}\left(\frac{r_d}{\rho+\phi-r_c}+1\right) + o(1)$$

5.2 Domain $\mathcal{R}_S$

This is the domain where creditors are electing not to roll over their debt claims and therefore to "run". On the interior of the domain $\mathcal{R}_S$, the value function v^* satisfies the following HJB equation:

$$\rho v^*(p,c) = r_d+(\mu+\lambda)p\partial_p v^*(p,c)+\frac{1}{2}\sigma^2p^2\partial_{pp}v^*(p,c)+\left((\rho-\mu)p+(r_c+\lambda)c-(r_d+\lambda)\right)\partial_c v^*(p,c)$$
$$+\lambda\max\left(0,1-v^*(p,c)\right)+\phi\left[\min(1,p+c)-v^*(p,c)\right]$$

On the interior of the domain $\mathcal{R}_S$, the value function e^* satisfies the following HJB:

$$\rho e^*(p,c) = (\mu+\lambda)p\partial_p e^*(p,c)+\frac{1}{2}\sigma^2p^2\partial_{pp}e^*(p,c)+\left((\rho-\mu)p+(r_c+\lambda)c-(r_d+\lambda)\right)\partial_c e^*(p,c)$$
$$+\phi\left[\max(0,p+c-1)-e^*(p,c)\right]$$

When $c=0$, Proposition 7 shows that if $\frac{1}{\alpha}\le\frac{r_d+\lambda}{\rho-\mu}$, the locus of points $\{(p,c):c=0,p<\frac{1}{\alpha}\}$ is in the run region $\mathcal{R}_{\mathcal{S}}$, whereas if $\frac{1}{\alpha}>\frac{r_d+\lambda}{\rho+\lambda}$, the locus of points $\{(p,c):c=0,p<\bar{p}\}$ (for some $\bar{p}>\frac{r_d+\lambda}{\rho-\mu}$) is in the run region $\mathcal{R}_{\mathcal{S}}$. On such boundary $\{(p,c)\in\mathcal{R}_S, c=0\}$, the value of the debt and equity claims is equal to:

$$v^*(p,0) = \min\left(\alpha p,1\right)$$
$$e^*(p,0) = \min\left(\left(\alpha p-1\right),0\right)$$
$$\lim_{c\downarrow 0}\partial_p v^*(p,c) = \alpha 1_{\{\alpha p<1\}}$$
$$\lim_{c\downarrow 0}\partial_p e^*(p,c) = \alpha 1_{\{\alpha p>1\}}$$

5.3 Boundary $\partial\mathcal{R}_S$

Symmetry and optimality of creditors' cutoff strategy means that for points on the boundary $\partial\mathcal{R}_S$ which are not on the vertical axis $c=0$ and which are not on the horizontal axis $p=0$

must satisfy:

$$v^*(p,c) = 1$$

In order to confirm that the cutoff stragegy s is optimal, I must verify that:

$$v^*(p,c) \geq 1 \text{ for all } (p,c) \in \mathcal{NR}_S$$
$$v^*(p,c) < 1 \text{ when } (p,c) \in \mathcal{R}_S$$

I will require the functions v^* and e^* to be continuously differentiable on the locus of points $\partial\mathcal{R}_S$ that are in the interior of the state space.

6 Numerical Implementation

6.1 Algorithm

I compute the value function v numerically over the compact set $[0,\bar{p}] \times [0,\bar{c}]$, by determining the value of v^* on a grid G_h, where $h > 0$ is my scalar appproximation parameter. I choose $\bar{c} = \frac{r_d}{r_c}$, and $\bar{p}$ large enough to ensure that v^* is close to its maximum at $p = \bar{p}$. I will use a Markov Chain approximation method, as explained in Kushner and Dupuis (2001), and solve the model assuming no dividends are payable. I start with a guess equilibrium map $S^{(1)}$, and a guess value function $v^{(1,1)}$. My guess functions will take the following form:

$$S^{(1)}(p,c) = 1_{\{p<\frac{1}{\alpha}(1-c/c^*)\}}$$
$$v^{(11)}(p,c) = \min\left(v_0(c) + \alpha p, \frac{r_d+\phi}{\rho+\phi}\right)$$

The initial guess equilibrium map corresponds to a run boundary that is a linear in the (p,c) space, intersecting the axis $p = 0$ at $c = c^*$, and intersecting the axis $c = 0$ at $p = \frac{1}{\alpha}$. My algorithm has an outer-loop, which updates the equilibrium map $S^{(i)}$, and an inner loop, which, for a given $S^{(i)}$, updates the function $v^{(i,j)}$. In the inner loop, I calculate the function $v^*(\cdot,\cdot;S^{(i)})$ as follows. Given the map $S^{(i)}$, the state space (p,c) evolves according to the following:

$$dp(t) = \left(\mu + \lambda 1_{\{S^{(i)}(p(t),c(t))=1\}}\right) p(t)dt + \sigma p(t)dB(t)$$
$$dc(t) = \left((\rho-\mu)p(t) + (r_c + \lambda 1_{\{S^{(i)}(p(t),c(t))=1\}})c(t) - (r_d + \lambda 1_{\{S^{(i)}(p(t),c(t))=1\}})\right) dt$$

I will use the following notation:

$$
\begin{aligned}
a_p(p,c) &= \sigma p \\
b_p(p,c) &= \left(\mu + \lambda 1_{\{S^{(i)}(p,c)=1\}}\right) p \\
b_c(p,c) &= (\rho - \mu)p + (r_c + \lambda 1_{\{S^{(i)}(p,c)=1\}})c - (r_d + \lambda 1_{\{S^{(i)}(p,c)=1\}})
\end{aligned}
$$

In the inner loop, I create a Markov Chain $\{\left(p_n^h, c_n^h\right), n < \infty\}$ that approximates the process $\{(p(t), c(t))\}_{t\geq 0}$. Let $\gamma > 0$ be an arbitrary constant. I introduce $Q^h(p,c)$ and $\Delta t^h(p,c)$ as follows:

$$
\begin{aligned}
Q^h(p,c) &\equiv a_p(p,c)^2 + h b_p(p,c) + h|b_c(p,c)| + h\gamma \\
\Delta t^h(p,c) &\equiv \frac{h^2}{Q^h(p,c)}
\end{aligned}
$$

Note that $\inf_{p,c} Q^h(p,c) > 0$, which means that $\Delta t^h(p,c)$ is well defined. Note also that I have for all (p,c):

$$
\lim_{h\to 0} \Delta t^h(p,c) = 0
$$

I then define the following transition probabilities:

$$
\begin{aligned}
\Pr\left(\left(p_{n+1}^h, c_{n+1}^h\right) = (p+h, c) \middle| \left(p_n^h, c_n^h\right) = (p,c)\right) &= \frac{a_p(p,c)^2/2 + h b_p(p,c)}{Q^h(p,c)} \\
\Pr\left(\left(p_{n+1}^h, c_{n+1}^h\right) = (p-h, c) \middle| \left(p_n^h, c_n^h\right) = (p,c)\right) &= \frac{a_p(p,c)^2/2}{Q^h(p,c)} \\
\Pr\left(\left(p_{n+1}^h, c_{n+1}^h\right) = (p, c) \middle| \left(p_n^h, c_n^h\right) = (p,c)\right) &= \frac{h\gamma}{Q^h(p,c)} \\
\Pr\left(\left(p_{n+1}^h, c_{n+1}^h\right) = (p, c+h) \middle| \left(p_n^h, c_n^h\right) = (p,c)\right) &= \frac{h \max\left(0, b_c(p,c)\right)}{Q^h(p,c)} \\
\Pr\left(\left(p_{n+1}^h, c_{n+1}^h\right) = (p, c-h) \middle| \left(p_n^h, c_n^h\right) = (p,c)\right) &= \frac{h \max\left(0, -b_c(p,c)\right)}{Q^h(p,c)}
\end{aligned}
$$

Notice that these transition probabilities are all greater than zero, less than 1, and they add up to 1. Noting $\Delta\left(p_n^h, c_n^h\right) \equiv \left(p_{n+1}^h, c_{n+1}^h\right) - \left(p_n^h, c_n^h\right)$, the Markov chain created satisfies the local consistency condition:

$$\mathbb{E}^{p,c}\left[\Delta\begin{pmatrix} p_n^h \\ c_n^h \end{pmatrix}\right] = \begin{pmatrix} b_p(p,c) \\ b_c(p,c) \end{pmatrix}\Delta t^h(p,c)$$
$$\text{var}^{p,c}\left[\Delta\begin{pmatrix} p_n^h \\ c_n^h \end{pmatrix}\right] = \begin{pmatrix} a_p(p,c)^2 & 0 \\ 0 & 0 \end{pmatrix}\Delta t^h(p,c) + o\left(\Delta t^h(p,c)\right)$$

For $\bar{p} > p > 0$ and $\bar{c} > c > 0$, and given a function $v^{(i,j)}$, I compute $v^{(i,j+1)}$ on the grid G_h as follows:

$$v^{(i,j+1)}(p,c) = r_d \Delta t^h(p,c)$$
$$+ e^{-\rho\Delta t^h(p,c)} \times \left\{ \begin{array}{c} \phi\Delta t^h(p,c)\min(1,p+c) + \lambda\Delta t^h(p,c)\max\left(1, v^{(i,j)}(p,c)\right) \\ + \left(1-(\lambda+\phi)\Delta t^h(p,c)\right)\sum_{(p',c')}\Pr\left((p',c')|(p,c)\right) \times v^{(i,j+1)}(p',c') \end{array} \right\}$$

For $c > 0$, when the Markov chain is in a state with $p = h$, the algorithm puts a non-zero probability onto the next state being such that $p = 0$. When that happens, I will assume that such next state is absorbing, with a terminal value $v_0(c)$. Similarly, for $p > 0$, when the Markov chain is in a state with $c = h$, our algorithm puts a non-zero probability onto the next state being such that $c = 0$ and $p \leq \frac{1}{\alpha}$. When that happens, I will assume that such next state is absorbing, with a terminal value $\alpha p \wedge 1$. Finally, when $c = \frac{r_d}{r_c} - h$ and the Markov chain transitions to a state where $c = \frac{r_d}{r_c}$, or when $p = \bar{p} - h$ and the Markov chain transitions to a state where $p = \bar{p}$, I assume that such state is absorbing, with value $\frac{r_d+\phi}{\rho+\phi}$.

So long as $||v^{(i,j+1)} - v^{(i,j)}||_\infty > \epsilon$, for ϵ small taken arbitrarily, I continue on the inner loop. When $||v^{(i,j+1)} - v^{(i,j)}||_\infty \leq \epsilon$, I have obtained $v^*(\cdot,\cdot;S^{(i)})$ as the limit of the shooting algorithm. I then set $S^{(i+1)}$ by solving for each (p,c) on the grid:

$$S^{(i+1)}(p,c) = 1_{\{v^*(p,c;S^{(i)})<1\}}$$

So long as $||S^{(i+1)} - S^{(i)}||_\infty > \hat{\epsilon}$, for $\hat{\epsilon}$ small taken arbitrarily, I continue on the outer loop, and stop when $||S^{(i+1)} - S^{(i)}||_\infty < \hat{\epsilon}$.

6.2 Numerical Results

6.2.1 Base Case Parameters

I first study the behavior of the model in the context of banks and broker dealers. The model parameters selected (displayed in Table 3) thus reflect some of the key characteristics of those financial intermediaries.

Parameter	Value	Description
ρ	0.04	Time preference rate
λ	2.00	1/Average debt maturity
ϕ	0.20	1/Average illiquid asset maturity
r_d	0.05	Interest rate on debt
r_c	0.01	Interest rate on internal cash
μ	0.025	Risk neutral illiquid asset growth rate
σ	0.10	Illiquid asset volatility
α	0.25	Recovery upon illiquid asset sale

Table 3: Calibration Parameters

Since the illiquid asset value follows the Gordon growth formula $P(t) = Y(t)/(\rho - \mu)$, I need to set $\rho > \mu$ to guarantee that the illiquid asset value is finite. r_d is the interest rate received by creditors. Since creditors need to have an incentive to roll-over their debt, I need to impose $r_d > \rho$. r_c is the yield on the cash internal to the firm. While the firm does not solve a portfolio choice problem in this version of the model, I want to set the parameter r_c so that it can be used in the version of the model with a dynamic porfolio choice. In other words, I need to choose $r_c < \rho$ in order to introduce a cost for the firm to hold liquid reserves, and I need to chose $r_c < \mu$ to give the firm an incentive to hold the illiquid asset. In other words, I need to chose the parameters ρ, r_c, r_d, μ so that $r_c < \mu < \rho < r_d$. Those considerations lead me to the parameter choices in Table 3. The parameter α is calibrated using Moody's historical recovery rate data[15]. The parameter ϕ influences the duration of the illiquid asset – its inverse $1/\phi$ represents the expected maturity of the asset. I choose $\phi = 0.2$, which means that I am assuming an illiquid asset average life of 5 years. This assumption differs from He and Xiong (2012): they focus on a firm that holds a mortgage portfolio, and thus set the expected life of their asset at 13 years. I have in mind a broker-dealer or investment bank, whose portfolio includes not only mortgages, but corporate bonds, asset-backed securities, non-agency mortgage-backed securities, and assets whose average lives are typically shorter than a standard agency mortgage expected life. The parameter λ drives the debt maturity structure – its inverse $1/\lambda$ represents the average debt maturity of the firm. The base case parameter $\lambda = 2$ thus means that the firm's average debt maturity is 6 months. This is consistent with the liability structure of many broker-dealers: firms such as Goldman Sachs or Morgan Stanley have a large tri-party repo book with a duration of approximately 3 months, combined with some long term debt with a duration of 5 to 10 years. The base case parameters lead to a value $c^* = 1.013909$. As expected, $1 < c^* < \frac{r_d+\lambda}{r_c+\lambda} = 1.0199$.

[15]See Moody's Special Report: *Corporate Default and Recovery Rates, 1920 – 2008*. According to the report, the historical recovery rate on senior unsecured bonds was 33.8% as of February 2009.

Figure 4: $v^*(\cdot,\cdot)$

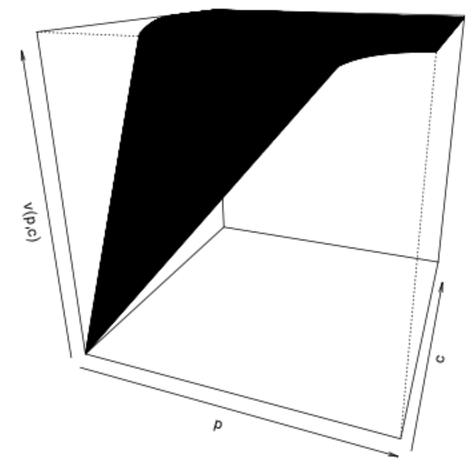
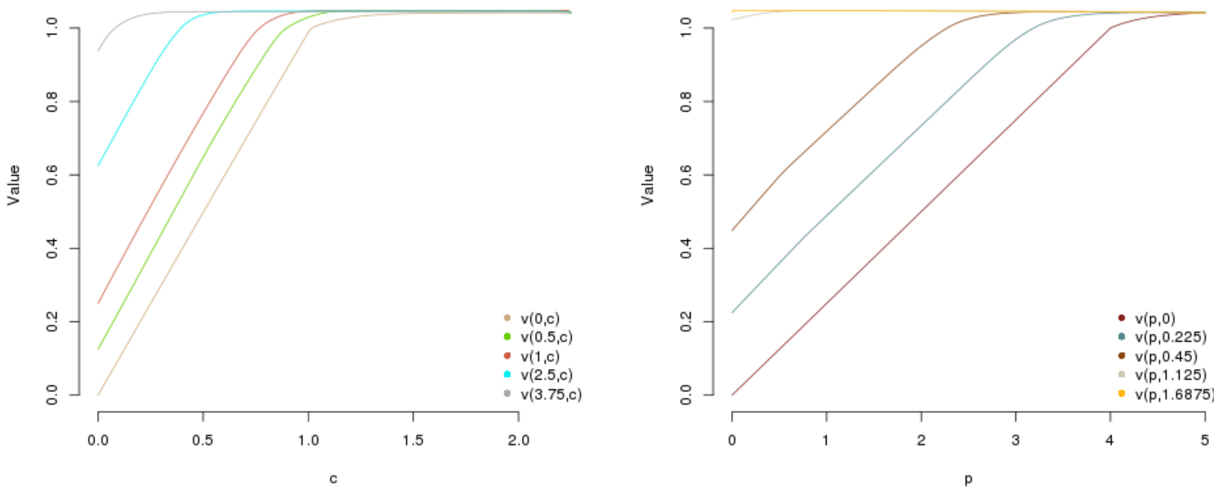


Figure 4 is the 3-D representation of the value funtion v^*. Figure 5a represents the value function $v^*(p,\cdot)$ for several price levels p arbitrarily chosen. $v^*(0,c) = v_0(c)$ is directly drawn from the expression obtained in Proposition 5. It exhibits kinks at $c = 1$ and $c = c^*$. For the selected values of p, $v^*(p,\cdot)$ is an increasing function of c, bounded above by $\frac{r_d+\phi}{\rho+\phi}$. Figure 5b represents the value function $v^*(\cdot,c)$ for several levels of cash c arbitrarily chosen. I focus on $v^*(\cdot,0)$ to start with – in other words the debt value function when the cash (per unit of debt outstanding) is zero. The function is linear for $p < 1/\alpha$, since for those values of p, the firm runs out of cash, sells its illiquid assets, with creditors realizing $\min(1,\alpha p)$. For $p \geq 1/\alpha$, the illiquid asset value is high enough that maturing creditors elect to roll-over their maturing debt claims, even if the bank has no cash resources available to it.

Figure 6 shows the endogenous equilibrium run boundary $\partial\mathcal{R}$ in the (p,c) space. For this specific parameter environment, the equilibrium run boundary is a decreasing function of the cash level. I have also included the locus of points (p,c) such that the cash drift is zero when a run occurs, and when creditors roll over[16]. Points of the state space below these lines are points where the cash drift is negative. As expected, the slope of equilibrium run boundary $\partial\mathcal{R}$ is steeper than -1, since cash has value as run-deterrent. Indeed, remember that leverage, which can be measured as total assets divided by total outstanding debt, is

[16]When creditors are rolling over their debt claims, the cash drift is equal to $(\rho-\mu)p + r_c c - r_d$. Thus, the locus of points (p,c) for which the cash drift is zero is characterized by $p = \frac{1}{\rho-\mu}(r_d - r_c c)$. Similarly, when a run is occuring, the locus of points (p,c) for which the cash drift is zero is characterized by $p = \frac{1}{\rho-\mu}(r_d + \lambda - (r_c+\lambda)c)$.

