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Regularity of generalized sphere valued p-harmonic maps with small mean oscillations

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Abstract. We prove (see Theorem 1.3 below) that a generalized harmonic map into a round sphere, i.e. a map $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ which solves the system

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0, \quad i, j = 1, \dots, n,$$

is smooth as soon as $|\nabla u| \in L^q$ for any $q > 1$, and the norm of u in BMO is sufficiently small. Here, $\Omega \subset \mathbb{R}^m$ is open, and m, n are arbitrary. This extends various earlier results of Almeida [1], Ge [15], and R. Moser [38].

A version of this result for generalized p-harmonic maps into spheres is also proved. The proofs rely on the duality of Hardy space and BMO combined with L^p stability of the Hodge decomposition and reverse Hölder inequalities.

The results of this note belong to the regularity theory of nonlinear elliptic systems with the right hand side growing critically with the gradient of solution. The particular system we have in mind may appear special but it is closely linked with vast parts of geometric analysis and PDE. In case $m = n = 2$ its solutions agree with asymptotic limits of solutions of a complex valued Ginzburg–Landau equation involving a small parameter, see Bethuel, Brezis and Hélein [4] and Hardt and Lin [22]. We consider solutions with weak integrability assumptions, weaker than those which are needed for a variational interpretration of the system. The proofs are based on a mixture of (well known) subtle hard analytic methods.

1. Introduction

In late 60's and early 70's famous examples of De Giorgi [6], Giusti and Miranda [19], Frehse [10] and others have ruined all hopes for a positive answer to Hilbert's 19th problem [28] in case of elliptic systems (see Giaquinta [17, Chap. II.3] for more information). Since then it is known that regularity requires either some smallness condition *or* a special structure of the nonlinear terms. This latter condition

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is satisfied for example in the case of harmonic and p-harmonic maps into (some) Riemannian manifolds. Here are the necessary definitions.

Let $\mathcal{N}$ be a compact closed Riemannian manifold, isometrically embedded in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^m$ be open and bounded, and let $p \in [2, m]$. Consider mappings $u : \Omega \to \mathcal{N}$ such that the p-Dirichlet energy of u, given by the functional

$$E_p[u] : = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx = \frac{1}{p}\int_\Omega \Bigl(\sum_{i,j}\Bigl(\frac{\partial u^i}{\partial x_j}\Bigr)^2\Bigr)^{p/2} dx\,, \tag{1.1}$$

is finite. Here, $u = (u^1, \dots, u^n)$ is a map into $\mathbb{R}^n$ with all coordinates $u^j \in W^{1,p}(\Omega)$, satisfying the additional constraint $u(x) \in \mathcal{N}$ a.e. The class of all such maps is traditionally denoted by $W^{1,p}(\Omega, \mathcal{N})$.

We say that $u \in W^{1,p}(\Omega, \mathcal{N})$ is *weakly p-harmonic* (or *weakly harmonic* when $p = 2$) iff u is a critical point of E_p with respect to variations in the range, i.e.

$$\frac{d}{dt}\Big|_{t=0} E_p\Bigl[\pi \circ (u + t\varphi)\Bigr] = 0 \quad \text{for each } \varphi \in W_0^{1,p}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n). \tag{1.2}$$

Here, π denotes the nearest point projection of a tubular neighbourhood of the manifold $\mathcal{N}$ onto $\mathcal{N}$.

A standard computation (see e.g. Fuchs [14] or Hélein [27]) shows that condition (1.2) leads to the Euler–Lagrange system

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) \perp T_u\mathcal{N} \qquad \text{in the sense of } \mathcal{D}'(\Omega, \mathbb{R}^n). \tag{1.3}$$

In what follows we consider only the case $\mathcal{N} = \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Condition (1.3) takes then the form

$$\int_\Omega |\nabla u|^{p-2}\nabla u^j \nabla\varphi^j \, dx = \int_\Omega |\nabla u|^p u^j \varphi^j \, dx, \qquad j = 1, \dots, n, \tag{1.4}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Regularity of weakly p-harmonic maps has a long history, dating back to the celebrated paper of Morrey on Plateau's problem. Morrey proved that for $m = p = 2$ mappings which minimize E_2 are of class C^∞. During the 1980's, a more or less complete partial regularity theory has been developed: for arbitrary $m \geq p$, mappings minimizing E_p are regular (i.e. C^∞ for $p = 2$ and $C^{1,\alpha}$ for $p \neq 2$) outside a closed singular set $\mathrm{Sing}\, u$ which has Hausdorff dimension at most $m - [p] - 1$; see Schoen and Uhlenbeck [41] for $p = 2$, and Hardt and Lin [21], Fuchs [11], [12] , and Luckhaus [36] for $p \neq 2$. The estimate of the dimension of the singular set is sharp. Moreover, for $p = 2$, $\mathrm{Sing}\, u$ is countably rectifiable (the proof, see Simon [43], is extremely technical).

When one drops the assumption that u (locally) minimizes E_p, there is no hope for a general partial regularity theory. Rivière [40] has proved that for any smooth nonconstant Dirichlet boundary data $\phi : \partial\mathbb{B}^3 \to \mathbb{S}^2$ there exists an everywhere discontinuous weakly harmonic $u : \mathbb{B}^3 \to \mathbb{S}^2$ with $u = \phi$ on $\partial\mathbb{B}^3$.

