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A Survey of Maneuvering Target Tracking—Part III: Measurement Models*

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Abstract

This is the third part of a series of papers that provide a comprehensive survey of the techniques for tracking maneuvering targets without addressing the so-called measurement-origin uncertainty. Part I [1] and Part II [2] deal with general target motion models and ballistic target motion models, respectively. This part surveys measurement models, including measurement model-based techniques, used in target tracking. Models in Cartesian, sensor measurement, their mixed, and other coordinates are covered. The stress is on more recent advances — topics that have received more attention recently are discussed in greater details.

Key Words: Target Tracking, Measurement Model, Survey

1 Introduction

This paper is the third part of a series that provides a comprehensive survey of techniques for maneuvering target tracking without addressing the so-called measurement-origin uncertainty.

Most maneuvering target tracking techniques are model based; that is, they rely on explicitly two descriptions: one for the behaviors of the target, usually in the form of a motion (or dynamics) model, and the other for our observations of the target, known as an observation model. A survey of target motion models in general and ballistic target motion models in particular has been reported in Part I [1] and Part II [2], respectively. This part surveys the measurement models and the relevant modeling techniques.

More precisely, this paper surveys the models of measurements characterized by the following: they are truly originated from the "point target" under track (i.e., there is no origin uncertainty); and they are measurements, rather than observations in a more general sense, which may contain other information, including target features as provided by an imaging sensor. Also, this survey is concerned with the mathematical models as a basis for maneuvering target tracking. For example, their other applications are not addressed and the actual sensor models are not of concern. Further, this survey includes some aspects of estimation and filtering techniques that are highly dependent on and thus hardly separable from the measurement models.

As in our survey of motion models [1, 2], we highlight underlying ideas and try to clarify both explicit and implicit assumptions involved in each model, in an attempt to reveal pros and cons of the models. A considerable amount of discussion is given towards this end, much of which cannot be found elsewhere. We remind the reader, however, that these discussions, although intended to be accurate and balanced, are obviously not necessarily free of our personal preference and bias. Also, we focus on more recent advances in measurement models — topics that have received more attention recently are discussed in greater details. Furthermore, some models and techniques presented in this paper have not yet appeared elsewhere.

As stated in Part I, we appreciate receiving comments and any missing material that should be included in this part.

The rest of the paper is organized as follows. Sec. 2 describes measurement models in the original sensor coordinates. Sec. 3 provides a general view of the roles of coordinate systems in maneuvering target tracking. Linearized measurement models in (Cartesian-sensor) mixed coordinates are presented in Sec. 4. Measurement models converted to Cartesian coordinates are covered in Sec. 5. Pseudomeasurement modeling techniques are surveyed in Sec. 6. Finally, concluding remarks are given in Sec. 7.

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2 Models in Sensor Coordinates

Sensors used for target tracking provide measurements of a target in a natural sensor coordinate system (CS) or frame. In many cases (e.g., with a dish radar), this CS is spherical in 3D or polar in 2D with range r, bearing (or azimuth) b, elevation e (Fig. 1)[1], and possibly range rate (or Doppler) $\dot{r}$. (We do not explicitly consider direct measurements of target height, as provided by, e.g., Mode C. Such measurements are usually not available for non-cooperative targets.) Not all these measurement components are available from all sensors. For example, some active sensors may not provide range rate or elevation angle, while passive sensors provide only angles (although passive ranging is possible). We consider generally the 3D case — the respective 2D case follows in a straightforward manner.

In the sensor coordinates, these measurements are generally modeled in the following form of additive noise

$$\begin{aligned} r &= \mathsf{r} + v_r &(1)\\ b &= \mathsf{b} + v_b &(2)\\ e &= \mathsf{e} + v_e &(3)\\ \dot{r} &= \dot{\mathsf{r}} + v_{\dot{r}} &(4) \end{aligned}$$

where $(\mathsf{r}, \mathsf{b}, \mathsf{e})$ denotes the *error-free true* target position in the sensor spherical coordinates, and $v_r, v_b, v_e, v_{\dot{r}}$ are the respective random measurement errors. We assume these measurements are made at time t_k (or k for short) but we will omit the time index k whenever possible without ambiguities. It is normally assumed that these measurement errors in the sensor CS are zero-mean, Gaussian distributed, and uncorrelated:

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k) \quad \text{with} \quad R_k = \text{cov}(\mathbf{v}_k) = \text{diag}(\sigma_r^2,\ \sigma_b^2,\ \sigma_e^2,\ \sigma_{\dot{r}}^2) \tag{5}$$

where $\mathbf{v}_k = [v_r, v_b, v_e, v_{\dot{r}}]'_k$ is the error vector[2] at time k and $\{\mathbf{v}_k\}$ is a white noise sequence.

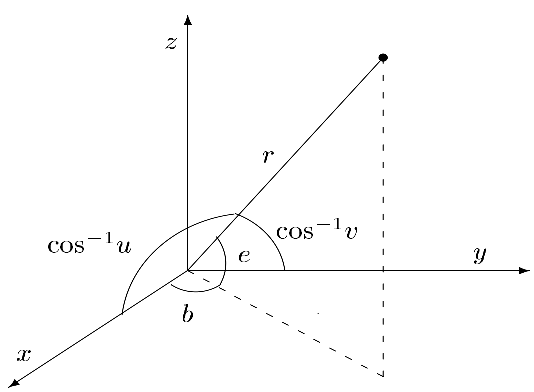

Fig. 1: Sensor coordinate systems.

The above measurement model in spherical coordinates is most common for track-while-systems, e.g., rotating surveillance radars [3, 4, 5]. For scan-while-track surveillance systems [6], such as phased array radars [7, 8, 9], the sensor provides measurements in terms of the direction cosines u and v — instead of the bearing b and elevation e — of the target position (Fig. 1) relative to reference axes. The RUV measurement model is as given above with (2)–(3) replaced by

$$\begin{aligned} u &= \mathsf{u} + v_u &(6)\\ v &= \mathsf{v} + v_v &(7) \end{aligned}$$

where u and v denote the *error-free* target position direction cosines, and v_u, v_v are the respective measurement errors. Sometimes a third direction cosine measurement $w = \mathsf{w} + v_w$ is used for convenience, albeit redundant ($w = \sqrt{1 - u^2 - v^2}$).

[1] Other conventions are also used. For example, the bearing may be defined as the angle from the y axis, rather than from the x axis.

[2] *We use Sans Serif letters (e.g.,* z*) to denote ideal error-free quantities and bold-face letters to denote vectors.*

It is also normally assumed that the errors are zero-mean, Gaussian distributed, and uncorrelated:

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k) \quad \text{with} \quad R_k = \text{cov}(\mathbf{v}_k) = \text{diag}(\sigma_r^2,\ \sigma_{\dot{r}}^2,\ \sigma_u^2,\ \sigma_v^2) \tag{8}$$

with $\mathbf{v}_k = [v_r, v_u, v_v, v_{\dot{r}}]_k'$.

Note that the RUV-CS is not orthogonal. Nevertheless, the above uncorrelatedness assumption cov($\mathbf{v}_k$) $=$ diag($\sigma_r^2,\ \sigma_u^2,\ \sigma_v^2,\ \sigma_{\dot{r}}^2$) is well justified by the fact that r, u, and v are measured by three physically independent systems.

The above two models arise naturally from the measurement process. They are linear and Gaussian and can be written compactly in vector-matrix notation

$$\mathbf{z} = H\mathbf{x} + \mathbf{v}, \qquad \mathbf{v} \sim \mathcal{N}(\mathbf{0}, R) \tag{9}$$

where $\mathbf{z} = [r,\ b,\ e,\ \dot{r}]'$ or $\mathbf{z} = [r,\ u,\ v,\ \dot{r}]'$, $\mathbf{x} = [\mathsf{r},\ \mathsf{b},\ \mathsf{e},\ \dot{\mathsf{r}},\ \ldots]'$ or $\mathbf{x} = [\mathsf{r},\ \mathsf{u},\ \mathsf{v},\ \dot{\mathsf{r}},\ \ldots]'$, $\mathbf{v} = [v_r, v_b, v_e, v_{\dot{r}}]'$ or $\mathbf{v} = [v_r, v_u, v_v, v_{\dot{r}}]'$, $H = [I, O]$, and I is an identity matrix and O is a zero matrix.

Range and angle measurements may have vastly different accuracies. For example, a phased array radar has range measurements much more accurate than angle measurements — its error ellipsoid looks like a pancake normal to the range vector; the situation is reversed for a continuous-wave radar, which often has a cigar-shaped error ellipsoid along the range vector [8].

This linear model is completely uncoupled across different coordinates. This is highly desirable for estimation and filtering in a number of aspects. For example, efficient parallel processing may be accomplished with little or no performance degradation. More important, coordinate-decoupled filters may be implemented that mitigate the possible ill conditioning arising from the vastly different accuracies in measuring range and angles [8]. These decoupled filters may possibly outperform the theoretically superior full-blown "optimal" filters in the presence of really ill conditioning.

3 Tracking in Various Coordinates

Various coordinate systems (CS) have been used in target tracking, including the Earth-centered inertial (ECI), Earth-centered (Earth) fixed (ECF, ECEF, or ECR), East-North-Up (ENU), and radar face (RF) coordinate systems. A concise description of these coordinate systems has been given in Part II [2]. Many factors affect the selection of a coordinate systems [8, 10, 11, 9]. The ENU-CS is a common choice for tactical systems with relatively limited sensor motion, such as in a platform-centric system. The ECI-CS, along with its variant ECF-CS, is a typical choice for a strategic system involving multiple platforms.