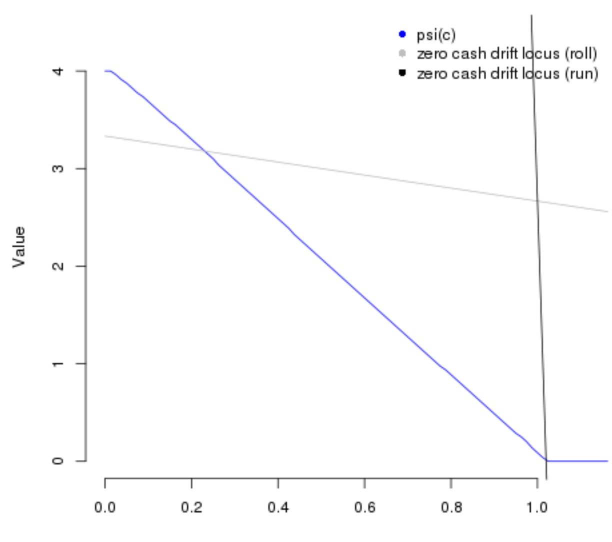
Figure 5: Value function v^*



(a) $v^*(p, \cdot)$ for several values of p

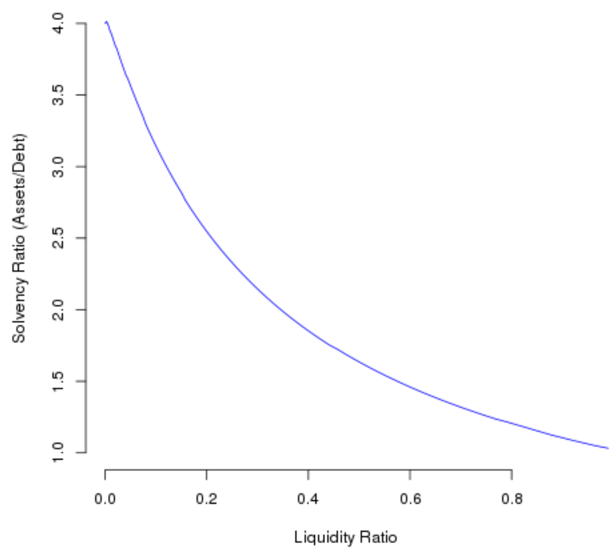
(b) $v^*(\cdot, c)$ for several values of c

Figure 6: Equilibrium threshold $\Psi(\cdot)$



simply equal to $p + c$ in my model. For a given leverage value $p + c$, creditors will be more inclined to run when the firm's liquid resources are low. When $c = 0$, unless the price of the illiquid asset (per unit of debt outstanding) is greater than $1/\alpha$, creditors start runnning. For this parameter environment, as the amount of cash (per unit of debt outstanding) increases, the minimum illiquid asset price level required to deter a run decreases. At the limit $p = 0$, a run is deterred when $c > c^*$.

Figure 7: Liquidity vs. Solvency

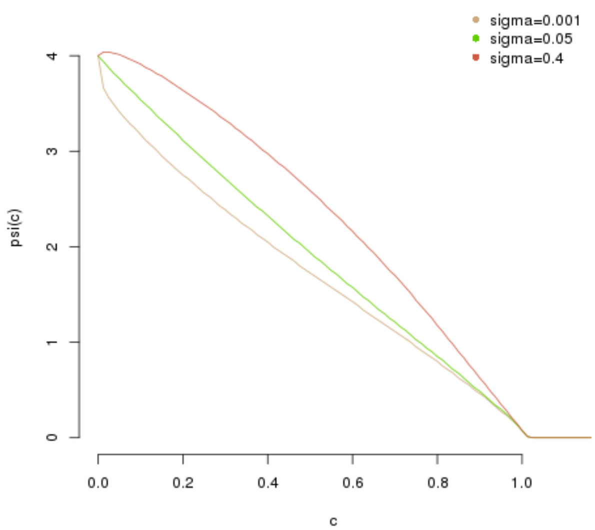


I then consider the following question: given a firm's asset liquidity composition, measured via $\frac{c}{p+c}$ (the fraction of the firm's assets that is liquid), what leverage (as measured by the asset to debt ratio $p + c$) does the firm need to maintain in order to deter a run. Figure 7 answers this question specifically: it plots the threshold leverage $c + p$ as a function of the percentage of the firm invested in liquid reserves $\frac{c}{c+p}$, for points $(p, c) \in \partial\mathcal{R}$. I can then relate those theoretical predictions to the empirical facts discussed in Section 2. For example, a firm that holds 20% of its balance-sheet in liquid resources needs to maintain an asset to debt leverage above 2 in order to deter a run.

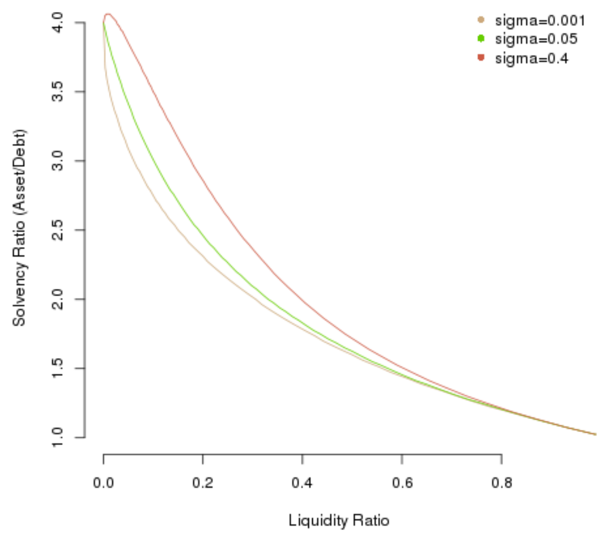
6.2.2 Sensitivity Analysis

How does the threshold strategy followed by creditors vary with some of the model parameters selected in the base case calibration? I first study the sensitivity of the threshold strategy to the illiquid asset volatility σ. For parameter values $\sigma = 0.1\%$, $\sigma = 5\%$ and $\sigma = 40\%$, Figure 8a shows the endogeneous run boundary $\partial\mathcal{R}$, while Figure 8b shows the trade-off between firm's leverage and the percentage of the balance-sheet needed to be invested in cash in order to deter a run. At the base case parameter values selected, the equilibrium run boundary $\partial\mathcal{R}$ increases as the illiquid asset volatility increases. This result is intuitive: for a given creditor, with higher illiquid asset volatility, there is a higher chance that a sequence of

Figure 8: Sensitivity to σ



(a) Equilibrium threshold $\Psi(\cdot)$



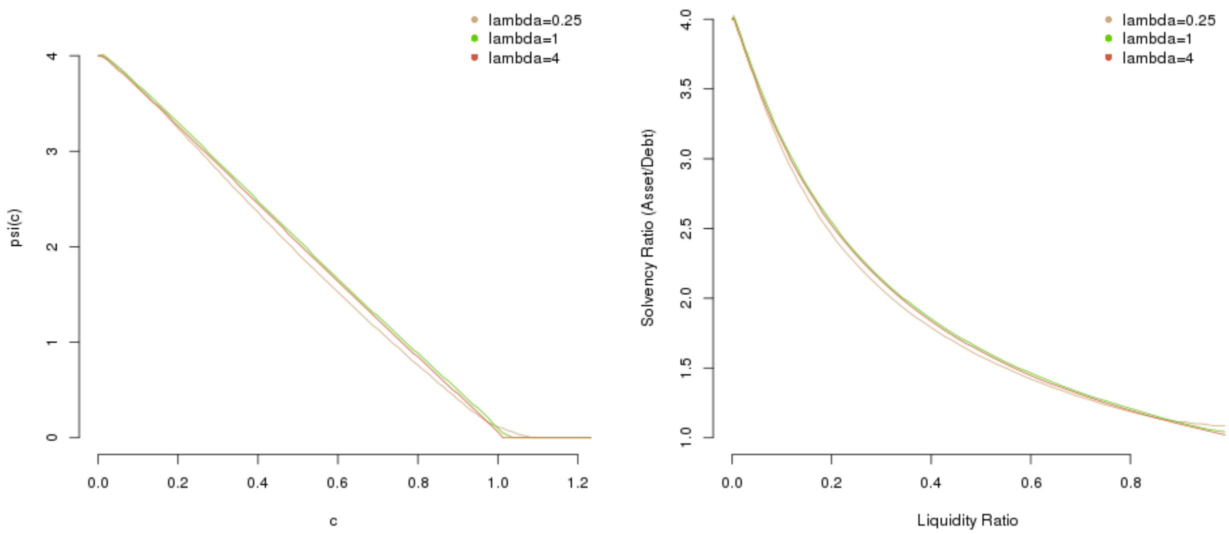
(b) Liquidity vs. Solvency

bad shocks occurs between two debt claim maturities, making such creditor more conservative in his run/roll strategy.

I do note however that the equilibrium run boundary $\partial\mathcal{R}$ is no longer monotone in c. For high volatility values σ and small values of the liquidity reserve c, the equilibrium run boundary $\partial\mathcal{R}$ first increases with c, before decreasing. This suggests that the value function v^* is not monotone in c for these parameter configurations. The intuition behind this surprising result is as follows. On the vertical axis $c = 0$, the debt value function is a concave function of p. At the point $(p, c) = (1/\alpha, 0)$, the debt value is exactly equal to 1. However, if at time t the state is $(p, c) = (1/\alpha, \epsilon)$, the game continues at least for a small time period $dt = \epsilon/|\text{drift}(c)|$. If the drift at that point of the state space is negative (which would be the case if creditors are running), then the state at time $t + dt$ will be $(1/\alpha + dp, 0)$, where dp is normally distributed with mean $\frac{\mu}{\alpha}dt$ and variance $\frac{\sigma^2}{\alpha^2}dt$. Given the concavity of $v^*(\cdot, 0)$, it is clear that for sufficiently high values of σ, Jensen's inequality will be such that $v^*(1/\alpha, \epsilon) < v^*(1/\alpha, 0)$, leading to a value function v^* that can be decreasing in c for values of p close to $1/\alpha$ and values of c close to zero.

I then analyze the sensitivity of the threshold strategy to the firm's debt maturity structure. For parameter values $\lambda = 0.25$, $\lambda = 1$ and $\lambda = 4$, corresponding to firm's debt average lives of 4 years, 1 year and 3 months respectively, Figure 9a shows the endogeneous run boundary $\partial\mathcal{R}$, while Figure 9b shows the trade-off between firm's leverage and the percent-

Figure 9: Sensitivity to λ when $\sigma = 10\%$



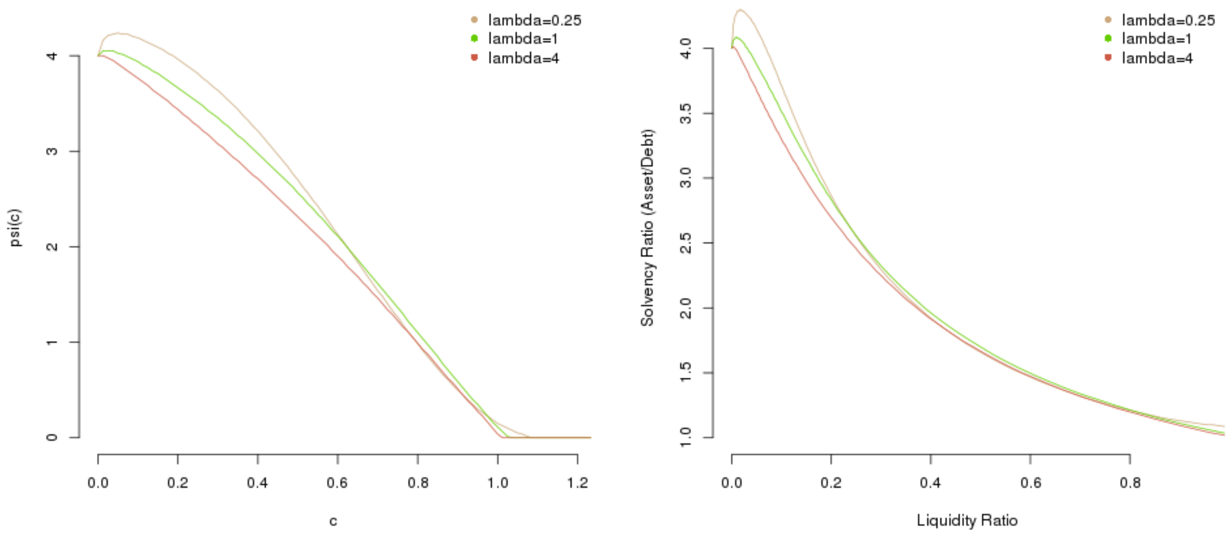
(a) Equilibrium threshold $\Psi(\cdot)$

(b) Liquidity vs. Solvency

age of the balance-sheet needed to be invested in cash in order to deter a run. At the base case parameter values selected, the run boundary $\partial\mathcal{R}$ is surprisingly insensitive to the term structure of the firm's liabilities. To gain some more intuition behind this result, I increase the base case volatility level from 10% to 30% and recompute the sensitivity analysis w.r.t. λ.

Figure 10 shows the result of this new analysis. It now appears that the longer the firm's average life, the sooner creditors run. This result seems to contradict the traditional belief that a firm with longer term debt is less run-prone than a firm with shorter term debt. This counterintuitive result, first highlighted by He and Xiong (2012), can be explained as follows. A smaller value for λ means a longer debt average life. Since the drift of the cash reserve depends on the term $-\lambda 1_{\{S(p,c)=1\}}(1-c)$, and since the values of interest for c are values $c < 1$, a lower λ leads to a lower downward pressure on the cash reserve, and therefore a longer period of time needed for the firm to go bankrupt. But a lower value of λ also means that a given creditor will need to wait a longer period of time between two roll-over decisions. If the volatility of the illiquid asset is high enough, a longer time period between two roll-over decisions means a greater probability that a bad sequence of shocks occur, making creditors potentially more conservative in their decision to roll-over or run. In the example of Figure 10, the volatility of the illiquid asset is sufficiently high for the second effect to dominate the first, leading to the counterintuitive result that creditors run sooner when the firm's debt

Figure 10: Sensitivity to λ when $\sigma = 30\%$



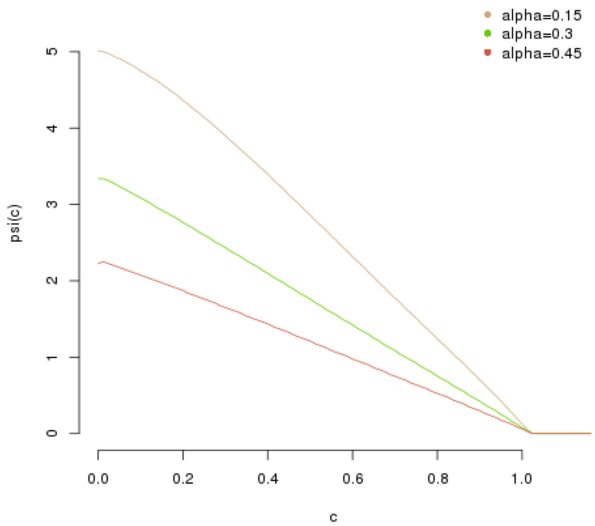
(a) Equilibrium threshold $\Psi(\cdot)$

(b) Liquidity vs. Solvency

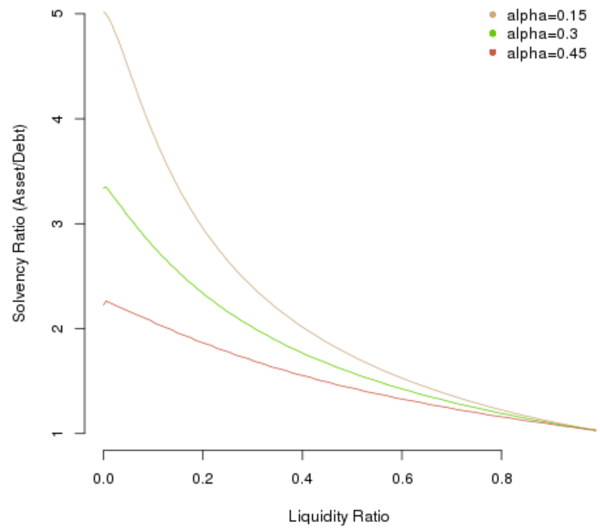
average life is longer. Instead of modeling the maturity of a creditor's debt contract as a Poisson arrival process, I could have elected to model the debt contract as a bullet bond with initial maturity T, as in Leland and Toft (1996). This choice would have led to an additional state variable for the creditor's problem: the time to maturity of the creditor's debt contract. However, I suspect that this alternative modeling strategy might attenuate if not entirely remove the counterintuitive result described above: a shorter average debt maturity T leads to greater liquidity pressure on the firm once it suffers a run, while a given creditor with a remaining maturity t will not benefit from greater arrival intensity of his debt maturity.

I continue my sensitivity analysis by looking at the role of the firm's illiquid asset recovery rate α. For parameter values $\alpha = 0.15$, $\alpha = 0.3$ and $\alpha = 0.45$, Figure 11a shows the endogeneous run boundary $\partial\mathcal{R}$, while Figure 11b shows the trade-off between firm's leverage and the percentage of the balance-sheet needed to be invested in cash in order to deter a run. The recovery rate α plays a crucial role in the determination of the run boundary $\partial\mathcal{R}$: as α increases, the propensity of creditors to run decreases. This phenomenon was to be expected, in light of Proposition 7: the run boundary $\partial\mathcal{R}$ cuts the axis $c = 0$ at $p = \frac{1}{\alpha}$. Therefore, the parameter α turns out to be a critical driver of the value function close to the boundary $c = 0$.

Figure 11: Sensitivity to α



(a) Equilibrium threshold $\Psi(\cdot)$



(b) Liquidity vs. Solvency

7 Externality - [Incomplete]

As pointed in He and Xiong (2012), maturing creditors who are electing to withdraw their funds from the firm increase the probability that the firm runs out of liquid resources and defaults, thus imposing an externality on remaining creditors. In this section, I quantify the loss in value, for both creditors and shareholders, due to this externality. In other words, I compute the debt value and the shareholder value of a firm that is still subject to run risk, but for which the externality imposed by running creditors on remaining creditors is absent. In order to achieve this, I change the debt ownership assumption of the base case model: instead of having a continuum of creditors invested in the debt issued by the firm, I now assume that a unique large creditor owns the entire debt stack of the firm. This large creditor controls whether to reinvest the maturing debt into new debt issued by the firm, or whether to withdraw funding. More specifically, in the infinitesimal time interval $[t, t+dt]$, an amount $\lambda D(t)dt$ of debt is maturing. The large creditor controls the fraction $\eta(t) \in [0, 1]$ of maturing debt that he decides to not roll over – and thus $1 - \eta(t)$ is the fraction of maturing debt reinvested into the firm.