However, for $m = p = 2$, on the borderline of Sobolev imbedding of $W^{1,p}(\Omega)$ into Hölder continuous functions, all weakly harmonic maps into arbitrary compact

Riemannian manifolds are smooth. This delicate result has been proved by Hélein in a series of papers [23], [24], [25] (see also Hélein's book [27] and the survey [26]), first for $\mathcal{N} = \mathbb{S}^{n-1}$, then for $\mathcal{N}$ being a homogeneous space, and finally for arbitrary compact targets. The proof was based on the duality of Hardy space and BMO (see next Section for definitions of these spaces).

Extending Hélein's methods, Evans [7] (for round spheres) and then Bethuel [3] (for arbitrary targets) have proved that the $(m-2)$-dimensional Hausdorff measure of the singular set of a stationary harmonic map is equal to zero. The key analytic indregedient of their results is the following.

Theorem 1.1. *Let $m \geq 2$, $\Omega \subset \mathbb{R}^m$. Assume that $\mathcal{N}$ is a compact Riemannian manifold. There exists a number $\varepsilon_0 = \varepsilon_0(m, \mathcal{N}) > 0$ such that if $u \colon \Omega \to \mathcal{N}$ is a weakly harmonic map with $\|u\|_{\mathrm{BMO}} < \varepsilon_0$, then u is of class C^∞.*

Many authors, see e.g. Fuchs [13], Takeuchi [48], Toro and Wang [49], and the author [46, 47], have observed that for symmetric targets this result can be generalized to p-harmonic maps. (There are also generalizations to the subellitpic setting, see [20], [50].)

Theorem 1.2. *Let $m \geq 2$, $\Omega \subset \mathbb{R}^m$, $p \in [2, m]$. There exists a number $\varepsilon_0 = \varepsilon_0(m, n, p) > 0$ such that if $u \in W^{1,p}(\Omega, \mathbb{S}^{n-1})$ is weakly p-harmonic and $\|u\|_{\mathrm{BMO}} < \varepsilon_0$, then u is of class $C^{1,\alpha}$ for some $\alpha > 0$.*

For $\mathcal{N} = \mathbb{S}^{n-1}$ Theorem 1.1 is grounded on the following observation: for $u \in W^{1,2}(\Omega, \mathbb{S}^{n-1})$ the harmonic map system $-\Delta u = |\nabla u|^2 u$ is equivalent to

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0, \quad i, j = 1, \ldots, n. \tag{1.5}$$

The reason behind that is Noether's theorem (see Hélein [27] for an explanation), but one can also give an easy straightforward proof based on the fact that $u \perp \frac{\partial u}{\partial x_k}$.

Now, to interpret (1.5) in the sense of distributions one does not have to know that $|\nabla u| \in L^2$; it is enough to assume $u \in L^\infty$ and $|\nabla u| \in L^1$. This motivates the following definition: $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ is a *generalized harmonic map* iff it is a weak solution of (1.5), i.e.

$$\int_\Omega (u^i \nabla u^j - u^j \nabla u^i) \nabla \varphi \, dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega), i, j = 1, 2, \ldots, n.$$

The map $x/|x|$ solves (1.5) and is of class $W^{1,1}_{\mathrm{loc}}$. (Incidentally: this is the example proposed — for a different system — by Giusti and Miranda in [19], and used in the theory of harmonic maps since Hildebrandt, Kaul and Widman [29].) Thus, generalized harmonic maps can have singularities. However, Almeida [1] has shown that for $\Omega \subset \mathbb{R}^2$ generalized harmonic maps with $\|\nabla u\|_{L^{2,\infty}} < \varepsilon$, where $\varepsilon = \varepsilon(m, n) > 0$, are smooth. Here, $L^{2,\infty}$ denotes the Lorentz space, see [45] for a definition. Ge [15] gave another proof of this result, and has shown that generalized harmonic maps with $|\nabla u| \in L^{2,r}$ for some $r \in [2, \infty)$ also are smooth. Both authors were using an extension of the compensated compactness results from Coifman, Lions, Meyer and Semmes [5] (cf. inequality (2.5) below) to the setting of Lorentz spaces.

Very recently, R. Moser has realised that the condition $\|\nabla u\|_{L^{2,\infty}} < \varepsilon$ was used by Almeida and Ge only to control the BMO norm of u, and extended their result to all dimensions, using the method of reverse Hölder inequalities and a nonlocal construction of test functions, relying on L^p estimates for Hodge decomposition due to Iwaniec and Martin [31]. Roughly speaking, he has shown that each generalized harmonic map $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ is of class C^∞ provided that $\|u\|_{\mathrm{BMO}}$ is small and p is sufficiently close to 2.

Let us now look back at the example of $u(x) = x/|x|$. This map belongs to $W^{1,p}(\mathbb{B}^m, \mathbb{S}^{n-1})$ for *every* $p \in [1, m)$; however, the BMO norm of u is not small (u has mean oscillation 1 on every ball centered at 0). One is tempted to suspect that the lack of regularity is caused *only* by the latter fact, and that the degree of integrability of $|\nabla u|$ plays a secondary role.

We show that this is indeed the case and sharpen Moser's result. It turns out that integrability of $|\nabla u|$ with *any* power > 1 leads to full regularity, if the BMO norm of the map is appropriately small.

Theorem 1.3. *For every $q \in (1, 2]$ there exists a constant $\mu_0 = \mu_0(m, n, q) > 0$ such that each generalized harmonic map $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ with $\|u\|_{\mathrm{BMO}} < \mu_0$ is of class $C^\infty(\Omega)$.*

It would be very interesting to know what is the best value of $\mu_0(m, n, q)$.

