Chordal digraphs*

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Abstract

Chordal graphs, also called triangulated graphs, are important in algorithmic graph theory. In this paper we generalise the definition of chordal graphs to the class of directed graphs. Several structural properties of chordal graphs that are crucial for algorithmic applications carry over to the directed setting, including notions like simplicial vertices, perfect elimination orderings, and characterisation by forbidden subgraphs resembling chordless cycles. Moreover, just as chordal graphs are related to treewidth, the chordal digraphs will be related to Kelly-width.

1 Introduction

Chordal graphs have many applications in algorithmic graph theory. In some cases the input graph itself is chordal, in other cases we work on a (minimal) triangulation of the input graph, with edges added so that we have a chordal graph. The algorithmic interest in triangulations and chordality is based on the many structural properties associated with chordal graphs. Since graphs can be regarded as a subclass of directed graphs (digraphs) a basic question is which algorithmic properties of graphs extend to digraphs. In this paper we generalise the definition of chordal graphs to digraphs and show that many of the properties that hold for chordal graphs carry over to the directed setting. Such properties are essential for algorithmic applications of chordal digraphs. Many problems that are NP-hard on general graphs become polynomial-time solvable on chordal graphs of bounded clique-size, and also on their subgraphs. The corresponding graph parameter, called treewidth, is in this way related to chordal graphs. The chordal digraphs will in an analogous way be related to the digraph parameter called Kelly-width, recently introduced by Hunter and Kreutzer [6]. The amount of research devoted to algorithms for digraphs is steadily increasing, see e.g. the recent monograph of Bang-Jensen and Gutin [1]. However, we have not found in the literature a generalisation of chordality to digraphs that succeeds in capturing the nice structural properties of chordal graphs. The structural properties we would like to capture include the following equivalent characterisations of chordal graphs:

(a) iteratively constructed by adding a new vertex adjacent to a clique

(b) have a perfect elimination ordering [8]

(c) have vertex layout such that if $u \prec v \prec w$, and $\{v,u\},\{u,w\}$ edges then $\{v,w\}$ also an edge [8]

(d) contains no chordless cycle, equivalently no induced cycle of length at least 4

(e) every minimal separator is a clique [3]

(f) are the intersection graphs of families of subtrees of a tree [2, 4, 10].

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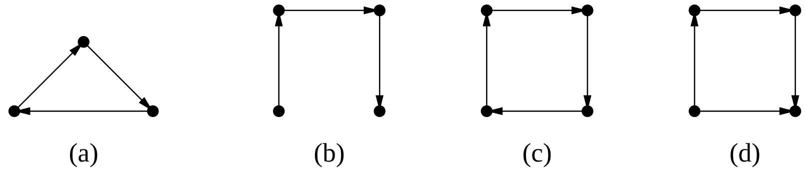

Figure 1: The forbidden induced subgraphs of uni(G) for G a chordal digraph.

Our definition of chordal digraphs in Section 3 is based on a generalisation of the iterative construction (a), which also serves to define Kelly-width [6, 7]. We then show that chordal digraphs allow equivalent characterisations in terms of natural generalisations of perfect elimination orderings (b) and also vertex layouts (c). Chordal digraphs also satisfy nice properties like closure under taking induced subgraphs and closure under reversal of all arcs. To generalise the characterisation by forbidden subgraphs (d) to chordal digraphs requires more work. Let us describe our results informally.

Partition the arcs of digraph G to define two digraphs uni(G) and bi(G), with uni(G) containing the arcs (u, v) for which (v, u) is not an arc, and bi(G) containing the arcs (u, v) for which (v, u) is also an arc. Our first result states that if bi(G) is empty (equivalently, if G is an orientation) then G is a chordal digraph if and only if G is acyclic (equivalently, contains no chordless directed cycle of length at least 3). On the other hand, if uni(G) is empty then G is chordal if and only if G contains no chordless directed cycle of length at least 4. Note that G could be viewed as an undirected graph precisely when uni(G) is empty, and hence the latter result shows that indeed the chordal digraphs are a generalisation of chordal (undirected) graphs. We also show that if G is a chordal digraph then bi(G) contains no chordless directed cycle of length at least 4. An important and large class of digraphs are the *semi-complete* digraphs where every pair of vertices have at least one arc. We show that a semi-complete digraph G is chordal if and only if bi(G) is chordal and uni(G) does not contain an induced subgraph isomorphic to a digraph in Figure 1.

Regarding the characterisation of chordal graphs by minimal separators (e) and by intersections of subtrees of a tree (f) it is an interesting open problem to provide a nice generalisation that will work for chordal digraphs. We discuss these and other open problems in the Conclusion section. For example, in this paper we give an algorithm taking as input a digraph G on n vertices and m arcs that in time $\mathcal{O}(n^2 m)$ decides if G is chordal, but faster algorithms should exist.