As far as tracking accuracy is concerned, the probably best choice of a coordinate system in principle is to align its coordinates to the principal axes of the tracking error ellipsoid [8]. This will avoid the corruption of an accurate estimate component by inaccurate ones. It also provides a good framework against ill conditioning. Since these principal axes usually vary with respect to time in a complex way, a sensible strategy is to align the coordinates to the principal axes of either the measurement error ellipsoid or dynamic error ellipsoid, depending on which error has more important directionality properties. Following this principle, several coordinate systems were discussed in [8] in the context of a single sensor, including radar-oriented, target-oriented, and their combinations.

A measurement is often described in a sensor reference frame, which is usually stabilized relative to the motion of the sensor, even if a different CS is selected for tracking purposes. It is in general different from a platform (or site) CS (e.g., the ENU-CS) when multiple sensors are involved in the platform (e.g., a ship). While the geographical ENU frame centered at a sensor is convenient for a rotating track-while-scan radar, the sensor-specific radar face CS is more often used with a phased array radar, where x and y axes are in the radar face plane and z axis along the boresight direction.

For detailed considerations of the coordinates systems and the respective transformations, the reader is referred to [7, 8, 10, 12, 11, 9].

In the sequel, by a Cartesian coordinate system, we mean a generic one unless otherwise is stated explicitly; and by a sensor frame (or CS), we mean non-Cartesian (spherical or RUV) CS in which the measurements are available directly without coordinate transformation.

Target motion is best described in a Cartesian CS, but measurements are available physically in a sensor CS. As such, there are basically four possibilities of do tracking: tracking in mixed coordinates, in Cartesian coordinates, in sensor coordinates, and in other coordinates. These are described next.

3.1 Tracking in Mixed Coordinates

This is the most popular approach. The target dynamics and measurements are modeled by

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{v} \tag{10}$$

where the target state $\mathbf{x}$ and process noise are in the *Cartesian* coordinates, but measurement $\mathbf{z}$ and its additive noise $\mathbf{v}$ are in the *sensor* coordinates[3]. Let (x, y, z) be the true position of the target in the Cartesian coordinates. For the case of spherical measurements, we have $\mathbf{z} = [r,\ b,\ e,\ \dot{r}]'$ and $\mathbf{h}(\mathbf{x}) = [\mathsf{r}, \mathsf{b}, \mathsf{e}, \dot{\mathsf{r}}]' = [h_r, h_b, h_e, h_{\dot{r}}]'$ with

$$h_r = \mathsf{r} = \sqrt{x^2 + y^2 + z^2} \tag{11}$$

$$h_b = \mathsf{b} = \tan^{-1}\frac{y}{x} \tag{12}$$

$$h_e = \mathsf{e} = \tan^{-1}\frac{z}{\sqrt{x^2 + y^2}} \tag{13}$$

$$h_{\dot{r}} = \dot{\mathsf{r}} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} \tag{14}$$

For RUV measurements, we have $\mathbf{z} = [r,\ u,\ v,\ \dot{r}]'$ and $\mathbf{h}(\mathbf{x}) = [\mathsf{r}, \mathsf{u}, \mathsf{v}, \dot{\mathsf{r}}]' = [h_r, h_u, h_v, h_{\dot{r}}]'$ with

$$h_u = \mathsf{u} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \tag{15}$$

$$h_v = \mathsf{v} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \tag{16}$$

(and $h_w = \mathsf{w} = \frac{z}{\sqrt{x^2+y^2+z^2}}$). Clearly the measurement models are nonlinear and coupled across Cartesian coordinates, although the measurement noise remains zero mean, Gaussian, and uncorrelated because measurements are in the sensor coordinates.

Most nonlinear estimation and filtering techniques, such as the extended Kalman filters (EKF), for maneuvering target tracking have been applied in this framework. Those based on measurement models are addressed in subsequent sections. Many other techniques are covered in subsequent parts of this survey series. A typical implementation of the EKF in mixed coordinates (Cartesian state and spherical measurements) can be found in [13].

3.2 Tracking in Cartesian Coordinates

In this approach, the measurements in the sensor coordinates are converted to the Cartesian coordinates for tracking. Clearly, any measurement expressed in the sensor coordinates has an exact and equivalent representation in the Cartesian coordinates. Let $\mathbf{x}_p = [x, y, z]' = H\mathbf{x}$ be the equivalent representation in the Cartesian coordinates of the *error-free* sensor measurement $(\mathsf{r}, \mathsf{b}, \mathsf{e})$ or $(\mathsf{r}, \mathsf{u}, \mathsf{v})$, with target state $\mathbf{x}$ and some H, for example, $H = [I,\ O]$ if $\mathbf{x} = [x, y, z, \ldots]'$. Clearly, $\mathbf{x}_p$ is in fact the true position of the target in the Cartesian coordinates, not known to us. Once the noisy measurements[4] of the target position are converted to the Cartesian coordinates (i.e., the noisy measurements originally available in sensor coordinates are expressed in the Cartesian coordinates), the measurement equation takes the following "linear" form in the Cartesian coordinates: $\mathbf{z}_c = \mathbf{x}_p + \mathbf{v}_c$, that is,

$$\mathbf{z}_c = H\mathbf{x} + \mathbf{v}_c \tag{17}$$

This measurement is sometimes referred to as a pseudolinear measurement. This model apparently "eliminates" the need to handle nonlinear measurements, in contrast to the above approach of tracking in mixed coordinates. The major advantage of this approach is that a linear Kalman filter then can be applied if the dynamics is linear.

Prior to [14] the measurement noise $\mathbf{v}_c$ was crudely treated to have zero mean and a covariance determined by a first-order Taylor series expansion. Since then several techniques have been developed to compute or account for the nonzero mean and the covariance more accurately (see, e.g., [14, 15, 16, 17, 18, 19]). These techniques are surveyed in Sec. 5.

We emphasize that the measurement noise $\mathbf{v}_c$ is in general not only coupled across coordinates, non-Gaussian, but also state dependent. This state dependency is probably more important but is largely ignored or overlooked in the literature

[3]This measurement model with additive noise corresponds to (9). The noise $\mathbf{v}$ is not necessarily in the sensor frame if the more general model $\mathbf{z} = \mathbf{h}(\mathbf{x}, \mathbf{v})$ is used.

[4]That is, the actual observed *values* (i.e., realizations), not the observables as (random) variables.

beyond its implicit use in the computation of the first two moments of $\mathbf{v}_c$. Due to the nonlinear dependency of $\mathbf{v}_c$ on the state $\mathbf{x}_p$, this measurement model is in fact nonlinear. As a result, even if the measurement conversion is done ideally with exact knowledge of the (state-dependent) first two moments of $\mathbf{v}_c$, it is still an illusion that the application of the Kalman filter here in the case of linear dynamics yields optimal results. Nevertheless, since the state dependence (i.e., nonlinearity) exists only in the measurement *noise* $\mathbf{v}_c$, rather than in the measurement *function* $\mathbf{h}(\cdot)$, it seems reasonable to expect that its impact on tracking performance is relatively smaller, as compared to the nonlinearity in $\mathbf{h}(\cdot)$ when handled by most popular nonlinear filtering techniques. On the other hand, a major drawback of this approach stems from a lack of available techniques to handle measurements with state-dependent, non-Gaussian errors, whereas abundant techniques are available for measurements with nonlinear $\mathbf{h}(\cdot)$. We have developed an extension of the Kalman filter for such problems that explicitly accounts for the state dependence of the measurement noise more effectively, as reported in [20].

Another weakness of this approach is that the conversion from sensor to Cartesian coordinates requires knowledge of range. For angle-only measurements, an estimated range can be used. However, the converted measurements have a degraded accuracy when an inaccurate range is used, such as for passive sensors or range-denial countermeasures. Indeed, angle-only measurements are rarely converted to the Cartesian coordinates with few exceptions, one of which is given in [21]. In addition, it is difficult to develop coordinate-decoupled filters in pure Cartesian coordinates for mitigating the possible ill conditioning due to the large difference in the accuracies of range and angle measurements, as well as for high efficiency.

The measurement conversion as described above does not deal with range rate measurements $\dot{r}$. It is substantially more complex when the range rate measurements are involved. In this case, the converted measurements are nonlinear (not even pseudolinear). The use of $d \stackrel{\Delta}{=} r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z} + v_d$ as a measurement of position (x, y, z) and velocity $(\dot{x}, \dot{y}, \dot{z})$ for tracking in the Cartesian coordinates was suggested in [22]. Note that this measurement is quadratic in the state, which is not highly nonlinear. This is clearly superior to converting the range rate measurements $\dot{r} = \frac{x\dot{x}+y\dot{y}+z\dot{z}}{\sqrt{x^2+y^2+z^2}} + v_{\dot{r}}$ directly, which is highly nonlinear. For uncorrelated range and range rate errors, however, the measurement d has an error $v_d = r\dot{r} - \mathrm{r}\dot{\mathrm{r}}$ with zero mean but variance $\mathrm{r}^2\sigma_{\dot{r}}^2 + \dot{\mathrm{r}}^2\sigma_r^2 + \sigma_r^2\sigma_{\dot{r}}^2$, which can be quite large for long-range targets.