The above theorem has the following immediate corollary.

Theorem 1.4. *If a generalized harmonic map $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{S}^{n-1})$ has vanishing mean oscillation and*

$$|\nabla u| \in L^q_{\mathrm{loc}}(\Omega) \textit{ for some } q > 1$$

then $|\nabla u| \in L^2_{\mathrm{loc}}(\Omega)$, and $u \in C^\infty(\Omega)$ is a classical solution of $-\Delta u = |\nabla u|^2 u$.

The proofs rely on a nonlocal construction of test functions via the Hodge decomposition, dating back to Iwaniec [30]. Critical nonlinearities in first order derivatives are estimated in a standard way, using the duality of Hardy space and BMO. We apply this method in a way slightly different from Moser [38], and obtain a reverse Hölder inequality which implies higher integrability of ∇u. What is more important, we show that the whole reasoning may be iterated, and each time the gain of integrability exceeds some fixed treshold level. One has to trace various constants carefully — this, however, is a rather elementary task. In fact, one can use this method to give a new, relatively simple proof of smoothness of harmonic maps into spheres (see the end of Section 3.1 for more details).

There is a version of these results also for $p \neq 2$. To state it, we adopt the following definition: $u \in W^{1,p-1}(\Omega, \mathbb{S}^{n-1})$ is a *generalized p-harmonic map* iff

$$\int_\Omega |\nabla u|^{p-2}(u^i \nabla u^j - u^j \nabla u^i)\nabla\varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega),\ i, j = 1, 2, \ldots, n. \tag{1.6}$$

Note that this condition, meaning formally that $\operatorname{div}\left(|\nabla u|^{p-2}(u^i \nabla u^j - u^j \nabla u^i)\right) = 0$, is equivalent to (1.4) — but only for $u \in W^{1,p}(\Omega, \mathbb{S}^{n-1})$.

Theorem 1.5. *Let $p \in (2, m]$. There exist two numbers $\varepsilon_0 = \varepsilon_0(m, n, p) > 0$ and $\mu_0 = \mu_0(m, n, p) > 0$ such that if $u \in W^{1,q}(\Omega, \mathbb{S}^{n-1})$ is a generalized p-harmonic map and*

$$q \in (p - \varepsilon_0, p), \qquad \|u\|_{\mathrm{BMO}} < \mu_0,$$

then $|\nabla u| \in L^p_{\mathrm{loc}}$ and u is locally Hölder continuous in Ω.

The proof is similar to that of Theorems 1.3 and 1.4. One has to use one more subtle ingredient, namely the stability theorem for Hodge decomposition, due to Iwaniec and Iwaniec–Sbordone (see Section 2.1 below).

The notation throughout the paper is standard. Barred integrals denote averages, i.e. $⨍_A g\,dx = |A|^{-1} \int_A g\,dx$, where $|A|$ is the Lebesgue measure. Sometimes we also write $g_A = ⨍_A g\,dx$. Hölder conjugates of various exponents p, q, s etc. $\in (1, \infty)$ are denoted by p', q', s' etc.; p_* is an exponent for which p is the Sobolev conjugate, i.e. $p_* = mp/(m+p)$ in dimension m. The letter C traditionally denotes a general constant which can change its value even in a single string of estimates. We write $C(a, b, c, \dots)$ when C depends only on $a, b, c, \dots$

2. The tools

To prove the results stated in the introduction, we need three different tools.

2.1. Hodge decomposition: L^p estimates and stability.

Hodge decomposition allows one to write a vector field X as $X = \nabla v + H$, where H is divergence free. In case of $X \in L^2(\mathbb{R}^m, \mathbb{R}^m)$ this is just the orthogonal projection onto gradient fields. In early 1990's, motivated mainly by beautiful applications to quasiconformal and quasiregular mappings, T. Iwaniec has obtained a series of sophisticated L^p estimates for the Hodge decomposition; see e.g. Iwaniec [30], Iwaniec and Martin [31], Iwaniec and Sbordone [34]. One of his main results [30, Thm. 8.1], improved and simplified in [34], is the so-called stability theorem. It expresses in a precise, quantitative way the following naive intuition: for $|\varepsilon| \approx 0$ the vector field $X = |\nabla w|^{\varepsilon} \nabla w$ is close to a gradient, hence its divergence free component should not be too large. Here is a precise statement, adapted to our purposes.

Theorem 2.1. *Let $w \in W^{1,r}(\mathbb{R}^m)$, $r > 1$, $\varepsilon \in (-1, r-1)$. Then there exist $v \in W^{1,r/(1+\varepsilon)}(\mathbb{R}^m)$ and $H \in L^{r/(1+\varepsilon)}(\mathbb{R}^m, \mathbb{R}^m)$ such that*

$$|\nabla w|^{\varepsilon} \nabla w = \nabla v + H, \qquad \operatorname{div} H = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m). \tag{2.1}$$

Moreover,

$$\|H\|_{L^{r/(1+\varepsilon)}} \le C(r)\, |\varepsilon|\, \|\nabla w\|_{L^r}^{1+\varepsilon}. \tag{2.2}$$

Finally, if $s_1 < s_2$ are two fixed numbers in $(1, \infty)$ such that both r and $r/1+\varepsilon$ belong to (s_1, s_2), then

$$C(r) \le \frac{2r(s_2 - s_1)}{(r - s_1)(s_2 - r)}\big(A(s_1) + A(s_2)\big), \tag{2.3}$$

where $A(s)$ stands for the norm of the operator $\mathrm{Id} + (R_{ij})\colon L^s(\mathbb{R}^m, \mathbb{R}^m) \to L^s(\mathbb{R}^m, \mathbb{R}^m)$, $R_{ij} = R_i \circ R_j$ being the second order Riesz transforms, $i, j = 1, \ldots, m$.