2 Preliminaries

We consider directed and undirected graphs. Directed graphs are called *digraphs.* All graphs are simple and finite. In particular, digraphs contain no loops. By "graph", we may refer to a directed or undirected graphs. Let G be a digraph. The vertex and arc set of G are denoted as respectively $V(G)$ and $A(G)$. To vertices u, v of G can be connected by one or two arcs, namely (u, v) or (v, u). If u and v are not connected by any of these arcs, we call u and v *non-adjacent*; otherwise, u and v are *adjacent.* If $(u, v) \in A(G)$ then u is *in-neighbour* of v and v is *out-neighbour* of u. By $N^{\text{in}}(u)$ and $N^{\text{out}}(u)$, we denote the respectively *in-neighbourhood* and *out-neighbourhood* of u. For a set $X \subseteq V(G)$, $G[X]$ denotes the *subgraph* of G *induced* by X, i.e., the digraph on vertex set X and $(u, v) \in A(G[X])$ if and only if $u, v \in X$ and $(u, v) \in A(G)$ for all $u, v \in V(G)$. For x a vertex of G, we denote $G[V(G) \setminus \{x\}]$ as $G-x$. A *directed path* in G is a sequence $(x_1, \ldots, x_k)$ of pairwise different vertices of G where $(x_i, x_{i+1}) \in A(G)$ for all $1 \leq i < k$. A *directed cycle* in G is a sequence $(x_1, \ldots, x_k)$ of pairwise different vertices of G that is a directed path and $(x_k, x_1) \in A(G)$. Note that $k \geq 2$, since G contains no loops.

Let H be an undirected graph. The vertex and edge set of H are denoted as $V(H)$ and $E(H)$. Edges of H are denoted as $\{u, v\}$. For undirected graphs, we use definitions analogous to the definitions for digraphs. For a set $E' \subseteq E(H)$, the undirected graph $H \setminus E'$ has vertex set $V(H)$ and edge set $E(H) \setminus E'$. A *cycle* C in H is a sequence $(x_1, \ldots, x_k)$ of pairwise different vertices of H with $k \geq 3$ where $\{x_i, x_{i+1}\} \in E(H)$ for all $1 \leq i < k$ and $\{x_1, x_k\} \in E(H)$; k denotes the *length* of C. If $\{x_i, x_j\} \in E(H)$ for some $i, j \in \{1, \ldots, k\}$ and $1 < |j - i| < k - 1$ then $\{x_i, x_j\}$ is a *chord* in C. A cycle without chord is called *chordless*. A *vertex layout* for a graph F is a linear order of its vertices, denoted as $\beta = \langle x_1, \ldots, x_n \rangle$. For a pair u, v of vertices of F, we write $u \prec_\beta v$ if $u = x_i$ and $v = x_j$ and $i < j$.

An undirected graph without chordless cycles of length at least 4 is called *chordal*. Chordal graphs have a large number of different characterisations, such as by properties of minimal separators [3] or as intersection graphs [2, 4, 10]. Another characterisation is by vertex layouts.

Theorem 2.1 ([8]). *An undirected graph H is chordal if and only if there is a vertex layout β for H such that for all vertex triples u, v, w of H with $u \prec_\beta v \prec_\beta w$, $\{v, u\}, \{u, w\} \in E(H)$ implies $\{v, w\} \in E(H)$.*

A digraph is called *acyclic* if it contains no directed cycle. Acyclic digraphs have at most one arc between every pair of vertices. The following characterisation is folklore: a digraph G is acyclic if and only if there is a vertex layout β for G such that for all arcs (u, v) of G, $u \prec_\beta v$.

The structure of digraphs is often studied by looking only at the adjacency relation. For a digraph G, the *underlying graph* is the undirected graph H on vertex set $V(G)$, and for every pair u, v of vertices of G, $\{u, v\} \in E(H)$ if and only if $(u, v) \in A(G)$ or $(v, u) \in A(G)$.

3 Definition of chordal digraphs and first results

We define a new class of digraphs. This class is defined inductively, in a manner similar to k-trees. We call our digraphs *chordal* and show by several characterisation results that the chordal digraphs and the chordal undirected graphs have analogous properties.

Definition 3.1.

1. *A* d-clique *of a digraph G is a pair (A, B) with $A, B \subseteq V(G)$ where for all $a \in A$ and $b \in B$ with $a \neq b$, $(a, b) \in A(G)$.*

2. *The class of* chordal digraphs *is inductively defined as follows:*

   – *a graph on a single vertex is a chordal digraph*

   – *let G be a chordal digraph and let u be a vertex that does not appear in G. Let (A, B) be a d-clique of G. The graph is also a chordal digraph that is obtained from G by adding u and the set of arcs $\{(a, u) : a \in A\} \cup \{(u, b) : b \in B\}$.*

In other words, a chordal digraph is extended by a new vertex u by chosing a d-clique and making the one set the in-neighbours of u and the other set the out-neighbours of u. In case of k-trees or general chordal undirected graphs, the chosen d-clique corresponds to the chosen clique and there is no distinction between in- and out-neighbours. For a digraph G that is constructed according to Definition 3.1, the vertex layout that lists the vertices in order they are added to build G where the leftmost vertex is the last added vertex is called *construction sequence*. Note that a chordal digraph can have different construction sequences, since the construction of a chordal digraph is no unique process.

The class of chordal digraphs is closely related to the notion of Kelly-width [6]. We can introduce a parameter k that bounds the size of the second component in a d-clique. A chordal digraph has *width k* if it can be constructed according to Definition 3.1 by only choosing d-cliques with size of the second component at most k. With the results in [6] and [7], it is easy to verify that a digraph has Kelly-width at most $k + 1$ if and only if it is subgraph of a chordal digraph of width at most k.

Chordal digraphs can be understood as a generalisation of chordal undirected graphs, since every chordal undirected graph can be seen as a chordal digraph in a natural sense: from an undirected graph H, obtain a digraph H by replacing every edge by the two arcs between the connected vertices. If we only use the adjacency matrix as graph representation, we can call H and G equivalent. We will see in the next section that G is chordal if and only if H is chordal. For chordal digraphs with all pairs of adjacent vertices connected by two arcs, we can make the following observation about the construction process: for every added vertex x, the in-neighbourhood and out-neighbourhood have to be the same, which means that the chosen d-clique is of the form (A, A). Such d-cliques induce complete subgraphs.