I first study the problem faced by the large creditor. As usual, $\tau_b = \inf\{t : C(t) = 0, (\rho - \mu)P(t) < (r_d + \lambda\eta(t))\, D(t)\}$ is the firm default time (potentially infinite), and τ_ϕ is the maturity date of the illiquid investment. Let $\tau \equiv \tau_\phi \wedge \tau_b$ be the earlier of (a) the illiquid asset maturity date, and (b) the firm's default. For a given adapted process $\{\eta(t)\}_{t\geq 0}$ taking values in the interval $[0,1]$, the large creditor's value is equal to:

$$W(P,D,C) = \mathbb{E}^{P,D,C}\left[\int_0^\tau e^{-\rho t} r_d D(t)dt + \int_0^\tau e^{-\rho t}\lambda\eta(t)D(t)dt\right] \\ + \mathbb{E}^{P,D,C}\left[e^{-\rho\tau}1_{\{\tau=\tau_\phi\}}\min\left(D(\tau), P(\tau)+C(\tau)\right) + e^{-\rho\tau}1_{\{\tau=\tau_b\}}\min\left(D(\tau),\alpha P(\tau)\right)\right]$$

In the expression above, the first integral represents interest collections on outstanding debt and the second integral respresents principal proceeds received on maturing debt that is not rolled. Upon the stopping time τ_ϕ, the large creditor receives the greater of (a) his outstanding debt holding $D(\tau_\phi)$ and (b) the fundamental value of the firm's assets $P(\tau_\phi)+C(\tau_\phi)$. Finally, upon the stopping time τ_b, the large creditor receives the greater of (a) his outstanding debt holding $D(\tau_\phi)$ and (b) the liquidation value of the firm's illiquid investment $\alpha P(\tau_b)$. The debt $D(t)$ evolves according to $dD(t) = -\lambda\eta(t)D(t)dt$, which can also be written:

$$D(t) = De^{-\lambda\int_0^t \eta(s)ds}$$

Using this expression, I can re-writte the value function W as follows:

$$W(P,D,C) = D\mathbb{E}^{P,D,C}\Bigg[\int_0^\tau e^{-\rho t} r_d e^{-\lambda\int_0^t \eta(s)ds}dt + \int_0^\tau e^{-\rho t}\lambda\eta(t)e^{-\lambda\int_0^t \eta(s)ds}dt \\ + e^{-\rho\tau}1_{\{\tau=\tau_\phi\}}e^{-\lambda\int_0^\tau \eta(s)ds}\min\left(1, \frac{P(\tau)+C(\tau)}{D(\tau)}\right) \\ + e^{-\rho\tau}1_{\{\tau=\tau_b\}}e^{-\lambda\int_0^\tau \eta(s)ds}\min\left(1, \alpha\frac{P(\tau)}{D(\tau)}\right)\Bigg]$$

Given the laws of motion for the state variables $\{D(t)\}_{t\geq 0}$, $\{P(t)\}_{t\geq 0}$ and $\{C(t)\}_{t\geq 0}$, it is thus clear that the value function W is homogeneous of degree 1 in (P,D,C), and can be expressed as:

$$W(P,D,C) = Dw(p,c)$$

For a given adapted process $\{\eta(t)\}_{t\geq 0}$ in $[0,1]$, the function w is equal to:

$$w(p,c) = \mathbb{E}^{p,c}\left[\int_0^\tau \left(r_d + \lambda\eta(t)\right)e^{-\int_0^t(\rho+\lambda\eta(s))ds}dt\right] \\ + \mathbb{E}^{p,c}\left[1_{\{\tau=\tau_\phi\}}e^{-\int_0^\tau(\rho+\lambda\eta(s))ds}\min\left(1, p(\tau)+c(\tau)\right) + 1_{\{\tau=\tau_b\}}e^{-\int_0^\tau(\rho+\lambda\eta(s))ds}\min\left(1,\alpha p(\tau)\right)\right]$$

The state variables $\{p(t)\}_{t\geq 0}$ and $\{c(t)\}_{t\geq 0}$ evolve according to:

$$dp(t) = (\mu + \lambda\eta(t))\, p(t)dt + \sigma p(t)dB(t)$$
$$dc(t) = ((\rho - \mu)p(t) + (r_c + \lambda\eta(t))c(t) - (r_d + \lambda\eta(t)))\, dt$$

The firm's large creditor maximizes the function $w(p, c)$ over all possible adapted processes $\{\eta(t)\}_{t\geq 0}$ in $[0, 1]$. In other words, he solves:

$$\max_{\eta:\forall t,\eta(t)\in[0,1]} \Bigg[\mathbb{E}^{p,c}\left[\int_0^\tau (r_d + \lambda\eta(t))\, e^{-\int_0^t(\rho+\lambda\eta(s))ds} dt\right] + \mathbb{E}^{p,c}\left[1_{\{\tau=\tau_\phi\}} e^{-\int_0^t(\rho+\lambda\eta(s))ds} \min(1, p(\tau) + c(\tau)) + 1_{\{\tau=\tau_b\}} e^{-\int_0^t(\rho+\lambda\eta(s))ds} \min(1, \alpha p(\tau))\right]\Bigg]$$

For $c > 0$, the corresponding Hamilton-Jacobi-Bellman equation is the following:

$$0 = \max_{\eta\in[0,1]} \Big[-(\rho + \lambda\eta)\, w + r_d + \lambda\eta + (\mu + \lambda\eta)p\partial_p w + \frac{1}{2}\sigma^2 p^2 \partial_{pp} w + [(\rho - \mu)p + (r_c + \lambda\eta)c - (r_d + \lambda\eta)]\, \partial_c w + \phi\, (\min(1, p + c) - w) \Big]$$

As expected, this is a bang-bang control problem. The optimal policy is the following:

$$\eta(p, c) = \begin{cases} 0 & \text{if } w(p, c) - 1 > p\partial_p w(p, c) + (c - 1)\partial_c w(p, c) \\ 1 & \text{if } w(p, c) - 1 < p\partial_p w(p, c) + (c - 1)\partial_c w(p, c) \end{cases} \tag{13}$$

The boundary $\{(p, c) : w(p, c) - 1 = p\partial_p w(p, c) + (c - 1)\partial_c w(p, c)\}$ is the set of points in the state space such that the single large creditor is indifferent between rolling over his debt claim or running. The large creditor's optimal decision can then be interpreted as follows. At each time t, a quantity of debt λdt comes due. The large creditor's flow benefit of reinvesting into the firm's debt is equal to $(w(p, c) - 1)\lambda dt$. Reinvesting into the firm's debt has a marginal cost, equal to the difference between the flow capital gains when running and the flow capital gains when rolling, which is equal to $(p\partial_p w(p, c) + (c - 1)\partial_c w(p, c))\, \lambda dt$. The reinvestement condition uncovered in equation (13) illustrates the fact that the large creditor internalizes the effect of his decision to run or roll onto the dynamics of the state variables, which is the key difference between this model and the model with the run externality discussed in the previous sections.

7.1 Large Creditor's Behavior when $p = 0$

I now derive the value function w in the special case where the illiquid asset cash flow – and thus the illiquid asset fundamental value – is zero. Similar to Section 4.8, this problem is deterministic with perfect foresight for the large creditor, who can predict perfectly the evolution of the state variable $c(t)$. Since c is the only state variable, I postulate a threshold $\hat{c}$ above which it will be optimal for the large creditor to continue rolling its debt claim, and below which it will be optimal to run. In what follows, I prove that the threshold $\hat{c}$ is unique, and that it verifies $\hat{c} > c^*$, where c^* is the corresponding threshold in the equilibrium with the run externality. I will note $w_0(c) \equiv w(0, c)$ the *optimal* value function of the large creditor of a game with no illiquid asset – in other words, the value function of the large creditor following the optimal cutoff strategy.

Proposition 9. In an economy without illiquid asset and with a unique large creditor, the creditor value function w_0 is equal to:

$$w_0(c) = \begin{cases} \left(\frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)}\right)\left[1-\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}\right]+\frac{\phi}{\phi+\rho-r_c}c & 0 \le c < 1 \\ H_1\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}+\frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} & 1 \le c < \hat{c} \\ H_2'\left(1-\frac{r_c}{r_d}c\right)^{\frac{\rho+\phi}{r_c}}+\frac{r_d+\phi}{\rho+\phi} & \hat{c} \le c < \frac{r_d}{r_c} \\ \frac{r_d+\phi}{\rho+\phi} & c \ge \frac{r_d}{r_c} \end{cases} \tag{14}$$

$\hat{c}$ defines the unique threshold equilibrium of this economy, and it satisfies:

$$w_0(c) - 1 < (c-1)w_0'(c) \quad \forall c < \hat{c}$$
$$w_0(c) - 1 > (c-1)w_0'(c) \quad \forall c > \hat{c}$$

The implicit equation satisfied by $\hat{c}$ is derived in the appendix. The value function w_0 is strictly increasing for $c < \frac{r_d}{r_c}$, and constant for $c > \frac{r_d}{r_c}$. H_1 and H_2' are constants described in the appendix.

8 Portfolio Choice - [Incomplete]

In this section, I analyze the portfolio choice problem facing the management of the firm, who is acting on behalf of the firm's shareholders. A portfolio policy for shareholders is now a non-decreasing adapted process $L(t)$, satisfying $L(0) = 0$, representing the cumulative dollar amount of additional illiquid assets purchased, per unit of debt outstanding at the time of purchase. For a given run strategy S employed by all creditors, and a given portfolio strategy $L(t)$ employed by shareholders, the state variables evolve according to:

$$dp(t) = \left(\mu + \lambda 1_{\{S(p(t),c(t))=1\}}\right) p(t)dt + \sigma p(t)dB(t) + dL(t) \tag{15}$$

$$dc(t) = \left((\rho - \mu)p(t) + (r_c + \lambda 1_{\{S(p(t),c(t))=1\}})c(t) - (r_d + \lambda 1_{\{S(p(t),c(t))=1\}})\right) dt - dL(t) \tag{16}$$

A portfolio policy $L(t)$ is admissible if $\Pr\left((p(t), c(t)) \in \mathbb{R}_+^2 \forall t \geq 0\right) = 1$. The optimization problem solved by equity holders takes as given creditors' behavior, as encoded by the strategy S:

$$\begin{aligned} e\left(p, c; S\right) = \sup_L \mathbb{E}^{p,c}\Big[e^{-\rho(\tau_b \wedge \tau_\phi)}\Big[&1_{\{\tau_\phi < \tau_b\}} \max\left(0, p(\tau_\phi) + c(\tau_\phi) - 1\right) \\ &+ 1_{\{\tau_b < \tau_\phi\}} \max\left(0, \alpha p(\tau_b) - 1\right)\Big]\Big] \end{aligned} \tag{17}$$

Noting $\lambda(p, c) = \lambda 1_{\{S(p,c)=1\}}$, the corresponding HJB equation is:

$$\begin{aligned} 0 = \max\Big[&-\rho e(p, c; S) - (\mu + \lambda(p, c))\, p\partial_p e(p, c; S) + \frac{1}{2}\sigma^2 p^2 \partial_{pp} e(p, c; S) \\ &+ ((\rho - \mu)p + (r_c + \lambda(p, c))\, c - (r_d + \lambda(p, c)))\, \partial_c e(p, c; S) + \phi\left[\max(0, p + c - 1) - e(p, c; S)\right], \\ &\partial_p e(p, c; S) - \partial_c e(p, c; S)\Big] \end{aligned}$$

As expected, the optimal portfolio choice problem is a bang bang control problem, featuring two regions of the state space: a region where the firm accumulates cash reserves (and characterized by $\partial_p e(p, c; S) < \partial_c e(p, c; S)$), and a region where the firm uses its cash reserve to purchase the illiquid asset. At the boundary of the two regions, one extra unit of cash yields the same additional value to shareholders than one extra unit of the illiquid asset: $\partial_p e(p, c; S) = \partial_c e(p, c; S)$.

I note also that the equity value $e(p, c; S)$ for high values of p or c calculated when the firm does not solve any portfolio choice problem indicates that the marginal condition for the firm to want to invest into the illiquid asset is satisfied. Indeed, when p or c is large, I established in Lemma 3 that:

$$e(p, c) = \frac{\phi}{\rho + \phi - r_c} c + \frac{\phi}{\rho + \phi - \mu}\left(1 + \frac{\rho - \mu}{\rho + \phi - r_c}\right) p - \frac{\phi}{\rho + \phi}\left(\frac{r_d}{\rho + \phi - r_c} + 1\right) + o(1)$$

Thus, when p or c is large, I have:

$$\begin{aligned}\partial_c e(p,c) &= \frac{\phi}{\rho+\phi-r_c} + o(1)\\ \partial_p e(p,c) &= \frac{\phi}{\rho+\phi-\mu}\left(1+\frac{\rho-\mu}{\rho+\phi-r_c}\right) + o(1)\\ &= \frac{\phi}{\rho+\phi-r_c} + \frac{\phi}{\rho+\phi-\mu}\left(1-\frac{\phi}{\rho+\phi-r_c}\right) + o(1) > \frac{\phi}{\rho+\phi-r_c}\end{aligned}$$

Thus, for large values of p or c, $\partial_p e(p,c) > \partial_c e(p,c)$. The discussion above supports the idea that any equilibrium of our new economy will feature the standard endogeneous run boundary characterized by the mapping S, as well as a new endogeneous reflecting boundary $\{(p,c) : \partial_p e(p,c;S) = \partial_c e(p,c;S)\}$ beyond which the firm invests its cash into the illiquid asset. I am now ready to define an equilibrium in the economy where the firm can invest into the illiquid asset.

Definition 3. A symmetric Markov perfect equilibrium of the game with portfolio choice is (i) a mapping $S : \mathbb{R}_+^2 \to \{0,1\}$ representing the run behavior of creditors, and (ii) a region $\mathcal{I} \subset \mathbb{R}_+^2$ (for "investmemt") such that:

- for any $(p,c) \in \mathbb{R}^2$, $v(p,c;S,S,\mathcal{I}) = \sup_s v(p,c;s,S,\mathcal{I})$;
- for any $(p,c) \notin \mathcal{I}$, $\partial_p e(p,c;S,\mathcal{I}) < \partial_c e(p,c;S,\mathcal{I})$;
- for any $(p,c) \in \partial\mathcal{I}$, $\partial_p e(p,c;S,\mathcal{I}) = \partial_c e(p,c;S,\mathcal{I})$.

9 Conclusion

[To be completed]

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A Proofs

Proof of Proposition 1: Strategic complementarity and supermodularity are typically defined for static games. I adapt the definition of strategic complementarity as follows: the payoff function $v(\cdot,\cdot;s,S)$ exhibits strategic complementarity if for any strategies S_1, S_2 such that $S_2(p,c) \geq S_1(p,c)$ for all (p,c), and for any strategy s, the incentive for a creditor to run are greater when responding to S_2 than when responding to S_1, in other words:

$$1 - v(p,c;s,S_2) \geq 1 - v(p,c;s,S_1)$$

In order to prove that the payoff function of the game studied in this paper does not exhibit supermodularity, I choose to focus on the region of the state space $p = 0$. Since the illiquid asset price is a geometric Brownian motion, when $p = 0$, $p(t) = 0$ for all $t \geq 0$. In the restricted domain $p = 0$, the firm holds cash yielding r_c on the asset side of its balance-sheet, and has debt yielding r_d on the liability side of its balance-sheet. The cash balance per unit of debt oustanding c is thus the only relevant state variable for creditors' roll-over decisions.

Consider an initial cash level $c(0) = c_0 < 1$. Let $c_f \in [0, c_0)$ be an arbitrary point on the state space, and define $\tau_{\mathcal{R}}(c_f; c_0)$ to be the c_f-hitting time assuming a run from $t = 0$ onwards and assuming the initial cash level is c_0, and define $\tau_{\mathcal{NR}}(c_f; c_0)$ to be the c_f-hitting time assuming creditors roll from $t = 0$ onwards and assuming the initial cash level is c_0. When $p = 0$ and when a run occurs, the cash balance evolves according to the ordinary differential equation $dc(t) = ((r_c + \lambda)c(t) - (r_d + \lambda))\,dt$. Instead, when creditors roll their debt claims into new debt claims, the cash balance evolves according to the ordinary differential equation $dc(t) = (r_c c(t) - r_d)\,dt$. In either regime, since $c_0 < 1$, the cash balance is decreasing as a function of time. Integrating the ordinary differential equations satisfied by $c(t)$ given the initial condition $c(0) = c_0$ leads to the following expressions for the stopping times $\tau_{\mathcal{R}}(c_f; c_0)$ and $\tau_{\mathcal{NR}}(c_f; c_0)$:

$$\tau_{\mathcal{R}}(c_f; c_0) = \frac{1}{r_c + \lambda} \ln\left(\frac{\frac{r_d+\lambda}{r_c+\lambda} - c_f}{\frac{r_d+\lambda}{r_c+\lambda} - c_0}\right)$$
$$\tau_{\mathcal{NR}}(c_f; c_0) = \frac{1}{r_c} \ln\left(\frac{\frac{r_d}{r_c} - c_f}{\frac{r_d}{r_c} - c_0}\right)$$

When $\tau_{\mathcal{R}}(c_f; c_0)$ is viewed as a function of λ, it can be showed that it is decreasing in λ, so long as $0 \leq c_f < c_0 < 1$. This is true since:

$$\frac{\partial \tau_{\mathcal{R}}(c_f; c_0)}{\partial \lambda} = \frac{1}{(r_c+\lambda)^2}\left[\frac{\frac{r_d+\lambda}{r_c+\lambda} - 1}{\frac{r_d+\lambda}{r_c+\lambda} - c_0} - \frac{\frac{r_d+\lambda}{r_c+\lambda} - 1}{\frac{r_d+\lambda}{r_c+\lambda} - c_f} - \ln \frac{\frac{r_d+\lambda}{r_c+\lambda} - c_f}{\frac{r_d+\lambda}{r_c+\lambda} - c_0}\right] < 0$$

This means that the cash balance converges to the state $c = c_f$ faster upon the occurrence of a run than when all creditors are rolling.