3.3 Tracking in Sensor Coordinates

Alternatively, target dynamics can be converted from the Cartesian to the sensor coordinates so that the desirable measurement structure is unaltered, in contrast to converting measurements from sensor to Cartesian coordinates. However, expressing typical target motions in sensor coordinates (spherical or RUV) leads to highly nonlinear, coordinate-coupled, and sometimes cumbersome models. For example, a constant-velocity (CV) motion has a simple Cartesian description with two or three independent two-state one-dimensional CV models. The same motion in the spherical coordinates is rather nonlinear and complicated, an explicit model of which can be found in, e.g., [3]. Nontrivial, variable accelerations (known as pseudoaccelerations) [10, 4, 11] are induced in the sensor coordinates by such a conversion, even for a perfect CV motion, and thus a state vector including acceleration components is needed. In short, it is impossible to describe typical target motions in the sensor coordinates in a simple, coordinate-uncoupled way. Further, the converted process noise is non-Gaussian and state dependent even if it is Gaussian and coordinate-uncorrelated in the original Cartesian coordinates.

Nevertheless, this approach has certain advantages. The foremost one is that the linear, uncoupled, Gaussian structure of the measurement model is maintained. A large number of tracking filters that operate purely in sensor coordinates have appeared in the literature. Their common feature is the use of the above linear-Gaussian measurement model. Their key difference lies in how the target dynamics are modeled. A detailed coverage of these models is beyond the scope of this part, which is supposed to cover *measurement* models. For completeness, however, we mention briefly below these techniques and direct the reader to the specific references.

A simplistic approach is to directly employ some decoupled 1D target dynamics models, such as the CV, CA, and Singer models (see Part I), for range (range rate) and other measurements (angles or direction cosines) separately. This approach accounts for the target dynamics in the sensor coordinates in a crude way; it does not really convert the target dynamics to the sensor coordinates. As explained above, however, high-order models that include accelerations are needed to "cover" the actual highly nonlinear dynamics in the sensor coordinates even for a truly CV motion. This leads to accuracy degradation. The geometry-induced pseudoaccelerations are clearly not accounted for if a two-state "CV" model is used in each of the sensor coordinates independently. An engineering fix is to compensate the resulting bias [23, 24].

A more effective approach is to use the target dynamics model actually converted in the sensor coordinates. This leads to a filtering problem with linear uncoupled measurements in Gaussian noise but nonlinear dynamics and non-Gaussian, state-dependent process noise. Albeit theoretically and computationally challenging, this approach is beneficial in a number of cases [7, 25, 26, 27, 5]. Decoupled first-order Markov motion models in polar coordinates for maneuvering aircraft tracking were used in [28, 29, 30]. A similar, decoupled motion model in spherical coordinates was proposed in [31]. Its completely coupled version was derived from the Cartesian version in [32]. For the developments in the context of ballistic target tracking, the reader is referred to Part II. For example, [5] reported the development of a tracking filter where the reentry vehicle dynamics are modeled directly in the spherical coordinates. These filters operate entirely in the sensor coordinates. Sec. 4.2.1 of Part II contains a more detailed discussion of comparison between reentry-vehicle tracking in Cartesian and in sensor coordinates.

3.4 Tracking in Other Coordinates

Although target motion and measurements are best described in Cartesian and sensor coordinates, respectively, it is clearly not necessary to do tracking entirely in one or both of these coordinates. The modified Cartesian coordinates [33, 34] and the better-known modified polar coordinates [35] for angle-only tracking are good examples. Also, it is fairly common to propagate the target state in a Cartesian frame and then convert the predicted state, along with the error covariance, to the sensor coordinates for state update there (see, e.g., [7, 36, 37, 12, 27]). As such, while state update is decoupled across coordinates, state prediction is in general coupled. This approach relies heavily on coordinate transformation: In addition to the conversion of the predicted state, the updated state and its error covariance must be converted back to the Cartesian coordinates. The covariance conversions usually rely on linearization of the error models and is possibly biased, not to emphasize the state dependency inherent in the approach.

Alternatively, the use of the so-called radar principal Cartesian coordinates was suggested in [8], which is an integration of the Cartesian and the original sensor coordinates in that the range vector in the original sensor frame is retained as an axis in this orthogonal Cartesian frame. (The other two axes quantify angular components, one parallel to the radar face, the other in the plane normal to the radar face and containing the range vector.) In the same spirit, a scheme based on range and angular models was developed in [10, 11] in the orthogonal range-horizontal-“vertical” (RHV) frame [5], which involve range (and range rate) and, in Cartesian (H and V) coordinates, angular velocity and acceleration. Similar approaches were also taken in [38, 22, 39, 40, 41, 42]. Thanks to the weak coupling between the range (range rate) and the non-range coordinates, a merit of this approach is that range and non-range coordinates are processed in a quasi-independent manner, capable of alleviating ill conditioning and having high efficiency. However, the measurement models in the non-range coordinates are no longer linear. In essence this approach combines, in a sensible way, the frameworks in purely sensor coordinates described above by using the range coordinate, and in mixed coordinates as described in Sec. 3.1 for the non-range coordinates. An additional advantage of using range as a coordinate is that range rate measurements can be incorporated nicely and easily.

4 Linearized Models in Mixed Coordinates

Throughout this paper, we consider only a generic filtering cycle from t_{k-1} to t_k (i.e., from $k-1$ to k) and write $\bar{\mathbf{x}}$ for the predicted state $\hat{\mathbf{x}}_{k|k-1}$ and $\hat{\mathbf{x}}$ for the updated state $\hat{\mathbf{x}}_{k|k}$. The associated error covariance matrices are denoted by $\bar{P}$ and P, respectively.

The “standard” technique for handling the nonlinear measurement model (10) is the extended Kalman filters (EKF) (see, e.g., [43, 44, 45])[6]. In general, it relies on approximating the nonlinear measurement by the first few terms of its Taylor series expansion. Specifically, the cornerstone of its first-order version, which is most widely used, is linearization of the nonlinear model, resulting in a derivative-based linearized model. Other linearized models have also been proposed to handle nonlinear measurements. We describe these linearized models next.

4.1 Derivative-Based

The most widely used technique for linearizing a nonlinear measurement model in the form of (10) is to expand the measurement function $\mathbf{h}(\mathbf{x})$ at the predicted state $\bar{\mathbf{x}}$ and ignore all nonlinear terms[7]:

$$\mathbf{h}(\mathbf{x}) \approx \mathbf{h}\left(\bar{\mathbf{x}}\right) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\bar{\mathbf{x}}} \left(\mathbf{x} - \bar{\mathbf{x}}\right) \tag{18}$$

[5]The H axis is really horizontal, but the V axis is not really vertical. They are both perpendicular to the range vector.

[6]The first application of the Kalman filter to a real-life problem was in fact in the form of an EKF (see, e.g., [46]).

[7]More generally, if a completely general nonlinear model $\mathbf{z} = \mathbf{h}\left(\mathbf{x}, \mathbf{v}\right)$ is considered, we would have

$$\mathbf{h}(\mathbf{x}, \mathbf{v}) \approx \mathbf{h}\left(\bar{\mathbf{x}}, \bar{\mathbf{v}}\right) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\bar{\mathbf{x}}} \left(\mathbf{x} - \bar{\mathbf{x}}\right) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{v}}\right|_{\mathbf{v}=\bar{\mathbf{v}}} \left(\mathbf{v} - \bar{\mathbf{v}}\right)$$

Such a model does not necessarily have additive noise in the sensor coordinates; for example, it may have additive noise in the Cartesian coordinates.

This amounts to approximating the nonlinear model (10) by the linear model

$$\mathbf{z} = H(\bar{\mathbf{x}})\,\mathbf{x} + \mathbf{d}(\bar{\mathbf{x}}) + \mathbf{v} \tag{19}$$

where $H(\bar{\mathbf{x}}) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\bar{\mathbf{x}}}$ is the Jacobian of $\mathbf{h}(\mathbf{x})$ and $\mathbf{d}(\bar{\mathbf{x}}) = \mathbf{h}(\bar{\mathbf{x}}) - H(\bar{\mathbf{x}})\,\bar{\mathbf{x}}$.

For this model the predicted state and its covariance are updated using the linear Kalman filter equations

$$\begin{aligned} K &= \bar{P}H'\left(H\bar{P}H' + R\right)^{-1} && (20) \\ \hat{\mathbf{x}} &= \bar{\mathbf{x}} + K(\mathbf{z} - \bar{\mathbf{z}}) && (21) \\ P &= (I - KH)\,\bar{P} && (22) \\ &= (I - KH)\,\bar{P}\,(I - KH) + KRK' && (23) \end{aligned}$$

where $H = H(\bar{\mathbf{x}})$ and $\bar{\mathbf{z}} = H(\bar{\mathbf{x}})\,\bar{\mathbf{x}} + \mathbf{d}(\bar{\mathbf{x}}) + E[\mathbf{v}] = \mathbf{h}(\bar{\mathbf{x}}) + E[\mathbf{v}]$. Note that H is used only in the covariance update and filter gain computation and that (23) is valid for arbitrary gain K and H. These facts are used in some techniques discussed later. Although appears almost everywhere, covariance update by (22) should be avoided for at least two reasons: It invites horrible numerical problems and it is theoretically valid only when the gain K is truly optimal, which is rarely the case in practice. It has been largely overlooked that the gain given by (20) is no longer optimal and thus can be improved since it ignores the linearization errors.

This linearized model is adequate only when $\mathbf{x}-\bar{\mathbf{x}}$ is sufficiently small, which can rarely be guaranteed since the accuracy of $\bar{\mathbf{x}} = \hat{\mathbf{x}}_{k|k-1}$ relies on that of target state propagation (i.e., dynamics model) and the previous state estimate $\hat{\mathbf{x}}_{k-1|k-1}$. This inaccuracy may build up and result in filtering divergence, as reported in numerous examples (see, e.g., [43, 47]). Techniques aimed at reducing linearization errors are discussed in Sec. 4.4.