The proof, including estimates of the constant $C(r)$, can be found in Iwaniec and Sbordone [34]. Iwaniec and Martin [33] show that $A(s)$ is dimension free.

Since $|\varepsilon| \le \max(1, r-1)$, a trivial application of the triangle inequality and (2.2) yields

$$\|\nabla v\|_{L^{r/(1+\varepsilon)}} \le C(r)\,\max(2, r)\,\|\nabla w\|_{L^r}^{1+\varepsilon}. \tag{2.4}$$

2.2. Hardy space and BMO.

Let us recall that $f \in L^1(\mathbb{R}^m)$ belongs to the *Hardy space* $\mathcal{H}^1(\mathbb{R}^m)$ if and only if

$$f_*\colon = \sup_{\varepsilon>0} |\varphi_\varepsilon * f| \in L^1(\mathbb{R}^m).$$

Here, $\varphi_\varepsilon(x) := \varepsilon^{-m}\varphi(x/\varepsilon)$ for a fixed $\varphi \in C_0^\infty(\mathbb{B}^m)$ with $\varphi \ge 0$ and $\int \varphi(y)\,dy = 1$. The definition does not depend on the choice of φ (see [9]).

Equivalently, one can define $\mathcal{H}^1(\mathbb{R}^m)$ as the space of those $f \in L^1(\mathbb{R}^m)$ for which all the Riesz transforms $R_j f$, $j = 1, 2, \ldots, n$, also belong to $L^1(\mathbb{R}^m)$. The third equivalent definition uses the notion of an atomic decomposition. The reader is referred to [42] and [44, Chapters 3 and 4] for more details; we just mention that $\mathcal{H}^1(\mathbb{R}^m)$ is a Banach space with the norm

$$\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|f_*\|_{L^1},$$

One proves that $\int f(y)\,dy = 0$ for all $f \in \mathcal{H}^1(\mathbb{R}^m)$. This is the primary reason of diverse cancellation phenomena.

Recall that a locally integrable function u belongs to the space of functions of *bounded mean oscillation*, BMO($\mathbb{R}^m$), iff

$$\|u\|_{\mathrm{BMO}}\colon = \sup\left(\fint_B |u(x) - u_B|\,dx\right) < +\infty,$$

the supremum being taken over all balls B in $\mathbb{R}^m$. If, in addition, $\fint_B |u - u_B|\,dx$ tends to zero uniformly as $\operatorname{diam} B \to 0$, then u is said to have *vanishing mean oscillation*, $u \in$ VMO for short. (We also write $\|u\|_{\mathrm{BMO}(\Omega)}$ if the supremum is taken only over balls $B \subset \Omega$ for some fixed domain Ω.) C. Fefferman [8], [9] proved that BMO($\mathbb{R}^m$) is the dual of $\mathcal{H}^1(\mathbb{R}^m)$.

Coifman, Lions, Meyer and Semmes [5] have shown that if $1 < q < \infty$, the vector field $E \in L^q(\mathbb{R}^m, \mathbb{R}^m)$ is divergence free, and $B \in L^{q'}(\mathbb{R}^m, \mathbb{R}^m)$ is rotation free, then the scalar product $E \cdot B$ belongs to $\mathcal{H}^1$. (The Jacobian det Dv of a map $v \in W^{1,m}(\mathbb{R}^m, \mathbb{R}^m)$ serves as a crucial example). Moreover,

$$\|E \cdot B\|_{\mathcal{H}^1} \le C(m,q)\|E\|_q \|B\|_{q'}. \tag{2.5}$$

Combining this inequality with Fefferman's duality theorem, and using standard extension methods, one easily obtains the following local result.

Theorem 2.2. *Let B be a ball compactly contained in $\Omega \subset \mathbb{R}^m$ and $q \in (1,\infty)$. Suppose that $E \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ satisfies* $\operatorname{div} E = 0$ *in the sense of distributions, $u \in W^{1,1}(\Omega, \mathbb{S}^{n-1}) \cap \mathrm{BMO}$, and $w \in W_0^{1,q'}(B)$. Then, for each $i = 1, 2, \dots, n$,*

$$\left| \int_B u^i E \cdot \nabla w \, dx \right| \le C(m,q) \|u\|_{\mathrm{BMO}(B)} \, \|E\|_{L^q(B)} \, \|\nabla w\|_{L^{q'}(B)} \tag{2.6}$$

for a constant $C(m,q)$ that depends only on m and q, and not on the size of B.

Moreover, if one assumes that $q \in [1+\delta_0, 1+1/\delta_0]$ for a fixed number $\delta_0 > 0$, then (2.6) holds with a constant $C(m,\delta_0)$ which does not depend on q.