Chordal undirected graphs have many equivalent characterisations. We show that some of these characterisations have analogues for chordal digraphs. We begin with a vertex layout characterisation for chordal digraphs, that is an analogue of Theorem 2.1.

Definition 3.2. *Let G be a digraph with vertex layout β. We say that β is* directed transitive *if for every triple u, v, w of pairwise different vertices of G with $u \prec_\beta v$, $u \prec_\beta w$, $(v, u) \in A$ and $(u, w) \in A$, it holds $(v, w) \in A$.*

Theorem 3.3. *Let G be a digraph with vertex layout β. G is a chordal digraph with construction sequence β if and only if the reverse of β is directed transitive.*

Proof. We show the theorem by induction over the number of vertices in a digraph. The statement is obviously true for digraphs on a single vertex. Now, let G be a digraph on at least two vertices, and let $\beta = \langle x_1, \ldots, x_n \rangle$ be a vertex layout for G. Let $G' =_{\text{def}} G[\{x_1, \ldots, x_{n-1}\}]$ and $\beta' =_{\text{def}} \langle x_1, \ldots, x_{n-1} \rangle$. Let G be a chordal digraph with construction sequence β. Then, G' is a chordal digraph with construction sequence β' according to Definition 3.1. Applying the induction hypothesis, β' is directed transitive for G'. It remains to prove that $(a, b) \in A$ for every pair a, b of vertices where $a \in N_G^{\text{in}}(x_n)$ and $b \in N_G^{\text{out}}(x_n)$. Let x_n be added to G' by chosing d-clique (A, B). By definition, every vertex in A is in-neighbour of every vertex in B. Let $a \in N_G^{\text{in}}(x_n)$ and $b \in N_G^{\text{out}}(x_n)$ with $a \neq b$, i.e., $a \in A$ and $b \in B$. Then, $(a, b) \in A(G)$. For the converse, let β be directed transitive for G. Then, β' is directed transitive for G', and by induction hypothesis, G' is a chordal digraph with construction sequence β'. Let $A =_{\text{def}} N_G^{\text{in}}(x_n)$ and $B =_{\text{def}} N_G^{\text{out}}(x_n)$. By definition of directed transitive, $(a, b) \in A(G)$ for every $a \in A$ and $b \in B$ and $a \neq b$. Hence, (A, B) is a d-clique in G', and G can be obtained from G' by adding x_n in the sense of Definition 3.1. Hence, G is a chordal digraph with construction sequence β. ∎

Since "directed transitive" generalises "topological ordering", it directly follows from Theorem 3.3 that acyclic digraphs are chordal. We will see later that acyclic digraphs are the only chordal digraphs that have at most one arc between every pair of vertices.

Lemma 3.4. *Every induced subgraph of a chordal digraph is chordal.*

For a digraph G, denote by $\text{rev}(G)$ the digraph on vertex set $V(G)$ and with arc set $A(\text{rev}(G))$ where for all $u, v \in V(G)$, $(u, v) \in A(\text{rev}(G))$ if and only if $(v, u) \in A(G)$. We call $\text{rev}(G)$ the *reverse graph* of G.

Lemma 3.5. *A digraph G is chordal if and only if* $\text{rev}(G)$ *is chordal.*

It is a natural question to ask whether chordality of a digraph can be determined by looking at the connected components separately. With the characterisation of Theorem 3.3, it is an easy observation that a digraph is chordal if and only if all its weakly connected components are chordal. A directed transitive vertex layout for the whole digraph can be constructed by concatenating directed transitive vertex layouts for the weakly connected components. However, the same is generally not true for strongly connected components. In fact, there is a non-chordal digraph on four vertices with only chordal strongly connected components.

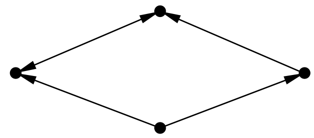

Figure 2: The dipicted digraph is chordal and has exactly one di-simplicial vertex.

Our first type of characterisations of chordal digraphs was by vertex layouts. The second type of characterisations involves vertices of special properties. In fact, we define and consider analogues of simplicial vertices for undirected chordal graphs.

Definition 3.6. *A vertex u of a digraph G is* di-simplicial *if* $(N_G^{\text{in}}(u), N_G^{\text{out}}(u))$ *is a d-clique of G.*

A vertex of an undirected graph is *simplicial* if its neighbourhood is a clique. It is an easy result that a vertex u of an undirected graph G is simplicial if and only if for every path P of G of length at least 1, $P - u$ is a path in G. An analogue characterisation holds for digraphs and di-simplicial vertices where paths are directed paths. In this sense, di-simplicial vertices **are** the directed analogue of simplicial vertices.

Lemma 3.7. *The first vertex of a directed transitive vertex layout is di-simplicial. In particular, every chordal digraph has a di-simplicial vertex.*

It is known that non-complete chordal undirected graph have two non-adjacent simplicial vertices [3]. The same is not generally true for chordal digraphs. The digraph in Figure 2 has exactly one di-simplicial vertex.

