Humps in the Volatility Structure of the Crude Oil Futures Market: New Evidence

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Abstract.
This paper analyzes the volatility structure of the commodity derivatives markets. The model encompasses stochastic volatility that may be unspanned by the futures contracts. A generalized hump-shaped volatility specification is assumed that entails a finite-dimensional affine model for the commodity futures curve and quasi-analytical prices for options on commodity futures. An empirical study of the crude oil futures volatility structure is carried out using an extensive database of futures prices as well as futures option prices spanning 21 years. The study supports hump-shaped, partially spanned stochastic volatility specification. Factor hedging, which takes into account shocks to both the volatility processes and the futures curve, depicts the out-performance of the hump-shaped volatility in comparison to the more popular exponential decaying volatility and the presence of unspanned components in the volatility of commodity futures.

Key words: Commodity derivatives, Crude oil derivatives, Unspanned stochastic volatility, Hump-shaped volatility, Pricing, Hedging

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1 Introduction

Commodity derivatives serve the very important role of helping to manage the volatility of commodity prices. However, those derivatives have their own volatility, of which the understanding and management is of paramount importance. In this paper, we will provide a tractable model for this volatility, and carry out empirical analysis for the most liquid commodity derivative market, namely the crude oil market.

The first contribution of the paper is to develop a model that focuses directly on the volatility of derivatives. The model is set up under the Heath, Jarrow & Morton (1992) framework that treats the entire term structure of futures prices as the primary modelling element. Due to the standard feature of commodity futures prices being a martingale under the risk-neutral measure, the model is completely identified by the volatility of futures prices. We model this volatility as a multifactor stochastic volatility, which may be partially unspanned by the futures contracts. Spot commodity prices and convenience yield are uniquely determined. Option prices can be obtained quasi-analytically and complex derivative prices can be determined via simulation.

Commodity derivatives have been previously studied under the Heath, Jarrow & Morton (1992) framework. However, previous work such as at of Miltersen & Schwartz (1998), Clewlow & Strickland (2000) and Miltersen (2003) were restricted to deterministic volatility. Trolle & Schwartz (2009b) extended the literature significantly by considering unspanned stochastic volatility. However, there are two differences between this paper and Trolle & Schwartz (2009b) paper. First, Trolle & Schwartz (2009b) started by modelling the spot commodity and convenience yield. Convenience yield is unobservable and therefore modelling it adds complexity to model assumptions and estimation. Moreover, sensitivity analysis has to rely on applying shocks to this unobserved convenience yield, which makes it less intuitive. Second, the volatility function in the Trolle & Schwartz (2009b) paper has an exponential decaying form, predicting that long term contracts will always be less volatile than short term contracts. Our model, on the other hand, uses a hump-shaped volatility (which can be reduced to exponential decaying one), and therefore allows for special activities and increasing volatility at the short end of the curve.

The second contribution of the paper is to re-examine the volatility structure of the crude oil derivatives market. We use a rich panel dataset of crude oil futures and options traded on the NYMEX, spanning 21-year from 1 January 1990 to 31 December 2010. We find that a three-factor stochastic volatility model works well. Two of the volatility functions have a hump shape that cannot be captured by the exponential decaying specification. The extent
to which the volatility can be spanned by futures contracts varies over time, with the lowest spanning the recent period of 2006-2010.

The fact that volatility in the market cannot be spanned by the futures contracts highlights the importance of options contracts for hedging purpose. We show that the hedging performance increases dramatically when options contracts are added to the hedging instrument set. The hedging performance is measured under various different factor hedging schemes, from delta-neutral to delta-vega and delta-gamma neutral.

An alternative approach to the HJM framework is by modelling the spot commodity prices directly. A representative list of relevant literature would include Gibson & Schwartz (1990), Litzenberger & Rabinowitz (1995), Schwartz (1997), Hilliard & Reis (1998) and Casassus & Collin-Dufresne (2005). These models have been successful on depicting essential and critical features of distinct commodity market prices, for instance, the mean-reversion of the agricultural commodity market, the seasonality of the natural gas market, the spikes and regime switching of the electricity market and the inverse leverage in the oil market. The disadvantage of the spot commodity models is the requirement to specify and estimate the unobservable convenience yield. The futures prices are then determined endogenously. In addition, spot commodity models in the literature so far have not considered unspanned stochastic volatility.

The paper is organized as follows. Section 2 presents a generalised unspanned stochastic volatility model for pricing commodity derivatives within the HJM framework. Section 3 describes and analyzes the data for crude oil derivatives and explains the estimation algorithm. Section 4 presents the results. Section 5 examines the hedging performance. Section 6 concludes. Technical details are presented at the Appendix.

2 The HJM framework for commodity futures prices

We consider a filtered probability space \((\Omega, \mathcal{A}_T, \mathcal{A}, P), T \in (0, \infty)\) with \(\mathcal{A} = (\mathcal{A}_t)_{t \in [0,T]}\), satisfying the usual conditions.\(^2\) We introduce \(V = \{V_t, t \in [0, T]\}\) a generic stochastic volatility process modelling the uncertainty in the commodity market. We denote as \(F(t, T, V_t)\), the futures price of the commodity at time \(t \geq 0\), for delivery at time \(T\), (for all maturities \(T \geq t\)). Consequently, the spot price at time \(t\) of the underlying commodity, denoted as \(S(t, V_t)\) satisfies \(S(t, V_t) = F(t, t, V_t), t \in [0, T]\). The futures price process is

\(^2\)The usual conditions satisfied by a filtered complete probability space are: (a) \(\mathcal{F}_0\) contains all the \(\mathbb{P}\)-null sets of \(\mathcal{F}\) and (b) the filtration is right continuous. See Protter (2004) for technical details.
equal to the expected future commodity spot price under an equivalent risk-neutral probability measure \( Q \), see Duffie (2001), namely

\[
F(t, T, V_t) = \mathbb{E}^Q[S(T, V_T)|\mathcal{A}_t].
\]

This leads to the well-known result that the futures price of a commodity is a martingale under the risk-neutral measure, thus the commodity futures price process follows a driftless stochastic differential equation. Let \( \mathcal{A}_t \) be an \( n \)-dimensional Wiener process driving the commodity futures prices and \( \mathcal{A}_t \) be the \( n \)-dimensional Wiener process driving the stochastic volatility process \( V_t \), for all \( t \in [0, T] \).

Assumption 2.1 The commodity futures price process follows a driftless stochastic differential equation under the risk-neutral measure of the form

\[
\frac{dF(t, T, V_t)}{F(t, T, V_t)} = \sum_{i=1}^{n} \sigma_i(t, T, V_t) dW_i(t),
\]

where \( \sigma_i(t, T, V_t) \) are the \( \mathcal{A} \)-adapted futures price volatility processes, for all \( T > t \). The volatility process \( V_t = \{V^1_t, \ldots, V^n_t\} \) is an \( n \)-dimensional well-behaved Markov process evolving as

\[
dV^i_t = a^V_i(t, V_t) dt + \sigma^V_i(t, V_t) dW^V_i(t),
\]

for \( i = 1, \ldots, n \), where \( a^V_i(t, V_t), \sigma^V_i(t, V_t) \) are \( \mathcal{A} \)-adapted stochastic processes and

\[
\mathbb{E}^Q[dW^i_t \cdot dW^j_t(t)] = \begin{cases} 
\rho_{ij} dt, & i = j; \\
0, & i \neq j.
\end{cases}
\]

Assume that all the above processes are \( \mathcal{A} \)-adapted bounded processes with drifts and diffusions to be regular and predictable so that the proposed SDEs admit unique strong solutions. The proposed volatility specification asserts naturally the feature of unspanned stochastic volatility in the model. The correlation structure of these innovations determines the extent in which the stochastic volatility is unspanned. If the Wiener processes \( W_i(t) \) are uncor-

\footnote{We essentially assume that the filtration \( \mathcal{A}_t \) includes \( \mathcal{A}_t = \mathcal{A}_t^f \vee \mathcal{A}_t^V \), where

\[
(\mathcal{A}_t^f)_{t \geq 0} = \{\sigma(W(s) : 0 \leq s \leq t)\}_{t \geq 0},
\]

\[
(\mathcal{A}_t^V)_{t \geq 0} = \{\sigma(W^V(s) : 0 \leq s \leq t)\}_{t \geq 0}.
\]
related with $W^V(t)$ then the volatility risk is unhedgeable by futures contracts. When the Wiener processes $W_i(t)$ are correlated with $W^V(t)$, then the volatility risk can be partially spanned by the futures contracts. Thus the volatility risk (and consequently options on futures contracts) cannot be completely hedged by using only futures contracts.