4.2 Difference-Based

We present now a new linearized model, proposed in [48], that is not only expectably more accurate but also potentially simpler than the above widely used derivative-based model.

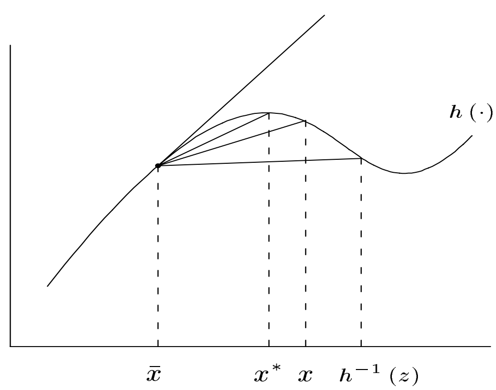

Fig. 2: Various linearizations.

Consider first a scalar nonlinear measurement $z = h(x) + v$ for simplicity. Let

$$H(x, \bar{x}) = \frac{h(x) - h(\bar{x})}{x - \bar{x}}, \quad \forall x \neq \bar{x} \tag{24}$$

Clearly, $H(x^*, \bar{x})$ is the slope of the straight line connecting $h(x^*)$ and $h(\bar{x})$ (see Fig. 2). For convenience, we denote

$$H(\bar{x}, \bar{x}) = \lim_{x \to \bar{x}} \frac{h(x) - h(\bar{x})}{x - \bar{x}} = H(\bar{x}) = \left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}} \tag{25}$$

which is the slope of the tangent of $h(x)$ at $\bar{x}$. If x^* is a better estimate than $\bar{x}$, it is reasonable to expect that

$$z = h(\bar{x}) + H(x^*, \bar{x})(x - \bar{x}) + v \tag{26}$$

is a better linearized model than the derivative-based model $z = h(\bar{x}) + H(\bar{x}, \bar{x})(x - \bar{x}) + v$.

In the vector case, this *difference-based linearized model* of (10) is

$$\mathbf{z} = \mathbf{h}(\bar{\mathbf{x}}) + H(\mathbf{x}^*, \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{v} \tag{27}$$

where

$$H(\mathbf{x}^*, \bar{\mathbf{x}}) = [H_{ij}], \quad H_{ij} = \frac{h_i(x_j^*, \bar{\mathbf{x}}) - h_i(\bar{\mathbf{x}})}{x_j^* - \bar{x}_j}, \quad h_i = i\text{th row of } \mathbf{h}, \quad h_i(x_j^*, \bar{\mathbf{x}}) = h_i(\mathbf{x})|_{\mathbf{x}=[\bar{x}_1,\ldots,\bar{x}_{j-1},x_j^*,\bar{x}_{j+1},\ldots,\bar{x}_n]'} \tag{28}$$

Clearly it is extremely easy to implement this linearized model. It does not involve computation of any Jacobian, which could be theoretically and/or computationally challenging for a complicated nonlinear function $\mathbf{h}$. It is expectably more accurate in general than the derivative-based linearized model, as widely used in the EKF, provided $\mathbf{x}^*$ is a more accurate estimate of $\mathbf{x}$ than $\bar{\mathbf{x}}$.

Several ways of determining $\mathbf{x}^*$ are possible. First, without loss of generality for tracking applications, assume that $\mathbf{h} = [\mathbf{h}_1', \mathbf{h}_2']'$, where $\mathbf{h}_1$ is invertible. Let $\mathbf{x}_1 = \mathbf{h}_1^{-1}(\mathbf{z})$. In the case of a 3D measurement of the target position, for example, $\mathbf{x}_1$ would be the 3D target position. We can then choose $\mathbf{x}^* = [\mathbf{x}_1', \bar{\mathbf{x}}_2']' = [\mathbf{h}_1^{-1}(\mathbf{z}), \bar{\mathbf{x}}_2']'$. For the components H_{ij} corresponding to $x_j^* = \bar{x}_j$, the derivatives $H_{ij} = \frac{\partial h_i}{\partial x}\big|_{x_j^*=\bar{x}_j}$ can be used. Alternatively, we can first update the state estimate from $\bar{\mathbf{x}}$ to $\hat{\mathbf{x}}$ as in an EKF (no need for covariance update here though) and then use $\mathbf{x}^* = \hat{\mathbf{x}}$ in the above linearized model. The use of this model will lead to at least an expectably more accurate covariance update for a state update in the form of $\hat{\mathbf{x}} = \bar{\mathbf{x}} + K(\mathbf{z} - \bar{\mathbf{z}})$ [48]:

$$P = [I - KH(\mathbf{x}^*, \bar{\mathbf{x}})]\,\bar{P}\,[I - KH(\mathbf{x}^*, \bar{\mathbf{x}})]' + KRK' \tag{29}$$

4.3 Optimally Linearized Model

The above derivative-based and difference-based linearization models in general have no optimality and can be quite bad in many cases. We now outline a linearized model that is optimal in the mean-square error (MSE) sense [44], as presented in [48] for tracking.

A nonlinear function $\mathbf{h}(\mathbf{x})$ can be approximated *optimally* around $\bar{\mathbf{x}}$ by a linear one:

$$\mathbf{h}(\mathbf{x}) \approx \mathbf{a} + H(\mathbf{x} - \bar{\mathbf{x}}) \tag{30}$$

in the sense of having the minimum MSE, denoting $\tilde{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}}$,

$$J = E[(\mathbf{h}(\mathbf{x}) - \mathbf{a} - H\tilde{\mathbf{x}})'(\mathbf{h}(\mathbf{x}) - \mathbf{a} - H\tilde{\mathbf{x}})] \tag{31}$$

It can be shown that [48]

$$\mathbf{a} = \{E[\mathbf{h}(\mathbf{x})] - E[\mathbf{h}(\mathbf{x})\tilde{\mathbf{x}}']\bar{P}^{-1}E[\tilde{\mathbf{x}}]\}/(1 - E[\tilde{\mathbf{x}}]'\bar{P}^{-1}E[\tilde{\mathbf{x}}]) \tag{32}$$

$$H = \{E[\mathbf{h}(\mathbf{x})\tilde{\mathbf{x}}'] - E[\mathbf{h}(\mathbf{x})]E[\tilde{\mathbf{x}}']\}\bar{P}^{-1}(I - E[\tilde{\mathbf{x}}]E[\tilde{\mathbf{x}}]'\bar{P}^{-1})^{-1} \tag{33}$$

where $\bar{P} = E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}']$. In the case of $E[\tilde{\mathbf{x}}] = \mathbf{0}$, it reduces to

$$\mathbf{a} = E[\mathbf{h}(\mathbf{x})], \qquad H = E[\mathbf{h}(\mathbf{x})\tilde{\mathbf{x}}']\bar{P}^{-1} \tag{34}$$

Consider an example of a scalar nonlinear measurement $z = x^3 + v$. Assume that $x \sim \mathcal{N}(\bar{x}, \bar{P})$. Then the optimally linearized model is

$$z = \bar{x}^3 + 3\bar{P}\bar{x} + (3\bar{x}^2 + 3\bar{P})(x - \bar{x}) + v \tag{35}$$

since $a = E[x^3] = \bar{x}^3 + 3\bar{P}\bar{x}$ and $H = E[x^3\tilde{x}]\bar{P}^{-1} = 3\bar{x}^2 + 3\bar{P}$. Compared with the derivative-based linearized model $z = \bar{x}^3 + 3\bar{x}^2(x - \bar{x}) + v$, which always underestimates the variation $h(x) - h(\bar{x})$, this model appears more appealing for many situations.

While the derivative-based linearization relies on truncation of the Taylor series, which will incur large errors if $\tilde{\mathbf{x}}$ is not small, this optimally linearized model accounts for large errors within the expectations by the probabilistic weights and thus tends to give a more conservative filter gain and better performance for cases involving large $\tilde{\mathbf{x}}$. Another possible advantage of this model is that $\mathbf{h}$ need not be differentiable. Basically, it trades integration with differentiation. This may be particularly useful in such cases where hard limiters (or saturations) are involved.

In the calculation of the required expectations, one may usually assume $\mathbf{x} \sim \mathcal{N}(\bar{\mathbf{x}}, \bar{P})$. It has a small tail probability, which is appealing because the goal is *local* linearization. In some situations, one may want to use a distribution flatter than the Gaussian, but the heavy tails should be truncated; that is, $\tilde{\mathbf{x}}$ may be assumed more evenly distributed than Gaussian, but only over a "small" neighborhood of $\bar{\mathbf{x}}$. In general, the larger the neighborhood, the more conservative the filter gain.

4.4 Linearization-Error Reduction Techniques

Sequential Processing. A well-known simple means to reduce linearization errors is sequential processing of the measurement components (see, e.g., [49, 50, 51]). It is well known that the nonlinear measurements should be processed in the order of their accuracy — more accurate first — (see, e.g., [51, 11]). A particular problem was considered in [52], where the spherical measurements were processed sequentially in the order of decreasing accuracy: azimuth, elevation, and range. Performance comparison results between sequential processing and the conventional vector processing were given in [53].