Theorem 3.8. *A digraph G is chordal if and only if G can be reduced to a digraph on a single vertex by repeatedly deleting an arbitrary di-simplicial vertex.*

Corollary 3.9. *There is an $\mathcal{O}(n^2m)$-time algorithm that recognises chordal digraphs.*

Proof. The algorithm applies Theorem 3.8. A vertex x is di-simplicial if every in-neighbour of x is in-neighbour of every out-neighbour of x. It takes $\mathcal{O}(m)$ time to check this condition. Thus, a di-simplicial vertex can be found in $\mathcal{O}(nm)$ time, which yields $\mathcal{O}(n^2m)$ time for the reduction algorithm. ∎

It is not difficult to see that there is a 1-to-1 correspondence between the directed transitive vertex layouts of a chordal digraph and the orderings defined by the elimination process in Theorem 3.8. Note that a result analogous to Theorem 3.8 exists for chordal undirected graphs [8].

4 Two classes of chordal digraphs

We consider two classes of digraphs and characterise their chordal digraphs. First, we consider digraphs that have at most one arc between every pair of vertices, and second, we consider digraphs that have two arcs between every pair of adjacent vertices. For a digraph G, we denote by $\text{uni}(G)$ the digraph on vertex set $V(G)$, and for all $u, v \in V(G)$, $(u, v) \in A(\text{uni}(G))$ if and only if $(u, v) \in A(G)$ and $(v, u) \notin A(G)$. In other words, $\text{uni}(G)$ is the restriction of G to the arcs that uniquely connected two vertices. We call $\text{uni}(G)$ the uni-*restriction* of G. As examples, if G is a tournament graph or an acyclic digraph then $\text{uni}(G) = G$.

Theorem 4.1. *A digraph G with* $\text{uni}(G) = G$ *is chordal if and only if G contains no chordless directed cycle of length at least 3.*

Proof. If G contains no chordless directed cycle of length at least 3 then G is acyclic (because of $\mathrm{uni}(G) = G$) and there is a topological ordering β for G. It is obvious that β is directed transitive for G. Now, let G contain a directed cycle $C = (x_1, \ldots, x_k)$. Let C be of shortest length. Note that $k \geq 3$. Suppose that G is chordal. Due to Theorem 3.3, there is a directed transitive vertex layout β for G. Without loss of generality, we can assume that $x_1 \prec_\beta x_2$ and $x_1 \prec_\beta x_k$. By definition of directed cycle, $(x_k, x_1), (x_1, x_2) \in A(G)$. Then, $(x_k, x_2) \in A(G)$, which contradicts the assumption $\mathrm{uni}(G) = G$ in case $k = 3$ or the choice of C as of shortest length in case $k \geq 4$. Hence, β does not exist, and G is not chordal. ∎

Corollary 4.2. *A digraph G with $\mathrm{uni}(G) = G$ is chordal if and only if G is acyclic.*

Every undirected graph is underlying graph of an acyclic digraph. So, every undirected graph is underlying graph of a chordal digraph. This also means that looking at underlying graphs of chordal digraphs cannot provide any insight into the structure of chordal digraphs.

The second class of digraphs in this section consists of the digraphs with always two arcs between every pair of adjacent vertices. For a digraph G, we denote by $\mathrm{bi}(G)$ the digraph on vertex set $V(G)$, and for all $u, v \in V(G)$, $(u, v) \in A(\mathrm{bi}(G))$ if and only if $(u, v) \in A(G)$ and $(v, u) \in A(G)$. We call $\mathrm{bi}(G)$ the bi-restriction of G. Clearly, $\mathrm{uni}(G)$ and $\mathrm{bi}(G)$ are complementary to each other. The digraphs G with $\mathrm{bi}(G)$ are the ones that are obtained from undirected graphs by replacing every edge $\{u, v\}$ by the two arcs (u, v) and (v, u). Note that for digraphs G with $\mathrm{bi}(G) = G$, the adjacency matrices of G and the underlying graph of G are equal.

Lemma 4.3. *If a digraph G is chordal then $\mathrm{bi}(G)$ contains no chordless directed cycle of length at least 4.*

Proof. Let G be chordal. Let β be a directed transitive vertex layout for G, that exists due to Theorem 3.3. Suppose that $\mathrm{bi}(G)$ contains a chordless directed cycle $(x_1, \ldots, x_k)$ of length at least 4. Without loss of generality, we can assume that $x_1 \prec_\beta x_i$ for all $2 \leq i \leq k$. By definition of directed cycle, $(x_k, x_1), (x_1, x_2) \in A(G)$, and by definition of directed transitive, $(x_k, x_2) \in A(G)$. Since C is a cycle in $\mathrm{bi}(G)$, it also holds that $(x_2, x_1), (x_1, x_k) \in A(G)$, and the definition of directed transitive yields $(x_2, x_k) \in A(G)$. This means that x_2 and x_k are adjacent in $\mathrm{bi}(G)$ and C cannot be chordless, a contradiction. ∎

In other words, Lemma 4.3 shows that for every chordal digraph G, the underlying graph of $\mathrm{bi}(G)$ is chordal.

Theorem 4.4. *A digraph G with $\mathrm{bi}(G) = G$ is chordal if and only if G contains no chordless directed cycle of length at least 4.*

Let H be a chordal undirected graph. Then, there is a digraph G with $\mathrm{bi}(G) = G$ such that H is the underlying graph of G. Since H is chordal, G contains no chordless directed cycle of length at least 4, and thus G is chordal due to Theorem 4.4. Hence, the chordal undirected graphs are exactly the underlying graphs of the bi-restriction of chordal digraphs.

5 Chordal semi-complete digraphs

A digraph is called *semi-complete* if every pair of vertices is adjacent. Pairs of vertices can be connected by one or two arcs. We give a complete characterisation of chordal semi-complete digraphs by forbidden induced subgraphs. We will obtain this result by mainly studying simplicial vertices in the underlying undirected graph of $\mathrm{bi}(G)$. Our approach to the forbidden induced subgraphs characterisation is to give a characterisation of semi-complete digraphs without di-simplicial vertices. Remember from Lemma 3.7 that every chordal digraph has a di-simplicial vertex.