Conveniently, the system (2.1) and (2.2) can be expressed in terms of independent Wiener processes. By considering the $-\text{dimensional independent Wiener processes}$ $W^1(t) = W(t)$ and $W^2(t)$, then one possible representation is

$$\frac{dF(t, T, V_t)}{F(t, T, V_t)} = \sum_{i=1}^{n} \sigma_i(t, T, V_t) dW^1_i(t), \quad (2.4)$$

$$dV^1_t = a^V_i(t, V^i_t)dt + \sigma^V_i(t, V^i_t) \left( \rho_i dW^1_i(t) + \sqrt{1 - \rho^2_i} dW^2_i(t) \right). \quad (2.5)$$

Clearly, volatility risk of any volatility factors $V^i_t$ with $\rho_i = 0$ cannot be spanned by futures contracts.

Let $X(t, T) = \ln F(t, T, V_t)$ be the logarithm of the futures prices process, then from (2.4) and an application of Ito’s formula, it follows that

$$dX(t, T) = -\frac{1}{2} \sum_{i=1}^{n} \sigma^2_i(t, T, V_t) dt + \sum_{i=1}^{n} \sigma_i(t, T, V_t) dW_i(t). \quad (2.6)$$

**Lemma 2.2** Under the Assumption 2.1 for the commodity futures price dynamics, the commodity spot prices satisfy the SDE

$$\frac{dS(t, V_t)}{S(t, V_t)} = \zeta(t) dt + \sum_{i=1}^{n} \sigma_i(t, t, V_t) dW_i(t), \quad (2.7)$$

with the instantaneous spot cost of carry $\zeta(t)$ satisfying the relationship

$$\zeta(t) = \frac{\partial}{\partial t} \ln F(0, t) - \frac{1}{2} \sum_{i=1}^{n} \sigma^2_i(t, t, V_t)$$

$$- \sum_{i=1}^{n} \int_0^t \sigma_i(u, t, V_u) \frac{\partial}{\partial t} \sigma_i(u, t, V_u) du + \sum_{i=1}^{n} \int_0^t \frac{\partial}{\partial t} \sigma_i(u, t, V_u) dW_i(u). \quad (2.8)$$

**Proof:** See Appendix A.
This commodity HJM model is Markovian in an infinite dimensional state space due to the fact that the futures price curve is an infinite dimensional object (one dimension for each maturity \( T \)). In addition, the path dependent nature of the integral terms in the drift (2.8) of the commodity spot prices illustrate the infinite dimensionality of the proposed model.

2.1 Finite Dimensional Realisations for a Commodity Forward Model

We specify functional forms for the futures price volatility functions \( \sigma_i(t, T, V_t) \) that will allow the proposed commodity forward model to admit finite dimensional realisations (FDR).

**Assumption 2.3** The commodity futures price volatility functions \( \sigma_i(t, T, V_t) \) are of the form

\[
\sigma_i(t, T, V_t) = \alpha_i(t, V_t) \varphi_i(T - t),
\]

(2.9)

where \( \alpha_i : \mathbb{R}^+ \rightarrow \mathbb{R} \) are \( \mathcal{F} \)-adapted square-integrable stochastic processes and \( \varphi_i : \mathbb{R} \rightarrow \mathbb{R} \) are quasi exponential functions. A quasi-exponential function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) has the general form

\[
\varphi(x) = \sum_i e^{m_i x} + \sum_j e^{n_j x}[p_j(x) \cos(k_j x) + q_j(x) \sin(k_j x)],
\]

(2.10)

where \( m_i, n_i \) and \( k_i \) are real numbers and \( p_j \) and \( q_j \) are real polynomials.

These very general volatility specifications have been proposed in Björk, Landén & Svensson (2004) and can be adapted for commodity forward models. Björk, Landén & Svensson (2004) have demonstrated, by employing methods of Lie algebra, that this functional form is a necessary condition for a forward interest rate model with stochastic volatility to admit FDR. Chiarella & Kwon (2000) have also investigated volatility functions to obtain FDR under stochastic volatility. The volatility conditions on the commodity futures prices through the \( \alpha_i \) can depend on stochastic state variables, for instance the spot commodity price \( S(t) \) and stochastic volatility. In the spirit of Chiarella & Kwon (2001b) and Björk, Landén & Svensson (2004), \( \alpha_i \) may also depend on a finite set of commodity futures prices with fixed tenors. When level dependent (or constant direction) volatility is considered, it becomes very difficult to obtain tractable analytical solutions for futures option prices. For this reason, even though FDR can be obtained for a level dependent stochastic volatility model (clearly with a higher dimensional state space), we consider the dependence of \( \alpha_i \) only on stochastic
volatility.

These volatility specifications have the flexibility of generating a wide range of shapes for the futures price volatility surface. Some typical examples of interest rate volatility curves include, the exponentially declining stochastic volatility structures of the Ritchken & Sankarabrahmanian (1995), and the hump-shaped volatility structures discussed in Chiarella & Kwon (2001a) and Trolle & Schwartz (2009a), which are special cases of these general specifications. Furthermore some special examples of commodity volatility curves include the exponentially declining stochastic volatility structures of Trolle & Schwartz (2009b) and the gas volatility structures following a regular pattern as discussed in Björk, Blix & Landen (2006). Note that the latter authors do not consider a stochastic volatility model.

2.2 Hump-Shaped Unspanned Stochastic Volatility

We propose next certain volatility specifications within the general functional form (2.9) which are not only multi-factor stochastic volatility of Heston (1993) type but also allow for humps.

**Assumption 2.4** The commodity futures price volatility processes \( \sigma_i(t, T, V_t) \) are of the form

\[
\sigma_i(t, T, V_t) = (\kappa_0 + \kappa_i(T - t)) e^{-\eta_i(T-t)} \sqrt{V_t},
\]  

(2.11)

where \( \kappa_0, \kappa_i \) and \( \eta_i \) are constants.

When the commodity futures prices volatilities are expressed in this functional form then finite dimensional realisations of the state space are possible.

**Proposition 2.5** Under the volatility specifications of Assumption 2.4, the logarithm of the instantaneous futures prices at time \( t \) with maturity \( T \), namely \( \ln F(t, T, V_t) \), is expressed in terms of \( 6n \) state variables as

\[
\ln F(t, T, V_t) = \ln F(0, T, V_0) - \sum_{i=1}^{n} \left( \frac{1}{2} (\gamma_{i1}(T - t) \phi_i(t) + \gamma_{i2}(T - t) \psi_i(t)) + (\beta_{i1}(T - t) \phi_i(t) + \beta_{i2}(T - t) \psi_i(t)) \right),
\]  

(2.12)
futures contracts can be priced. These results are summarised in the following proposition.

The price of options on futures can be obtained in closed form as a tractable expression for the characteristic function exists. By employing Fourier transforms, call and put options on futures contracts can be priced. These results are summarised in the following proposition.

where for $i = 1, 2, \ldots, n$

$$\beta_{i1}(T - t) = (\kappa_{0i} + \kappa_{i}(T - t))e^{-\eta_{i}(T-t)}, \quad (2.13)$$

$$\beta_{i2}(T - t) = \kappa_{i}e^{-\eta_{i}(T-t)}, \quad (2.14)$$

$$\gamma_{i1}(T - t) = \beta_{i1}(T - t)^2, \quad (2.15)$$

$$\gamma_{i2}(T - t) = 2\beta_{i1}(T - t)\beta_{i2}(T - t), \quad (2.16)$$

$$\gamma_{i3}(T - t) = \beta_{i2}(T - t)^2. \quad (2.17)$$

The state variables $x_i(t), y_i(t), z_i(t), \phi_i(t)$ and $\psi_i(t), i = 1, 2, \ldots, n$ evolve according to

$$dx_i(t) = (-2\eta x_i(t) + V^i_t)dt,$$

$$dy_i(t) = (-2\eta y_i(t) + x_i(t))dt,$$

$$dz_i(t) = (-2\eta z_i(t) + 2y_i(t))dt,$$

$$d\phi_i(t) = -\eta \phi_i(t)dt + \sqrt{V^i_t}dW_i(t),$$

$$d\psi_i(t) = (-\eta \psi_i(t) + \phi_i(t))dt, \quad i = 1, 2, 3. \quad (2.18)$$

subject to $x_i(0) = y_i(0) = z_i(0) = \phi_i(0) = \psi_i(0) = 0$. The above-mentioned $5n$ state variables are associated with the stochastic volatility process $V_t = \{V^1_t, \ldots, V^n_t\}$ which is assumed to be an $n$-dimensional of Heston (1993) type process such as

$$dV^i_t = \mu^V_i(\nu^V_i - V^i_t)dt + \varepsilon^V_i \sqrt{V^i_t}dW^V_i(t), \quad (2.19)$$

where $\mu^V_i, \nu^V_i, \text{ and } \varepsilon^V_i$ are constants (they can also be deterministic functions).

Proof: See Appendix B for technical details.