A uniform estimate $C(m,q) \le C(m,\delta_0)$ for $q \in [1+\delta_0, 1+1/\delta_0]$ follows from a simple analysis of the argument in [5, Section 2]. One has to bound the constants in Hardy–Littlewood maximal theorem (for p in a compact subinterval of $(1,\infty)$) and Sobolev imbedding theorem (for $p_* = mp/(m+p)$ in a compact subinterval of $[1,m)$). Uniform bounds follow from the Riesz–Thorin theorem, or from an elementary analysis of known values of these constants.

2.3. Reverse Hölder inequalities.

Gehring's lemma [16], often used in a local variant given by Giaquinta and Modica [18], is a standard tool applied to prove higher integrability of derivatives of solutions to elliptic equations and systems. (See also Iwaniec's survey [32].) We shall use the following version.

Proposition 2.3. *Let $0 \le f \in L^q_{\mathrm{loc}}(\Omega)$ for some open domain $\Omega \subset \mathbb{R}^m$. Assume there exist two constants b, θ such that*

$$\fint_{B_{\rho/2}} f^q \, dx \le b\left(\fint_{B_\rho} f \, dx\right)^q + \theta \fint_{B_\rho} f^q \, dx$$

for each pair of concentric balls $B_{\rho/2} = B(a, \frac{\rho}{2}) \subset B_\rho = B(a,\rho)$ compactly contained in Ω.

There exists a constant $\theta_0 = \theta_0(q,m)$ such that if $0 \le \theta < \min(\theta_0, b)$, then $f \in L^p_{\mathrm{loc}}(\Omega)$ for all $p < q + \gamma_0$, where $\gamma_0 = \gamma_0(q,m,\theta,b) > 0$.

Moreover, if $q \in (1, 2m]$, then one can take e.g.

$$\theta_0 := \theta_0(m) = 2^{-10m} m^{-2m} > 0, \tag{2.7}$$

and the gain of integrability satisfies

$$\gamma_0 = \gamma_0(m,b,q) > \theta_0 \cdot \frac{q-1}{2b}. \tag{2.8}$$

We refer to Bensoussan and Frehse [2, pages 25–36] for a proof, with cubes instead of balls. A. Zatorska–Goldstein [51] has recently adapted the proof to a general setting of a metric space equipped with a doubling measure, replacing the traditional Calderon–Zygmund subdivision of cubes by a tricky application of Vitali's lemma. The values (not best possible, of course) of θ_0 and γ_0 given above come from her proof. In fact, if q belongs to an arbitrary finite interval $(1, K)$, then one can find a $\theta_0 = \theta_0(m, K)$ so that (2.8) is still satisfied.

3. The proofs

3.1. Generalized harmonic maps.

We begin with the following lemma which provides the crucial tool for the proof of Theorem 1.3 (and Theorem 1.4).

Lemma 3.1. *Fix an arbitrary* $\delta_0 \in (0, 1)$. *There exist two numbers*

$$\gamma_0 = \gamma_0(\delta_0, m, n) > 0, \qquad \mu_0 = \mu_0(\delta_0, m, n) > 0$$

such that each generalized harmonic map $u \in W^{1,1}(\Omega, \mathbb{S}^{n-1})$ *with*

$$|\nabla u| \in L^q, \quad 1 + \delta_0 \le q \le 2, \qquad \|u\|_{\mathrm{BMO}(\Omega)} < \mu_0\,,$$

satisfies $|\nabla u| \in L^{q+\gamma_0}_{\mathrm{loc}}(\Omega)$.

Proof. We use the constraint $|u|^2 = 1$ a.e. to write

$$\nabla u^j = \sum_{i=1}^n u^i E^{ij}, \tag{3.1}$$

where $E^{ij} := u^i \nabla u^j - u^j \nabla u^i$. Next, we multiply this identity by $\nabla\varphi$ and integrate over Ω.

The test function φ is constructed as follows. We fix two concentric balls $B_\rho \subset B_{2\rho} \subset \Omega$, a cutoff function $\zeta \in C_0^\infty(B_{2\rho})$ with $\zeta = 1$ on B_ρ and $|\nabla\zeta| \le C(m)/\rho$, and set $\tilde{u} := \zeta(u - u_{B_{2\rho}})$. Using the Hodge decomposition, we write

$$|\nabla\tilde{u}|^{q-2}\nabla\tilde{u}^j = \nabla v^j + H^j, \qquad j = 1, \dots, n, \tag{3.2}$$

where $\operatorname{div} H^j = 0$ for each j. Applying Theorem 2.1 with $\varepsilon = q - 2$ and $r = q = q'(1 + \varepsilon)$, we check that

$$\begin{aligned} \|\nabla v^j\|_{L^{q'}(\mathbb{R}^m)} + \|H^j\|_{L^{q'}(\mathbb{R}^m)} &\le C\,\|\nabla\tilde{u}\|^{q-1}_{L^q(\mathbb{R}^m)} \\ &\le C\big\|\nabla u\big\|^{q-1}_{L^q(B_{2\rho})} \end{aligned} \tag{3.3}$$

for some constant $C = C(m, \delta_0)$. (One may select $s_1 = 1 + \delta_0/2$, $s_2 = s_1'$ in (2.3). The second line in (3.3) follows from Minkowski and Poincaré inequalities.) We set $\varphi^j = \zeta(v^j - v^j_{B_{2\rho}})$. By Poincaré inequality,

$$\|\nabla\varphi^j\|_{L^{q'}(\mathbb{R}^m)} \le C(m, \delta_0)\|\nabla u\|^{q-1}_{L^q(B_{2\rho})}. \tag{3.4}$$

