Three-dimensional analysis of a functionally graded coating/substrate system of finite thickness

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The concept of functionally graded material (FGM) is currently actively explored in coating design for the purpose of eliminating the mismatch of thermomechanical properties at the interfaces and thus increasing the resistance of coatings to functional failure. In the present paper, three-dimensional elastic deformation of a functionally graded coating/substrate system of finite thickness subjected to mechanical loading is investigated. A comparative study of FGM versus homogeneous coating is conducted to examine the effect of the coating type on stress and displacement fields in the system.

Keywords: functionally graded material; coating; three-dimensional elasticity theory

1. Introduction

‘It is only when we contemplate, not matter in itself, but the form in which it actually exists, that our mind finds something on which it can lay hold’ (Maxwell 1873). These words surprisingly well summarize the driving force behind the fundamental change in the relationship between human beings and materials, which led to the idea of specifically designing materials for particular applications. One class of such ‘engineered’ materials are functionally graded materials (FGMs), i.e. heterogeneous composite materials with gradient compositional variation of the constituents from one surface of the material to the other which results in continuously varying material properties (Miyamoto et al. 1999). The concept of FGMs is currently actively explored in coating design for the purpose of increasing the resistance of coatings to functional failure. In conventional multi-layered coatings, the mismatch of thermomechanical properties between coating and substrate results in high interfacial stresses, especially in the presence of a temperature gradient, which may cause cracking and debonding of the coating. Owing to the importance and wide engineering applications, thermomechanical behaviour of FGM coatings has been investigated by many researchers. Most recent studies of functionally graded coating/substrate systems of finite thickness have focused on their fracture behaviour (Guo et al. 2004; Pindera et al. 2005; Wang et al. 2005), vibrations (Liew et al. 2006) and

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thermoelastic behaviour (Shodja et al. 2007). It is worth noting that all the above investigations were carried out in the context of two-dimensional elasticity theory. The aim of the present paper is to investigate the three-dimensional elastic deformation of a functionally graded coating/substrate system of finite thickness subjected to mechanical loading.

2. Formulation

A coating/substrate system considered in this paper is referred to as a Cartesian coordinate system $Oxyz$ and consists of substrate $S$ ($0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq h^{(1)}$) and the coating $C$ ($0 \leq x \leq a$, $0 \leq y \leq b$, $h^{(1)} \leq z \leq h$). All quantities associated with the substrate and the coating will be denoted with the superscripts (1) and (2), respectively. Both the substrate and the coating are assumed to be functionally graded, with the Poisson’s ratios $\nu^{(1)} = \text{const.}$ and $\nu^{(2)} = \text{const.}$, and the shear moduli that vary continuously through the thickness as

$$G^{(k)}(z) = G^{(k)}(h)\exp\left[\gamma^{(k)} \left(\frac{z}{h} - 1\right)\right], \quad k = 1, 2,$$

where $\gamma^{(k)}$ are inhomogeneity parameters. The coating and the substrate are assumed to be perfectly bonded to each other so that stress and displacement continuity conditions are satisfied at the coating/substrate interface $z = h^{(1)}$. The top and bottom surfaces of the coating/substrate system are subjected to mechanical loading

$$z = 0, h : \quad \sigma_{iz}^{(k)} = q_{iz}^{(k)}(x, y), \quad i = x, y, z.$$  \hfill (2.2)

If the Poisson’s ratios are constant, the stresses and displacements in the substrate and the coating can be expressed in terms of the functions $L^{(k)} = L^{(k)}(x, y, z)$ and $N^{(k)} = N^{(k)}(x, y, z)$, $k = 1, 2$, as follows (Plevako 1971; Kashtalyan 2004):

$$u_x^{(k)} = -\frac{1}{2G^{(k)}} \left(\nu^{(k)} \Delta - \frac{\partial^2}{\partial z^2}\right) \frac{\partial L^{(k)}}{\partial x} + \frac{\partial N^{(k)}}{\partial y},$$  \hfill (2.3)

$$u_y^{(k)} = -\frac{1}{2G^{(k)}} \left(\nu^{(k)} \Delta - \frac{\partial^2}{\partial z^2}\right) \frac{\partial L^{(k)}}{\partial y} - \frac{\partial N^{(k)}}{\partial x},$$  \hfill (2.4)

$$u_z^{(k)} = -\frac{1}{2G^{(k)}} \left(\Delta - \frac{\partial^2}{\partial z^2}\right) \frac{\partial L^{(k)}}{\partial z} + \frac{\partial}{\partial z} \left[\frac{1}{2G^{(k)}} \left(\nu^{(k)} \Delta - \frac{\partial^2}{\partial z^2}\right) L^{(k)}\right].$$  \hfill (2.5)