Note that the model admits FDR within the affine class of Duffie & Kan (1996). Additionally, the model is consistent, by construction, with the currently observed futures price curve, consequently it is a time-inhomogeneous model. However for estimation purposes, it is necessary to reduce the model to a time-homogeneous one as it is presented in Section 3.2. Note that in Andersen (2010), the proposed volatility conditions lead to time-inhomogeneous models, which cannot directly be applied for estimation purposes.

The price of options on futures can be obtained in closed form as a tractable expression for the characteristic function exists. By employing Fourier transforms, call and put options on futures contracts can be priced. These results are summarised in the following proposition.

where for $i = 1, 2, \ldots, n$
which is a natural extensions of existing literature and are quoted here for completeness.

**Proposition 2.6** Under the stochastic volatility specifications (2.19) and for \( t \leq T_0 \leq T \), the transform \( \phi(t; \omega, T_0, T) =: \mathbb{E}_t[\exp\{\omega \ln F(T_0, T, V_{T_0})\}] \) is expressed as

\[
\phi(t; \omega, T_0, T) = \exp\{M(t; \omega, T_0) + \sum_{i=1}^{n} N_i(t; \omega, T_0)V_t^i + \omega \ln F(t, T, V_t)\}, \tag{2.20}
\]

where \( M(t) = M(t; \omega, T_0) \) and for \( i = 1, \ldots, n \), \( N_i(t) = N_i(t; \omega, T_0) \) satisfy the Ricatti ordinary differential equations

\[
\frac{dM(t)}{dt} = - \sum_{i=1}^{n} \mu_i^Y \nu_i^Y N_i(t), \tag{2.21}
\]

\[
\frac{dN_i(t)}{dt} = \frac{\omega^2 - \omega}{2} \left( \varphi_i \right)^2 - \left( \varepsilon_i^Y \omega \rho_i \varphi_i - \mu_i^Y \right) N_i(t) - \frac{1}{2} \varepsilon_i^Y N_i^2(t), \tag{2.22}
\]

subject to the terminal conditions \( M(T_0) = N_i(T_0) = 0 \), where \( \varphi_i = (\kappa_i + \kappa_i(T - t))e^{-\kappa_i(T-t)} \).

The price at time \( t \) of a European put option maturing at \( T_0 \) with strike \( K \) on a futures contract maturing at time \( T \), is given by

\[
\mathcal{P}(t, T_0, T, K) = \mathbb{E}_t^Q[\exp\left\{-\int_{T_0}^{T} r_s ds \ (K - F(T_0, T))^+\right\} = P(t, T_0)[KG_{0,1}(\log(K)) - G_{1,1}(\log(K))] \tag{2.23}
\]

where \( P(t, T_0) \) is the price of a zero-coupon bond maturing at \( T_0 \) and \( G_{a,b}(y) \) is given by

\[
G_{a,b}(y) = \frac{\phi(a, T_0, T) - 1}{\pi} \int_0^{\infty} \frac{\text{Im}[\phi(a + i\nu, T_0, T)e^{-\nu y}]}{\nu} d\nu. \tag{2.24}
\]

Note that \( i^2 = -1 \).

**Proof:** Follows along the lines of Duffie, Pan & Singleton (2000) and Collin-Dufresne & Goldstein (2002). Technical details on the characteristic function are also presented in Appendix C.

For the market price of volatility risk, a “complete” affine specification is assumed, see Doran & Ronn (2008) (where they have shown that market price of volatility risk is negative) and
in particular Dai & Singleton (2000). Accordingly, the market price of risk is specified as,

\[ dW^P_i(t) = dW_i(t) - \lambda_i \sqrt{V_i} dt, \]
\[ dW^{PV}_i(t) = dW^V_i(t) - \lambda^V_i \sqrt{V^V_i} dt, \tag{2.25} \]

for \( i = 1, \ldots, n \), where \( W^P_i(t) \) and \( W^{PV}_i(t) \) are Wiener processes under the physical measure \( \mathbb{P} \). Note that under these specifications, the model parameters are \( 9n \), namely; \( \lambda_i, \lambda^V_i, \kappa_0, \kappa_i, \eta_i, \mu^V_i, \nu^V_i, \sigma^V_i, \rho_i \) which we will estimate next by fitting the proposed model to crude oil derivative prices.

## 3 Data and estimation method

### 3.1 Data

We estimate the model using an extended dataset of crude oil futures and options traded on the NYMEX\(^4\). The database spans 21-year from 1 January 1990 to 31 December 2010. This is one of the richest databases available on commodity derivatives. In addition, over this period, noteworthy financial market events with extreme market movements, for instance the oil price crisis in 1990 and the financial crisis in 2008, have occurred.

Throughout the sample period, the number of available futures contracts with positive open interest per day has increased from 17 on 1\(^{st}\) of January 1990 to 67 on 31\(^{st}\) of December 2010. The maximum maturity of futures contracts with positive open interest has also increased from 499 (calendar) days to 3128 days. Figure 3.1 and Figure 3.2 plot the futures prices of the sample period on Wednesdays. We can see that the price surfaces are very different during different periods. The maximum futures price was US$40 per barrel in 1990 reaching US$140 per barrel in 2008.

Given the large number of available futures contracts per day, we make a selection of contracts for estimation purposes based on their liquidity. Liquidity has increased across the sample. For instance, the open interest for the futures contract with 6 months to maturity has increased from 13,208 to 38,766 contracts. For contracts with less than 14 days to expiration, liquidity is very low, while for contracts with more that 14 days to expiration, liquidity increases significantly. We begin with the first seven monthly contracts, near to the trade date, namely \( m_1, m_2, m_3, m_4, m_5, m_6, \) and \( m_7 \). Note that the first contract should have

\(^4\)The database has been provided by CME.
more than 14 days to maturity. After that liquidity is mostly concentrated in the contracts expiring in March, June, September and December. Thus the first seven monthly contracts are followed by the three contracts which have either March, June, September or December expiration months. We name them $q_1$, $q_2$ and $q_3$. Beyond that, liquidity is concentrated in December contracts only, therefore the next five December contracts, namely $y_1$, $y_2$, $y_3$, $y_4$ and $y_5$, are included. As a result, the total number of futures contracts to be used in our analysis is 70,735, with the number of contracts to be used on a daily basis varying between a maximum of 15 and a minimum of 8.

Figure 3.1: Futures curve - 1 January 1990 to December 1999

Figure 3.2: Futures curve - 1 January 2000 to December 2010

With regard to option data, we consider the options on the first ten futures contracts only, namely the futures contracts $m_1$-$m_7$ and $q_1$-$q_3$. We avoid the use of longer maturities because in the proposed model we have not taken into account stochastic interest rates. Due to this model constraint, the option pricing equation (2.23) is not very accurate for longer maturities. Furthermore, the option pricing equation (2.23) provides the price for European options, not
American options that are the options of our database. For the conversion of American prices to European prices, including the approximation of the early exercise premium, we follow the same approach proposed by Broadie, Chernov & Johannes (2007) for equity options and applied by Trolle & Schwartz (2009b) for commodity options.

For each option maturity, we consider six moneyness intervals, 0.86—0.90, 0.91—0.95, 0.96—1.00, 1.01—1.05, 1.06—1.10, 1.11—1.15. Note that moneyness is defined as option strike divided by the price of the underlying futures contract. In each moneyness interval, we use only the out-of-the-money (OTM) and at-the-money (ATM) options that are closest to the interval mean. OTM options are generally more liquid and we also benefit by a reduction in the errors that occurred in the early exercise approximation.

Based on this selection criteria, we consider 433,137 option contracts over the 21 years, with the daily range varying between 29 and 100 contracts (per trading day). Note that the total number of trading days where both futures and options data are available is 5272. Figure 3.3 displays the ATM implied volatilities for options on the first ten futures contracts.

Figure 3.3: Implied Volatilities
3.2 Estimation method

The estimation approach is quasi-maximum likelihood in combination with the extended Kalman filter. The model is cast into a state-space form, which consists of the system equations and the observation equations.

For estimation purposes, a time-homogeneous version of the model (2.12) is considered, by assuming for all \( T, F(0, T) = f_o \), where \( f_o \) is a constant representing the long-term futures rate (at infinite maturity). This constant is an additional parameter that is also estimated subsequently.

The system equations describe the evolution of the underlying state variables. In our case, the state vector is \( X_t = \{ X_t^i, i = 1, 2, \ldots, n \} \) where \( X_t^i \) consists of six state variables: \( x_i(t), y_i(t), z_i(t), \psi_i(t), \phi_i(t) \) and \( V_i(t) \). The continuous time dynamics (under the physical probability measure) of these state variables are defined by (2.18), (2.19) and (2.25). The corresponding discrete evolution is

\[
X_{t+1} = \Phi_0 + \Phi_X X_t + w_{t+1}, \quad w_{t+1} \sim iid \ N(0, Q_t), \tag{3.26}
\]

where \( \Phi_0, \Phi_X \) and \( Q_t \) can be computed in closed form. Details can be found in Appendix D.