Consider the family $\{S_{\hat{c}}\}$ of cutoff strategies followed by all other creditors – in other words, strategies such that creditors run when $c \in [0, \hat{c})$ and creditors roll over when $c \in [\hat{c}, +\infty)$, for some $\hat{c} \in \mathbb{R}_+$. First, take $1 > \hat{c}_2 > \hat{c}_1 \geq 0$, which means that $S_{\hat{c}_2}(0, c) \geq S_{\hat{c}_1}(0, c)$

for all c. Given our previous observation, the cash per unit of debt oustanding verifies:

$$c^{c_0}(t; S_{\hat{c}_1}) \geq c^{c_0}(t; S_{\hat{c}_2})$$

In the above, $c^{c_0}(t; S_{\hat{c}})$ represents the cash level at time t, assuming that the cash level at $t = 0$ is c_0, and assuming that all creditors play a cutoff strategy $S_{\hat{c}}$. The inequality above is an equality when $\hat{c}_1 > c_0$. The inequality is strict for $t > 0$ when $\hat{c}_2 > c_0 > \hat{c}_1$, and for $t \geq \tau_{\mathcal{NR}}(\hat{c}_2; c_0)$ when $c_0 > \hat{c}_2$. Given the fact that a specific agent i's flow payoff does not depend on other creditors' strategy but that such specific creditor's terminal payoff is increasing in the cash balance $c(\tau)$ (where τ is the earlier of (a) the default stopping time τ_b or (b) the maturity stopping time τ_ϕ), I must have, for any strategy s followed by a specific creditor i:

$$v(0, c; s, S_{\hat{c}_2}) \leq v(0, c; s, S_{\hat{c}_1})$$

Thus, when considering creditors' strategies $S_{\hat{c}}$ for $\hat{c} < 1$, the game's payoff function exhibits strategic complementarity. Unfortunately, the strategic complementarity uncovered above is local instead of global. Consider now (c_0, c_f) with $\frac{r_d+\lambda}{r_c+\lambda} > c_0 > c_f > 1$. A reasoning similar to the previous one shows that:

$$\tau_{\mathcal{R}}(c_f; c_0) \geq \tau_{\mathcal{NR}}(c_f; c_0)$$

This is due to the fact that on that interval,

$$\frac{\partial \tau_{\mathcal{R}}(c_f; c_0)}{\partial \lambda} > 0$$

Consider then two cutoff strategies $S_{\hat{c}_1}$ and $S_{\hat{c}_2}$, with $\frac{r_d+\lambda}{r_c+\lambda} > \hat{c}_2 > \hat{c}_1 > 1$. The cash per unit of debt oustanding verifies:

$$c^{c_0}(t; S_{\hat{c}_1}) \leq c^{c_0}(t; S_{\hat{c}_2})$$

The inequality above is an equality when $\hat{c}_1 > c_0$. The inequality is strict for $t > 0$ when $\hat{c}_2 > c_0 > \hat{c}_1$, and for $t \geq \tau_{\mathcal{NR}}(\hat{c}_2; c_0)$ when $\frac{r_d+\lambda}{r_c+\lambda} > c_0 > \hat{c}_2$. When the firm starts with an amount of cash $\hat{c}_2 > c_0 > \hat{c}_1$, the cash balance decreases more rapidly when all creditors are rolling. Using a reasoning similar to the one in the previous paragraph, it means that for any strategy s followed by a specific creditor:

$$v(0, c; s, S_{\hat{c}_2}) \geq v(0, c; s, S_{\hat{c}_1})$$

In other words, the game's payoff exhibits strategic substitutability for cutoff strategies $\hat{c} \in \left(1, \frac{r_d+\lambda}{r_c+\lambda}\right)$. Intuitively, when $\frac{r_d+\lambda}{r_c+\lambda} > c > 1$, a creditor run decreases the speed at which the cash reserve is depleted. Upon a run, expensive debt yielding r_d per unit of time ends up being paid down, which helps improve the financial health of the firm (since it can only rely on an asset yielding $r_c < r_d$). Thus, the earlier creditors run (in other words the higher the threshold $\hat{c}$), the better off a single creditor is. $\square$

Proof of Proposition 2: For a given strategy $S:\mathbb{R}_+^2\to\{0,1\}$ followed by creditors, the best achievable payoff for a given creditor is to be rolling over all the time, until the time τ_ϕ at which the illiquid asset matures and creditors are fully paid off. In other words, I have:

$$\begin{aligned} v(p,c;s,S) &\leq \mathbb{E}^{p,c}\left[\int_0^{\tau_\phi} e^{-\rho t} r_d dt + e^{-\rho\tau_\phi}\right] \\ &\leq \frac{r_d+\phi}{\rho+\phi} \end{aligned}$$

□

Proof of Proposition 3: I want to establish that for (p,c) small enough, it is optimal for a given creditor to run when he gets the chance to do so, irrespective of the strategy S followed by other creditors. In order to achieve that, take $S:\mathbb{R}_+^2\to\{0,1\}$ and $s:\mathbb{R}_+^2\to\{0,1\}$ arbitrary. When I integrate the stochastic differential equation for $c(t)$, I obtain:

$$e^{-\int_0^t (r_c+\lambda(s))ds}c(t) = c + p\int_0^t (\rho-\mu)e^{\left(\mu-\frac{1}{2}\sigma^2-r_c\right)s+\sigma B(s)}ds - \int_0^t (r_d+\lambda(s))e^{-\int_0^s(r_c+\lambda(u))du}ds$$

In the above, I have noted $\lambda(t)\equiv\lambda 1_{\{S(p(t),c(t))=1\}}$. For a given realization $\{B(t,\omega)\}_{t\geq 0}$ of the Brownian motion, $\tau_b(\omega)$ is the smallest time that solves:

$$c + p\int_0^{\tau_b(\omega)} (\rho-\mu)e^{\left(\mu-\frac{1}{2}\sigma^2-r_c\right)s+\sigma B(s,\omega)}ds = \int_0^{\tau_b(\omega)} (r_d+\lambda(s,\omega))e^{-\int_0^s(r_c+\lambda(u,\omega))du}ds \qquad (18)$$

Note that the equation above might not have a solution, in which case $\tau_b(\omega)=\infty$. Let τ_λ be an exponentially distributed time, with arrival intensity λ, and $\tau\equiv\tau_\lambda\wedge\tau_b\wedge\tau_\phi$. The value function can be re-written:

$$\begin{aligned} v(p,c;s,S) = \mathbb{E}^{p,c}\Bigg[\int_0^{\tau} e^{-\rho t} r_d dt + e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}\left[(1-s(p(\tau),c(\tau)))\,v(p(\tau),c(\tau);s,S) + s(p(\tau),c(\tau))\right] \\ + e^{-\rho\tau}1_{\{\tau=\tau_\phi\}}\min(1,p(\tau)+c(\tau)) + e^{-\rho\tau}1_{\{\tau=\tau_b\}}\min(1,\alpha p(\tau))\Bigg] \end{aligned}$$

Using the law of iterated expectations, and making explicit the dependence on the Brownian motion realization (via ω), the value function for creditor i can then be simplified as follows:

$$\begin{aligned} v(p,c;s,S) = \mathbb{E}^{p,c}\Bigg[& e^{-(\lambda+\phi)\tau_b(\omega)}\frac{r_d}{\rho}\left(1-e^{-\rho\tau_b(\omega)}\right) + e^{-(\lambda+\phi)\tau_b(\omega)}e^{-\rho\tau_b(\omega)}\min(1,\alpha p(\tau_b(\omega))) \\ &+ \Pr(\tau_\lambda\leq\tau_b(\omega)\wedge\tau_\phi)\times\mathbb{E}\Bigg[\frac{r_d}{\rho}\left(1-e^{-\rho\tau_\lambda}\right) + e^{-\rho\tau_\lambda}\left[(1-s(p(\tau),c(\tau)))\,v(p(\tau_\lambda),c(\tau_\lambda);s,S)\right. \\ &\left. + s(p(\tau),c(\tau))\right]|\tau_\lambda\leq\tau_b(\omega)\wedge\tau_\phi\Bigg] \\ &+ \Pr(\tau_\phi\leq\tau_b(\omega)\wedge\tau_\lambda)\times\mathbb{E}\left[\frac{r_d}{\rho}\left(1-e^{-\rho\tau_\phi}\right) + e^{-\rho\tau_\phi}\min(1,p(\tau_\phi)+c(\tau_\phi))\,|\tau_\phi\leq\tau_b(\omega)\wedge\tau_\lambda\right]\Bigg] \end{aligned}$$

For a fixed and given $\tau_b(\omega)$, simple algebra gives me the following:

$$\begin{aligned}
\Pr\left(\tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi\right) &= \frac{\lambda}{\lambda+\phi}\left(1-e^{-(\lambda+\phi)\tau_b(\omega)}\right) \\
\Pr\left(\tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi\right) \times \mathbb{E}\left[e^{-\rho\tau_\lambda}|\tau_\lambda \leq \tau_b(\omega) \wedge \tau_\phi\right] &= \frac{\lambda}{\rho+\lambda+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right) \\
\Pr\left(\tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda\right) &= \frac{\phi}{\lambda+\phi}\left(1-e^{-(\lambda+\phi)\tau_b(\omega)}\right) \\
\Pr\left(\tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda\right) \times \mathbb{E}\left[e^{-\rho\tau_\phi}|\tau_\phi \leq \tau_b(\omega) \wedge \tau_\lambda\right] &= \frac{\phi}{\rho+\lambda+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right)
\end{aligned}$$

Note that given $p(0) = p$, I have $p(t) \leq pe^{\left(\mu-\frac{1}{2}\sigma^2+\lambda\right)t+\sigma B(t)}$ irrespective of creditors' strategy. Given the upper bound computed for v and the previous comment, I have the following inequality:

$$\begin{aligned}
v\left(p,c;s,S\right) \leq \mathbb{E}^{p,c}\Bigg[&e^{-(\lambda+\phi)\tau_b(\omega)}\left(\frac{r_d}{\rho}\left(1-e^{-\rho\tau_b(\omega)}\right)+e^{-\rho\tau_b(\omega)}\min\left(1,\alpha pe^{\left(\mu+\lambda-\frac{1}{2}\sigma^2\right)\tau_b(\omega)+\sigma B(\tau_b(\omega))}\right)\right) \\
&+\frac{r_d}{\rho}\frac{\lambda}{\lambda+\phi}\left(1-e^{-(\lambda+\phi)\tau_b(\omega)}\right)+\left(\frac{r_d+\phi}{\rho+\phi}-\frac{r_d}{\rho}\right)\frac{\lambda}{\rho+\lambda+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right) \\
&+\frac{r_d}{\rho}\frac{\phi}{\lambda+\phi}\left(1-e^{-(\lambda+\phi)\tau_b(\omega)}\right)+\left(1-\frac{r_d}{\rho}\right)\frac{\phi}{\rho+\lambda+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right)\Bigg]
\end{aligned}$$

The inequality can be simplified further:

$$v\left(p,c;s,S\right) \leq \mathbb{E}^{p,c}\left[\frac{r_d+\phi}{\rho+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right)+e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\min\left(1,\alpha pe^{\left(\mu+\lambda-\frac{1}{2}\sigma^2\right)\tau_b(\omega)+\sigma B(\tau_b(\omega))}\right)\right]$$

The expectation above is taken over the random variable $\tau_b(\omega)$, defined implicitly by equation (18). The key idea is that by choosing (p,c) "small enough", I can make $\tau_b(\omega)$ "small", which means that the term $\frac{r_d+\phi}{\rho+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right)$ in the expectation above is dominated by the term $e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\min\left(1,\alpha pe^{\left(\mu+\lambda-\frac{1}{2}\sigma^2\right)\tau_b(\omega)+\sigma B(\tau_b(\omega))}\right)$. Using Doob's optional sampling theorem and Jensen's inequality (on the concave function $x \to \min(1,x)$), for any finite T, I have:

$$\begin{aligned}
&\mathbb{E}^{p,c}\left[\frac{r_d+\phi}{\rho+\phi}\left(1-e^{-(\rho+\lambda+\phi)(\tau_b(\omega)\wedge T)}\right)+e^{-(\rho+\lambda+\phi)(\tau_b(\omega)\wedge T)}\min\left(1,\alpha pe^{\left(\mu+\lambda-\frac{1}{2}\sigma^2\right)(\tau_b(\omega)\wedge T)+\sigma B((\tau_b(\omega)\wedge T))}\right)\right] \leq \\
&\qquad \mathbb{E}^{p,c}\left[\frac{r_d+\phi}{\rho+\phi}\left(1-e^{-(\rho+\lambda+\phi)(\tau_b(\omega)\wedge T)}\right)+\min\left(e^{-(\rho+\lambda+\phi)(\tau_b(\omega)\wedge T)},\alpha pe^{-(\rho+\phi-\mu)(\tau_b(\omega)\wedge T)}\right)\right]
\end{aligned}$$

Taking $T \to +\infty$, I obtain the inequality:

$$v\left(p,c;s,S\right) \leq \mathbb{E}^{p,c}\left[\frac{r_d+\phi}{\rho+\phi}\left(1-e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\right)+e^{-(\rho+\lambda+\phi)\tau_b(\omega)}\min\left(1,\alpha pe^{(\mu+\lambda)\tau_b(\omega)}\right)\right]$$

Note that the right handside of equation (18) verifies the following inequality, for any t, and any strategy S:

$$\int_0^t (r_d + \lambda(s,\omega)) e^{-\int_0^s (r_c+\lambda(u,\omega))du} ds \geq \int_0^t r_d e^{-(r_c+\lambda)s} ds$$
$$\geq \frac{r_d}{r_c+\lambda}\left(1 - e^{-(r_c+\lambda)t}\right)$$

Thus, the stopping time $\tau_b(\omega)$ is almost surely less than or equal to the stopping time $\bar{\tau}_b(\omega)$, defined implicitly as the smallest time solving:

$$c + p\int_0^{\bar{\tau}_b(\omega)} (\rho-\mu) e^{\left(\mu - \frac{1}{2}\sigma^2 - r_c\right)s + \sigma B(s,\omega)} ds = \frac{r_d}{r_c+\lambda}\left(1 - e^{-(r_c+\lambda)\bar{\tau}_b(\omega)}\right) \tag{19}$$

Note that the stopping time $\bar{\tau}_b$ does not depend on creditors' strategies. When the equation above does not have a solution, I set $\bar{\tau}_b(\omega) = +\infty$. Since the function $t \to \frac{r_d+\phi}{\rho+\phi}\left(1 - e^{-(\rho+\lambda+\phi)t}\right) + e^{-(\rho+\lambda+\phi)t}\min\left(1, \alpha p e^{(\mu+\lambda)t}\right)$ is increasing in t for positive values of t, I can write:

$$v(p,c;s,S) \leq \mathbb{E}^{p,c}\left[\frac{r_d+\phi}{\rho+\phi}\left(1 - e^{-(\rho+\lambda+\phi)\bar{\tau}_b(\omega)}\right) + e^{-(\rho+\lambda+\phi)\bar{\tau}_b(\omega)}\min\left(1, \alpha p e^{(\mu+\lambda)\bar{\tau}_b(\omega)}\right)\right] \equiv g(p,c)$$

$g(p,c)$ is thus an upper bound of $v(p,c;s,S)$ that does not depend on the strategies (s,S). It remains to be shown that for (p,c) small enough, g is strictly less than 1. Notice that $g(0,0) = 0$, and that g is continuous in a neighborhood of $(0,0)$. The continuitity of g means that there exists a neighborhood $\mathcal{D}_l \subset \mathbb{R}_+^2$ such that for $(p,c) \in \mathcal{D}_l$, $g(p,c) < 1$, which is the desired result. As an aside, the stopping time $\bar{\tau}_b$ is increasing in p and in c. Since $t \to \frac{r_d+\phi}{\rho+\phi}\left(1 - e^{-(\rho+\lambda+\phi)t}\right) + e^{-(\rho+\lambda+\phi)t}\min\left(1, \alpha p e^{(\mu+\lambda)t}\right)$ is increasing in t, this means that the function g defined above is increasing in p and c, with value zero at $(0,0)$. In other words, the region $\{(p,c) : g(p,c) < 1\}$ includes the point $(0,0)$ and is path-connected, implying that the largest possible dominance region (which can be defined as $\{(p,c) : v(p,c;s,S) < 1 \forall (s,S)\}$) is path-connected and contains $(0,0)$. □

I now want to establish that for (p,c) large enough, it is optimal for a given creditor to roll over his maturing debt claim into a new debt claim when he gets the chance to do so, irrespective of the strategy S followed by other creditors. Note that for $c > \frac{r_d}{r_c}$, the cash reserve $c(t)$ is strictly increasing, irrespective of S. Thus the default time τ_b is infinite almost surely, which means that it is dominant for creditors to always roll over. I can also show that for p high enough, it is dominant for a given creditor to roll, irrespective of other creditors' strategy. The idea is that for an arbitrarily small $\epsilon > 0$ given, I can find a p high enough such that:

$$((\rho-\mu)p + r_c c - (r_d+\lambda))\,\epsilon > \frac{r_d}{r_c}$$

In other words, irrespective of creditors' strategies, I can find a p large enough such that after an arbitrarily small time interval ϵ, the state variable c ends up above r_d/r_c, value at which I know it becomes dominant for creditors to roll. □

Proof of Proposition 4: Let me first introduce some notation:

$$\mu_p(p,c) = \left(\mu + \lambda 1_{\{(p,c)\in\mathcal{R}_S\}}\right) p$$
$$\sigma_p(p,c) = \sigma p$$
$$\mu_c(p,c) = (\rho - \mu)\, p + \left(r_c + \lambda 1_{\{(p,c)\in\mathcal{R}_S\}}\right) c - \left(r_c + \lambda 1_{\{(p,c)\in\mathcal{R}_S\}}\right)$$

Given the drift coefficients μ_p, μ_c are sublinear and the volatility coefficient σ_p is Lipschitz, Karatzas (1991) (page 303) provides for the existence of a weak solution to the stochastic differential equation governing the evolution of the state variables.