Since div $E^{ij} = 0$, the estimate of the integral which results from the right hand side of (3.1) follows immediately from Theorem 2.2 and (3.3). Namely,

$$\begin{aligned} \left|\sum_{i=1}^{n}\int_\Omega u^i E^{ij}\nabla\varphi^j\,dx\right| &\le C(m,\delta_0)n\|u\|_{\mathrm{BMO}(\Omega)}\|\nabla u\|_{L^q(B_{2\rho})}\|\nabla\varphi\|_{L^{q'}(B_{2\rho})} \\ &\overset{(3.4)}{\le} C(m,n,\delta_0)\|u\|_{\mathrm{BMO}(\Omega)}\|\nabla u\|^q_{L^q(B_{2\rho})}\,. \end{aligned} \tag{3.5}$$

To deal with the left hand side of (3.1), note that

$$\begin{aligned} \int_\Omega \nabla u^j\nabla\varphi^j\,dx &= \int_\Omega \nabla u^j\Big(\zeta\nabla v^j + (v^j - v^j_{B_{2\rho}})\nabla\zeta\Big)\,dx \\ &= \int_{B_{2\rho}} \nabla\tilde{u}^j\nabla v^j\,dx \\ &\quad + \int_{B_{2\rho}}\big((v^j - v^j_{B_{2\rho}})\nabla u^j - (u^j - u^j_{B_{2\rho}})\nabla v^j\big)\nabla\zeta\,dx \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 :\, = \int_{B_{2\rho}} \nabla\tilde{u}^j(\nabla v^j + H^j)\,dx \ge \int_{B_\rho}|\nabla u|^{q-2}|\nabla u^j|^2\,dx\,, \tag{3.6}$$

and

$$I_2 :\, = \int_{B_{2\rho}}\Big((v^j - v^j_{B_{2\rho}})\nabla u^j - (u^j - u^j_{B_{2\rho}})\nabla v^j\Big)\nabla\zeta\,dx\,. \tag{3.7}$$

Our aim is now to obtain an estimate of integral (3.7) by an integral of $|\nabla u|^s$ with some exponent $s < q$. To this end we apply Hölder and Sobolev inequalities, splitting I_2 into two terms. We have

$$\begin{aligned} |I_{2,1}| &= \left|\int_{B_{2\rho}}(v^j - v^j_{B_{2\rho}})\nabla u^j\nabla\zeta\,dx\right| \\ &\le C\rho^{m-1}\left(\fint_{B_{2\rho}}|v - v_{B_{2\rho}}|^t\,dx\right)^{1/t}\left(\fint_{B_{2\rho}}|\nabla u|^{t'}\,dx\right)^{1/t'} \\ &\le C\rho^m\left(\fint_{B_{2\rho}}|\nabla v|^{t_*}\,dx\right)^{1/t_*}\left(\fint_{B_{2\rho}}|\nabla u|^{t'}\,dx\right)^{1/t'} \\ &\overset{(3.3)}{\le} C\rho^m\left(\fint_{B_{2\rho}}|\nabla u|^{t_*(q-1)}\,dx\right)^{1/t_*}\left(\fint_{B_{2\rho}}|\nabla u|^{t'}\,dx\right)^{1/t'}\,. \end{aligned} \tag{3.8}$$

To choose the exponent t, we consider now two cases.

Case 1. If $q > 1 + \frac{1}{m}$, set $t = qm/(qm - m - 1)$. Then $1 < t' = qm/(m+1) < q$, and $t_* = tm/(t+m) = q'm/(m+1)$. Thus, we have $t_*(q-1) = t'$, and both exponents in (3.8) are equal.

Case 2. If $1 < q \le 1 + \frac{1}{m}$, set $t = (q+1)/(q-1)$, so that $t' = (q+1)/2$. We have then

$$\frac{1}{t_*(q-1)} = \frac{1}{q+1} + \frac{1}{m(q-1)} \ge \frac{1}{q+1} + 1 \ge \frac{2}{q+1} = \frac{1}{t'} \,.$$

It is clear that in both cases $t < C(\delta_0) < \infty$ and one applies Sobolev inequality with $|t_* - m| > \theta(\delta_0) > 0$, i.e. 'far away' from the borderline case $t \to \infty$, $t_* \to m$. Therefore (3.8) holds with a constant $C = C(m, \delta_0)$ independent from the particular value of $q \in [1+\delta_0, 2]$. Thus (applying Hölder inequality in the second case to obtain two identical integrands) we arrive at the estimate

$$|I_{2,1}| \le C(m, \delta_0)\rho^m \left(\fint_{B_{2\rho}} |\nabla u|^{q\lambda}\, dx \right)^{1/\lambda}, \tag{3.9}$$

where

$$\lambda = \max\left(\frac{m}{m+1}, \frac{q+1}{2q}\right) \le \max\left(\frac{m}{m+1}, \frac{2+\delta_0}{2+2\delta_0}\right) =: \lambda_0 < 1\,. \tag{3.10}$$

The second term resulting from I_2, i.e. the integral

$$|I_{2,2}| :\, = \left| \int_{B_{2\rho}} (u^j - u^j_{B_{2\rho}}) \nabla v^j \nabla \zeta \, dx \right| ,$$

does not exceed the right hand side of (3.9). The details of this computation are left to the reader.