The observation equations describe how observed options and futures prices are related to the state variables

\[
z_t = h(X_t) + u_t, \quad u_t \sim iid \ N(0, \Omega). \tag{3.27}
\]

In particular, log futures prices are linear functions of the state variables (as described in (2.12)) and the options prices are nonlinear functions of the state variables (as described in (2.23) and (2.24)).

3.3 Other consideration

3.3.1 Number of stochastic factors

The number of driving stochastic factors affecting the evolution of futures curve can be determined by performing a principal component analysis (PCA) of futures price returns. Table 3.1 and Table 3.2 show that the crude oil futures markets have become more sophisticated. In the 1990s, two factors are sufficient to capture the evolution of the futures prices,
whereas in 2000s, three factors are needed. We therefore will estimate 3-factor models for all sample periods in our empirical analysis and check their performance against corresponding 2-factor models.

<table>
<thead>
<tr>
<th>Time Period</th>
<th>One factor</th>
<th>Two factor</th>
<th>Three factor</th>
<th>Four factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990 - 1999</td>
<td>0.9042</td>
<td>0.9822</td>
<td>0.9972</td>
<td>0.9995</td>
</tr>
<tr>
<td>1990 - 1994</td>
<td>0.9056</td>
<td>0.9805</td>
<td>0.9961</td>
<td>0.9995</td>
</tr>
<tr>
<td>1995 - 1999</td>
<td>0.8913</td>
<td>0.9667</td>
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Table 3.1: Accumulated percentage of contribution.

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<th>Time Period</th>
<th>One factor</th>
<th>Two factor</th>
<th>Three factor</th>
<th>Four factor</th>
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<tr>
<td>2000 - 2010</td>
<td>0.8761</td>
<td>0.9402</td>
<td>0.9719</td>
<td>0.9928</td>
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<td>2000 - 2005</td>
<td>0.8229</td>
<td>0.9059</td>
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<td>2006 - 2010</td>
<td>0.9275</td>
<td>0.9715</td>
<td>0.9887</td>
<td>0.9977</td>
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</tbody>
</table>

Table 3.2: Accumulated percentage of contribution.

### 3.3.2 The discount function

The discount function $P(t, T)$ is obtained by fitting a Nelson & Siegel (1987) curve each trading day to LIBOR and swap data consisting of 1-, 3-, 6-, 9- and 12-month LIBOR rates and the 2-year swap rate, similar to Trolle & Schwartz (2009b).

Let $f(t, T)$ denote the time–$t$ instantaneous forward interest rate to time $T$. Nelson and Siegel (1987) parameterize the forward interest rate curve as

$$f(t, T) = \beta_0 + \beta_1 e^{-\theta(T-t)} + \beta_2\theta(T-t)e^{-\theta(T-t)} \tag{3.28}$$

from which we can price LIBOR and swap rates. This also yields the following expression for zero-coupon bond prices

$$P(t, T) = \exp \left\{ \beta_0(T-t) + (\beta_1 + \beta_2) \frac{1}{\theta} \left( 1 - e^{-\theta(T-t)} \right) + \beta_2(T-t)e^{-\theta(T-t)} \right\}. \tag{3.29}$$

On each observation date, we recalibrate the parameters $\beta_0, \beta_1, \beta_2$ and $\theta$, by minimizing the mean squared percentage differences between the model implied forward rates (3.28) and the observed LIBOR and swap curve consisting of the 1-, 3-, 6-, 9- and 12-month LIBOR rates and the 2-year swap rate on that date.
3.3.3 Computational details

The loglikelihood function is maximised by using the NAG optimization library. We begin with several different initial hypothetical parameter values, firstly on monthly data, then on weekly data and finally on daily data, aimed to obtain global optima.

The ODE’s (2.21) and (2.22) are solved by a standard fourth-order Runge-Kutta algorithm. The integral in (2.24), is approximated by the Gauss-Legendre quadrature formula with 30 integration points and truncating the integral at 400.

4 Empirical Results

4.1 Parameter Estimation

Parameter estimates for our three-factor unspanned stochastic volatility model can be found in Table 4.3 and Table 4.4. Estimation is carried out for four different subsamples due to the marked difference in their price behaviour, as can be seen in Figure 3.1 and Figure 3.2.

One of the main improvements of our models compared to the existing ones in the literature is that we allow for a hump shape in the volatility curve. The usual exponential decaying specification implies that volatility should die out as the maturity of futures increases, whereas the hump-shaped volatility allows volatility to increases first then decrease. The combination of different hump-shaped curves can result in a rich pattern for volatility behaviour. From the parameter estimates, the significance of $\kappa$ in all subsamples indicates the existence of this hump shape. Figure 4.4 shows the shape of each volatility component and the total volatility of the futures prices. The hump of the first volatility factor occurs at around 20 months to maturity, whereas the second volatility factor is increasing at the relevant maturity range. The third volatility factor, on the other hand, can be captured by an exponential decay function. Table 4.5 shows the contribution of each volatility factor to the total variance. The first two factors, where the hump shape is present, are very significant.

All of the volatility factors are highly persistent (evidenced by the very low value of $\mu$), suggesting that they are important for the pricing of futures and options of all maturities. For

---

5We also estimate the corresponding 2-factor models for each sub-sample. The likelihood ratio tests strongly reject the 2-factor models in favour of the 3-factor ones.
Table 4.3: 1990 - 1999, three factor model

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
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<tr>
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Table 4.4: 2000 - 2010, three factor model.

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<td>-10502.37</td>
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</tr>
</tbody>
</table>

each of the subsamples, the innovation to at least one of the volatility factors has a very low correlation (around 5%) with the innovations to the futures prices, implying the large extent to which the volatility is unspanned by the futures contracts.

We note that there were three major events that affect the volatility of the crude oil market, namely the Gulf War 1990-1991, the Iraq War 2003 and the Global Financial Crisis 2008. The implied volatility especially for short-dated options increased by more than 100% over the 1991 and 2003 crises while implied volatilities for both short-dated and long-dated options increased by 90% and 50%, respectively, over the 2008 crisis. Furthermore the effect of the shock to the implied volatility was more persistent over the 2008 crisis. We certainly expect the parameter estimates to be affected by these extreme market conditions. Nevertheless, Figure 4.4 shows that our estimates did pick up some, if not all, of these effects.
4.2 Pricing performance

Figure 4.5 graphs the RMSEs of the percentage differences between actual and fitted futures prices as well as of the difference between actual and fitted implied option volatilities, whereas Table 4.6 gives the average values. The overall fitness is very good, except during the special events of 1991, 2003 and 2008. Table 4.6 also compares the fitness of the hump-shaped volatility specification to the exponential decay specification. The log likelihood ratio tests clearly favour the hump volatility specification. The improvement for the fit of futures prices averages at 2.5%. The improvement for the fit of option implied volatility is not much for the period 1995-2005, but very significant during the period of 1990-1994 and 2006-2010 at 4.05% and 10.23% respectively.

<table>
<thead>
<tr>
<th>Subsample</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_3$</th>
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<td>1990 - 1994</td>
<td>23.69%</td>
<td>69.61%</td>
<td>6.70%</td>
</tr>
<tr>
<td>1995 - 1999</td>
<td>2.51%</td>
<td>84.81%</td>
<td>12.68%</td>
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<td>2000 - 2005</td>
<td>22.55%</td>
<td>61.41%</td>
<td>16.04%</td>
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<tr>
<td>2006 - 2010</td>
<td>30.95%</td>
<td>68.96%</td>
<td>0.08%</td>
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Table 4.5: Contribution of each volatility factor to the total variance.

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<th>improvement compared to exponential decaying</th>
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<td>option</td>
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<tr>
<td>1990 - 1994</td>
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</table>

Table 4.6: Fitness of the model - RMSEs

RMSEs for futures are the percentage differences between actual and fitted futures prices. RMSEs for options are the differences between actual and fitted implied option volatilities.

Figure 4.6 displays time series of implied volatilities and fit to the three-factor model. There were a lot of fluctuations on the implied volatilities over the last 21 years. The model does well in capturing these changes, as well as the special periods of the Gulf War 1990-1991, the Iraq War 2003 and the Global Financial Crisis 2008. The only exception is spike of the 16th November 2001 that our model cannot match.
5 Hedging Performance

To gauge the impact of the hump-shaped volatility specification compared to exponential decaying only volatility specifications, we assess the hedging performance of option portfolios on crude oil futures by using the hedge ratios implied by the corresponding models. The various factors of the model manifested by the empirical analysis represent different dimensions of risk to which a portfolio of oil derivatives is exposed. In our stochastic volatility modelling framework, the variation in the crude oil forward curve is instigated by random changes of these forward curve volatility factors as well as random changes in a general stochastic volatility factor. By extending the traditional factor hedging method to accommodate the stochastic volatility specifications, a set of futures and futures options are used to hedge the risk associated to the forward curve variation. The technical details of the extended factor hedging are presented next.