Stresses in the substrate and the coating can be expressed in terms of functions $L^{(k)}$ and $N^{(k)}$ using Hooke’s Law. Functions $L^{(k)}$ and $N^{(k)}$ in equations (2.3)–(2.5) must satisfy the fourth- and second-order partial differential equations, respectively. Upon separating variables in the functions $L^{(k)}$ and $N^{(k)}$ in the form

$$L^{(k)}(x, y, z) = \psi_1^{(k)}(x, y)\phi_1^{(k)}(z), \quad N^{(k)}(x, y, z) = \psi_2^{(k)}(x, y)\phi_2^{(k)}(z),$$  \hfill (2.6)

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and choosing functions \( \psi_1^{(k)}(x, y) \) and \( \psi_2^{(k)}(x, y) \) as (Kashtalyan 2004)

\[
\psi_1^{(k)}(x, y) = \sin \frac{\pi mx}{a} \sin \frac{\pi nx}{b}, \quad \psi_2^{(k)}(x, y) = \cos \frac{\pi mx}{a} \cos \frac{\pi nx}{b}, \quad k = 1, 2,
\]

(2.7)

the partial differential equations for \( L^{(k)} \) and \( N^{(k)} \) are reduced to the ordinary differential equations (Kashtalyan 2004) with respect to functions \( \phi_1^{(k)}(z) \) and \( \phi_2^{(k)}(z) \). Solutions to them are found as

\[
a\phi_1^{(k)}(z) = h^4 \exp \left( \frac{\gamma^{(k)} z}{2h} \right) \left[ A_1^{(k)} \cosh \frac{\lambda^{(k)} z}{h} \cos \frac{\mu^{(k)} z}{h} + A_2^{(k)} \sinh \frac{\lambda^{(k)} z}{h} \cos \frac{\mu^{(k)} z}{h} \right] + A_3^{(k)} \cosh \frac{\lambda^{(k)} z}{h} \sin \frac{\mu^{(k)} z}{h} + A_4^{(k)} \sinh \frac{\lambda^{(k)} z}{h} \sin \frac{\mu^{(k)} z}{h} \right],
\]

(2.8)

\[
\phi_2^{(k)}(z) = h^2 \exp \left( -\frac{\gamma^{(k)} z}{2h} \right) \left[ A_5^{(k)} \cosh \frac{\beta^{(k)} z}{h} + A_6^{(k)} \sinh \frac{\beta^{(k)} z}{h} \right].
\]

(2.9)

Unknown constants \( A_j^{(k)} (j=1, \ldots, 6; k=1, 2) \) are determined from the stress and displacement continuity conditions at the substrate/coating interface and the boundary conditions on the top and bottom surfaces of the coating/substrate system (equation (2.2)). Coefficients \( \lambda^{(k)} \) and \( \mu^{(k)} \), and \( \beta^{(k)} \) are equal to

\[
\begin{pmatrix} \lambda^{(k)} \\ \mu^{(k)} \end{pmatrix} = \sqrt{\frac{1}{2} \left( \pm \beta^{(k)} + \sqrt{\beta^{(k)^2 + \gamma^{(k)^2} \alpha^2 h^2 \frac{\mu^{(k)}}{1-\mu^{(k)}}} \right)},
\]

(2.10)

\[
\beta^{(k)} = \sqrt{\frac{\gamma^{(k)^2}}{4} + \alpha^2 h^2}, \quad \alpha = \pi \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2}.
\]

(2.11)

Substituting functions \( \psi_1^{(k)}(x, y) \) and \( \psi_2^{(k)}(x, y) \) (equation (2.7)), and functions \( \phi_1^{(k)}(z) \) and \( \phi_2^{(k)}(z) \) (equations (2.8) and (2.9)) into equation (2.6), and then into equations (2.3)–(2.5), the displacement field over the entire coating/substrate system is determined. Details of the stress field in the coating/substrate system are published elsewhere (Kashtalyan & Menshykova 2007).