Let $\mathcal{C}(\mathbb{R}^2_+)$ be the set of functions that are continuous and bounded on $\mathbb{R}^2_+$. For the cutoff strategy $S : \mathbb{R}^2_+ \to \{0,1\}$ followed by other creditors, I define the operator $\mathbb{T}_S$ that maps any arbitrary continuous bounded function $f : \mathbb{R}^2_+ \to \mathbb{R}$ into a function $\mathbb{T}_S(f)$ defined as follows:

$$\mathbb{T}_S(f)(p,c) = \mathbb{E}^{p,c}\left\{\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} \max\left(1, f\left(p(\tau), c(\tau)\right)\right)\right\}$$
$$+ \mathbb{E}^{p,c}\left\{e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \min\left(1, p(\tau) + c(\tau)\right) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \min\left(1, \alpha p(\tau)\right)\right\}$$

Where p and c evolve according to:

$$dp(t) = \mu_p\left(p(t), c(t)\right) dt + \sigma_p\left(p(t), c(t)\right) dB(t)$$
$$dc(t) = \mu_c\left(p(t), c(t)\right) dt$$

I want to establish that $\mathbb{T}_S$ maps the set of continuous and bounded functions on $\mathbb{R}^2$ into itself. Let $f \in \mathcal{C}(\mathbb{R}^2_+)$ be given, and let M be its upper bound. For any $(p,c) \in \mathbb{R}^2_+$, I have:

$$\mathbb{T}_S(f)(p,c) \le \mathbb{E}^{p,c}\left\{\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} M + e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} + e^{-\rho\tau} 1_{\{\tau=\tau_b\}}\right\}$$
$$\le \max\left(\frac{r_d}{\rho}, M\right)$$

Thus $\mathbb{T}_S$ maps bounded functions into bounded functions. Let me pick $(p,c) \in \mathbb{R}^2_+$ and a sequence $\{(p_\epsilon, c_\epsilon)\}_\epsilon \in \mathbb{R}^2_+$, where $\lim_{\epsilon\to 0}(p_\epsilon, c_\epsilon) = (p,c)$. I want to prove that $\lim_{\epsilon\to 0} \mathbb{T}_S(f)(p_\epsilon, c_\epsilon) = \mathbb{T}_S(f)(p,c)$. Let $\tau_{b,\epsilon}$ be the firm default time conditioned on the starting values (p_ϵ, c_ϵ). Similarly, let $(p_\epsilon(t), c_\epsilon(t))$ be the random variables associated with the values of the state variables (p,c) at time t, conditioned on initial values (p_ϵ, c_ϵ). Let $\tau = \tau_{b,\epsilon} \wedge \tau_b \wedge \tau_\lambda \wedge \tau_\phi$. To simplify notation, I define the following events:

$$
\begin{aligned}
E_1 &\equiv \{\tau_\lambda < \tau_b \wedge \tau_{b,\epsilon} \wedge \tau_\phi\} \\
E_2 &\equiv \{\tau_\phi < \tau_b \wedge \tau_{b,\epsilon} \wedge \tau_\lambda\} \\
E_3 &\equiv \{\tau_{b,\epsilon} < \tau_b < \tau_\lambda \wedge \tau_\phi\} \\
E_4 &\equiv \{\tau_b < \tau_{b,\epsilon} < \tau_\lambda \wedge \tau_\phi\} \\
E_5 &\equiv \{\tau_{b,\epsilon} < \tau_\phi < \tau_\lambda \wedge \tau_b\} \\
E_6 &\equiv \{\tau_b < \tau_\phi < \tau_\lambda \wedge \tau_{b,\epsilon}\} \\
E_7 &\equiv \{\tau_{b,\epsilon} < \tau_\lambda < \tau_\phi \wedge \tau_b\} \\
E_8 &\equiv \{\tau_b < \tau_\lambda < \tau_\phi \wedge \tau_{b,\epsilon}\}
\end{aligned}
$$

Note that those events do not intersect, and $\Pr\left(\bigcup_{i=1}^{8} E_i\right) = 1$. I have the following:

$$
\begin{aligned}
&\mathbb{T}_S(f)(x_\epsilon, y_\epsilon) - \mathbb{T}_S(f)(x,y) = \\
&\qquad \mathbb{E}\left[e^{-\rho\tau_\lambda}\left(\max\left(1, f(p_\epsilon(\tau_\lambda), c_\epsilon(\tau_\lambda))\right) - \max\left(1, f(p(\tau_\lambda), c(\tau_\lambda))\right)\right)|E_1\right]\Pr(E_1) \\
&\qquad + \mathbb{E}\left[e^{-\rho\tau_\phi}\left(\min\left(1, p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)\right) - \min\left(1, p(\tau_\phi) + c(\tau_\phi)\right)\right)|E_2\right]\Pr(E_2) \\
&\qquad + \mathbb{E}\left[\int_{\tau_b}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}}\min\left(1, \alpha p_\epsilon(\tau_{b,\epsilon})\right) - e^{-\rho\tau_b}\min\left(1, \alpha p(\tau_b)\right)|E_3\right]\Pr(E_3) \\
&\qquad + \mathbb{E}\left[\int_{\tau_{b,\epsilon}}^{\tau_b} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}}\min\left(1, \alpha p_\epsilon(\tau_{b,\epsilon})\right) - e^{-\rho\tau_b}\min\left(1, \alpha p(\tau_b)\right)|E_4\right]\Pr(E_4) \\
&\qquad + \mathbb{E}\left[\int_{\tau_\phi}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}}\min\left(1, \alpha p_\epsilon(\tau_{b,\epsilon})\right) - e^{-\rho\tau_\phi}\min\left(1, p(\tau_\phi) + c(\tau_\phi)\right)|E_5\right]\Pr(E_5) \\
&\qquad + \mathbb{E}\left[\int_{\tau_b}^{\tau_\phi} e^{-\rho t} r_d dt + e^{-\rho\tau_\phi}\min\left(1, p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)\right) - e^{-\rho\tau_b}\min\left(1, \alpha p(\tau_b)\right)|E_6\right]\Pr(E_6) \\
&\qquad + \mathbb{E}\left[\int_{\tau_\lambda}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}}\min\left(1, \alpha p_\epsilon(\tau_{b,\epsilon})\right) - e^{-\rho\tau_\lambda}\max\left(1, f\left(p(\tau_\lambda), c(\tau_\lambda)\right)\right)|E_7\right]\Pr(E_7) \\
&\qquad + \mathbb{E}\left[\int_{\tau_b}^{\tau_\lambda} e^{-\rho t} r_d dt + e^{-\rho\tau_\lambda}\max\left(1, f\left(p_\epsilon(\tau_\phi), c_\epsilon(\tau_\phi)\right)\right) - e^{-\rho\tau_b}\min\left(1, \alpha p(\tau_b)\right)|E_8\right]\Pr(E_8)
\end{aligned}
$$

From this expression, I have the following inequality:

$$
\begin{aligned}
|\mathbb{T}_S(f)(x_\epsilon, y_\epsilon) - \mathbb{T}_S(f)(x,y)| \leq & \\
& \mathbb{E}\left[e^{-\rho\tau_\lambda}|f(p_\epsilon(\tau_\lambda), c_\epsilon(\tau_\lambda)) - f(p(\tau_\lambda), c(\tau_\lambda))||E_1\right] \Pr(E_1) \\
& + \mathbb{E}\left[e^{-\rho\tau_\phi}|(p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)) - (p(\tau_\phi) + c(\tau_\phi))||E_2\right] \Pr(E_2) \\
& + \mathbb{E}\left[\left(\frac{r_d}{\rho} + \min(1, \alpha p(\tau_b))\right)|e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_{b,\epsilon}}|p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)||E_3\right] \Pr(E_3) \\
& + \mathbb{E}\left[\left(\frac{r_d}{\rho} + \min(1, \alpha p(\tau_{b,\epsilon}))\right)|e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_b}|p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)||E_4\right] \Pr(E_4) \\
& + \mathbb{E}\left[\int_{\tau_\phi}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}}|E_5\right] \Pr(E_5) \\
& + \mathbb{E}\left[\int_{\tau_b}^{\tau_\phi} e^{-\rho t} r_d dt + e^{-\rho\tau_b}|E_6\right] \Pr(E_6) \\
& + \mathbb{E}\left[\int_{\tau_\lambda}^{\tau_{b,\epsilon}} e^{-\rho t} r_d dt + e^{-\rho\tau_{b,\epsilon}}|E_7\right] \Pr(E_7) \\
& + \mathbb{E}\left[\int_{\tau_b}^{\tau_\lambda} e^{-\rho t} r_d dt + e^{-\rho\tau_b}|E_8\right] \Pr(E_8)
\end{aligned}
$$

Given that S is cutoff and given the drift and diffusion coefficients of $(p(t), c(t))$, Nilssen (2012) provides for the continuous differentiability of the process $(p_\epsilon(t), c_\epsilon(t))$ – in other words, the vector $(p_\epsilon(t), c_\epsilon(t))$ converges pathwise to the vector $(p(t), c(t))$. This means that:

$$\lim_{\epsilon\to 0} \mathbb{E}\left[e^{-\rho\tau_\phi}|(p_\epsilon(\tau_\phi) + c_\epsilon(\tau_\phi)) - (p(\tau_\phi) + c(\tau_\phi))||E_2\right] = 0$$

Given the pathwise convergence of $(p_\epsilon(t), c_\epsilon(t))$ to $(p(t), c(t))$, the stopping time $\tau_{b,\epsilon}$ converges in probability to τ_b. Thus I have:

$$
\begin{aligned}
& \lim_{\epsilon\to 0} \mathbb{E}\left[\left(\frac{r_d}{\rho} + \alpha p(\tau_b)\right)|e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_{b,\epsilon}}|p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)||E_3\right] = 0 \\
& \lim_{\epsilon\to 0} \mathbb{E}\left[\left(\frac{r_d}{\rho} + \alpha p(\tau_{b,\epsilon})\right)|e^{-\rho\tau_{b,\epsilon}} - e^{\tau_b}| + \alpha e^{-\rho\tau_b}|p_\epsilon(\tau_{b,\epsilon}) - p(\tau_b)||E_4\right] = 0 \\
& \lim_{\epsilon\to 0} \Pr(E_5) = 0 \\
& \lim_{\epsilon\to 0} \Pr(E_6) = 0 \\
& \lim_{\epsilon\to 0} \Pr(E_7) = 0 \\
& \lim_{\epsilon\to 0} \Pr(E_8) = 0
\end{aligned}
$$

Finally, the continuitity of the function f provides for:

$$\lim_{\epsilon\to 0} \mathbb{E}\left[e^{-\rho\tau_\lambda}|f(p_\epsilon(\tau_\lambda), c_\epsilon(\tau_\lambda)) - f(p(\tau_\lambda), c(\tau_\lambda))||E_1\right] = 0$$

In other words, I have just established that:

$$\lim_{\epsilon\to 0}|\mathbb{T}_S(f)(x_\epsilon,y_\epsilon)-\mathbb{T}_S(f)(x,y)|=0$$

This enables me to conclude that $\mathbb{T}_S$ maps the set of continuous and bounded functions on $\mathbb{R}^2$ into itself. Finally, I want to show that $\mathbb{T}_S$ is a contraction. For any pair of functions (f,g) in $\mathcal{C}(\mathbb{R}^2_+)$, and for any $(p,c)\in\mathbb{R}^2_+$, I have:

$$\begin{aligned}|\mathbb{T}_S(f)(p,c)-\mathbb{T}_S(g)(p,c)| &= |\mathbb{E}^{p,c}\left[e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}\left(\max\left(1,f\left(p(\tau),c(\tau)\right)\right)-\max\left(1,g\left(p(\tau),c(\tau)\right)\right)\right)\right]| \\ &\leq \mathbb{E}^{p,c}\left[e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}|f\left(p(\tau),c(\tau)\right)-g\left(p(\tau),c(\tau)\right)|\right]\end{aligned}$$

Noting with a slight abuse of notation $f(t)\equiv f(p(t),c(t))$ and $g(t)\equiv g(p(t),c(t))$, I can condition on the stopping time τ_λ being less than or greater than ϵ, for some fixed $\epsilon>0$:

$$\begin{aligned}&\mathbb{E}^{p,c}\left[e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}|f\left(p(\tau),c(\tau)\right)-g\left(p(\tau),c(\tau)\right)|\right] \\ &=\mathbb{E}^{p,c}\left[e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}|f(\tau)-g(\tau)|\middle|\tau_\lambda\leq\epsilon\right]\Pr(\tau_\lambda\leq\epsilon)+\mathbb{E}^{p,c}\left[e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}|f(\tau)-g(\tau)|\middle|\tau_\lambda>\epsilon\right]\Pr(\tau_\lambda>\epsilon) \\ &\leq ||f-g||_\infty\times\left(1-e^{-\lambda\epsilon}\right)+e^{-\rho\epsilon}||f-g||_\infty\times e^{-\lambda\epsilon} \\ &\leq\left(1-e^{-\lambda\epsilon}+e^{-(\rho+\lambda)\epsilon}\right)||f-g||_\infty\end{aligned}$$

Since $0<1-e^{-\lambda\epsilon}+e^{-(\rho+\lambda)\epsilon}<1$ for any strictly positive ϵ, $\mathbb{T}_S$ is a contraction map. Thus, for a given cutoff strategy S, a solution $v^*(\cdot,\cdot;S)$, fixed point of the mapping $\mathbb{T}_S$ defined above, exists and is unique. □

Proof of Proposition 5: I will assume that there is a threshold c^* such that for $c\leq c^*$, it is optimal for creditors to run, while for $c\geq c^*$, it is optimal for creditors to continue rolling. The threshold c^* will need to verify $v_0(c^*)=1$. I will establish that $c^*\in(1,\frac{r_d+\lambda}{r_c+\lambda})$, but for the time being, no specific assumption is made on the value of c^*. Finally, I will assume that the value functions v and e are continuous and continuously differentiable at $p=0$.

1. $c\in(0,1\wedge c^*)$

   On this interval, none of the maturing creditors are rolling over their debt. The value functions v_0 and e_0 must satisfy:

   $$\begin{aligned}\rho v_0(c)&=r_d+\left((r_c+\lambda)c-(r_d+\lambda)\right)v_0'(c)+\lambda\left(1-v_0(c)\right)+\phi\left(\min(1,c)-v_0(c)\right) \\ \rho e_0(c)&=\left((r_c+\lambda)c-(r_d+\lambda)\right)e_0'(c)+\phi\left(\max(0,c-1)-e_0(c)\right)\end{aligned}$$

   Since on this interval, $c<1$, I have:

   $$\begin{aligned}v_0'(c)&=\frac{\rho+\lambda+\phi}{(r_c+\lambda)c-(r_d+\lambda)}v_0(c)-\frac{r_d+\lambda+\phi c}{(r_c+\lambda)c-(r_d+\lambda)} \\ e_0'(c)&=\frac{\rho+\phi}{(r_c+\lambda)c-(r_d+\lambda)}e_0(c)\end{aligned}$$

Given the boundary $e_0(0) = v_0(0) = 0$, these ODEs admit the following solutions:

$$v_0(c) = \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)}\right)\left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}\right] + \frac{\phi}{\phi + \rho - r_c}c$$
$$e_0(c) = 0$$

The expression for $v_0(c)$ on this interval admits a natural interpretation. When $c < (1 \wedge c^*)$, the optimal strategy for creditors is to stop rolling their debt claims. Thus, the evolution of $c(t)$ is the following:

$$c'(t) = (r_c + \lambda)c(t) - (r_d + \lambda)$$

This means that $c(t)$ evolves as follows:

$$c(t) = \left(c(0) - \frac{r_d + \lambda}{r_c + \lambda}\right)e^{(r_c+\lambda)t} + \frac{r_d + \lambda}{r_c + \lambda}$$

Taking $c(0) = c < 1 < \frac{r_d+\lambda}{r_c+\lambda}$, the cash (per unit of debt) is a strictly decreasing function of time, which hits zero at time τ_0:

$$\tau_0 = \frac{-1}{r_c + \lambda}\ln\left(1 - \frac{r_c + \lambda}{r_d + \lambda}c\right)$$

Thus, for a given creditor i, if within a period of time of length τ_0 neither τ_λ (the creditor's claim maturing), nor τ_ϕ (the asset maturing) occurs, the creditor is guaranteed to only receive interest payment on its debt and lose its principal balance. This means that I can think of the creditor's value as the sum of interest collections until a stopping time $\tau = \tau_0 \wedge \tau_\lambda \wedge \tau_\phi$, plus principal collections that depend on whether τ_0, τ_λ or τ_ϕ occurs first:

$$\begin{aligned} v_0(c) = &\Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi) \times \int_0^{\tau_0} e^{-\rho t} r_d dt \\ &+ [1 - \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi)] \times \mathbb{E}^c\left[\int_0^{\tau_\lambda \wedge \tau_\phi} e^{-\rho t} r_d dt | \tau_\lambda \wedge \tau_\phi < \tau_0\right] \\ &+ [1 - \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi)] \times \Pr(\tau_\lambda < \tau_\phi | \tau_0 < \tau_\lambda \wedge \tau_\phi) \times \mathbb{E}^c\left[e^{-\rho\tau_\lambda} | \tau_\lambda < \tau_\phi < \tau_0\right] \\ &+ [1 - \Pr(\tau_0 < \tau_\lambda \wedge \tau_\phi)] \times \Pr(\tau_\phi < \tau_\lambda | \tau_0 < \tau_\lambda \wedge \tau_\phi) \times \mathbb{E}^c\left[e^{-\rho\tau_\phi} c(\tau_\phi) | \tau_\phi < \tau_\lambda < \tau_0\right] \end{aligned}$$

A separate calculation of all the components above, taking into account the fact that $\tau_\lambda \wedge \tau_\phi$ is exponentially distributed with parameter $\lambda + \phi$, and taking into account the deterministic value of τ_0 calculated above, enables me to verify that the expression for $v_0(\cdot)$ previously obtained corresponds to the decompositon above. Given the expression for v_0, I can immediately conclude that v_0 is strictly increasing on $(0, 1 \wedge c^*)$. The derivative v_0' on the interval $[0,1]$ takes the following form:

$$v_0'(c) = \left(1 - \frac{\phi}{\phi+\rho-r_c}\right) \times \underbrace{\left(1 - \frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}-1}}_{\in(0,1)} + \left(\frac{\phi}{\phi+\rho-r_c}\right) \times 1$$

Since this is a weighted average of $\left(1 - \frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}-1}$ (which is strictly less than 1 for $c > 0$ and equal to 1 for $c = 0$) and 1, I have $v_0'(c) < 1$ for all $c \in (0,1)$. Since $v_0(0) = 0$ and $v_0'(c) < 1$ for all $c < 1$, it must be the case that $v_0(1) < 1$. Since c^* must satisfy $v_0(c^*) = 1$, I must have $c^* > 1$: when the cash (per unit of debt outstanding) falls below unity, it is already "too late": creditors have started running for cash levels greater than 1. This is intuitive: debt interest r_d is strictly greater than the discount rate ρ, while the cash only earns $r_c < \rho$, which means that the cash available plus its interest collections are insufficient to cover interest and principal repayments on the debt.