5.1 Factor hedging for a multi-factor stochastic volatility model

Factor hedging is a broad hedging method that allows to hedge simultaneously the multiple factors impacting the forward curve of commodities and subsequently the value of commodity derivative portfolios. By considering the $n$ factor stochastic volatility model developed in Section 2, the forward curve should be shocked by each of the $n$ forward curve volatility processes. However, by using a stochastic volatility model, initially an appropriate shock to the variance process is applied, see equation (2.2),

$$
\Delta V^i_t = a_i^V(t, V_t) \Delta t + \sigma_i^V(t, V_t) \Delta W^V_i; \quad i = 1, \ldots, n.
$$

(5.30)

Then, a shock to each volatility factor of the multi-factor model (2.1) is applied, namely for $i = 1, \ldots, n$,

$$
\Delta F_i(t, T, V_t) = F(t, T, V_t) \sigma_i(t, T, V_t) \Delta W_i,
$$

(5.31)

where $\Delta W_i$ is specified through its correlation structure with $\Delta W^V_i$, as given by (2.3). By allowing for both positive and negative changes, the corresponding shocks to the forward curve are obtained. The size of the shocks $\Delta W_i$ and $\Delta W^V_i$ should be chosen to give a typical movement of the curve and the variance over the hedging period, respectively. If $\Upsilon$ denotes the value of a portfolio, then the changes $\Delta \Upsilon_i$ in the value of the portfolio between the downward and upward shifts of the forward curve for each volatility factor $i$ are computed.
\[ \Delta \Upsilon_i = \Upsilon(F_{i,U}(t, T, \mathbf{V}_t)) - \Upsilon(F_{i,D}(t, T, \mathbf{V}_t)); \ i = 1, \ldots, n, \]  

(5.32)

where the subscript $U$ indicates an up movement of the forward curve embedding the impact of the change in the variance and subscript $D$ indicates the corresponding down movement of the forward curve.

### 5.1.1 Delta Hedging

For an $n$ factor model, factor delta hedging necessitates $n$ hedging instruments. The hedging instruments could be futures contracts or options contracts, yet with different maturities, thus their values are denote as $\Psi(t, T_j)$ for $j = 1, \ldots, n$. By selecting appropriate positions $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ in these hedging instruments such that, for each factor, the change in the hedged portfolio $\Upsilon$ is zero, the following conditions are obtained, for $i = 1, \ldots, n$,

\[ \Delta \Upsilon_{H,i} = \Delta \Upsilon_i + \delta_1 \Delta \Psi_i(t, T_1) + \delta_2 \Delta \Psi_i(t, T_2) + \ldots + \delta_n \Delta \Psi_i(t, T_n) = 0. \]  

(5.33)

The system of equations (5.33) is a system of $n$ linear equations with $n$ unknowns and the $\delta_i, i = 1, \ldots, n$, that can be easily obtained explicitly. The $\Delta \Psi_i(t, T_j)$ can be specified as follows; if the hedging instrument is a futures contract with maturity $T_j$ then from (5.31)

\[ \Delta \Psi_i(t, T_j) = \Psi(t, T_j) \sigma_i(t, T_j, \mathbf{V}_t) \Delta \mathbf{W}_i. \]  

(5.34)

If the hedging instrument is an option on a futures contract with value $F(t, T_j, \mathbf{V}_t)$ then $\Delta \Psi_i(t, T_j) = \Psi(F_{i,U}(t, T_j, \mathbf{V}_t)) - \Psi(F_{i,D}(t, T_j, \mathbf{V}_t))$. The conditions (5.33) eliminate only risk generated by small changes in the forward curve without directly account for the impact of the changes in the volatility, which are crucial in the stochastic volatility setup of our model.

### 5.1.2 Delta-Vega Hedging

In order to account also for the variation in the volatility process, $n$ additional hedging instruments are required to make the portfolio $\Upsilon$ simultaneously delta-vega neutral. $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ denotes the positions held in these hedging instruments that have values of $\Lambda(t, T_j, \mathbf{V}_t)$ for $j = 1, \ldots, n$. The positions $\beta$ are selected such that, for each factor, the overall change of the hedged portfolio is zero, after applying a shock $\Delta \mathbf{W}_i^\nu$ to the variance
process. Thus the following conditions should hold for $i = 1, \ldots, n$

$$\Delta \Upsilon_{H,i} = \Delta \Upsilon_i + \beta_1 \Delta \Lambda_i(t, T_1) + \beta_2 \Delta \Lambda_i(t, T_2) + \ldots + \beta_n \Delta \Lambda_i(t, T_n) = 0,$$  \hspace{1cm} (5.35)

where

$$\Delta \Lambda_i(t, T_j) = \Lambda_i(t, T_j, V_i^U) - \Lambda_i(t, T_j, V_i^P).$$  \hspace{1cm} (5.36)

The change $\Delta \Upsilon_i$ as a result of this shock is computed by equation (5.32). By initially solving equation (5.35), the position $\beta$ in the $n$ hedging instruments is determined. For these positions, the portfolio combining $\Upsilon$ and the $n$ hedging instruments, by construction, has a vega of zero but in general, a non-zero residual delta. We can neutralise the delta of the combined portfolio by taking positions in $n$ additional hedging instruments to satisfy condition (5.33) in which $\Delta \Upsilon_i$ is now the changes of the combined portfolio for each factor $i$.

### 5.1.3 Delta-Gamma Hedging

Sensitivity to large price changes can be controlled by gamma hedging. For the portfolio $\Upsilon$ to be gamma neutral, $n$ hedging instruments are required and more specifically $n$ options (as the gamma of a forward or futures contract is zero). The positions $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ in these hedging instruments, with values $\Phi(t, T_j, V_i)$ for $j = 1, \ldots, n$, are selected such that the gamma of the hedged portfolio is zero with respect to each factor $i$, that is

$$\Gamma \Upsilon_{H,i} = \gamma_1 \Gamma \Phi_i(t, T_1) + \gamma_2 \Gamma \Phi_i(t, T_2) + \ldots + \gamma_n \Gamma \Phi_i(t, T_n) = 0,$$  \hspace{1cm} (5.37)

where for $i = 1, \ldots, n$,

$$\Gamma \Upsilon_i = \Upsilon(F_{i,U}(t, T, V_U)) - 2 \times \Upsilon(F_i(t, T, V)) + \Upsilon(F_{i,D}(t, T, V_D)); \hspace{1cm} (5.38)$$

$$\Gamma \Phi_i = \Phi(F_{i,U}(t, T, V_U)) - 2 \times \Phi(F_i(t, T, V)) + \Phi(F_{i,D}(t, T, V_D)). \hspace{1cm} (5.39)$$

For these positions, the portfolio combining $\Upsilon$ and the $n$ hedging instruments have a non-zero residual delta. For the portfolio $\Upsilon$ to be simultaneously delta-gamma neutral, we must neutralise also the delta of the combined portfolio by taking positions in $n$ additional hedging instruments to satisfy condition (5.33), as it was done for vega-delta hedging, in which $\Delta \Upsilon_i$ is now the changes of the combined portfolio for each factor $i$. 

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5.2 An Example: Hedging a straddle

The impact of the hump-shaped volatility specifications on hedging commodity derivative portfolios that are sensitive to volatility is examined. Straddle is a typical options portfolio that is sensitive to volatility. A long straddle consisting of a call and a put with same strike of 130 and the same maturity of February 2009\(^6\) is constructed and hedged by using weights implied by the three-factor models prevailed in Section 4.

Different sets of hedging instruments have been considered, where different combination of futures and options contracts are used. Motivated by the compelling empirical evidence of unspanned components in the oil futures volatility structure, we increase the number of options contracts used to demonstrate that futures contracts cannot not sufficiently hedge variation in the futures curve. Thus delta-vega and delta-gamma hedging is undertaken by using the two alternative hedging portfolios; three futures and three options or two futures and four options. The three futures contracts used as hedging instruments are futures contracts with maturities of six-months, nine-months and one-year. The three options contracts used as hedging instruments have the same maturities as the (three) futures contracts but different strikes from the target option. More specifically, the maturities of the target option and three futures contract in the hedging instruments are: February 2009, May 2009 and August 2009. The strikes of three options used as hedging instruments are 133, 128, 132.5 respectively. When four options are used then at the above three options, an option with strike of 131 and maturity of July 2009 is added. The two futures contracts are used which are the two futures contracts with the two shortest maturities.