Iterative EKF. Once the updated state $\hat{\mathbf{x}}$ is obtained, the nonlinear measurement model can be re-linearized at $\hat{\mathbf{x}}$. This will in general reduce the linearization error compared with linearization at $\bar{\mathbf{x}}$. The state and its error covariance can then be re-updated based on the re-linearized model. This process can be repeated, resulting in an algorithm known as an iterated EKF (IEKF) in the context of Kalman filtering [43]. Following Theorem 8.2 of [43] the iteration algorithm is

$$\hat{\mathbf{x}}^0 = \bar{\mathbf{x}} \tag{36}$$

$$\hat{\mathbf{x}}^{i+1} = \bar{\mathbf{x}} + K(\hat{\mathbf{x}}^i)\left[\mathbf{z} - \mathbf{h}(\hat{\mathbf{x}}^i) - H(\hat{\mathbf{x}}^i)(\mathbf{x} - \hat{\mathbf{x}}^i)\right], \quad i = 0, 1, \ldots, L \tag{37}$$

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^{L+1} \tag{38}$$

$$P = \left[I - K(\hat{\mathbf{x}}^L)H(\hat{\mathbf{x}}^L)\right]\bar{P}\left[I - K(\hat{\mathbf{x}}^L)H(\hat{\mathbf{x}}^L)\right]' + K(\hat{\mathbf{x}}^L)RK(\hat{\mathbf{x}}^L)' \tag{39}$$

where $H(\hat{\mathbf{x}}^i) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\hat{\mathbf{x}}^i}$, $K(\hat{\mathbf{x}}^i) = \bar{P}H(\hat{\mathbf{x}}^i)'[H(\hat{\mathbf{x}}^i)\bar{P}H(\hat{\mathbf{x}}^i)' + R]^{-1}$, $i = 0, 1, \ldots, L$ if a Kalman filtering is used with a derivative-based linearized model (i.e., in a first-order EKF). A similar iteration can be written if a difference-based model (Sec. 4.2) is used. In contrast to some other iteration schemes (see, e.g., [54, 47]), the computation of the updated covariance P by the Joseph's form (39) is better to be outside of the iteration loop. This was used in [43], and emphasized and discussed in [55]. A theoretical consideration of the EKF and IEKF measurement updates as Gauss-Newton iteration schemes and a demonstration of the superiority of the IEKF were presented in [56]. The simulations of [7] (Table I) and [55] show that such re-linearization iterations can indeed improve accuracy at a level that is scenario dependent [55]. It should be warned, however, that an improvement is not guaranteed. There are reports that the relinearization iteration degrades the performance. The reader is referred to [57] for some insight that substantiates this warning.

Higher-Order Polynomial Models. Another straightforward idea to increase the accuracy of a polynomial approximation of a nonlinear measurement model is to use quadratic term (and possibly higher-order terms) in the Taylor series expansion. In the Kalman filtering context, this leads to what is sometimes referred to as a second-order (and higher-order, respectively) EKF [43, 47, 58]. The simulation results reported in [7] show a considerable improvement in performance of a second-order EKF over a first-order EKF. However, second-order EKFs are not very often used in practice mainly because of their rather burdensome computation and limited or marginal performance improvement.

Following the same ideas it is also possible to develop higher-order versions of the difference-based and optimally linearized models.

Many other techniques are available for mitigating the performance degradation due to linearization, such as artificial inflation of error covariance [58]. [8] includes a short list of such techniques and a brief discussion.

5 Models in Cartesian Coordinates

Since target motion is best described in Cartesian coordinates but measurements are available in sensor coordinates, as explained in Sec. 3.2, a commonly used method is to convert measurements from sensor to Cartesian coordinates, and do tracking entirely in the Cartesian coordinates. As before, we will assume a generic Cartesian frame since measurements between Cartesian frames can be converted easily and exactly (assuming no sensor registration or gridlock errors).

5.1 Conversion of Measured Positions

The spherical-to-Cartesian transformation $\boldsymbol{\varphi} = \mathbf{h}^{-1}$ with $\mathbf{h} = [h_r, h_b, h_e]'$ is given by

$$\mathbf{z}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \boldsymbol{\varphi}(\mathbf{z}) = \boldsymbol{\varphi}(r, b, e) = \begin{bmatrix} r\cos b\cos e \\ r\sin b\cos e \\ r\sin e \end{bmatrix} \tag{40}$$

where $\mathbf{z} = [r, b, e]'$ and $\mathbf{z}_c = [x_c, y_c, z_c]'$ are one and the same noisy measurement, expressed in the original spherical coordinates and the converted Cartesian coordinates, respectively. The RUV-to-Cartesian transformation $\boldsymbol{\phi} = \mathbf{h}^{-1}$ with $\mathbf{h} = [h_r, h_u, h_v, h_w]'$ is given by

$$\mathbf{z}_c = \boldsymbol{\phi}(\mathbf{z}) = \boldsymbol{\phi}(r, u, v, w) = \begin{bmatrix} ru \\ rv \\ rw \end{bmatrix} \tag{41}$$

If range measurements are not available, as in the case of passive sensors and range-denial countermeasures, the above range measurement r can be replaced by an estimated range $\hat{r}$ [21, 11] and the range measurement error $r - \mathsf{r}$ (and its bias and variance) should be replaced by the range estimation error $\hat{r} - \mathsf{r}$ (and its bias and variance).

In the sequel, we describe techniques for converting measurements from spherical to Cartesian coordinates. The reader is referred to [59, 20] for conversion from RUV to Cartesian coordinates.

5.2 Standard Model of Converted Measurements

After conversion, the measurement model in Cartesian coordinates has the form

$$\mathbf{z}_c = H\mathbf{x} + \mathbf{v}_c \tag{42}$$

where $\mathbf{x}_p := H\mathbf{x}$ is the position subvector of the state vector $\mathbf{x}$ and $\mathbf{v}_c$ stands for the resulting measurement error.

By Taylor series expansion of $\boldsymbol{\varphi}(\mathsf{z})$ around the noisy measurement $\mathbf{z}$, we have

$$\mathbf{x}_p = \boldsymbol{\varphi}(\mathsf{z}) = \boldsymbol{\varphi}(\mathbf{z} - \mathbf{v}) = \boldsymbol{\varphi}(\mathbf{z}) - J(\mathbf{z})\mathbf{v} + \text{HOT}(\mathbf{v}) \tag{43}$$

where $\mathsf{z} = [\mathsf{r}, \mathsf{b}, \mathsf{e}]'$ is the error-free true target position in spherical coordinates and HOT$(\mathbf{v})$ stands for the higher order (≥ 2) terms, and the Jacobian $J(\mathbf{z})$ is evaluated at the noisy measurement $\mathbf{z}$

$$J(\mathbf{z}) = \left.\frac{\partial \boldsymbol{\varphi}}{\partial \mathsf{z}}\right|_{\mathsf{z}=\mathbf{z}} = \begin{bmatrix} \cos b \cos e & -r \sin b \cos e & -r \cos b \sin e \\ \sin b \cos e & r \cos b \cos e & -r \sin b \sin e \\ \sin e & 0 & r \cos e \end{bmatrix} \tag{44}$$

Then the exactly converted measurement model (42) can be written as

$$\mathbf{z}_c = \boldsymbol{\varphi}(\mathbf{z}) = \mathbf{x}_p + \mathbf{v}_c = \mathbf{x}_p + \underbrace{J(\mathbf{z})\mathbf{v} - \text{HOT}(\mathbf{v})}_{\mathbf{v}_c} \tag{45}$$

Clearly, this expansion is superior to expanding $\boldsymbol{\varphi}(\mathbf{z})$ around the error-free measurement z

$$\mathbf{z}_c = \boldsymbol{\varphi}(\mathbf{z}) = \boldsymbol{\varphi}(\mathsf{z} + \mathbf{v}) = \mathbf{x}_p + \mathbf{v}_c = \mathbf{x}_p + \underbrace{J(\mathsf{z})\mathbf{v} + \text{HOT}(\mathbf{v})}_{\mathbf{v}_c} \tag{46}$$

which involves the unknown z.

Evidently, the true converted measurement error $\mathbf{v}_c$ is *measurement dependent* (or *state dependent*), *non-Gaussian*, correlated across coordinates, and has *nonzero mean*. Thus, the converted measurement $\mathbf{z}_c$ has a *bias* $E[\mathbf{v}_c]$ and a conditional bias $E[\mathbf{v}_c|\mathbf{z}]$.

Clearly, conversion of the measured position values per se is straightforward and nothing is really subtle here. The main task in the so-called *measurement conversion*, which is better called *measurement model conversion*, really lies in the conversion of the associated noise statistics. As explained in Sec. 3.2, the major advantage of expressing measurements in Cartesian coordinates is the attractive "linear" structure of the corresponding measurement model (42). In order for a linear filter (e.g., the Kalman filter) to take advantage of this "linear" structure, the first two moments of $\mathbf{v}_c$ must be determined.

5.3 Linearized Conversion

The "standard" approach is to treat $\mathbf{v}_c$ (approximately) as zero-mean with covariance [50, 22, 10, 45, 11]

$$R^L = J(\mathbf{z})RJ(\mathbf{z})' \tag{47}$$

where superscript L stands for "linearized." This is usually justified by ignoring HOT$(\mathbf{v})$ in the exact model (45) to yield

$$\mathbf{z}_c = \varphi(\mathbf{z}) \approx \mathbf{x}_p + \mathbf{v}^L = \mathbf{x}_p + \underbrace{J(\mathbf{z})\,\mathbf{v}}_{\mathbf{v}^L} \tag{48}$$

where $\mathbf{v}^L = J(\mathbf{z})\,\mathbf{v}$. In other words, the true converted measurement error $\mathbf{v}_c$ is approximated by the "linearized" one $\mathbf{v}^L$. In a similar manner (45) can be used to obtain second- and higher-order approximations of the statistics of $\mathbf{v}_c$ if necessary.