2. $\underline{c \in (1, c^* \wedge \frac{r_d+\lambda}{r_c+\lambda})}$

On this interval, the value functions v_0 and e_0 satisfy:

$$\rho v_0(c) = r_d + ((r_c+\lambda)c - (r_d+\lambda))\, v_0'(c) + \lambda\,(1 - v_0(c)) + \phi\,(\min(1,c) - v_0(c))$$
$$\rho e_0(c) = ((r_c+\lambda)c - (r_d+\lambda))\, e_0'(c) + \phi\,(\max(0, c-1) - e_0(c))$$

Since I am now focused on $c^* > c > 1$, I have:

$$v_0'(c) = \frac{\rho+\lambda+\phi}{(r_c+\lambda)c-(r_d+\lambda)} v_0(c) - \frac{r_d+\lambda+\phi}{(r_c+\lambda)c-(r_d+\lambda)}$$
$$e_0'(c) = \frac{\rho+\phi}{(r_c+\lambda)c-(r_d+\lambda)} e_0(c) - \frac{\phi(c-1)}{(r_c+\lambda)c-(r_d+\lambda)}$$

These ODEs admit the following solutions:

$$v_0(c) = H_1 \left(1 - \frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d+\lambda+\phi}{\rho+\lambda+\phi}$$
$$e_0(c) = K_1 \left(1 - \frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\phi}{r_c+\lambda}} + \frac{\phi}{\phi+\rho-(r_c+\lambda)}c - \frac{\phi}{\rho+\phi}\left(1 + \frac{r_d+\lambda}{\phi+\rho-(r_c+\lambda)}\right)$$

K_1, H_1 are constants to be determined. Let me first focus on the expression for v_0 on this interval. Value matching at $c = 1$ gives us:

$$v_0(1) = H_1 \left(1 - \frac{r_c+\lambda}{r_d+\lambda}\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d+\lambda+\phi}{\rho+\lambda+\phi}$$

I know from the previous section that $v_0(1) < 1$. Since $\frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} > 1$, it must be the case that $H_1 < 0$, meaning that $v_0(\cdot)$ is strictly increasing on $(1, \frac{r_d+\lambda}{r_c+\lambda})$. I also know that I must have $v_0(c^*) = 1$. Since $v_0\left(\frac{r_d+\lambda}{r_c+\lambda}\right) = \frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} > 1$, and since $v_0(\cdot)$ is strictly increasing for $c < \frac{r_d+\lambda}{r_c+\lambda}$, it must be the case that $c^* < \frac{r_d+\lambda}{r_c+\lambda}$. Using value matching at $c = 1$, I can conclude that H_1 is equal to:

$$H_1 = -\frac{\phi(r_d - r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)}\left(1 - \frac{r_c+\lambda}{r_d+\lambda}\right)^{-\frac{\rho+\lambda+\phi}{r_c+\lambda}} - \frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)}$$

I thus obtain the following functional form for v_0 on $(1, c^*)$:

$$v_0(c) = H_1\left(1 - \frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d+\lambda+\phi}{\rho+\lambda+\phi}$$

The threshold c^* is the unique value that satisfies:

$$1 = H_1\left(1 - \frac{r_c+\lambda}{r_d+\lambda}c^*\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d+\lambda+\phi}{\rho+\lambda+\phi}$$

In other words, I have:

$$c^* = \frac{r_d+\lambda}{r_c+\lambda}\left(1 - \left(\frac{1}{H_1}\frac{\rho-r_d}{\rho+\phi+\lambda}\right)^{\frac{r_c+\lambda}{\rho+\lambda+\phi}}\right)$$

Similarly, value matching at $c = 1$ means that I must have $e_0(1) = 0$, implying that the constant K_1 is equal to:

$$K_1 = \frac{\phi(r_d - r_c)}{(\rho+\phi)(\phi+\rho-(r_c+\lambda))}\left(1 - \frac{r_c+\lambda}{r_d+\lambda}\right)^{-\frac{\rho+\phi}{r_c+\lambda}}$$

Some algebra enables me to compute $e_0'(c)$, for $c \in (1, c^*)$:

$$e_0'(c) = \frac{\phi}{\rho+\phi-(r_c+\lambda)}\left[1 - \left(\frac{1 - \frac{r_c+\lambda}{r_d+\lambda}c}{1 - \frac{r_c+\lambda}{r_d+\lambda}}\right)^{\frac{\rho+\phi-(r_c+\lambda)}{r_c+\lambda}}\right] > 0$$

The value function $e_0(\cdot)$ is strictly increasing on the interval $(1, c^*)$ and is equal to:

$$e_0(c) = K_1\left(1 - \frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\phi}{r_c+\lambda}} + \frac{\phi}{\phi+\rho-(r_c+\lambda)}c - \frac{\phi}{\rho+\phi}\left(1 + \frac{r_d+\lambda}{\phi+\rho-(r_c+\lambda)}\right)$$

Once again, the equation above has a natural interpretation. Indeed, during a run (i.e. when $c < c^*$), I know the cash level (per unit of outstanding debt) is strictly decreasing. The equity value is only positive due to the fact that when $c \in (1, c^*)$, there is a chance that the stopping time τ_ϕ occurs, at which point cash is distributed to creditors and the remainder $c-1$ is distributed to the equity holders. The cash $c(t)$ evolves according to:

$$c(t) = \left(c(0) - \frac{r_d + \lambda}{r_c + \lambda}\right) e^{(r_c+\lambda)t} + \frac{r_d + \lambda}{r_c + \lambda}$$

When $c(0) = c$, I know that $c(t)$ is above 1 as long as $t \leq \tau$, where τ satisfies $c(\tau) = 1$:

$$\tau = \frac{-1}{r_c + \lambda} \ln \left(\frac{1 - \frac{r_c+\lambda}{r_d+\lambda} c}{1 - \frac{r_c+\lambda}{r_d+\lambda}} \right)$$

This means that I can think of the shareholder's value as the probability of τ_ϕ occuring before τ, multiplied by the expected discounted value of $c(\tau_\phi) - 1$ conditioned on $\tau_\phi < \tau$:

$$\begin{aligned} e_0(c) &= \Pr\left(\tau_\phi < \tau\right) \times \mathbb{E}^c \left[e^{-\rho\tau_\phi} \left(c(\tau_\phi) - 1\right) | \tau_\phi < \tau \right] \\ &= \int_0^\tau e^{-\rho x} \left(c(x) - 1\right) \phi e^{-\phi x} dx \end{aligned}$$

Further algebra enables me to recover the expression derived previously using the HJB equation for e_0.

3. $c \in (c^*, \frac{r_d}{r_c})$

On this interval, the value functions v_0 and e_0 satisfy:

$$\begin{aligned} \rho v_0(c) &= r_d + (r_c c - r_d) v_0'(c) + \phi \left(\min(1, c) - v_0(c)\right) \\ \rho e_0(c) &= (r_c c - r_d) e_0'(c) + \phi \left(\max(0, c - 1) - e_0(c)\right) \end{aligned}$$

Since $c^* > 1$, I have for $c \in (c^*, \frac{r_d}{r_c})$:

$$\begin{aligned} v_0'(c) &= \frac{\rho + \phi}{r_c c - r_d} v_0(c) - \frac{r_d + \phi}{r_c c - r_d} \\ e_0'(c) &= \frac{\rho + \phi}{r_c c - r_d} e_0(c) - \frac{\phi(c - 1)}{r_c c - r_d} \end{aligned}$$

These ODEs admit the following solutions:

$$\begin{aligned} v_0(c) &= H_2 \left(1 - \frac{r_c}{r_d} c\right)^{\frac{\rho+\phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi} \\ e_0(c) &= K_2 \left(1 - \frac{r_c}{r_d} c\right)^{\frac{\rho+\phi}{r_c}} + \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c}\right) \end{aligned}$$

K_2, H_2 are constants to be determined. Value matching at $c = c^*$ gives me the following equation for H_2:

$$1 = H_2 \left(1 - \frac{r_c}{r_d} c^*\right)^{\frac{\rho+\phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi}$$

$$\Rightarrow H_2 = \frac{\rho - r_d}{\rho + \phi} \left(1 - \frac{r_c}{r_d} c^*\right)^{-\frac{\rho+\phi}{r_c}}$$

Since $r_d > \rho$, it is clear that $H_2 < 0$, meaning that $v(\cdot)$ is increasing on $(c^*, \frac{r_d}{r_c})$. Let me now look at the function e_0. Value matching at $c = c^*$ gives the following:

$$K_2 \left(1 - \frac{r_c}{r_d} c^*\right)^{\frac{\rho+\phi}{r_c}} + \frac{\phi}{\phi + \rho - r_c} c^* - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c}\right)$$
$$= K_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c^*\right)^{\frac{\rho+\phi}{r_c+\lambda}} + \frac{\phi}{\phi + \rho - (r_c + \lambda)} c^* - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d + \lambda}{\phi + \rho - (r_c + \lambda)}\right)$$

I conclude this section by highlighting that on this interval $(c^*, \frac{r_d}{r_c})$, creditors are not running but the cash available at the firm is strictly decreasing with time. When noting $c(0) = c$, the evolution of $c(t)$ is as follows:

$$c(t) = \left(c - \frac{r_d}{r_c}\right) e^{r_c t} + \frac{r_d}{r_c}$$

Which is strictly decreasing since $c < \frac{r_d}{r_c}$. Thus, while creditors are not running, the cash reserves are decreasing, and will for sure reach the level c^* at which point it starts becoming optimal for creditors to stop rolling over.

4. $c \in (\frac{r_d}{r_c}, +\infty)$

Since shareholders do not receive dividends, $c(t)$ is a strictly increasing function of time since the initial cash reserve is above $\frac{r_d}{r_c}$. Creditors constantly roll over their debt, and their value function is constant, equal to:

$$v_0(c) = \frac{r_d + \phi}{\rho + \phi}$$

The equity value is then equal to the expected discounted value of $c(\tau_\phi) - 1$, in other words:

$$e_0(c) = \mathbb{E}^{p,c}\left[e^{-\rho \tau_\phi} \left(c(\tau_\phi) - 1\right)\right]$$
$$= \int_0^{+\infty} \phi e^{-\rho x} \left(\left(c - \frac{r_d}{r_c}\right) e^{r_c x} + \frac{r_d}{r_c} - 1\right) e^{-\phi x} dx$$
$$= \frac{\phi}{\phi + \rho - r_c} c - \frac{\phi}{\rho + \phi} \left(1 + \frac{r_d}{\phi + \rho - r_c}\right)$$

To summarize, the value function of creditors is equal to:

$$v_0(c) = \begin{cases} \left(\frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)}\right)\left[1-\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}\right]+\frac{\phi}{\phi+\rho-r_c}c & \text{for } 0<c<1 \\ H_1\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}+\frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} & \text{for } 1<c<c^* \\ H_2\left(1-\frac{r_c}{r_d}c\right)^{\frac{\rho+\phi}{r_c}}+\frac{r_d+\phi}{\rho+\phi} & \text{for } c^*<c<\frac{r_d}{r_c} \\ \frac{r_d+\phi}{\rho+\phi} & \text{for } c>\frac{r_d}{r_c} \end{cases}$$

Similarly, the value function for shareholders is equal to:

$$e_0(c) = \begin{cases} 0 & \text{for } 0<c<1 \\ K_1\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\phi}{r_c+\lambda}}+\frac{\phi}{\phi+\rho-(r_c+\lambda)}c-\frac{\phi}{\rho+\phi}\left(1+\frac{r_d+\lambda}{\phi+\rho-(r_c+\lambda)}\right) & \text{for } 1<c<c^* \\ K_2\left(1-\frac{r_c}{r_d}c\right)^{\frac{\rho+\phi}{r_c}}+\frac{\phi}{\phi+\rho-r_c}c-\frac{\phi}{\rho+\phi}\left(1+\frac{r_d}{\phi+\rho-r_c}\right) & \text{for } c^*<c<\frac{r_d}{r_c} \\ \frac{\phi}{\phi+\rho-r_c}c-\frac{\phi}{\rho+\phi}\left(1+\frac{r_d}{\phi+\rho-r_c}\right) & \text{for } c>\frac{r_d}{r_c} \end{cases}$$

□

Proof of Proposition 6: I have established in Proposition 5 that the threshold c^* must be between 1 and $\frac{r_d+\lambda}{r_c+\lambda}$. I now establish that this cutoff is decreasing as λ increases. In order to do this, I will leverage the strategic substitutability property of the value function $v(0,\cdot;s,S)$ on the interval $[1,\frac{r_d+\lambda}{r_c+\lambda}]$. I first establish a preliminary result.

Lemma 4. *Consider the economy without illiquid asset. For any cutoff strategy $S_{\hat{c}}$ played by all other creditors, creditor i's optimal value function $v^*(0,\cdot;S_{\hat{c}})$ is continuous, bounded and monotone in c.*

The proof of Lemma 4 is straightforward. Take the cutoff strategy $S_{\hat{c}}$, and define the operator $\mathbb{T}_{S_{\hat{c}}}$ that maps any arbitrary continuous bounded function $f:\mathbb{R}_+\to\mathbb{R}$ into a function $\mathbb{T}_{S_{\hat{c}}}(f)$ defined as follows:

$$\mathbb{T}_{S_{\hat{c}}}(f)(c) \;=\; \mathbb{E}^c\left[\int_0^\tau e^{-\rho t}r_d dt + e^{-\rho\tau}1_{\{\tau=\tau_\lambda\}}\max\left(1,f\left(c(\tau)\right)\right)+e^{-\rho\tau}1_{\{\tau=\tau_\phi\}}\min\left(1,c(\tau)\right)\right],$$

with τ_λ exponentially distributed (with intensity λ), and with c evolving according to:

$$dc(t) = \left[\left(r_c+\lambda 1_{\{c(t)<\hat{c}\}}\right)c(t)-\left(r_d+\lambda 1_{\{c(t)<\hat{c}\}}\right)\right]dt$$

The operator $\mathbb{T}_{S_{\hat{c}}}$ maps the space of continuous bounded functions into itself, and given that such space is complete, $\mathbb{T}_{S_{\hat{c}}}$ admits a unique fixed point – the function $v^*(0,\cdot;S_{\hat{c}})$. Then note that the operator $\mathbb{T}_{S_{\hat{c}}}$ maps weakly increasing functions into weakly increasing functions. Using a corollary of the contraction mapping theorem, I conclude that $v^*(0,\cdot;S_{\hat{c}})$ is weakly increasing.

Lemma 4 then enables me to conclude that creditor i's best response to a cutoff strategy $S_{\hat{c}}$ employed by other creditors is also cutoff. Let $c^*(\hat{c})$ be the best response of a creditor i, when all other creditors use a cutoff strategy $S_{\hat{c}}$, with $\hat{c} \in [1, \frac{r_d+\lambda}{r_c+\lambda}]$. $c^*(\hat{c})$ is a well defined function – indeed, if there were several points $c_1 < ... < c_n$ that satisfy $v^*(0, c_k; S_{\hat{c}}) = 1$ for all k, then $v^*(0, c; S_{\hat{c}}) = 1$ on the full non-empty interval $[c_1, c_n]$, since I established that $v^*(0, \cdot; S_{\hat{c}})$ is monotone. But that would mean that $\partial_c v^*(0, c; S_{\hat{c}}) = 0$ on such interval, and using the HJB equation satisfied by v^*, would suggest that $v^*(0, c; S_{\hat{c}}) = \frac{r_d+\phi}{\rho+\phi} > 1$ for $c \in [c_1, c_n]$, which is a contradiction.

Now that the best response function $c^*(\hat{c})$ is properly defined, consider $\hat{c} = \frac{r_d+\lambda}{r_c+\lambda}$. When $c(0) = \hat{c} = \frac{r_d+\lambda}{r_c+\lambda}$, the cash per unit of debt outstanding $c(t)$ is constant and equal to $c(0)$. In other words, $v^*(0, \frac{r_d+\lambda}{r_c+\lambda}; S_{\frac{r_d+\lambda}{r_c+\lambda}}) = \frac{r_d+\phi}{\rho+\phi} > 1$. This means that the best response $c^*(\frac{r_d+\lambda}{r_c+\lambda}) < \frac{r_d+\lambda}{r_c+\lambda}$. Consider then $\hat{c} = 1$. The proof of Proposition 5 shows that $v^*(0, 1; S_1) < 1$, which means that $c^*(1) > 1$. Standard arguments can show that the function $c^*(\cdot)$ is continuous on $[1, \frac{r_d+\lambda}{r_c+\lambda}]$. A straight application of the intermediate value theorem, using the continuity of $c^*(\cdot)$ with the fact that $c^*(1) > 1$ and $c^*(\frac{r_d+\lambda}{r_c+\lambda}) < \frac{r_d+\lambda}{r_c+\lambda}$, delivers the existence of at least one fixed point of the function $c^*(\cdot)$.

Finally, the strategic substitutability property of the payoff function on $[1, \frac{r_d+\lambda}{r_c+\lambda}]$ leads to the result that the function $c^*(\cdot)$ is decreasing on the interval $[1, \frac{r_d+\lambda}{r_c+\lambda}]$, which means that the fixed point of the best response function is unique and on this interval. So far, I have established (using different arguments) the exact result that was proven analytically in the proof of Lemma 4. But the strategic substitutability property can also be leveraged further. Indeed, as I established in the proof of Proposition 1, when $\frac{r_d+\lambda}{r_c+\lambda} > c_0 > c_f \geq 1$, the stopping time function $\tau_{\mathcal{R}}(c_f; c_0)$ is increasing in the parameter λ. In other words, when the cutoff strategy $S_{\hat{c}}$ is such that $\hat{c} \in [1, \frac{r_d+\lambda}{r_c+\lambda}]$, the cash per unit of debt outstanding $c(t)$ decreases (for $c(0) < [1, \frac{r_d+\lambda}{r_c+\lambda}]$) faster, the smaller λ is. This leads to the conclusion that if $\lambda' > \lambda$, for any $x \in [1, \frac{r_d+\lambda}{r_c+\lambda}]$:

$$c^*(x; \lambda') \leq c^*(x; \lambda),$$

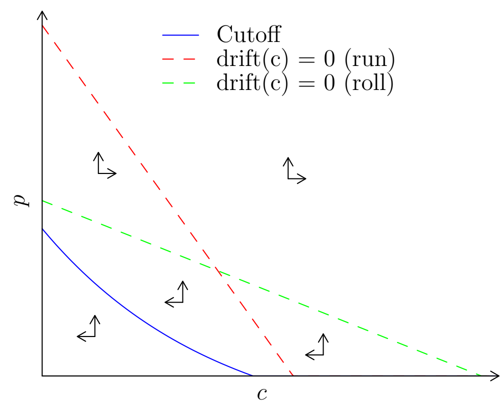
where $c^*(\cdot; \lambda)$ is the best response function for parameter λ. This leads to the conclusion that the fixed point c^* is decreasing in λ. □

Proof of Proposition 7: The analysis of the boundary $c = 0$ needs to be split between several cases. I assume the existence of a symmetric cutoff Markov perfect equilibrium, with related run region $\mathcal{R}$ and roll region $\mathcal{NR}$. As a reminder, the locus of points where the cash drift is zero when creditors are rolling satisfies $p = \frac{r_d}{\rho-\mu}(1 - \frac{r_c}{r_d}c)$. This locus of points is a straight line in the (p, c) space, intersecting $c = 0$ at $p = \frac{r_d}{\rho-\mu}$. The locus of points where the cash drift is zero when creditors are running satisfies $p = \frac{r_d+\lambda}{\rho-\mu}(1 - \frac{r_c+\lambda}{r_d+\lambda}c)$. This locus of points is also a straight line in the (p, c) space, intersecting $c = 0$ at $p = \frac{r_d+\lambda}{\rho-\mu}$.