We use the notion of uni- and bi-restriction defined in the previous section. The uni-restrictions will be our main study objects. Let F be a semi-complete digraph and let $G =_{\text{def}} \text{uni}(F)$. Let u, v, w be a vertex triple of F. We call (u, v, w) a *witness triple for u in G* if one of the following three conditions is satisfied:

– $(u, v), (v, u), (u, w), (w, u) \notin A(G)$ and $(v, w) \in A(G)$

– $(u, v), (v, w) \in A(G)$ and $(u, w), (w, u) \notin A(G)$, or
  $(w, v), (v, u) \in A(G)$ and $(u, w), (w, u) \notin A(G)$

– $(v, u), (u, w), (w, v) \in A(G)$.

We refer to the different schemes as "witness triple of the first, second or third type".

Lemma 5.1. *1) A vertex u of a semi-complete digraph F is di-simplicial if and only if there is no witness triple for u in* $\text{uni}(F)$.

2) If a vertex u of a digraph F is di-simplicial in F then u is simplicial in the underlying graph of $\text{bi}(F)$.

For the proof of our main result, we need two properties of chordal undirected graphs. Let x and y be adjacent simplicial vertices of an undirected graph H. Then, $N_H[x] = N_H[y]$. Equivalently, x and y have the same non-neighbours. The second tool property is the following: for H a chordal undirected graph u a vertex of H that is not universal, every connected component of $H \setminus N_H[u]$ contains a vertex that is simplicial in H. We are ready for the main result of this section.

Lemma 5.2. *Let F be a semi-complete digraph with the underlying graph of* $\text{bi}(F)$ *being chordal. If F is not chordal then* $\text{uni}(F)$ *contains one of the digraphs of Figure 1 as induced subgraph.*

Proof. Let F not be chordal. We first consider the case that F contains no di-simplicial vertex; the other case is discussed at the end of the proof. Let $G =_{\text{def}} \text{uni}(F)$ and let H be the underlying graph of $\text{bi}(F)$. By assumption, H is chordal. Due to Lemma 5.1, a vertex that is not simplicial in H is not di-simplicial in F. If a simplicial vertex of H is not di-simplicial in F then it is not di-simplicial because of orientations of arcs in G. Applying Lemma 5.1, the assumption that F contains no di-simplicial vertex means that every vertex of G has a witness triple in G. In particular, every simplicial vertex of H has a witness triple in G. If there is a vertex with a witness triple of the third type then G contains a copy of digraph (a) of Figure 1 as induced subgraph. Now, assume that all witness triples in G are of the first or second type. Let u be a simplicial vertex of H. Then, u cannot have a witness triple of the first type. Hence, all simplicial vertices of H have only witness triples of the second type in G. We construct an auxiliary digraph based on these witness triples and show the existence of a digraph of Figure 1 as induced subgraph.

Let S be the set of vertices of F that are simplicial in H. We construct a digraph D that has vertex set S, and there is an arc (u, v) if and only if there is a vertex w of G such that (u, v, w) is a witness triple for u in G. Notice for later that (u, v) arc in D particularly means that u and v are adjacent in G. We show that every vertex of D has an out-neighbour, which particularly implies that D is not acyclic. Let u be a vertex of S. By the above considerations, there is a witness triple (u, y, z) for u in G. Since u and z are adjacent to y in G (recall the definition of witness triple of the second type), u and z are non-adjacent to y in H. Furthermore, u and z are non-adjacent in G, thus adjacent in H. So, y is vertex in a connected component K of $H \setminus (N_H(u) \cup N_H(z))$. Since u is simplicial and z is a neighbour of u in H, $N_H(u) \subseteq N_H[z]$. There exists a vertex v from S in K. We want to show that (u, v, z) is a witness triple for u in G or G contains a copy of a digraph of Figure 1 as induced subgraph. Since K is connected, there is a spanning tree T for K. Note now that u and z are adjacent to all vertices of K in G; this directly follows from F being semi-complete.

Let all vertices of T be unmarked. We show by induction that (u, x, z) is a witness triple for u in G for all vertices x of T. Mark y in T. By assumption, the claim is true for every marked vertex of T. Let x' be an unmarked vertex of T that has a marked neighbour x'' in T. Note that x' and x'' are non-adjacent in G, since K is a connected induced subgraph of H. Mark x'. We consider the triple (u, x', z). Remember that u and z are non-adjacent in G and x' is adjacent to both u and z in G. Assume that (u, x', z) is not a witness triple for u in G. Then, $(u, x'), (z, x') \in A(G)$ or $(x', u), (x', z) \in A(G)$. This implies that $\{u, z, x', x''\}$ induces a copy of digraph (d) of Figure 1 in G. If no copy of digraph (d) has been detected, (u, v, z) is a witness triple for u in G, since all vertices of T have been marked, and thus D contains arc (u, v). We conclude that every vertex of D has an out-neighbour or G contains a copy of digraph (d) as induced subgraph. This completes the construction of D.