The daily P&L of the hedged and unhedged positions are computed, by using the root mean squared error (RMSE) to assess the hedging performance. The daily P&L of a perfect hedge should be 0, hence the RMSE of our hedged position is computed as

\[
RMSE_{hedge} = \sqrt{\sum_{day} (P&L - 0)^2}_{day} = \sqrt{\sum_{day} (P&L)^2}_{day}.
\]

The shocks that will provide the optimal hedging performance are selected. Theses shocks then change the volatility and the futures prices according to (5.30) and (5.31). Recall that the volatility process (5.30) is specified as a Heston type (2.19) for \(n = 3\).

\(^6\)Seeking a representative example of a period that the market was very volatile, the hedging performance of an option during the financial crisis in 2008 has been selected.
\[ \Delta V_t^i = \mu_t^i (\nu_t^i - V_t^i) \Delta t + \varepsilon_t^i \sqrt{V_t^i \Delta W_t^V}; \ i = 1, \ldots, 3. \] (5.40)

Table 5.7 presents the hedging error for three different hedging schemes; delta hedging, delta-gamma hedging and delta-vega hedging. Additionally the table compares the hedging error generated by the model allowing for humps in the volatility structure with the hedging error generated by the exponential volatility model.

For a static factor hedging, Table 5.7 clearly shows that the hump-shaped stochastic volatility model performs consistently better than the model with only exponential volatility functions for all three hedging schemes namely delta, delta-gamma and delta-vega hedging.\(^7\) Delta hedging is not as effective with futures only, accentuating the feature of unspanned futures volatility. Increasing the number of options in the hedged portfolio significantly reduces the hedging error for all three hedging schemes. The delta-vega hedge is the most effective. Furthermore, the stability of the factor hedging is much better with hump-shaped volatility than those with the exponential one. In fact, the hedging performance of a model with hump-shaped volatility is affected less by the size of the shocks than the model with exponential volatility functions.\(^8\) The static hedge with the hump-shaped volatility specifications explains only 36.60\% of the variance of the unhedged residuals, improving though the exponential volatility specification explanatory power by approximately 14%.

<table>
<thead>
<tr>
<th>RMSE</th>
<th>(R^2(%))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>hump</td>
</tr>
<tr>
<td>Unhedged</td>
<td>2.6170</td>
</tr>
<tr>
<td>Delta Hedge (3 futures)</td>
<td>2.6013</td>
</tr>
<tr>
<td>Delta Hedge (2 futures + 1 option)</td>
<td>2.5781</td>
</tr>
<tr>
<td>Delta-Gamma Hedge (3 futures + 3 options)</td>
<td>2.4951</td>
</tr>
<tr>
<td>Delta-Gamma Hedge (2 futures + 4 options)</td>
<td>2.4238</td>
</tr>
<tr>
<td>Delta-Vega Hedge (3 futures + 3 options)</td>
<td>2.0877</td>
</tr>
<tr>
<td>Delta-Vega Hedge (2 futures + 4 options)</td>
<td>2.0838</td>
</tr>
</tbody>
</table>

Table 5.7: Example: Hedging performance of static factor hedging for straddles

By increasing the hedging frequency, the hedging performance improves consistently for all schemes, as Table 5.8 demonstrates. This table displays the hedging error by undertak-\(^7\) Over periods of stable markets, the magnitude of improvement of the hump-shaped volatility specification over the exponential volatility specification is decreasing, leading to similar conclusions as in Trolle & Schwartz (2009b).
\(^8\) Note that, the factor hedging allows to determine the size of the shocks that requires rebalancing of the factor hedge, a feature that can be very convenient for applications.
ing biweekly rebalancing. The hump-shaped volatility model outperforms at all hedging schemes. The hump-shaped volatility specifications now explain 65.98% of the variance of the unhedged residuals, while the exponential volatility specifications explains only 38.18%.

<table>
<thead>
<tr>
<th></th>
<th>RMSE</th>
<th>$R^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>hump</td>
<td>exp</td>
</tr>
<tr>
<td>Unhedged</td>
<td>2.617</td>
<td>2.617</td>
</tr>
<tr>
<td>Delta Hedge (3 futures)</td>
<td>2.215</td>
<td>2.746</td>
</tr>
<tr>
<td>Delta Hedge (2 futures + 1 option)</td>
<td>2.081</td>
<td>2.608</td>
</tr>
<tr>
<td>Delta-Gamma Hedge (3 futures + 3 options)</td>
<td>1.751</td>
<td>2.456</td>
</tr>
<tr>
<td>Delta-Gamma Hedge (2 futures + 4 options)</td>
<td>1.748</td>
<td>2.423</td>
</tr>
<tr>
<td>Delta-Vega Hedge (3 futures + 3 options)</td>
<td>1.530</td>
<td>2.057</td>
</tr>
<tr>
<td>Delta-Vega Hedge (2 futures + 4 options)</td>
<td>1.526</td>
<td>2.057</td>
</tr>
</tbody>
</table>

Table 5.8: Example: Hedging performance of factor hedging for straddles with biweekly re-balancing

6 Conclusion

A multi-factor stochastic volatility model for studying commodity futures curves within the Heath, Jarrow & Morton (1992) framework is proposed. The model aims to capture the main characteristics of the volatility structure in commodity futures markets. The model accommodates exogenous stochastic volatility processes that may be partially unspanned by futures contracts. We specify a hump component for the volatility of the futures curves, which can generate a finite dimensional Markovian forward model. The resulting model is highly tractable with quasi-analytical prices of European options on futures contracts.

The model was fitted to an extensive database of crude oil futures prices and option prices traded in the NYSE over 21 years. We find supporting evidence for three volatility factors, two of which exhibit a hump. This provides new evidence on the volatility structure in crude oil futures markets, which has been traditionally modelled with exponentially declining volatility functions. Finally, by using hedge ratios implied by the proposed unspanned hump-shaped stochastic volatility model, the hedging performance of factor hedging schemes is examined. The results favour the proposed model compared to the exponential decaying only volatility model.

The current work instigates new developments in commodity market modelling. Firstly,
it will be interesting to verify the existence of humps in the volatility structure of other commodities. Our methodology is generic and can be adapted to any commodity futures market. Additionally, the current model can be adjusted to accommodate stochastic convenience yield and stochastic interest rates. This direction has the potential of providing useful insights on the features of convenience yields in commodity markets.
A Proof of Lemma 2.2

Define the process \( X(t, T) = \ln F(t, T, V_t) \). Then an application of the Ito’s formula derives

\[
dX(t, T) = -\frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(t, T, V_t)dt + \sum_{i=1}^{n} \sigma_i(t, T, V_t)dW_i(t). \tag{A.41}
\]

By integrating (A.41) we obtain

\[
F(t, T, V_t) = F(0, T) \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} \int_0^t \sigma_i^2(s, T, V_s)ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s, T, V_s)dW_i(s) \right]. \tag{A.42}
\]

For \( t = T \), (A.42) derives the dynamics of the commodity spot price as

\[
S(t, V_t) = F(0, t) \exp \left[ -\frac{1}{2} \sum_{i=1}^{n} \int_0^t \sigma_i^2(s, t, V_s)ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s, t, V_s)dW_i(s) \right], \tag{A.43}
\]

or equivalently,

\[
\ln S(t, V_t) = \ln F(0, t) + \left[ -\frac{1}{2} \sum_{i=1}^{n} \int_0^t \sigma_i^2(s, t, V_s)ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s, t, V_s)dW_i(s) \right].
\]

By differentiating, it follows that \( S(t, V_t) \) satisfies the stochastic differential equation (2.7).

B Proof of Lemma 2.5

We consider the process \( X(t, T) = \ln F(t, T, V_t) \) and by integrating (A.41) we obtain (A.42). We need to calculate

\[
I = \int_0^t \sigma_i(u, T, V_u) dW_i(u) \tag{B.44}
\]

\[
II = \int_0^t \sigma_i^2(u, T, V_u) du \tag{B.45}
\]
We substitute the volatility specifications (2.9) to obtain

\[ I = \int_0^t (\kappa_0i + \kappa_i(T - u))e^{-\eta_i(T - u)}\sqrt{\mathbf{V}_u}dW_i(u) \]

\[ = \int_0^t (\kappa_0i + \kappa_i(T - t + t - u))e^{-\eta_i(T - t + t - u)}\sqrt{\mathbf{V}_u}dW_i(u) \]

\[ = \beta_{i1}(T - t)\int_0^t e^{-\eta_i(t - u)}\sqrt{\mathbf{V}_u}dW_i(u) + \beta_{i2}(T - t)\int_0^t (t - u)e^{-\eta_i(t - u)}\sqrt{\mathbf{V}_u}dW_i(u) \]

\[ = \beta_{i1}(T - t)\phi_i(t) + \beta_{i2}(T - t)\psi_i(t) \]

where

\[ \beta_{i1}(T - t) = (\kappa_0i + \kappa_i(T - t))e^{-\eta_i(T - t)} \]

\[ \beta_{i2}(T - t) = \kappa_i e^{-\eta_i(T - t)} \]

and the state variable are defined by

\[ \phi_i(t) = \int_0^t e^{-\eta_i(t - u)}\sqrt{\mathbf{V}_u}dW_i(u), \]

\[ \psi_i(t) = \int_0^t (t - u)e^{-\eta_i(t - u)}\sqrt{\mathbf{V}_u}dW_i(u). \]  