Combining the estimates of I_1 and I_2 with (3.5), and summing w.r.t. j, we finally obtain

$$\begin{aligned} \fint_{B_\rho} |\nabla u|^q\, dx \le C_1(m, n, \delta_0) \left(\fint_{B_{2\rho}} |\nabla u|^{q\lambda}\, dx \right)^{1/\lambda} & \\ + C_2(m, n, \delta_0) \|u\|_{\mathrm{BMO}(\Omega)} \fint_{B_{2\rho}} |\nabla u|^q\, dx\,. & \end{aligned} \tag{3.11}$$

Both constants C_1 and C_2 above do not depend on $q \in [1+\delta_0, 2]$ (but they both blow up to ∞ as $\delta_0 \to 0$ since then estimates of singular integral operators in L^p for p near 1 are involved in the Hodge decomposition). Thus, for $\|u\|_{\mathrm{BMO}}$ appropriately small, the assumptions of Proposition 2.3 are satisfied. The lemma follows. It also follows from (2.7)–(2.8) that γ_0, measuring the gain of integrability, depends only on n, m and δ_0. $\square$

The proof of Theorem 1.3 is now standard. Iterating the above lemma finitely many times, one concludes that $|\nabla u| \in L^2$. Thus, u is a weakly harmonic map, and smoothness follows from Theorem 1.1.

In fact, Lemma 3.1 may be used to obtain a relatively simple proof of smoothness of harmonic maps into spheres, different from the one based on arguments by Ladyzhenskaya and Ural'tseva and described by Hélein [27, Chap. 1.5] or Jost

[35, Chap. 8.5]. The advantage is that one does not have to pass through tedious considerations of difference quotients; special structure of the harmonic map system can be exploited instead.

Here is a sketch of this argument. It is clear that the constraint $q \le 2$ plays no significant role in the proof above. What matters is that q does not approach either 1 or ∞; the statement remains unchanged, one only has to adjust the constants. Thus, iterating Lemma 3.1 finitely many times, we conclude that $|\nabla u| \in L^{2r}_{\text{loc}}$ for some $r > m =$ dimension. Since $-\Delta u = |\nabla u|^2 u$, we obtain $u \in W^{2,r}_{\text{loc}}$ by a standard argument using Calderon–Zygmund L^p theory. Hence, $|\nabla u|$ is Hölder continuous by Sobolev imbedding. By bootstrap, smoothness follows now from classical Schauder theory.

3.2. Generalized p-harmonic maps.

Lemma 3.2. *Let $u \in W^{1,p-1}_{\text{loc}}(\Omega, \mathbb{S}^{n-1})$ be a generalized p-harmonic map. There exist two numbers $\varepsilon_0 = \varepsilon_0(m, n, p) > 0$ and $\mu_0 = \mu_0(m, n, p) > 0$ with the following property: if $|\nabla u| \in L^q_{\text{loc}}(\Omega)$ for some $q \in (p - \varepsilon_0, p)$ and $\|u\|_{\text{BMO}} < \mu_0$, then $|\nabla u| \in L^{p+\varepsilon_0}_{\text{loc}}(\Omega)$.*

It is clear that Theorem 1.5 follows immediately from this lemma; one can use the higher integrability of ∇u to apply Theorem 1.2.

Proof of the lemma. As before, we apply the method of reverse Hölder inequalities. Set, for sake of brevity, $E^{ij} = |\nabla u|^{p-2}(u^i \nabla u^j - u^j \nabla u^i)$. Since $|u|^2 = 1$ a.e., we have $u \perp \frac{\partial u}{\partial x_k}$. Hence

$$|\nabla u|^{p-2} \nabla u^j = \sum_{i=1}^{n} u^i E^{ij}, \tag{3.12}$$

We construct test functions, using the Hodge decomposition again. Fix two concentric balls $B_\rho = B(a, \rho) \subset B_{2\rho} = B(a, 2\rho)$ compactly contained in Ω and pick ζ with $\zeta \equiv 1$ on B_ρ, $\operatorname{supp} \zeta \subset B_{2\rho}$ and $|\nabla \zeta| \le C\rho^{-1}$. Set $\tilde{u} = \zeta(u - u_{B_{2\rho}})$. For $\varepsilon = p - q$ we write

$$|\nabla \tilde{u}^j|^{-\varepsilon} \nabla \tilde{u}^j = \nabla v^j + H^j, \qquad j = 1, \dots, n, \tag{3.13}$$

where $\operatorname{div} H^j = 0$ for each j. By Theorem 2.1 and Poincaré inequality, for all exponents $s \in (1, \frac{q}{1-\varepsilon}]$ we have

$$\|H^j\|_{L^s(\mathbb{R}^m)} \le C(m, s)\, \varepsilon\, \|\nabla u\|^{1-\varepsilon}_{L^{s(1-\varepsilon)}(B_{2\rho})}, \tag{3.14}$$

and moreover

$$\|\nabla v^j\|_{L^s(\mathbb{R}^m)} \le C(m, s) \|\nabla u\|^{1-\varepsilon}_{L^{s(1-\varepsilon)}(B_{2\rho})}. \tag{3.15}$$

In the sequel, we shall apply both these inequalities only for $s \in [\frac{m}{m+1} \frac{p-\varepsilon}{1-\varepsilon}, \frac{p-\varepsilon}{1-\varepsilon}]$, where $\varepsilon \in (0, \frac{1}{2})$ is small. It is clear that for such s both (3.14) and (3.15) hold with constants $C(m, p)$ instead of $C(m, s)$.