Assume that every vertex of D has an out-neighbour in D. Then, D is not acyclic and therefore contains a directed cycle. Let $C = (u_1, \ldots, u_k)$ be a directed cycle in D of shortest length. In particular, D has no chords. Assume $k = 2$. Without loss of generality, we can assume that $(u_1, u_2) \in A(G)$. Remember that u_1 and u_2 are indeed adjacent in G. By definition of D, there are vertices a, b such that (u_1, u_2, a) is a witness triple for u_1 and (u_2, u_1, b) is a witness triple for u_2 in G. Remember also that u_1 and a are non-adjacent and u_2 and b are non-adjacent in G. In particular, $a \neq b$. Then, $\{b, u_1, u_2, a\}$ induces a copy of digraph (b), (c) or (d) of Figure 1 in G, depending on how a and b are connected in F. Thus, the case $k = 2$ is done. Henceforth, let $k \geq 3$.

If $(u_1, u_2), \ldots, (u_{k-1}, u_k), (u_k, u_1) \in A(G)$ or $(u_1, u_k), (u_k, u_{k-1}), \ldots, (u_2, u_1) \in A(G)$ then G contains a directed cycle as subgraph, and therefore, G contains digraph (a) or (c) of Figure 1 as induced subgraph (depending on k and chords). Note that we cannot obtain a directed and chordless cycle of length more than 4 in the present case, since H is chordal and thus the complement of H contains no chordless cycles of length more than 4. Now, assume that $(u_1, \ldots, u_k)$ does not define a directed cycle in G. We can assume without loss of generality that $(u_1, u_2), (u_1, u_k) \in A(G)$. Suppose that there are $1 \leq i, i', i'' \leq k$ with $(u_i, u_{i'}, u_{i''})$ a witness triple for u_i in G. Then, $(u_{i''}, u_{i'}, u_i)$ is a witness triple for $u_{i''}$, and $(u_i, u_{i'})$ and $(u_{i''}, u_{i'})$ are arcs of D. By the definition of C, neither u_i nor $u_{i''}$ can be the vertex following $u_{i'}$ in C. Thus, C has a chord in D. But a chord contradicts the choice of C as of shortest length. Hence, no witness triple has all its vertices on C.

Suppose that $k \geq 4$. We consider the vertices u_1, u_2, u_k. Let a, b be vertices such that (u_1, u_2, b) is a witness triple for u_1 and (u_k, u_1, a) is a witness triple for u_k in G. Remember with the results from the previous paragraph that $(u_1, u_2), (u_1, u_k) \in A(G)$. Thus, $(u_1, u_2), (u_2, b) \in A(G)$ and $(a, u_1), (u_1, u_k) \in A(G)$. By the considerations above, a and b are not vertices on C. Furthermore, since a is adjacent to u_1 and b is non-adjacent to u_1 in G, $a \neq b$. If a and u_2 are non-adjacent in G then (u_2, u_1, a) is a witness triple for u_2 in G, which means $(u_2, u_1) \in A(D)$, and D has a cycle of length 2. This contradicts our assumption $k \geq 4$. Thus, a and u_2 are adjacent in G. If $(u_2, a) \in A(G)$ then $\{u_1, u_2, a\}$ induces a copy of digraph (a) of Figure 1 in G, and we are done. Otherwise, let $(a, u_2) \in A(G)$. Observe that a is adjacent to u_2 and non-adjacent to u_k in G. Since u_2 and u_k are simplicial in H and have different neighbourhoods, they are adjacent in G. If $(u_2, u_k) \in A(G)$ then (u_k, u_2, a) is a witness triple for u_k in G, which means $(u_k, u_2) \in A(D)$, and C has a chord. This contradicts the above results. Hence, $(u_k, u_2) \in A(G)$. If u_k and b are non-adjacent in G then (u_k, u_2, b) is a witness triple for u_k in G, and $(u_k, u_2) \in A(G)$. Since this yields a contradiction, u_k and b must be adjacent. If $(b, u_k) \in A(G)$ then $\{u_k, u_2, b\}$ induces a copy of digraph (a) of Figure 1 in G. Otherwise, if $(u_k, b) \in A(G)$ then (u_1, u_k, b) is a witness triple for u_1 in G, which means $(u_1, u_k) \in A(D)$, thus a contradiction. Since all cases lead to contradictions, we conclude that $k \not\geq 4$, i.e., $k = 3$.

We distinguish between two cases: $(u_2, u_3) \in A(G)$ and $(u_3, u_2) \in A(G)$. Let a, b, c be vertices such that (u_1, u_2, b), (u_2, u_3, c) and (u_3, u_1, a) are witness triples for respectively u_1, u_2, u_3 in G. Note that a, b, c are pairwise different vertices; for instance, a and b are different since a is adjacent to u_1 and b is non-adjacent to u_1. And since u_1, u_2, u_3 are pairwise adjacent in G,

$\{a, b, c\} \cap \{u_1, u_2, u_3\} = \emptyset$. Hence, u_1, u_2, u_3, a, b, c are pairwise different. For the first case, let $(u_2, u_3) \in A(G)$. This means that $(u_3, c) \in A(G)$. We consider a and u_2. If a and u_2 are non-adjacent in G then $\{a, u_1, u_2, b\}$ induces a copy of digraph (b) or (c) or (d) of Figure 1 in G. If $(u_2, a) \in A(G)$ then $\{a, u_1, u_2\}$ induces a copy of digraph (a) of Figure 1 in G. And if $(a, u_2) \in A(G)$ then $\{a, u_2, u_3, c\}$ induces a copy of digraph (b) or (c) or (d) of Figure 1 in G. For the second case, let $(u_3, u_2) \in A(G)$. This means that $(c, u_3) \in A(G)$. We consider b and u_3 and conclude in the above manner that $\{c, u_3, u_2, b\}$ (in case u_3 and b non-adjacent in G) or $\{u_3, u_2, b\}$ (in case $(b, u_3) \in A(G)$) or $\{a, u_1, u_3, b\}$ (in case $(u_3, b) \in A(G)$) induces a copy of a digraph of Figure 1 in G. Hence, we have found a copy of a digraph of Figure 1 as induced subgraph in G in every case, so that we can conclude the case when G has no di-simplicial vertex.