(B.46)

Next

\[ II = \int_0^t \sigma_i^2(u, T, \mathbf{V}_u)du \]

\[ = \int_0^t (\kappa_0i + \kappa_i(T - u))^2 e^{-2\eta_i(T - u)}\mathbf{V}_u du \]

\[ = \int_0^t (\beta_{i1}(T - t)e^{-\eta_i(t - u)} + \beta_{i2}(T - t)(t - u)e^{-\eta_i(t - u)})^2 \mathbf{V}_u du \]

\[ = \int_0^t (\gamma_{i1}(T - t) + \gamma_{i2}(T - t)(t - u) + \gamma_{i3}(T - t)(t - u)^2)e^{-2\eta_i(t - u)}\mathbf{V}_u du \]

\[ = \gamma_{i1}(T - t)x_i(t) + \gamma_{i2}(T - t)y_i(t) + \gamma_{i3}(T - t)z_i(t) \]

where

\[ \gamma_{i1}(T - t) = \beta_{i1}(T - t)^2, \quad \gamma_{i2}(T - t) = 2\beta_{i1}(T - t)\beta_{i2}(T - t), \quad \gamma_{i3} = \beta_{i2}(T - t)^2. \]
We define the state variables

\[ x_i(t) = \int_0^t e^{-2\eta_i(t-u)} V_u^{i \phi} du, \]

\[ y_i(t) = \int_0^t (t-u)e^{-2\eta_i(t-u)} V_u^{i \phi} du, \]

\[ z_i(t) = \int_0^t (t-u)^2 e^{-2\eta_i(t-u)} V_u^{i \phi} du. \]  

(B.47)

Hence by differentiating

\[ dx_i(t) = (-2\eta_i x_i(t) + V_t^{i \phi}) dt, \]

\[ dy_i(t) = (-2\eta_i y_i(t) + x_i(t)) dt, \]

\[ dz_i(t) = (-2\eta_i z_i(t) + 2y_i(t)) dt. \]

C Characteristic Function

We consider the characteristic function

\[ \phi(t; \omega, T_0, T) =: \mathbb{E}_t[\exp\{\omega \ln F(T_0, T)\}] \]

\[ = \mathbb{E}_t[\mathbb{E}_{T_0}[\exp\{\omega \ln F(T_0, T)\}]] = \mathbb{E}_t[\phi(T_0; \omega, T_0, T)]. \]

Therefore the process \( k(t) = \phi(t; \omega, T_0, T) \) is a martingale under the risk-neutral measure. Given that \( k(t) \) should be of the form (2.20), an application of Ito’s lemma yields to

\[ \frac{dk(t)}{k(t)} = \left( \frac{dM(t)}{dt} + \sum_{i=1}^n \frac{dN_i(t)}{dt} V_t^{i \phi} \right) dt + \sum_{i=1}^n N_i(t)dV_t^{i \phi} + \omega \frac{dF(t, T)}{F(t, T)} \]

\[ + \frac{1}{2} \sum_{i=1}^n N_i^2(t)(dV_t^{i \phi})^2 + \frac{\omega^2 - \omega}{2} \left( \frac{dF(t, T)}{F(t, T)} \right)^2 + \omega \sum_{i=1}^n N_i(t)dV_t^{i \phi} \frac{dF(t, T)}{F(t, T)} \]  

(C.48)

\[ + \sum_{j \neq i}^n N_i(t)N_j(t)dV_t^i dV_t^j. \]
The drift of this SDE should be zero, thus

\[ 0 = \frac{dM(t)}{dt} + \sum_{i=1}^{n} \frac{dN_i(t)}{dt} V_t^i + \sum_{i=1}^{n} N_i(t) \mu_i^V (\nu_t^Y - V_t^i) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} N_i^2(t)(\varepsilon_t^Y)^2 V_t^i + \frac{\omega^2 - \omega}{2} \sum_{i=1}^{n} \left( (\kappa_{0i} + \kappa_i(T-t)) e^{-\eta(T-t)} \sqrt{V_t^i} \right)^2 \]  

\[ + \omega \sum_{i=1}^{n} N_i(t) \varepsilon_t^Y \sqrt{V_t} \rho_i(\kappa_{0i} + \kappa_i(T-t)) e^{-\eta(T-t)} \sqrt{V_t}. \]  

(C.49)

By using \( \varphi_i = (\kappa_{0i} + \kappa_i(T-t)) e^{-\eta(T-t)} \) then from (C.49) we obtain the ODE (2.21) and (2.22) for \( M(t) \) and \( N_i(t) \) respectively.

**D Appendix: Extended Kalman Filter**

**D.1 The extended Kalman filter**

Our model consists of 2 sets of equations. The first one is the system equation that describes the evolution of the state variables:

\[ X_{t+1} = \Phi_0 + \Phi X_t + w_{t+1}, \quad w_{t+1} \sim iid N(0, Q_t), \]  

(D.50)

whereas the second one is the observation equation that links the state variables with the market-observable variables

\[ z_t = h(X_t) + u_t, \quad u_t \sim iid N(0, \Omega). \]  

(D.51)

It is noted that the \( h \) function is nonlinear here.

Let \( \hat{X}_t = E_t[X_t] \) and \( \hat{X}_{t|t-1} = E_{t-1}[X_t] \) denote the expectations of \( X_t \), and let \( P_t \) and \( P_{t|t-1} \) denote the corresponding estimation error covariance matrices. Linearizing the \( h \) function around \( \hat{X}_{t|t-1} \) we obtain,

\[ z_t = (h(\hat{X}_{t|t-1}) - H_t^' \hat{X}_{t|t-1}) + H_t^' X_t + u_t, \quad u_t \sim iid N(0, \Omega), \]  

(D.52)

where

\[ H_t^' = \frac{\partial h(X_t)}{\partial X_t} \bigg|_{X_t = \hat{X}_{t|t-1}}. \]  

(D.53)
The Kalman filter yields:

\[
\dot{X}_{t+1|t} = \Phi_0 + \Phi_X \dot{X}_t, \quad (D.54)
\]

\[
P_{t+1|t} = \Phi_X P_t \Phi_X' + Q_t, \quad (D.55)
\]

and

\[
\dot{X}_{t+1} = \dot{X}_{t+1|t} + P_{t+1|t} H_t^{-1} \epsilon_t, \quad (D.56)
\]

\[
P_{t+1} = P_{t+1|t} - P_{t+1|t} H_t^{-1} H_t P_{t+1|t}, \quad (D.57)
\]

where

\[
\epsilon_t = z_{t+1} - h(\hat{X}_{t+1|t}), \quad (D.58)
\]

\[
F_t = H_t P_{t+1|t} H_t' + \Omega. \quad (D.59)
\]

The log-likelihood function is constructed as:

\[
\log L = -\frac{1}{2} \log(2\pi) \sum_{t=1}^{T} N_t - \frac{1}{2} \sum_{t=1}^{T} \log |F_t| - \frac{1}{2} \sum_{t=1}^{T} \epsilon_t' F^{-1}_t \epsilon_t. \quad (D.60)
\]

### D.2 The system equation

The dynamics of the state vector under the physical measure can be written as:

\[
dX^i_t = (\Psi_i - \mathcal{K}_i X^i_t)dt + \sqrt{V^i_t} \Sigma_i dW^P_i(t)
\]

where \(X^i_t = (x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \mathbf{V}_x^i)'\), and \(W^P_i(t) = (W^1_i(t), W^2_i(t))'\), and

\[
\Psi_i = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\mu_x^i \nu_x^i
\end{pmatrix}, \quad \mathcal{K}_i = \begin{pmatrix}
2\eta_i & 0 & 0 & 0 & 0 & -1 \\
-1 & 2\eta_i & 0 & 0 & 0 & 0 \\
0 & -2 & 2\eta_i & 0 & 0 & 0 \\
0 & 0 & 0 & \eta_i & 0 & 0 \\
0 & 0 & 0 & -1 & \eta_i & 0 \\
0 & 0 & 0 & 0 & \mu_x^i & \nu_x^i
\end{pmatrix}, \quad \Sigma_i = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \epsilon_l^i
\end{pmatrix} \cdot R_i
\]
where $R_i$ is the correlation matrix for the Wiener processes, i.e. $dW^P_i(t) dW^P_i(t') = R_i dt$ with

$$R_i = \begin{pmatrix} 1 & 0 \\ \rho_i & \sqrt{1 - \rho_i^2} \end{pmatrix}.$$ 

Applying Ito’s Lemma to $e^{K_i t} X^i_t$, we have

$$d(e^{K_i t} X^i_t) = e^{K_i t} K_i X^i_t dt + e^{K_i t} dX^i_t$$

$$= e^{K_i t} \Psi dt + e^{K_i t} \sqrt{V_i \Sigma_i dW^P_t}.$$  \hfill (D.61)