### 3. Numerical results and discussion

In this section, a comparative study of two coating/substrate systems is presented and discussed. The first system (H system) has homogeneous substrate with the shear modulus \( G_H^{(1)} = \text{const.} \) and homogeneous coating with the shear modulus \( G_H^{(2)} = \text{const.} \). The second system (FGM system) has a homogeneous substrate with the shear modulus \( G_{\text{FGM}}^{(1)} = G_H^{(1)} \). The coating in the FGM system is functionally graded, with a shear modulus that varies exponentially from \( G_H^{(1)} \) to \( G_H^{(2)} \) (figure 1) according to equation (2.1).
The coating in the H system and the homogeneous substrates in the H and FGM systems are simulated as FGMs with the inhomogeneity parameters $g^k$ close to zero. It was shown by Kashtalyan (2004) that, when the inhomogeneity parameter approaches zero, the three-dimensional solution for an FGM rapidly converges to that for a homogeneous material. The inhomogeneity parameter of the coating of the FGM system is $g^2_{\text{FGM}} = h/\ln(d^2/\delta^2)$, where $d^2$ is a stiffness gradient and $h^2$ is the coating thickness.

Figure 2 shows through-thickness variation of the normalized stresses $\tilde{\sigma}_{ij} = \sigma_{ij}/q$ and the normalized displacements $\tilde{u}_i = G^2_H u_i (qh)^{-1}$ in the two coating/substrate systems subjected to transverse loading $q_{zz}(x,y) = -q \sin(\pi x/a) \sin(\pi y/b)$ on the coating surface, while the bottom surface of the substrate is stress-free. The thickness of the coating is taken as $h^2/h = 0.2$, and the stiffness gradient in the FGM system as $\delta^2 = 10$ (stiff coating protecting soft substrate). The chosen value of $\delta^2$ does not necessarily represent a certain material, but is rather used to show the effect of coating type on the stress and displacement fields. The Poisson’s ratios are taken as $\nu^1 = \nu^2 = 0.3$. The results are shown for thick ($a/h = b/h = 3$; figure 2a,c,e) and thin ($a/h = b/h = 10$; figure 2b,d,f) systems. Numerical results reveal that through-thickness variation of the in-plane normal stress $\tilde{\sigma}_{xx}$ and the transverse shear stress $\tilde{\sigma}_{xz}$ is strongly influenced by the type of coating: in the FGM system, as opposed to the H system, the in-plane normal stress $\tilde{\sigma}_{xx}$ is continuous across the coating/substrate interface, although its value at the top surface is higher (figure 2a,b). It also becomes compressive throughout the FGM coating. For the thicker H system (figure 2c), the transverse shear stress $\tilde{\sigma}_{xz}$ has a peak in the coating, while for FGM systems it is close to parabolic profile, typical for homogeneous plates (figure 2c,d). Through-thickness variation of the out-of-plane displacement $\tilde{u}_z$ is highly nonlinear in the thicker system (figure 2c,e), and almost constant through the thickness in the thinner system (figure 2f).

4. Conclusions

In the present paper, three-dimensional elastic deformation of a functionally graded coating/substrate system of finite thickness is investigated in the context of elasticity theory. The Young’s modulus of an FGM coating varies exponentially through its thickness, while the Poisson’s ratio is assumed.
constant. The general solution of the equilibrium equations for inhomogeneous isotropic media (Plevako 1971) and its recent application to functionally graded plates with exponential variation of the Young’s modulus through thickness (Kashtalyan 2004) are used to solve the problem. A comparative study of two coating/substrate systems, one with a homogeneous coating and the other with a functionally graded one, has shown that use of the functionally
graded coating eliminates discontinuity of the in-plane normal stress across the coating/substrate interface, and it can become compressive throughout the FGM coating.

References


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