Clearly, the linearized error $\mathbf{v}^L$ is *measurement dependent* (or *state dependent*) and *non-Gaussian* because $J(\mathbf{z})\,\mathbf{v}$ is a nonlinear transformation of the Gaussian vector $\mathbf{v}$ due to the dependence of $\mathbf{z}$ on $\mathbf{v}$. It has zero mean $E[\mathbf{v}^L] = E\left[E(\mathbf{v}^L|\mathbf{z})\right] = \mathbf{0}$ and measurement-conditional cov$\left(\mathbf{v}^L|\mathbf{z}\right) = R^L = J(\mathbf{z})\,RJ(\mathbf{z})'$, as given by (47). However, the Kalman filter requires in theory knowledge of *unconditional* covariance, given by

$$\text{cov}\left(\mathbf{v}^L\right) = E\left[\text{cov}\left(\mathbf{v}^L|\mathbf{z}\right)\right] + \text{cov}\left[E(\mathbf{v}^L|\mathbf{z})\right] = E[J(\mathbf{z})\,RJ(\mathbf{z})'] \tag{49}$$

which is quite involved. It appears that this fact has been overlooked in the literature. As a result, pretending cov$\left(\mathbf{v}^L\right) =$ cov$\left(\mathbf{v}^L|\mathbf{z}\right) = R^L$ is another possible error source for tracking based on this measurement conversion. What is tricky here is that linear filters also assume that measurement error is not measurement or state dependent. Therefore, errors caused by pretending cov$\left(\mathbf{v}^L\right) =$ cov$\left(\mathbf{v}^L|\mathbf{z}\right) = R^L$ and ignoring the state dependency of $\mathbf{v}^L$ *may* possibly cancel each other to some extent because the use of the state-dependent covariance cov$\left(\mathbf{v}^L|\mathbf{z}\right)$ in place of cov$\left(\mathbf{v}^L\right)$ provides a means of accounting for the otherwise-ignored state dependency of $\mathbf{v}^L$. A similar analysis applies to the other conversions described later.

What is just described is the so-called *credibility* problem of this simple conversion [60]: the actual covariance is statistically different from the covariance used. More seriously, this linearized conversion ignores the bias in the converted measurements, which may lead to substantially degraded performance and even filtering divergence.

5.4 Toward Better Conversions

As recognized in [14], the bias of the above linearized conversion can be considerable in some tracking applications and needs to be compensated. Since then, a number of improved techniques for measurement conversion have been proposed [14, 51, 61, 16, 17, 18, 19, 62]. This topic was treated well in [18, 19] in a comprehensive way.

The probably most natural approach to counter the bias problem is to find the bias $\mathbf{b}$ and then remove it from the converted measurements. This leads to what is known as a *debiased conversion*. A straightforward way to accomplish this is the *additive debiasing* [14, 51, 61, 18, 19]: If $\mathbf{z}_c$ has bias $\mathbf{b}$, then $\mathbf{z}_* = \mathbf{z}_c - \mathbf{b}$ would be unbiased. In theory, a filter needs only to know the exact bias amount and there is no need to de-bias the converted measurement. Nonetheless, a debiased conversion is more convenient in practice than a conversion with a known bias.

The *multiplicative debiasing* provides an alternative to the additive debiasing. Consider first a scalar converted measurement x_c of a true state component x. Let $x_* = \lambda x_c$. If we choose $\lambda = E[x]/E[x_c]$, then x_* is clearly an guaranteed unbiased measurement of x. For the vector case, $\mathbf{z}_* = \text{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{z}_c$ would certainly be an unbiased measurement of $\mathbf{x}$ provided we choose $(\lambda_1, \lambda_2, \lambda_3) = (E[x]/E[x_c], E[y]/E[y_c], E[z]/E[z_c])$.

We now show that as long as the bias can be eventually expressed as a scaled version of noisy measurement $\mathbf{z}_c$, additive debiasing and multiplicative debiasing are completely equivalent. Consider again a scalar converted measurement x_c of a true state component x. Let $b = (1-\lambda)x_c$ be its bias expressed as a scaled x_c. Then the converted measurement using additive debiasing is $x_* = x_c - b = \lambda x_c$, which is seen to be equivalent to multiplicative debiasing with $\lambda = E[x]/E[x_c]$. Both are guaranteed to be unbiased provided the mean and bias are computed exactly. This is clearly true also for the vector case, where we use $\mathbf{z}_* = \text{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{z}_c$ and $\mathbf{b} = \text{diag}(1-\lambda_1, 1-\lambda_2, 1-\lambda_3)\mathbf{z}_c$.

The fundamental idea underlying all measurement conversion techniques developed so far in the literature is the following: Convert the measurements $\mathbf{z}$ in the original sensor coordinates to Cartesian coordinates such that $\mathbf{z}_* = \mathbf{x}_p + \mathbf{v}_*$, where $\mathbf{x}_p$ is the true position of the target. All of these techniques seek to have an unbiased conversion. Their differences lie in the choice of $\mathbf{z}_*$ since there are infinitely many such unbiased conversions. The ultimate judgment on these different conversions is the performance of the filter using them. It is, however, hard to isolate the impact of these conversions on the filter performance in a realistic scenario. In view of this, it may suffice to evaluate the size of the covariance of the converted measurement noise for different unbiased conversions.

Debiased conversion is not the only conversion that can effectively counter the bias problem in measurement conversion. The equivalent measurement approach proposed in [20] is a more general technique. The reader is referred to [20] for details.

To have a credible covariance for the error of the converted measurements, we also need to compute the covariance as accurately as possible [60].

5.5 Conditioning in Measurement Conversion

Clearly, the key to all debiased conversions is finding mean of the converted measurement error. The question is: conditional mean or unconditional mean? If conditional, conditioned on what?

Nested Conditioning. The earliest approach, started from [14, 51], is to find the mean and covariance conditioned on the *unknown* ideal measurement z first [i.e., $E\left[\mathbf{v}_c|\mathrm{z}\right]$ and $\text{cov}(\mathbf{v}_c|\mathrm{z})$] and then find their averages (expectations) conditioned on the noisy measurement $\mathbf{z}$. In effect, this approach computes $E[E(\mathbf{v}_c|\mathrm{z})|\mathbf{z}]$ and $E[\text{cov}(\mathbf{v}_c|\mathrm{z})|\mathbf{z}]$. This was referred to as "fixed truth" in [18, 19], which we think is not precise because nothing is really fixed here.

This approach has a fundamental flaw in that the nested conditional expectations are fundamentally incompatible: The uncertainty in the noisy measurement $\mathbf{z}$ (relative to the ideal measurement z) is removed in the inner expectations, but retained in the outer expectations. As a result, part but not all of the dependence on $\mathbf{z}$ is retained. This flaw is more serious for the covariance computation. While such a less solid approach is not uncommon in applied research mainly for its potential tractability, it should be avoided when a more solid approach is readily available.

Measurement Conditioned. More appealing is to compute the mean and covariance of the converted measurement error conditioned on something known directly. The current noisy measurement is clearly a good choice. Specifically, we find $E[\mathbf{v}_c|\mathbf{z}]$ and $\text{cov}(\mathbf{v}_c|\mathbf{z})$. The covariance part of this approach was first used in [17], which has a compatibility problem with the corresponding mean part, though. The mean part was developed in [18, 19] and referred to as a "fixed measurement" approach there. This name is again not precise because the real situation is that the mean and covariance of the converted measurement error are functions of the current noisy measurement being converted, which can well be actually a deterministic or random variable. In other words, the expectation operations never assumes invariance of the current noisy measurement.

Estimate Conditioned. Another possibility is to have the noise mean and covariance conditioned on the best estimate $\hat{\mathbf{x}}$ known at the time: $E[\mathbf{v}_c|\hat{\mathbf{x}}]$ and $\text{cov}(\mathbf{v}_c|\hat{\mathbf{x}})$, as presented in [18]. It can be justified by

$$\mathbf{x}_p = \varphi\left(\mathrm{z}\right) = \varphi\left(\hat{\mathbf{z}} + \tilde{\mathbf{z}}\right) = \varphi\left(\hat{\mathbf{z}}\right) + J\left(\hat{\mathbf{z}}\right)\tilde{\mathbf{z}} + \text{ HOT}\left(\tilde{\mathbf{z}}\right) = \hat{\mathbf{x}}_p + J\left(\hat{\mathbf{z}}\right)\tilde{\mathbf{z}} + \text{ HOT}\left(\tilde{\mathbf{z}}\right) \tag{50}$$

where $\tilde{\mathbf{z}}$ is the error of the estimate $\hat{\mathbf{z}}$ of the true position z in spherical coordinates. This approach has the potential of improved performance compared to the above measurement-conditioned approach when the estimate $\hat{\mathbf{x}}$ is more accurate than the measurement $\mathbf{z}$. However, the exact impact of using such estimate-dependent mean and covariance of measurement noise in the Kalman filter is hard to predict. This approach may run into the risk of narcissism — the conversion may be too optimistic as a result of an optimistic estimate. The estimate-conditioned conversion presented in [18] is based on fairly simplistic assumptions. No performance results are available. Of course, more sophisticated schemes may be developed.

Estimate and Measurement Conditioned. We can also condition the noise mean and covariance on both measurement and estimate. This should have good potential at the cost of sophistication. Nothing has yet been done along this line.

Unconditional. In fact, what is required in the Kalman filter is the *unconditional* mean and covariance of the measurement noise. As a result, it appears that we should provide such mean and covariance of the converted measurement error. The problem is that the Kalman filter also assumes that the measurement errors are not state dependent, which is unfortunately not the case for converted measurements. For this reason, as explained before, use of a conditional mean and covariance of the error may improve performance since they provide a means to account for the state dependency. That is possibly why in the literature no unconditional mean and covariance have been computed for the converted measurement and used in the corresponding Kalman filters.