It follows that $X^i_s$, $s > t$ is given as

$$X^i_s = e^{-K_i(s-t)} X^i_t + \int_t^s e^{-K_i(s-u)} \Psi_i du + \int_t^s e^{-K_i(s-u)} \sqrt{V_i \Sigma_i dW^P_t(u)}.$$ 

The conditional mean of $X^i_t$, given time $t$ information, is given by

$$E_t[X^i_t] = \int_t^s e^{-K_i(s-u)} \Psi_i du + e^{-K_i(s-t)} X^i_t.$$  \hfill (D.62)

and the conditional covariance matrix of $X^i_s$, given time- $t$ information, is given by

$$Cov_t[X^i_s] = E_t \left[ \left( \int_t^s e^{-K_i(s-u)} \sqrt{V_i \Sigma_i dW^P_t(u)} \right) \left( \int_t^s e^{-K_i(s-u)} \sqrt{V_i \Sigma_i dW^P_t(u)} \right)' \right]$$

$$= \int_t^s E_t[V_i] e^{-K_i(s-u)} \Sigma_i R_i \Sigma_i' e^{-K_i(s-u)} du$$

$$= \int_t^s \left( 1 - e^{-\rho_i V(s-t)} \right) \nu_i V e^{-K_i(s-u)} \Sigma_i R_i \Sigma_i' e^{-K_i(s-u)} du$$

$$+ \int_t^s e^{-\rho_i V(s-t)} e^{-K_i(s-u)} \Sigma_i R_i \Sigma_i' e^{-K_i(s-u)} du.$$  \hfill (D.63)

Putting the three factors together, we obtain

$$X_t = \begin{pmatrix} X^1_t \\ X^2_t \\ X^3_t \end{pmatrix}, W^P(t) = \begin{pmatrix} W^P_1(t) \\ W^P_2(t) \\ W^P_3(t) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, K = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},$$
\[
\text{Cov}_t[X_s] = \begin{pmatrix}
\text{Cov}_t[X^1_s] & 0 & 0 \\
0 & \text{Cov}_t[X^2_s] & 0 \\
0 & 0 & \text{Cov}_t[X^3_s]
\end{pmatrix}
\]

The system equation, therefore, can be written in discrete form as

\[
X_{t+1} = \Phi_0 + \Phi_X X_t + w_{t+1}, \quad w_{t+1} \sim iid N(0, Q_t),
\]

(D.64)

where

\[
\Phi_0 = \int_t^{t+dt} e^{-\kappa(t+dt-u)} \Psi du, \quad \Phi_X = e^{-\kappa dt},
\]

and \(Q_t\) can be derived directly from Equ.(D.63)

## E Models with exponential decaying volatility

### E.1 Volatility functions without hump

**Proposition** If the volatility function \(\sigma_i(t, T, V^i_t) = \kappa_0 e^{-\eta_i(t-t)} \sqrt{V_i^T(t-t)}\) the logarithm of the time—\(t\) instantaneous futures prices at time \(T\), \(\ln F(t, T)\), is given by

\[
\ln F(t, T, V_i) = \ln F(0, T, V_0) + \sum_{i=1}^{3} \left( \beta_{i1}(T-t)x_i(t) - \frac{1}{2} \beta_{i2}(T-t)y_i(t) \right)
\]

(E.65)

where \(x_i(t), y_i(t)\) evolve according to

\[
dx_i(t) &= -\eta_i x_i(t) dt + \sqrt{V_i^t} dW_i(t), \quad (E.66) \\
dy_i(t) &= (-2\eta_i y_i(t) + V_i^t) dt, \quad (E.67)
\]

(E.68)

subject to \(x_i(0) = y_i(0) = 0\). We also have, for \(i = 1, 2, 3\),

\[
\beta_{i1}(T-t) = \kappa_0 e^{-\eta_i(T-t)}, \quad (E.69) \\
\beta_{i2}(T-t) = \frac{1}{2} \kappa_0 e^{-2\eta_i(T-t)}. \quad (E.70)
\]

(E.71)
Proof. Similar to the proofs of the case with hump.

E.2 Transition density

The dynamic of the state vector under the actual measure can be written as:

$$dX_i^t = (Ψ_i - K_i X_i^t) dt + \sqrt{V_i^t} \Sigma_i dW^P(t)$$

where, $X_i^t = (x_i(t), y_i(t), V_i^t)'$, and $W^P_i(t) = (W^1_i(t), W^2_i(t))'$, and

$$Ψ_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu^V_i \nu_i^V \end{pmatrix}, K_i = \begin{pmatrix} η_i & 0 & 0 \\ 0 & 2η_i & -1 \\ 0 & 0 & \mu^V_i \end{pmatrix}, \Sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \rho_i \sqrt{1 - \rho_i^2} & \sqrt{1 - \rho_i^2} \end{pmatrix},$$

where $R_i$ is the correlation matrix for the Wiener processes, i.e. $dW_i^P(t) dW_i^P(t)' = R_i dt$ and

$$R_i = \begin{pmatrix} 1 \\ 0 \\ \rho_i \sqrt{1 - \rho_i^2} \end{pmatrix}.$$

Applying Ito’s Lemma to $e^{K_i t} X_i^t$, we have

$$d(e^{K_i t} X_i^t) = e^{K_i t} K_i X_i^t dt + e^{K_i t} dX_i^t = e^{K_i t} Ψ_i dt + e^{K_i t} \sqrt{V_i^t} \Sigma_i dW^P(t). \quad (E.72)$$

It follows that $X_i^s, s > t$ is given as

$$X_i^s = e^{-K_i(s-t)} X_i^t + \int_t^s e^{-K_i(s-u)} Ψ_i du + \int_t^s e^{-K_i(s-u)} \sqrt{V_i^u} \Sigma_i dW^P(u)$$

The conditional mean of $X_i^s$, given time $t$ information, is given by

$$E_t[X_i^s] = \int_t^s e^{-K_i(s-u)} Ψ_i du + e^{-K_i(s-t)} X_i^t. \quad (E.73)$$
and the conditional covariance matrix of $X^i_t$, given time- $t$ information, is given by

\[
Cov_t[X^i_s] = E_t \left[ \left( \int_t^s e^{-\mathcal{K}_i(s-u)} \sqrt{V^i_u} \Sigma_i dW^P(u) \right) \left( \int_t^s e^{-\mathcal{K}_i(s-u)} \sqrt{V^i_u} \Sigma_i dW^P(u) \right) \right]
\]

\[
= \int_t^s E_t[V^i_u] e^{-\mathcal{K}_i(s-u)} \Sigma_i R_i \Sigma'_i e^{-\mathcal{K}'_i(s-u)} du
\]

\[
= \int_t^s \left( 1 - e^{-\nu^V_i(u-t)} \right) \nu^V_i e^{-\mathcal{K}_i(s-u)} \Sigma_i R_i \Sigma'_i e^{-\mathcal{K}'_i(s-u)} du
\]

\[
+ \left( \int_t^s e^{-\nu^V_i(u-t)} e^{-\mathcal{K}_i(s-u)} \Sigma_i R_i \Sigma'_i e^{-\mathcal{K}'_i(s-u)} du \right) V^i_t.
\]  

(E.74)

Putting three factors together, we would get

\[
X_t = \begin{pmatrix} X^1_t \\ X^2_t \\ X^3_t \end{pmatrix}, \quad W^P(t) = \begin{pmatrix} W^P_1(t) \\ W^P_2(t) \\ W^P_3(t) \end{pmatrix},
\]

\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & \mathcal{K}_3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},
\]

\[
Cov_t[X^i_s] = \begin{pmatrix} Cov_t[X^i_s] & 0 & 0 \\ 0 & Cov_t[X^i_s] & 0 \\ 0 & 0 & Cov_t[X^i_s] \end{pmatrix}
\]

References


Figure 4.4: $v_1(T - t)$ and $V_t^i$ for a three-factor model - Top panel: January 1990 to December 1994; Second panel: January 1995 to December 1999; Third panel: January 2000 to December 2005; Bottom Panel: January 2006 to December 2010
Figure 4.5: RMSEs of the percentage differences between actual and fitted futures prices as well as of the difference between actual and fitted implied option volatilities for a three-factor model from January 1990 to December 2010

Figure 4.6: Time series of implied volatilities and fit to the three-factor model