Now, we consider the case that F contains di-simplicial vertices. We apply Theorem 3.8 and conclude that F contains an induced subgraph F' on at least two vertices without di-simplicial vertex. Then, F' is not chordal and satisfies the conditions of the above case. Note that the underlying graph of $\mathrm{bi}(F')$ is chordal, since it is an induced subgraph of a chordal undirected graph. We apply the above case and conclude that $\mathrm{uni}(F')$ contains a copy of a digraph of Figure 1 as induced subgraph. This completes the proof. ■

Theorem 5.3. *A semi-complete digraph F is chordal if and only if the underlying graph of* $\mathrm{bi}(F)$ *is chordal and* $\mathrm{uni}(F)$ *does not contain a copy of any of the digraphs of Figure 1 as induced subgraph.*

Proof. Let H be the underlying graph of $\mathrm{bi}(F)$. If H is not chordal then $\mathrm{bi}(F)$ contains a chordless directed cycle of length at least 4. With Lemma 4.3, F is not chordal. Now, let $\mathrm{uni}(F)$ contain a copy C of a digraph of Figure 1 as induced subgraph. It is easy to check that every vertex of the digraphs of Figure 1 has a witness triple, so that C has no di-simplicial vertex due to Lemma 5.1. According to Lemma 3.7, C is not chordal. Hence, F contains a non-chordal induced subgraph, so that F is not chordal due to Lemma 3.4. For the converse, let F not be chordal and let H be chordal. Then, $\mathrm{uni}(F)$ contains a copy of one of the digraphs of Figure 1 as induced subgraph due to Lemma 5.2. This completes the proof. ■

The actual set of minimal forbidden induced subgraphs for chordal semi-complete digraphs is the following: all semi-complete digraphs F with the underlying graph of $\mathrm{bi}(F)$ a chordless cycle of length at least 4 and the four digraphs that are obtained from the digraphs of Figure 1 by adding the two arcs between every pair of non-adjacent vertices (of each digraph). We want to conclude with two remarks. Firstly, note that even though the actual set of minimal forbidden induced subgraphs for chordal semi-complete digraphs is much bigger than the set of minimal forbidden induced subgraphs for chordal undirected graphs (due to the many different orientations), the structure of chordal semi-complete digraph is already much richer than the structure of the whole class of chordal undirected graphs. This can give a first impression of the significant difference between directed and undirected graphs. Secondly, it is an interesting observation that each digraph of Figure 1 is isomorphic to its own reverse graph.

6 Conclusion

In Section 3, we have given two characterisations of chordal digraphs (Theorem 3.3 and Theorem 3.8), which are analogues of characterisations of chordal undirected graphs. We have also introduced the notion of *d-clique* as a directed analogue or even generalisation of the undirected notion of *clique*. A famous characterisation of chordal undirected graphs gives a connection between cliques and minimal separators [3]. Is there a directed notion of minimal separator that is connected to d-cliques in a similar way for chordal digraphs?

Another famous characterisation of chordal undirected graphs is as intersection graphs of subtrees of a tree. Can this be generalised to chordal digraphs? Let us remark that there is a generalisation of intersection graphs to 'intersection digraphs' that results in an interesting directed analogue of interval graphs [9, 11]. However, if using this definition of 'intersection digraphs' then *all digraphs* become representable by 'intersection subtrees' [5], see also Bang-Jensen and Gutin [1] [Proposition 4.13.2]. Thus, a different approach is needed to define the proper directed analogue of 'intersection graph of subtrees of a tree'.

On the structural side, the main open problem for chordal digraphs is a characterisation by forbidden induced subgraphs. We have given such characterisations for large subclasses of digraphs, such as digraphs G with $\mathrm{bi}(G)$ empty and semi-complete digraphs. The case of semi-complete digraphs shows that a forbidden induced subgraphs characterisation for the whole class of chordal digraphs is challenging. Are there other interesting classes of digraphs for which one can give a characterisation of its chordal digraphs by forbidden induced subgraphs?

On the algorithmic side, there are many interesting problems. In Corollary 3.9, we have given a chordal digraph recognition algorithm whose running time mainly depends on the time for finding a di-simplicial vertex. Our time for this problem is $\mathcal{O}(nm)$. Can this be reduced to $\mathcal{O}(n^2)$ or even linear time? For chordal undirected graphs, there are linear-time recognition algorithms that do not rely on finding simplicial vertices; instead they generate a vertex layout that has the property of Theorem 2.1 if and only if the input graph is chordal. Does there exist a recognition algorithm for chordal digraphs with a similar approach? A subproblem for such a recognition algorithm is the verification that a given vertex layout is directed transitive. Does there exist an $\mathcal{O}(n^2)$-time algorithm for this verificaton problem?

A main motivation for the study of chordal digraphs is their connection to Kelly-width. Can the structural properties exhibited by chordal digraphs be exploited algorithmically for graphs of bounded Kelly-width?