Prediction Conditioned. More fundamentally, the Kalman filter actually does not need mean and covariance of the measurement noise once the predicted measurement and the measurement residual covariance are known. In view of this, we need only to calculate the predicted measurement $E[\mathbf{z}_c|\mathbf{z}^{k-1}]$ and the residual covariance $\text{cov}(\mathbf{z}_c|\mathbf{z}^{k-1})$, where $\mathbf{z}^{k-1}$ stands for the sequence of all past measurements, as proposed in [20]. Note that this is different from $E[\mathbf{v}_c|\mathbf{z}^{k-1}]$ and the residual covariance $\text{cov}(\mathbf{v}_c|\mathbf{z}^{k-1})$, as used in [63, 59], because the true state is random and unknown. In our opinion, this is the most solid approach to measurement conversion, but has been largely overlooked in measurement conversion research. It circumvents the ambiguity as which conditioning to use for the mean and covariance of the measurement error.

Covariance versus Mean-Square Matrix. What is needed in the Kalman filter for the measurement noise $\mathbf{v}_c$ is its mean and *covariance*. As such, in addition to the mean, we should compute covariance cov($\mathbf{v}_c$), rather than the mean-square matrix $E[\mathbf{v}_c\mathbf{v}_c']$, as was done in [18, 19] (it was called mean-squared error there because $\mathbf{v}_* = \mathbf{z}_* - \mathbf{x}_p$). Of course, this difference vanishes in the case where the debiased converted measurement $\mathbf{z}_*$ is truly unbiased since cov$(\mathbf{y}) = E[\mathbf{y}\mathbf{y}'] - E[\mathbf{y}]E[\mathbf{y}]'$. Use of a mean-square matrix calculated before debiasing would be a mistake. For credibility evaluation, we may use either the (theoretical and sample) mean-square matrices, as in [18, 19], or the (theoretical and sample) covariance and bias separately.

5.6 Debiased Conversions

All debiased conversions attempt to build a converted measurement model in Cartesian coordinates in the following form

$$\mathbf{z}_* = H\mathbf{x} + \mathbf{v}_* \tag{51}$$

where $\mathbf{x}_p = H\mathbf{x}$ is the position subvector of the state vector $\mathbf{x}$ and $\mathbf{v}_*$ stands for the measurement error.

In the sequel, we consider only the 3D case. The corresponding 2D formulas are obtained by simply setting $e = 0$ and $\sigma_e = 0$ in the 3D formulas. Unless stated otherwise explicitly, the measurement errors in the spherical coordinates are assumed zero-mean, Gaussian distributed, and uncorrelated with covariance diag($\sigma_r^2,\ \sigma_b^2,\ \sigma_e^2$).

No matter whether additive or multiplicative debiasing is used, all debiased conversions developed so far (except the estimate-conditioned one of [18]) turn out to have the following multiplicative form

$$\mathbf{z}_* = \begin{bmatrix} x_* \\ y_* \\ z_* \end{bmatrix} = \text{diag}(\lambda, \lambda, \mu)\mathbf{z}_c = \begin{bmatrix} \lambda r \cos b \cos e \\ \lambda r \sin b \cos e \\ \mu r \sin e \end{bmatrix} \tag{52}$$

*where the coefficients λ and μ are determined depending on the conditioning used. This is because the conditional bias $\mathbf{b}$ used in all additive debiasing techniques developed so far turns out to have the form $\mathbf{b} = \text{diag}(1-\lambda, 1-\lambda, 1-\mu)\mathbf{z}_c$, and thus, as shown before, additive debiasing is equivalent to multiplicative debiasing.

The first debiased conversion, proposed in [14] for the 2D case and later extended in [61] to the 3D case, uses additive debiasing. It is, however, equivalent to the multiplicative debiasing (52) with

$$\lambda = 1 - \exp\left(-\sigma_b^2 - \sigma_e^2\right) + \exp\left(-\sigma_b^2/2 - \sigma_e^2/2\right) \tag{53}$$

$$\mu = 1 - \exp\left(-\sigma_e^2\right) + \exp\left(-\sigma_e^2/2\right) \tag{54}$$

The bias was obtained by the nested conditioning $E[E(\mathbf{v}_c|\text{z})|\mathbf{z}]$. The corresponding covariance $E[\text{cov}(\mathbf{v}_c|\text{z})|\mathbf{z}]$ is more involved and can be found in [14, 51, 61]. The covariance was replaced in [18, 19] by the mean-square matrices $E\left[E\left(\mathbf{v}_*\mathbf{v}_*'|\text{z}\right)|\mathbf{z}\right]$ and $E\left[\mathbf{v}_*\mathbf{v}_*'|\mathbf{z}\right]$. As explained before, covariance should be used, but conditioning on measurement $\mathbf{z}$ is better than the nested conditioning. Due to the undesirable nested conditioning, it should come as no surprise that this conversion is not truly unbiased in the sense of $E[\mathbf{z}_* - \mathbf{x}_p] = \mathbf{0}$, as shown explicitly in [17, 18], and use of the exact cov$(\mathbf{v}_*|\mathbf{z})$ is preferable to any of the covariance or mean-square matrices above.

This conversion serves as the basis in [63, 59] for the development of an EKF-type filter in the mixed coordinates, which compensates the EKF's bias and processes measurement components sequentially in the original spherical coordinates along the line of [52]. An improvement in performance over the standard EKF was demonstrated in [63, 59], which for the 2D case is close to that of the measurement conversion based filter of [14]. No such comparison with debiased conversion based filter regarding accuracy and computational cost is made for the 3D case in [59].

The first explicit multiplicative debiasing based conversion was proposed in [16, 17] using $\mathbf{z}_* = \text{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{z}_c$. The scaling factors $\lambda_1, \lambda_2, \lambda_3$ are chosen such that $E[\mathbf{z}_* - \mathbf{x}_p] = \mathbf{0}$, leading to the following [16, 17], in the form of (52),

$$\lambda_1 = \lambda_2 = \lambda = \frac{1}{E\left[\cos v_b\right]}\frac{1}{E\left[\cos v_e\right]}, \qquad \lambda_3 = \mu = \frac{1}{E\left[\cos v_e\right]} \tag{55}$$

under the assumption that the measurement errors in spherical coordinates have a symmetric distribution with independent components that is independent of the ideal measurement z. For Gaussian and uniform measurement errors, they are given by

$$\lambda = \exp\left(\sigma_b^2/2 + \sigma_e^2/2\right), \quad \mu = \exp\left(\sigma_e^2/2\right) \quad \text{(for Gaussian errors)} \tag{56}$$

$$\lambda = \sin a/a, \quad \mu = \sin(2a)/(2a) \quad \text{(for uniform errors over } [-a, a]) \tag{57}$$

This conversion is truly unbiased (i.e., $E[\mathbf{z}_* - \mathbf{x}_p] = \mathbf{0}$) regardless whether the true $\mathbf{x}_p$ is fixed or random. The corresponding cov$(\mathbf{v}_*|\mathbf{z})$ was also derived in [17] without the Gaussian distribution assumption and formulas of $E\left[E\left(\mathbf{v}_*\mathbf{v}_*'|\mathbf{z}\right)|\mathbf{z}\right]$ (less desirable) can be found in [16, 19]. Note that cov$(\mathbf{v}_*|\mathbf{z})$ and $\mathbf{z}_* = \text{diag}(\lambda, \lambda, \mu)\mathbf{z}_c$ are not completely compatible because $\mathbf{b} = \text{diag}(1-\lambda, 1-\lambda, 1-\mu)\mathbf{z}_c \neq E[\mathbf{v}_c|\mathbf{z}]$, and thus this conversion is not perfectly credible, as verified by simulation in [18, 19] and by analysis for small values of r/σ_r in [62].

A measurement-conditioned additive debiased conversion was developed in [18], given by (52) with

$$\lambda = \exp(-\sigma_b^2/2 - \sigma_e^2/2), \quad \mu = \exp(-\sigma_e^2/2) \tag{58}$$

It is truly unbiased for every $\mathbf{z}$ in that $E[\mathbf{z}_* - \mathbf{x}_p|\mathbf{z}] = \mathbf{0}$, as expected. The mean-square matrix $E\left[\mathbf{v}_*\mathbf{v}_*'|\mathbf{z}\right]$ (i.e., the cov$(\mathbf{v}_*|\mathbf{z})$ since it is unbiased for every $\mathbf{z}$) is also given in [18, 19]. Note that $E[\mathbf{z}_* - \mathbf{x}_p|\mathbf{z}] = \mathbf{0}$ (unbiased for every $\mathbf{z}$) implies $E[\mathbf{z}_* - \mathbf{x}_p] = \mathbf{0}$ (unbiased on average). Therefore, this conversion has a more solid foundation (e.g., no incompatibility between the first two moments).

Performance comparison results of all these debiased conversions were reported in [18, 19] based on computer simulations using several scenarios, including fixed-truth and fixed-measurement scenarios. Both debiasing performance and credibility were examined. In reality, measurement conversions are usually used in the context of dynamic filtering, where both the state and the measurements are random. The somewhat artificial fixed-truth and fixed-measurement scenarios verify that the nested conditioning and measurement-conditioned debiased conversions perform relatively better in the two scenarios, respectively, as expected. Beyond this capability of checking the correctness of the formulas, their value in evaluating the true performance in the filtering context is limited. For the dynamic scenarios with the Kalman filter, the results are somewhat dependent on the performance criterion and do not indicate a convincing clear-cut superiority of any single conversion. This is not surprising because the Kalman filter was actually developed to use *unconditional* mean and covariance of measurement noise that is *independent* of the state and measurements. However, the additive debiased conversion with nested conditioning of [14, 51, 61] appears to be inferior, as expected, due to its residual bias and noncredibility. All debiased conversions demonstrate a significant performance improvement over the conventional linearized conversion. For more details, the reader is referred to [18, 19].