1. $\underline{\frac{r_d}{\rho-\mu} > \frac{1}{\alpha}}$
   This case corresponds to a relatively high recovery rate. Figure 12 illustrates the parameter configuration studied. For any $p < \frac{1}{\alpha} < \frac{r_d}{\rho-\mu}$, the drift of the cash reserve is

negative at $c = 0$, meaning that the firm has to sell its illiquid asset and distribute the proceeds to creditors. Since $p < 1/\alpha$, creditors take a loss, and the value function at any such point $(p, 0)$ is strictly less than 1. Thus I must have $\{(p, 0) : p < 1/\alpha\} \subset \mathcal{R}$. In other words, the segment of the vertical axis that is below $p = 1/\alpha$ must be part of the run region.

Figure 12: Case $\frac{r_d}{\rho-\mu} > \frac{1}{\alpha}$

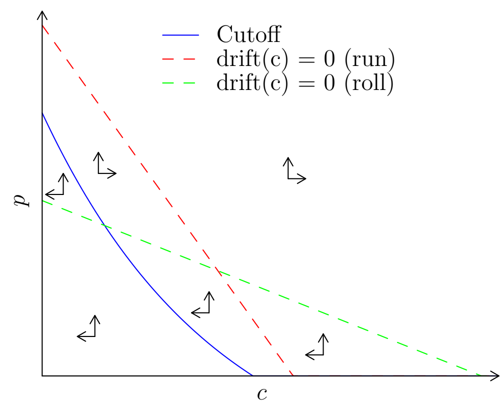


For any $\frac{r_d}{\rho-\mu} > p > \frac{1}{\alpha}$, the drift of the cash reserve is also negative at $c = 0$, meaning that the firm has to sell its illiquid asset and distribute the proceeds to creditors. But at those points of the state space, the recovery rate realized upon the asset sale is greater than 1, meaning that creditors' value must be exactly equal to 1 (since liquidation proceeds in excess of the outstanding debt are paid to shareholders). Since I am assuming that agents indifferent between running and rolling will chose to roll, I must have $\{(p, 0) : p \geq 1/\alpha\} \subset \mathcal{NR}$.

2. $\underline{\frac{r_d}{\rho-\mu} < \frac{1}{\alpha} < \frac{r_d+\lambda}{\rho-\mu}}$
   This case corresponds to intermediate recovery values. Figure 13 illustrates the parameter configuration studied. I will prove (by contradiction) that I must have $\{(p, 0) : p < 1/\alpha\} \subset \mathcal{R}$, and $\{(p, 0) : p \geq 1/\alpha\} \subset \mathcal{NR}$. First, assume that there exists $\bar{p} > 1/\alpha$ such that $\{(p, 0) : p < \bar{p}\} \subset \mathcal{R}$. If that was the case, for any point of the state space $(p, 0)$ where $p \in (1/\alpha, \bar{p})$, since $p < \frac{r_d+\lambda}{\rho-\mu}$ the drift of cash is negative and the firm is forced to sell its illiquid asset. The liquidation proceeds are sufficient for creditors to be fully paid back, meaning that creditors' value function at that point of the state space has to be equal to 1. In other words, creditors must be rolling over their debt at those points of the state space, leading to a contradiction.

   Now assume that there exists $\bar{p} < 1/\alpha$ such that $\{(p, 0) : p \geq \bar{p}\} \subset \mathcal{NR}$. Without loss

Figure 13: Case $\frac{r_d}{\rho-\mu} < \frac{1}{\alpha} < \frac{r_d+\lambda}{\rho-\mu}$



of generality, assume that $\bar{p} = \inf\{\tilde{p} : \{(p, 0) : p \geq \tilde{p}\} \subset \mathcal{NR}\}$. Take any arbitrary $\epsilon > 0$, since $\bar{p} - \epsilon < \frac{r_d+\lambda}{\rho-\mu}$, at the point $(\bar{p} - \epsilon, 0)$ the cash drift is negative, meaning that the firm sells its illiquid assets and distributes the proceeds to creditors. Since $\bar{p} - \epsilon < 1/\alpha$, creditors realize a loss, and their value function at that point is equal to $\alpha(\bar{p} - \epsilon)$. Take $\epsilon \to 0$, since v^* is continuous on $\mathbb{R}_+^2$, I must have $\lim_{\epsilon\to 0} v^*(\bar{p} - \epsilon, 0) = v^*(\bar{p}, 0) = \alpha\bar{p} < 1$. But by construction, $(\bar{p}, 0) \in \mathcal{NR}$, which means that $v(\bar{p}, 0) \geq 1$. This is the contradition I was looking for.

3. $\frac{1}{\alpha} > \frac{r_d+\lambda}{\rho-\mu}$

   This case corresponds to low recovery values. Since I established the existence of an upper dominance region, it must be the case that $\{p > 0 : (p, 0) \in \mathcal{NR}\}$ is non-empty. Let $\bar{p} = \inf\{p > 0 : (p, 0) \in \mathcal{NR}\}$. I will show by contradiction that I must have $\bar{p} \geq \frac{r_d+\lambda}{\rho-\mu}$. Assume the opposite – i.e. assume that $\bar{p} < \frac{r_d+\lambda}{\rho-\mu}$. Take any arbitrary $\epsilon > 0$, since $\bar{p} - \epsilon < \frac{r_d+\lambda}{\rho-\mu}$, at the point $(\bar{p} - \epsilon, 0)$ the cash drift is negative, meaning that the firm sells its illiquid assets and distributes the proceeds to creditors. Since $\bar{p} - \epsilon < 1/\alpha$, creditors realize a loss, and their value function at that point is equal to $\alpha(\bar{p} - \epsilon)$. Take $\epsilon \to 0$, since v^* is continuous on $\mathbb{R}_+^2$, I must have $\lim v^*(\bar{p} - \epsilon, 0) = v^*(\bar{p}, 0) = \alpha\bar{p} < 1$. But by construction, $(\bar{p}, 0) \in \mathcal{NR}$, which means that $v(\bar{p}, 0) \geq 1$. This is the contradition I was looking for.

□

Proof of Lemma 3: When p or c are very large, the probability that a run occurs before the illiquid asset matures converges to zero. It is thus clear that when p or c tends to $+\infty$, the value function e verifies:

$$e(p,c) = \mathbb{E}^{p,c}\left[e^{-\rho\tau_\phi}\left(p(\tau_\phi) + c(\tau_\phi) - 1\right)\right] + o(1)$$

Where the expectation is taken under the following dynamics for p and c:

$$\begin{aligned}
dp(t) &= \mu p(t)dt + \sigma p(t)dB(t) \\
dc(t) &= \left((\rho-\mu)p(t) + r_c c(t) - r_d\right)dt
\end{aligned}$$

Given $p(0) = p$, since $e^{-\rho t}p(t) = pe^{(\mu-\rho-\frac{1}{2}\sigma^2)t+\sigma B(t)}$, I have:

$$\mathbb{E}^{p,c}\left[e^{-\rho\tau_\phi}p(\tau_\phi)\right] = p\int_0^{+\infty}\phi e^{-\phi x}e^{(\mu-\rho)x}dx = \frac{\phi}{\rho+\phi-\mu}p$$

Note also that for any fixed time t, I have:

$$\begin{aligned}
\mathbb{E}^{p,c}\left[e^{-\rho t}c(t)\right] &= c + \mathbb{E}^{p,c}\left[\int_0^t \left((\rho-\mu)p(s) + (r_c-\rho)c(s) - r_d\right)e^{-\rho s}ds\right] \\
&= c + \left(1 - e^{-(\rho-\mu)t}\right)p + (r_c-\rho)\int_0^t \mathbb{E}^{p,c}\left[e^{-\rho s}c(s)\right]ds - \frac{r_d}{\rho}\left(1-e^{-\rho t}\right)
\end{aligned}$$

Thus, I have:

$$\frac{d}{dt}\left(\mathbb{E}^{p,c}\left[e^{-\rho t}c(t)\right]\right) = (\rho-\mu)pe^{-(\rho-\mu)t} + (r_c-\rho)\mathbb{E}^{p,c}\left[e^{-\rho t}c(t)\right] - r_d e^{-\rho t}$$

This enables me to conclude that:

$$\mathbb{E}^{p,c}\left[e^{-\rho t}c(t)\right] = \left(c + \frac{\rho-\mu}{\mu-r_c}\left(e^{(\mu-r_c)t}-1\right)p - \frac{r_d}{r_c}\left(1-e^{-r_c t}\right)\right)e^{-(\rho-r_c)t}$$

And finaly:

$$\begin{aligned}
\mathbb{E}^{p,c}\left[e^{-\rho\tau_\phi}c(\tau_\phi)\right] &= \int_0^{+\infty}\phi\left[\left(c + \frac{\rho-\mu}{\mu-r_c}\left(e^{(\mu-r_c)t}-1\right)p - \frac{r_d}{r_c}\left(1-e^{-r_c t}\right)\right)e^{-(\rho-r_c)t}\right]e^{-\phi t}dt \\
&= \frac{\phi}{\rho+\phi-r_c}\left(c + \frac{\rho-\mu}{r_c-\mu}p - \frac{r_d}{r_c}\right) + \frac{\phi}{\rho+\phi-\mu}\frac{\rho-\mu}{\mu-r_c}p + \frac{\phi}{\rho+\phi}\frac{r_d}{r_c}
\end{aligned}$$

I can then conclude that when p, c are large, I have:

$$e(p,c) = \frac{\phi}{\rho+\phi-r_c}c + \frac{\phi}{\rho+\phi-\mu}\left(1 + \frac{\rho-\mu}{\rho+\phi-r_c}\right)p - \frac{\phi}{\rho+\phi}\left(\frac{r_d}{\rho+\phi-r_c}+1\right) + o(1)$$

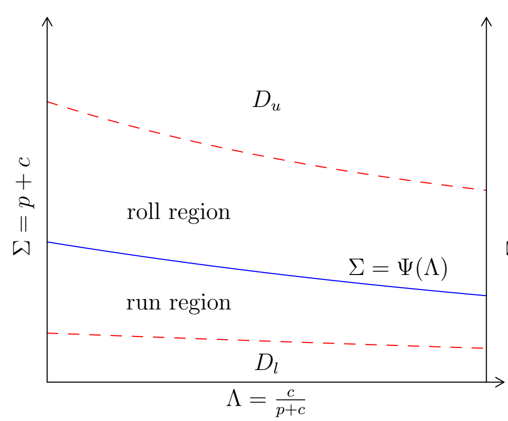
Note finally that when $c > \frac{r_d}{r_c}$, since c is monotone and increasing irrespective of the stragegy followed by creditors, I know that creditors never run thereafter, which means that the approximation above is actually an equality. □

Proof of Proposition 8[Incomplete]: For the purpose of this proof, I find it appropriate to do a change in variable. Instead of working with (p, c), I will work with (Σ, Λ) defined as follows:

$$\Sigma = p + c$$
$$\Lambda = \frac{c}{p+c}$$

In other words, Σ represents the total asset to debt ratio of the firm (or solvency ratio), while Λ represents the liquidity ratio of the firm – in other words, the fraction of the firm's total assets invested in cash. Note that $\Lambda \in [0,1]$ while $\Sigma \in \mathbb{R}_+$. The proof will be divided in several components. First, I will prove that there exists a function $\Psi : [0,1] \to \mathbb{R}_+$ that satisfies $v^*(\Psi(\Lambda), \Lambda; \Psi) = 1$, where I note (with a slight abuse of notation) $v^*(\cdot, \cdot; \Psi)$ the optimal value function of a given creditor, given that other creditors play a cutoff strategy encoded by the function Ψ (i.e. creditors run when $\Sigma < \Psi(\Lambda)$ and roll otherwise). In other words, points (Σ, Λ) of the state space satisfying $\Sigma = \Psi(\Lambda)$ are indifference points for creditors. Since I will be looking for a fixed point Ψ in the space of functions, I will want to use Schauder's fixed point theorem. Second, I will prove that points (Σ, Λ) of the state space satisfying $\Sigma \geq \Psi(\Lambda)$ are such that $v^*(\Sigma, \Lambda; \Psi) \geq 1$, establishing that above the cutoff boundary $\Sigma = \Psi(\Lambda)$, creditors roll over their debt claims when they have the opportunity to do so. Finally, I will prove that points (Σ, Λ) of the state space satisfying $\Sigma < \Psi(\Lambda)$ are such that $v^*(\Sigma, \Lambda; \Psi) < 1$, establishing that below the cutoff boundary $\Sigma = \Psi(\Lambda)$, creditors run.

Figure 14: Reparametrized State Space (Λ, Σ)



Proof – Part A[Incomplete]: In this section I prove that there exists a function $\Psi : [0,1] \to \mathbb{R}_+$ that satisfies $v^*(\Psi(\Lambda), \Lambda; \Psi) = 1$. The dynamics of the state variables (Σ, Λ) can be computed using Ito's lemma, for any strategy S. Indeed, noting $\lambda 1_{\{S(\Sigma(t),\Lambda(t))=1\}} = \lambda(t)$, I have:

$$d\Sigma(t) = ((\lambda(t)+\rho)\Sigma(t) - (\rho - r_c)\Lambda(t)\Sigma(t) - (r_d + \lambda(t)))\,dt + (1-\Lambda(t))\Sigma(t)\sigma dB_t$$
$$d\Lambda(t) = (1-\Lambda(t))\left((\rho-\mu) - (\rho - r_c)\Lambda(t) - \frac{r_d+\lambda(t)}{\Sigma(t)} + \Lambda(t)(1-\Lambda(t))\sigma^2\right)dt - \Lambda(t)(1-\Lambda(t))\sigma dB_t$$

In the reparametrized state space, the dominance regions are now located as indicated in Figure 14. I now focus on functions $\Psi : [0,1] \to \mathbb{R}_+$ such that creditors run whenever $\Sigma < \Psi(\Lambda)$, and roll over otherwise. Going forward, I will therefore refer to creditor i's strategy as the function ψ, and all other creditors' strategy as the function Ψ.

The existence of dominance regions imply that there exists $\underline{\Sigma}, \bar{\Sigma}$, both strictly positive, such that for any (ψ, Ψ), and for any Λ, I have:

$$v(\Sigma, \Lambda; \psi, \Psi) < 1 \text{ if } \Sigma < \underline{\Sigma}$$
$$v(\Sigma, \Lambda; \psi, \Psi) > 1 \text{ if } \Sigma > \bar{\Sigma}$$

Let $A > \bar{\Sigma}$. Note $\mathbb{C}$ the space of continuous functions $\psi : [0,1] \to [0, A]$. This space of functions is not compact when equipped with the sup norm, which means that I will not be able to use it when applying Schauder's fixed point theorem. A more restrictive space of functions would be any subset of $\tilde{\mathbb{C}} \subset \mathbb{C}$ that is closed, bounded and equicontinuous. Indeed, since $[0,1]$ is compact, Arzela-Ascoli's theorem guarantees that any such subspace is compact. I am considering functions that are bounded (i.e. with images in $[0, A]$). I thus need to design a subspace $\tilde{\mathbb{C}} \subset \mathbb{C}$ that is closed and equicontinuous. Let me take for example the space of Lipschitz continuous functions that have the same Lipschitz constant K. In other words, $\tilde{\mathbb{C}} = \{\psi \in \mathbb{C} : |\psi(x_1) - \psi(x_2)| < K|x_1 - x_2|, \forall (x_1, x_2) \in [0,1]^2\}$.

For $\Psi \in \mathbb{C}$, I define $v^*(\cdot,\cdot;\Psi)$ as the optimal creditor value function, solution of the fixed point problem:

$$v^*(\Sigma, \Lambda; \Psi) = \mathbb{E}^{\Sigma,\Lambda}\left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} \max(1, v^*(\Sigma(\tau), \Lambda(\tau); \Psi)) \right.$$
$$\left. + e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \min(1, \Sigma(\tau)) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \min(1, \alpha\Sigma(\tau))\right] \quad (20)$$

v^* denote the value function for a given creditor, evaluated using the creditor's best response, when all other creditors use strategy Ψ. Proposition 4 shows that v^* is appropriately defined, and is continous and bounded. Let $\kappa > 0$ be "small enough" (precise statement to follow). Let $\mathbb{T}$ be the following functional map:

$$\mathbb{T} : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$$
$$\Psi \to (\mathbb{T}\Psi)(\Lambda) = \Psi(\Lambda) + \kappa(1 - v^*(\Psi(\Lambda), \Lambda; \Psi))$$

I need to show that this functional map is properly defined. In other words, I need to show that for any $\Psi \in \tilde{\mathbb{C}}$, (a) $\mathbb{T}\Psi$ is Lipschitz with constant K, and (b) the image of $\mathbb{T}\Psi$ is in $[0, A]$.

Part (a) is difficult to prove. Since I have established (via the contraction mapping theorem) that $v^*(\cdot,\cdot;\Psi)$ is continuous in both argument, it means that if $\Psi \in \mathbb{C}$, it must also be the case that $\Lambda \to v^*(\Psi(\Lambda), \Lambda; \Psi)$ is continuous on $[0,1]$. In other words, the mapping $\mathbb{T}$ maps continuous functions into continuous functions. Does it map Lipschitz functions with Lipschitz constant K into Lipschitz functions with Lipschitz constant K? More work needs to be done on this.