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Appendix

Proof of Lemma 3.4. Let G be a chordal digraph with directed transitive vertex layout β. Note that β exists due to Theorem 3.3. Let $U \subseteq V(G)$. Let $G' =_{\text{def}} G[U]$ and let β' be the restriction of β to the vertices in U. We claim that β' is directed transitive for G'. Let u, v, w be vertices of G' with $u \prec_{\beta'} v$ and $u \prec_{\beta'} w$ such that $(v, u), (u, w) \in A(G')$. By construction of β', $u \prec_\beta v$ and $u \prec_\beta w$, and $(v, u), (u, w) \in A(G)$. Since β is directed transitive for G, $(v, w) \in A(G)$, and since G' is an induced subgraph of G containing v and w, $(v, w) \in A(G')$. Hence, G' is chordal with construction sequence β' due to Theorem 3.3. ■

Proof of Lemma 3.5. Since $\text{rev}(\text{rev}(G)) = G$ for any digraph G, it suffices to show one implication. We even show a stronger result, namely that the two digraphs have the same directed transitive vertex layouts. So, let G be a chordal digraph with directed transitive vertex layout β. We show that β is directed transitive for $\text{rev}(G)$. Let u, v, w be a vertex triple of $\text{rev}(G)$ with $u \prec_\beta v$ and $u \prec_\beta w$ and $(v, u), (u, w) \in A(\text{rev}(G))$. By definition of $\text{rev}(G)$, $(w, u), (u, v) \in A(G)$, and since β is directed transitive for G, $(w, v) \in A(G)$. Then, $(v, w) \in A(\text{rev}(G))$. Hence, β is directed transitive for $\text{rev}(G)$, and $\text{rev}(G)$ is chordal due to Theorem 3.3. ■

Proof of Theorem 3.8. We prove the statement by induction over the number of vertices of G. If G is digraph on a single vertex then the statement is obviously true due to Definition 3.1. Now, let the statement by true for all digraphs on at most n vertices for some $n \geq 1$. Let G be a digraph on $n + 1$ vertices. Let G be chordal. Let u be an arbitrary di-simplicial vertex of G. Note that u exists due to Lemma 3.7. Due to Lemma 3.4, $G - u$ is chordal, and by application of the induction hypothesis, $G - u$ can be reduced in the sense of the statement. Hence, G can be reduced in the sense of the statement by the choice of u as an arbitrary di-simplicial vertex. For the converse, let G be reducible in the sense of the statement. Let u be the first vertex that is deleted. Note that u is di-simplicial in G. By induction hypothesis, $G - u$ is chordal. Let β' be an arbitrary directed transitive vertex layout for $G - u$, that exists due to Theorem 3.3. Obtain β from β' by adding u as first vertex. We show that β is directed transitive for G. It suffices to consider vertex triples u, x, y. Let $(x, u), (u, y) \in A(G)$. Then, $(x, y) \in A(G)$. Hence, G is chordal due to Theorem 3.3. ■

Proof of Theorem 4.4. If G is chordal then the statement follows from Lemma 4.3. For the converse, let G not contain a chordless directed cycle of length at least 4. Then, the underlying undirected graph H of G is chordal and thus has a vertex layout β that satisfies Theorem 2.1. We show that β is directed transitive for G. Let u, v, w be a vertex triple of G with $u \prec_\beta v$ and $u \prec_\beta w$ and $(v, u), (u, w) \in A(G)$. By definition of H, $\{v, u\}, \{u, w\} \in E(H)$, and thus $\{v, w\} \in E(H)$. Hence, $(v, w) \in A(G)$. We conclude that β is directed transitive for G and thus G is chordal due to Theorem 3.3. ■

Proof of Lemma 5.1. For statement 1, let $G =_{\text{def}} \text{uni}(F)$. Let u, v, w be a vertex triple of F such that (u, v, w) is a witness triple for u in G. We distinguish between the three cases. Let (u, v, w) be of the third type. This means that $(v, u), (u, w), (w, v) \in A(G)$. By definition of uni-restriction, it follows that $(v, w) \notin A(F)$, so that u is not di-simplicial in F. If (u, v, w) is witness triple of the first type then, by F semi-complete, $(w, u), (u, v) \in A(F)$ and $(w, v) \notin A(F)$. Note that this follows from $(v, w) \in A(G)$. Hence, u is not di-simplicial. For (u, v, w) witness triple of the second type, one of the two cases holds: $(w, u), (u, v) \in A(F)$ and $(w, v) \notin A(F)$ or $(v, u), (u, w) \in A(F)$ and $(v, w) \notin A(F)$. In both cases, it is important to observe that v and w are adjacent in G and therefore only one of the two arcs (v, w) or (w, v) is in F. Hence, u is not di-simplicial in F. For the converse, let u not be di-simplicial in F. Then, there are vertices v, w of F such that $(v, u), (u, w) \in A(F)$ and $(v, w) \notin A(F)$. Since F is semi-complete, $(w, v) \in A(F)$ and therefore $(w, v) \in A(G)$. Depending on whether $(u, v) \in A(F)$ or $(w, u) \in A(F)$ or not, $(v, u) \in A(G)$ or v and u are non-adjacent in G and $(u, w) \in A(G)$ or w and u are non-adjacent in G. Thus, (u, v, w) or (u, w, v) is a witness triple for u in G.

For statement 2, let H be the underlying graph of $\text{bi}(F)$. Let u be a vertex of H that is not simplicial in H. Due to definition, u has two non-adjacent neighbours v and w in H. By definition of H, this means that $(u, v), (v, u), (u, w), (w, u) \in A(F)$. Since $\{v, w\} \notin E(H)$, $(v, w) \notin A(F)$ or $(w, v) \notin A(F)$. Then, $(N_F^{\text{in}}(u), N_F^{\text{out}}(u))$ is not a d-clique in F because of $v, w \in N_F^{\text{in}}(u) \cap N_F^{\text{out}}(u)$. Hence, u is not di-simplicial in F. ∎