Multiplying the left-hand side of (3.12) by $\nabla\big(\zeta(v^j - v^j_{B_{2\rho}})\big)$ and integrating over $B_{2\rho}$, we obtain

$$\begin{aligned}
&\int_{B_{2\rho}} |\nabla u|^{p-2}\nabla u^j \nabla(\zeta(v^j - v^j_{B_{2\rho}}))\,dx \\
&\quad= \int_{B_{2\rho}} |\nabla u|^{p-2}\nabla\big(\zeta(u^j - u^j_{B_{2\rho}})\big)\big(\nabla v^j + H^j\big)\,dx \\
&\qquad - \int_{B_{2\rho}} |\nabla u|^{p-2}\nabla \tilde{u}^j\, H^j\,dx \\
&\qquad - \int_{B_{2\rho}} |\nabla u|^{p-2}\big((u^j - u^j_{B_{2\rho}})\nabla v^j - (v^j - v^j_{B_{2\rho}})\nabla u^j\big)\nabla\zeta\,dx \\
&\quad =: I_1 - I_2 - I_3\,.
\end{aligned}$$

We have

$$I_1 = \int_{B_{2\rho}} |\nabla u|^{p-2}|\nabla \tilde{u}^j|^{2-\varepsilon}\,dx \ge \int_{B_\rho} |\nabla u|^{p-2}|\nabla u^j|^{2-\varepsilon}\,dx\,, \tag{3.16}$$

whereas

$$|I_2| \le C\,\varepsilon \int_{B_{2\rho}} |\nabla u|^{p-\varepsilon}\,dx\,. \tag{3.17}$$

(To check this, apply Hölder inequality with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, and next invoke the stability estimate (3.14).) The last integral, I_3, is a lower order term. We bound it, applying Hölder inequality, Sobolev inequality and (3.15) in the following way:

$$\begin{aligned}
|I_{3,1}| :&= \left|\int_{B_{2\rho}} |\nabla u|^{p-2}(u^j - u^j_{B_{2\rho}})\nabla\zeta\,\nabla v^j\,dx\right| \\
&\le C\rho^{m-1}\left(\fint_{B_{2\rho}} |\nabla u|^{(p-2)s_1}\,dx\right)^{1/s_1} \\
&\quad\times\left(\fint_{B_{2\rho}} |u - u_{B_{2\rho}}|^{s_2}\,dx\right)^{1/s_2}\left(\fint_{B_{2\rho}} |\nabla v|^{s_3}\,dx\right)^{1/s_3} \\
&\le C\rho^{m}\left(\fint_{B_{2\rho}} |\nabla u|^{(p-2)s_1}\,dx\right)^{1/s_1} \\
&\quad\times\left(\fint_{B_{2\rho}} |\nabla u|^{s_{2*}}\,dx\right)^{1/s_{2*}}\left(\fint_{B_{2\rho}} |\nabla u|^{s_3(1-\varepsilon)}\,dx\right)^{1/s_3}.
\end{aligned}$$

It is convenient to set $\lambda = \frac{m}{m+1}$ and choose $s_1 = \lambda\frac{p-\varepsilon}{p-2}$, $s_3 = \lambda\frac{p-\varepsilon}{1-\varepsilon}$, since then $s_{2*} = (p-\varepsilon)\lambda$ and all integrands become equal. The second term, $I_{3,2} :\, = I_3 - I_{3,1}$, can be estimated in a similar way. Thus,

$$|I_3| \le C\rho^m\left(\fint_{B_{2\rho}} |\nabla u|^{(p-\varepsilon)\lambda}\,dx\right)^{1/\lambda}. \tag{3.18}$$

To cope with the right hand side of (3.12), note that

$$\left|\sum_{i=1}^{n}\int u^i E^{ij}\nabla\big(\zeta(v^j - v^j_{B_{2\rho}})\big)\,dx\right| \le C\|u\|_{\mathrm{BMO}}\int_{B_{2\rho}}|\nabla u|^{p-\varepsilon}\,dx \tag{3.19}$$

by Theorem 2.2, (3.15) and Poincaré inequality. Gathering the estimates (3.16), (3.17), (3.18), and (3.19), and summing w.r.t. to $j = 1, \dots, n$, we obtain

$$\begin{aligned}\fint_{B_\rho}|\nabla u|^{p-\varepsilon}\,dx \le C_1\left(\fint_{B_{2\rho}}|\nabla u|^{(p-\varepsilon)\lambda}\,dx\right)^{1/\lambda}\\ + C_2(\|u\|_{\mathrm{BMO}}+\varepsilon)\fint_{B_{2\rho}}|\nabla u|^{p-\varepsilon}\,dx\,,\end{aligned}$$

with constants C_1 and C_2 that depend only on m, n, p. Thus, if $\|u\|_{\mathrm{BMO}} + \varepsilon$ is small enough, Proposition 2.3 can be applied, to yield higher integrability of $|\nabla u|$. Iterating the whole reasoning finitely many times, we finish the proof. (Note that each time the gain of integrability does exceed some fixed positive constant.) □

Remark. Contrary to the case $p = 2$, one cannot use this proof to go down with q *almost* to $p - 1$. The problem is caused by the error term

$$\int_{B_{2\rho}}|\nabla u|^{p-2}\nabla\tilde{u}^j\, H^j\,dx\,,$$

which vanishes for $p = 2$. For $p \ne 2$ it must be absorbed via an application of the stability theorem, and this forces $|p - q|$ small.

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