Part (b) requires to show that the image of the mapped function $\mathbb{T}\Psi$ is in $[0, A]$. For an arbitrary $\Psi \in \mathbb{C}$ and an arbitrary $\Lambda \in [0,1]$, if $0 \leq \Psi(\Lambda) < \underline{\Sigma}$, I know that $\kappa(1 - v^*(\Psi(\Lambda), \Lambda; \Psi)) > 0$ due to the existence of the lower dominance region; in other words it must be the case that $(\mathbb{T}\Psi)(\Lambda) > 0$. For an arbitrary $\Psi \in \mathbb{C}$ and an arbitrary $\Lambda \in [0,1]$, if $A \geq \Psi(\Lambda) > \bar{\Sigma}$, I know that $\kappa(1 - v^*(\Psi(\Lambda), \Lambda; \Psi)) < 0$ due to the existence of the higher dominance region; in other words it must be the case that $(\mathbb{T}\Psi)(\Lambda) < A$. Finally, note that for any Λ, and any Ψ, $\kappa(1 - v^*(\Psi(\Lambda), \Lambda; \Psi)) \in [-\kappa\frac{r_d - \rho}{\rho + \phi}, \kappa]$. I can thus pick κ small enough such that for any $\Psi \in \mathbb{C}$, the image of $\mathbb{T}\Psi$ is in $[0, A]$, which is what I need to establish that $\mathbb{T}$ is properly defined.

The next step is to show that $\mathbb{T}$ is a continuous mapping, in other words, for any sequence of functions $\{\Psi_n\}_{n\geq 1}$ such that $\Psi_n \in \tilde{\mathbb{C}}$ for any n, if $\Psi_n \to \Psi$, then $\mathbb{T}\Psi_n \to \mathbb{T}\Psi$. Once this is established, I can use Schauder's fixed point theorem to conclude: since $\mathbb{T}$ is a continuous map from $\tilde{\mathbb{C}}$ into itself, $\mathbb{T}$ must have a fixed point, i.e. there must exist a function Ψ such that $\Psi = \mathbb{T}\Psi$. For such function, for any Λ, I have:

$$1 = v^*(\Psi(\Lambda), \Lambda; \Psi) \tag{21}$$

□

Proof – Part B[Incomplete]: In this section I prove that for any point (Σ, Λ) such that $\Sigma \geq \Psi(\Lambda)$, I must have $v^*(\Sigma, \Lambda; \Psi) \geq 1$. First, note that Proposition 7 provides the two boundary points of any function Ψ satisfying equations (21):

$$\Psi(0) = 1/\alpha$$
$$\Psi(1) = c^*$$

Now, take any point (Σ, Λ) such that $\Sigma > \Psi(\Lambda)$. Let $\tilde{\tau} = \inf\{t \geq 0 : \Sigma(t) = \Psi(\Lambda(t))\}$ – in other words, the first time at which the state reaches the boundary $\Sigma = \Psi(\Lambda)$. $\tilde{\tau} = \infty$ if the stopping time occurs after a debt maturity τ_λ, an asset maturity τ_ϕ, or a default τ_b. Note $\tau = \tilde{\tau} \wedge \tau_\lambda \wedge \tau_\phi \wedge \tau_b$. The value function v^* can be written:

$$v^*(\Sigma, \Lambda; \Psi) = \mathbb{E}^{\Sigma,\Lambda}\left[\int_0^\tau e^{-\rho t} r_d dt + e^{-\rho\tau} 1_{\{\tau=\tau_\lambda\}} \max(1, v^*(\Sigma(\tau), \Lambda(\tau); \Psi)) + e^{-\rho\tau} 1_{\{\tau=\tilde{\tau}\}} \right.$$
$$\left. + e^{-\rho\tau} 1_{\{\tau=\tau_\phi\}} \min(1, \Sigma(\tau)) + e^{-\rho\tau} 1_{\{\tau=\tau_b\}} \min(1, \alpha\Sigma(\tau))\right]$$

The equality above is obtained after noticing that $v^*(\Sigma(\tilde{\tau}), \Lambda(\tilde{\tau}); \Psi) = 1$.

Proof of Proposition 9: I will assume that there is a threshold $\hat{c}$ such that for $c \leq \hat{c}$, it is optimal for the atomistic creditor to run, while for $c \geq \hat{c}$, it is optimal for the atomistic creditor to continue rolling. The threshold $\hat{c}$ will need to verify $w_0(c) - 1 < (c-1)w_0'(c)$ for $c < \hat{c}$, and $w_0(c) - 1 > (c-1)w_0'(c)$ for $c > \hat{c}$. Note that I do not necessarily impose the condition $w_0(\hat{c}) - 1 = (\hat{c} - 1)w_0'(\hat{c})$ since w_0 is a-posteriori not continuously differentiable at

$c = \hat{c}$. I will establish that $\hat{c} \in (1, \frac{r_d+\lambda}{r_c+\lambda})$, but for the time being, no specific assumption is made on the value of $\hat{c}$.

1. $\underline{c \in (0, 1 \wedge \hat{c})}$

   On this interval, the atomistic creditor is not rolling over its debt claim. The value function w_0 must satisfy:

$$(\rho + \lambda) w_0(c) = r_d + \lambda + ((r_c + \lambda)c - (r_d + \lambda)) \, w_0'(c) + \phi \left(\min(1, c) - w_0(c)\right)$$

   Since on this interval, $c < 1$, I have:

$$w_0'(c) = \frac{\rho + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)} w_0(c) - \frac{r_d + \lambda + \phi c}{(r_c + \lambda)c - (r_d + \lambda)}$$

   Given the boundary $w_0(0) = 0$, this ODE admits the following solution:

$$w_0(c) = \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right] + \frac{\phi}{\phi + \rho - r_c} c$$

   Note that the expression for $w_0(c)$ on this interval is identical to the expression of v_0. Given the expression for w_0, I can immediately conclude that w_0 is strictly increasing on $(0, 1 \wedge \hat{c})$. I then compute $w_0(c) - 1 - (c-1)w_0'(c)$:

$$w_0(c) - 1 - (c-1)w_0'(c) = \left(\frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \right) \left[1 - \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \right]$$
$$+ \frac{\phi}{\phi + \rho - r_c} c - 1 - (c-1) \left[\frac{\rho - r_c}{\rho + \phi - r_c} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1} + \frac{\phi}{\phi + \rho - r_c} \right]$$

   This simplifies to:

$$\left(\frac{\rho - r_c}{r_c + \lambda} \right) \left[\frac{r_d + \lambda}{\rho + \phi + \lambda} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \hat{c} \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} - \frac{r_d - r_c}{\rho + \phi - r_c} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \hat{c} \right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1} - \frac{(\rho + \phi - r_d)(r_c + \lambda)}{(\rho + \phi + \lambda)(\rho + \phi - r_c)} \right]$$

   It can then be showed that $w_0(c) - 1 - (c-1)w_0'(c)$ is strictly decreasing on $[0, 1]$, with value zero when $c = 0$. In other words, on $(0, 1]$, $w_0(c) - 1 < (c-1)w_0'(c)$, justifying a posteriori our assumption that the atomistic creditor runs on this interval.

2. $\underline{c \in (1, \hat{c} \wedge \frac{r_d+\lambda}{r_c+\lambda})}$

   On this interval, the value function w_0 satisfies:

$$(\rho + \lambda) w_0(c) = r_d + \lambda + ((r_c + \lambda)c - (r_d + \lambda)) \, w_0'(c) + \phi \left(\min(1, c) - w_0(c)\right)$$

Since I am now focused on $\hat{c} > c > 1$, I have:

$$w_0'(c) = \frac{\rho + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)} w_0(c) - \frac{r_d + \lambda + \phi}{(r_c + \lambda)c - (r_d + \lambda)}$$

Using value matching at $c = 1$, this ODE admits the following solution:

$$w_0(c) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

Where the constant H_1 was determined previously to be equal to:

$$H_1 = -\frac{\phi(r_d - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)} \left(1 - \frac{r_c + \lambda}{r_d + \lambda}\right)^{-\frac{\rho + \lambda + \phi}{r_c + \lambda}} - \frac{(r_d + \lambda)(\rho - r_c)}{(\rho + \lambda + \phi)(\phi + \rho - r_c)}$$

I thus obtain the following functional form for w_0 on $(1, \hat{c})$:

$$w_0(c) = H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi}$$

Note that once again the expression for $w_0(c)$ on this interval is identical to the expression for v_0. Since $H_1 < 0$, it must also be the case that w_0 is strictly increasing on this interval. I then find a convenient form for the function $w_0(c) - 1 - (c-1)w_0'(c)$ on $(1, \hat{c} \wedge \frac{r_d + \lambda}{r_c + \lambda})$:

$$\begin{aligned} w_0(c) - 1 - (c-1)w_0'(c) = &-H_1 \frac{\rho + \phi - r_c}{r_c + \lambda} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda}} \\ &+ H_1 \frac{(r_d - r_c)(\rho + \phi + \lambda)}{(r_c + \lambda)(r_d + \lambda)} \left(1 - \frac{r_c + \lambda}{r_d + \lambda} c\right)^{\frac{\rho + \lambda + \phi}{r_c + \lambda} - 1} + \frac{r_d - \rho}{\rho + \lambda + \phi} \end{aligned}$$

This function of c can be showed to be strictly increasing on $(1, \frac{r_d + \lambda}{r_c + \lambda})$. For $c = 1$, it takes value $w_0(1) - 1 < 0$, inequality already established in my study of the funtion v_0. For $c = \frac{r_d + \lambda}{r_c + \lambda}$, the function $w_0(c) - 1 - (c-1)w_0'(c)$ is equal to $\frac{r_d - \rho}{\rho + \lambda + \phi} > 0$. By the intermediate value theorem, there must exists a $\hat{c} \in (1, \frac{r_d + \lambda}{r_c + \lambda})$ such that $w_0(\hat{c}) - 1 = (\hat{c} - 1)w_0'(\hat{c})$. Note finally that that $\hat{c} > c^*$. This inequality is due to the fact that:

$$\begin{aligned} v_0(c^*) - 1 &= 0 \\ w_0(\hat{c}) - 1 &= (\hat{c} - 1)w_0'(\hat{c}) > 0 \end{aligned}$$

3. $c \in (\hat{c}, \frac{r_d}{r_c})$

   On this interval, the value function w_0 satisfies:

$$\rho w_0(c) = r_d + (r_c c - r_d)w_0'(c) + \phi \left(\min(1, c) - w_0(c)\right)$$

Since $\hat{c} > 1$, I have for $c \in (\hat{c}, \frac{r_d}{r_c})$:

$$w_0'(c) = \frac{\rho + \phi}{r_c c - r_d} w_0(c) - \frac{r_d + \phi}{r_c c - r_d}$$

This ODE admits the following solution:

$$w_0(c) = H_2' \left(1 - \frac{r_c}{r_d} c\right)^{\frac{\rho+\phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi}$$

H_2' is a constant to be determined. Value matching at $c = \hat{c}$ gives me the following equation for H_2':

$$H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \hat{c}\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} + \frac{r_d + \lambda + \phi}{\rho + \lambda + \phi} = H_2' \left(1 - \frac{r_c}{r_d} \hat{c}\right)^{\frac{\rho+\phi}{r_c}} + \frac{r_d + \phi}{\rho + \phi}$$

$$\Rightarrow H_2' = \left(1 - \frac{r_c}{r_d} \hat{c}\right)^{-\frac{\rho+\phi}{r_c}} \left[H_1 \left(1 - \frac{r_c + \lambda}{r_d + \lambda} \hat{c}\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}} - \frac{\lambda(r_d - \rho)}{(\rho + \phi)(\rho + \lambda + \phi)}\right]$$

$H_2' < 0$, meaning that $w_0(\cdot)$ is increasing on $(\hat{c}, \frac{r_d}{r_c})$. It now remains to verify that on this interval $(\hat{c}, \frac{r_d}{r_c})$, $w_0(c) - 1 > (c - 1) w_0'(c)$. I compute:

$$\begin{aligned} w_0(c) - 1 - (c-1) w_0'(c) = -H_2' \frac{\rho + \phi - r_c}{r_c} \left(1 - \frac{r_c}{r_d} c\right)^{\frac{\rho+\phi}{r_c}} \\ + H_2 \frac{(r_d - r_c)(\rho + \phi)}{r_c r_d} \left(1 - \frac{r_c}{r_d} c\right)^{\frac{\rho+\phi}{r_c} - 1} + \frac{r_d - \rho}{\rho + \phi} \end{aligned}$$

This function of c can be showed to be strictly increasing on $(\hat{c}, \frac{r_d}{r_c})$, with value zero for $c = \hat{c}$. In other words, I verify a-posteriori that the atomistic creditor's choice of run boundary $\hat{c}$ is optimal. I also obtain a result that was a-priori not imposed: the function w_0 is continuously differentiable at $c = \hat{c}$.

4. $c \in (\frac{r_d}{r_c}, +\infty)$

   $c(t)$ is a strictly increasing function of time since the initial cash reserve is above $\frac{r_d}{r_c}$. The atomistic creditor constantly rolls over its debt claim, and the value function w_0 is constant, equal to:

$$w_0(c) = \frac{r_d + \phi}{\rho + \phi}$$

   Of course on that interval, $w_0'(c) = 0$ and $w_0(c) > 1$, meaning that the optimality condition $w_0(c) - 1 > (c - 1) w_0'(c)$ is satisfied.

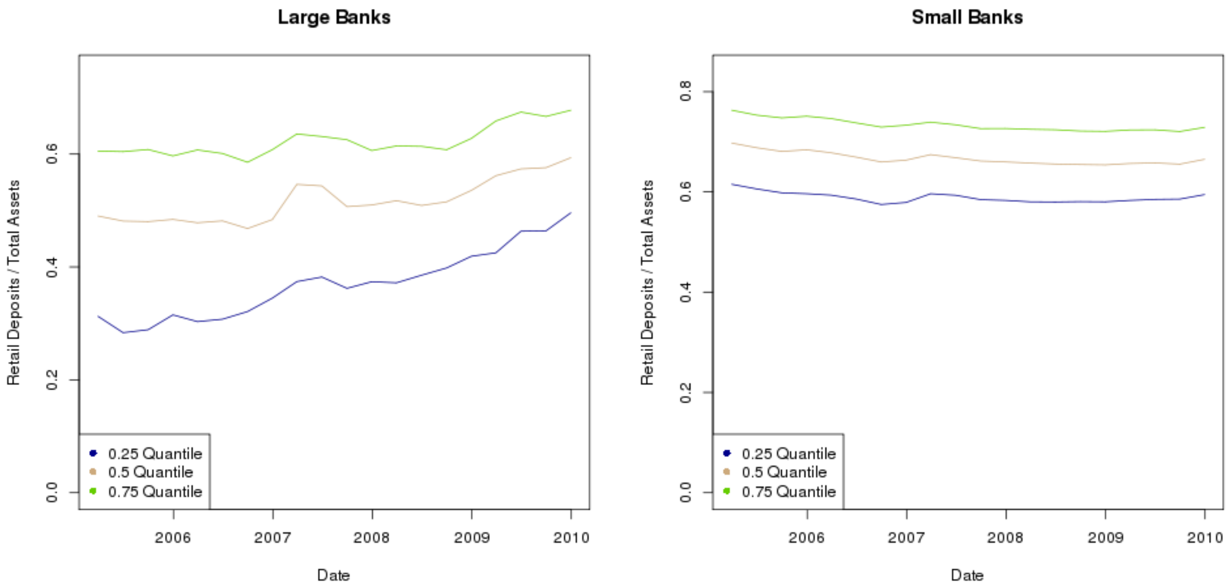
To summarize, the value function of the atomistic creditor w_0 is equal to:

$$w_0(c) = \begin{cases} \left(\frac{(r_d+\lambda)(\rho-r_c)}{(\rho+\lambda+\phi)(\phi+\rho-r_c)}\right)\left[1-\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}\right]+\frac{\phi}{\phi+\rho-r_c}c & \text{for } 0<c<1 \\ H_1\left(1-\frac{r_c+\lambda}{r_d+\lambda}c\right)^{\frac{\rho+\lambda+\phi}{r_c+\lambda}}+\frac{r_d+\lambda+\phi}{\rho+\lambda+\phi} & \text{for } 1<c<\hat{c} \\ H_2'\left(1-\frac{r_c}{r_d}c\right)^{\frac{\rho+\phi}{r_c}}+\frac{r_d+\phi}{\rho+\phi} & \text{for } \hat{c}<c<\frac{r_d}{r_c} \\ \frac{r_d+\phi}{\rho+\phi} & \text{for } c>\frac{r_d}{r_c} \end{cases}$$

□

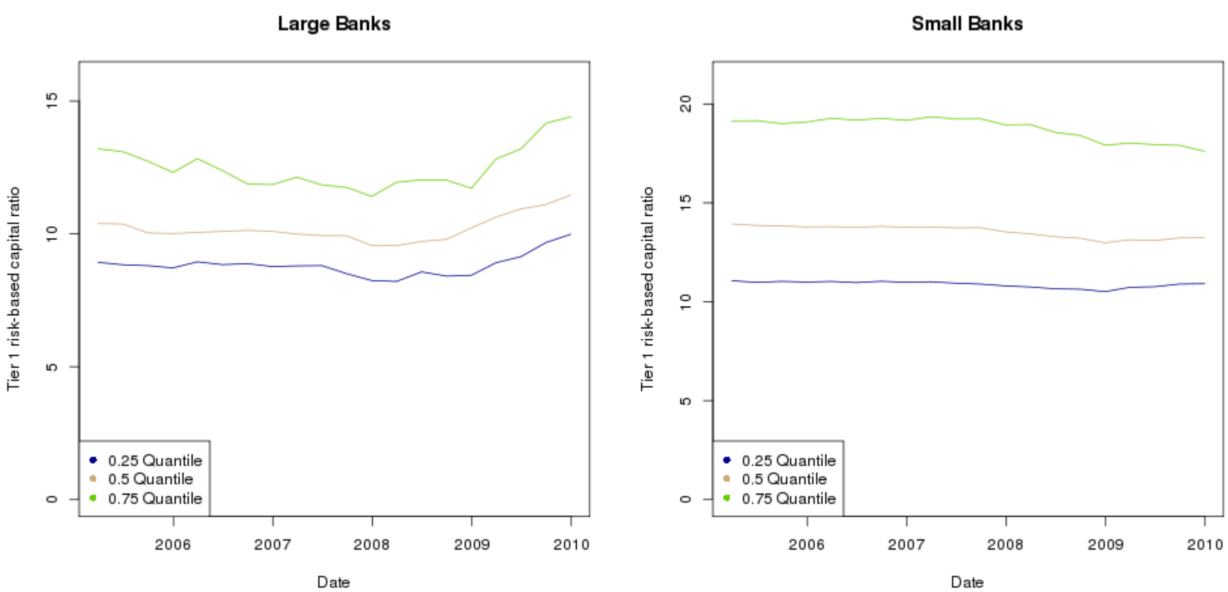
B Key Figures

Figure 15: Retail Deposits - Total Assets Ratio



(a) Banks with Total Assets > \$10bn (b) Banks with Total Assets < \$10bn

Figure 16: Tier 1 Risk Based Capital Ratio



(a) Banks with Total Assets > \$10bn (b) Banks with Total Assets < \$10bn