5.7 Quasi-Monte-Carlo Transformations

The above debiasing techniques rely on explicit formulas of the mean and covariance of the converted measurement errors. Such an analytic method is clearly limited. For more complex situations (e.g., involving range rate measurements, higher moments, or less tractable distributions), the Monte-Carlo (random sampling) method (see, e.g., [64]) may be used to obtain the required noise statistics. To our knowledge, however, there is no report of any such application most likely because it is computationally demanding and probably represents an overkill for measurement conversion.

A recently-developed computationally much more efficient sampling-based method appears to be quite suitable for measurement conversion. This method, referred to as *unscented transformation* by its inventors [65, 15, 66], belongs to what is known as *quasi-Monte-Carlo* methods in statistics and some other disciplines [67], which differs from the Monte-Carlo method in that the sample points are well-designed and "deterministic." This approach is particularly suited for the problem of finding the statistics of a random vector $\boldsymbol{\eta} = \boldsymbol{\varphi}(\boldsymbol{\xi})$ that is a (nonlinear) function of a random vector $\boldsymbol{\xi}$ with known statistics. It has many potential applications in target tracking, especially as a general nonlinear filtering method [65, 68, 69, 70]. In the context of the measurement conversion, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are the original and converted measurements, respectively.

Briefly, the main idea of this approach is the following.

- Given the probability distribution $p_{\boldsymbol{\xi}}(\cdot)$ of $\boldsymbol{\xi}$, select a set of well-designed "deterministic" sample points (referred to as *sigma points*) and associated weights $\mathcal{S} = \{\boldsymbol{\xi}_i, w_i : i = 0, 1, \ldots, l\}$ such that $\sum_{i=0}^{l} w_i = 1$.
- Transform the samples $\boldsymbol{\eta}_i = \boldsymbol{\varphi}(\boldsymbol{\xi}_i), i = 0, 1, \ldots, l$.
- Approximate the true moments of $\boldsymbol{\eta}$ by their sample moments:

$$E[\boldsymbol{\eta}] \approx \bar{\boldsymbol{\eta}} = \sum_{i=0}^{l} w_i \boldsymbol{\eta}_i, \quad \text{cov}(\boldsymbol{\eta}) \approx C_{\boldsymbol{\eta}} = \sum_{i=0}^{l} w_i \left(\boldsymbol{\eta}_i - \bar{\boldsymbol{\eta}}\right)\left(\boldsymbol{\eta}_i - \bar{\boldsymbol{\eta}}\right)' \tag{59}$$

The key issue of this approach is: given $p_{\boldsymbol{\xi}}(\cdot)$, how to specify a *small* set of "deterministic" sample points and weights $\mathcal{S} = \{\boldsymbol{\xi}_i, w_i : i = 0, 1, \ldots, l\}$ so that the true moments of $\boldsymbol{\eta}$ are well approximated by their sample moments using samples

$\varphi(\boldsymbol{\xi}_i)$. To this end, expand $\varphi(\boldsymbol{\xi})$ around the mean $\bar{\boldsymbol{\xi}}$ by Taylor series:

$$\boldsymbol{\eta} = \varphi(\boldsymbol{\xi}) = \varphi(\bar{\boldsymbol{\xi}}) + D_{\tilde{\boldsymbol{\xi}}}\varphi + D^2_{\tilde{\boldsymbol{\xi}}}\varphi/2! + \cdots + D^n_{\tilde{\boldsymbol{\xi}}}\varphi/n! + \cdots \tag{60}$$

where the first-order term $D_{\tilde{\boldsymbol{\xi}}}\varphi = \frac{\partial\varphi}{\partial\boldsymbol{\xi}}\Big|_{\boldsymbol{\xi}=\bar{\boldsymbol{\xi}}}\tilde{\boldsymbol{\xi}}$ (with $\tilde{\boldsymbol{\xi}} = \xi - \bar{\boldsymbol{\xi}}$) is the total differential of $\varphi(\cdot)$ perturbed around $\bar{\boldsymbol{\xi}}$ by $\boldsymbol{\xi}$, and similarly the nth-order term $D^n_{\tilde{\boldsymbol{\xi}}}\varphi/n!$ is an nth-order polynomial in the components of $\tilde{\boldsymbol{\xi}}$. Thus, $E[\boldsymbol{\eta}]$ can be approximated by up to the second-order terms with linear and second-order polynomial terms of the components of $\tilde{\boldsymbol{\xi}}$ replaced by the corresponding mean and mean-square value. The error of this approximation arises only from the third and higher order terms. Therefore, if we guarantee

$$E[\boldsymbol{\xi}] = \sum_{i=0}^{l} w_i\boldsymbol{\xi}_i, \quad \mathrm{cov}(\boldsymbol{\xi}) = \sum_{i=0}^{l} w_i (\boldsymbol{\xi}_i - E[\boldsymbol{\xi}])(\boldsymbol{\xi}_i - E[\boldsymbol{\xi}])' \tag{61}$$

then $E[\boldsymbol{\eta}] \approx \bar{\boldsymbol{\eta}} = \sum_{i=0}^{l} w_i\varphi(\boldsymbol{\xi}_i)$ will have a guaranteed accuracy up to the second order. This turns out true for $\mathrm{cov}(\boldsymbol{\eta}) \approx C_{\boldsymbol{\eta}}$ as well [65]. This leads to a *second-order transformation*.

A symmetric second-order transformation [matching $E[\boldsymbol{\xi}]$ and $\mathrm{cov}(\boldsymbol{\xi})$] for an n-dimensional distribution is given by the following $2n+1$ sigma points [65]:

$$\mathcal{S} = \begin{cases} \boldsymbol{\xi}_0 = E[\boldsymbol{\xi}] & w_0 = \kappa/(n+\kappa) \\ \boldsymbol{\xi}_{\pm i} = E[\boldsymbol{\xi}] \pm \left[\sqrt{(n+\kappa)\,\mathrm{cov}(\boldsymbol{\xi})}\right]_i & w_{\pm i} = 1/2(n+\kappa) \end{cases} \tag{62}$$

where $\left[\sqrt{(n+\kappa)\,\mathrm{cov}(\boldsymbol{\xi})}\right]_i$ is the ith row (or column) of a square root of the matrix $(n+\kappa)\mathrm{cov}(\boldsymbol{\xi})$, which may be obtained by, e.g., the Cholesky decomposition, and κ is a free parameter that can be designed to minimize higher-order errors. It can be easily verified by a direct substitution that this $\mathcal{S}$ satisfies the constraints in (61) [70].

A measurement conversion using the above symmetric second-order transformation [15] can be implemented with $\kappa = 0$ as[8]

$$\mathbf{z}^{(\pm1)} = \mathbf{z}^{(0)} \pm \sqrt{3}\begin{bmatrix}\sigma_r\\0\\0\end{bmatrix}, \ \mathbf{z}^{(\pm2)} = \mathbf{z}^{(0)} \pm \sqrt{3}\begin{bmatrix}0\\\sigma_b\\0\end{bmatrix}, \ \mathbf{z}^{(\pm3)} = \mathbf{z}^{(0)} \pm \sqrt{3}\begin{bmatrix}0\\0\\\sigma_e\end{bmatrix} \tag{63}$$

$$\mathbf{z}_c^{(i)} = \varphi\left(\mathbf{z}^{(i)}\right), \quad i = \pm1, \pm2, \pm3 \quad \text{with } \varphi \text{ of (40)} \tag{64}$$

$$\bar{\mathbf{z}}_c = \frac{1}{6}\sum_{i=\pm1}^{\pm3}\mathbf{z}_c^{(i)}, \quad C_{\mathbf{z}_c} = \frac{1}{6}\sum_{i=\pm1}^{\pm3}\left(\mathbf{z}_c^{(i)} - \bar{\mathbf{z}}_c\right)\left(\mathbf{z}_c^{(i)} - \bar{\mathbf{z}}_c\right)' \tag{65}$$

Note that only *six* well-designed points are actually used and that $\bar{\mathbf{z}}_c$ and $C_{\mathbf{z}_c}$ are actually conditioned on $\mathbf{z}^{(0)}$, which can be random or deterministic. As such,

$$\begin{array}{lll} E[\mathbf{z}_c|\bar{\mathbf{z}}] \approx \bar{\mathbf{z}}_c, & \mathrm{cov}(\mathbf{z}_c|\bar{\mathbf{z}}) \approx C_{\mathbf{z}_c} & \text{if } \mathbf{z}^{(0)} = \bar{\mathbf{z}} \text{ is the \textit{predicted} spherical measurement} \\ E[\mathbf{z}_c|\mathbf{z}] \approx \bar{\mathbf{z}}_c, & \mathrm{cov}(\mathbf{z}_c|\mathbf{z}) \approx C_{\mathbf{z}_c} & \text{if } \mathbf{z}^{(0)} = \mathbf{z} \text{ is the \textit{actual} spherical measurement} \\ E[\mathbf{z}_c|\mathrm{z}] \approx \bar{\mathbf{z}}_c, & \mathrm{cov}(\mathbf{z}_c|\mathrm{z}) \approx C_{\mathbf{z}_c} & \text{if } \mathbf{z}^{(0)} = \mathrm{z} \text{ is the \textit{ideal} spherical measurement} \end{array} \tag{66}$$

It is clumsy, although possible, to use this approach to obtain the converted measurement noise statistics directly — for example, the transformation from the spherical measurement error $\mathbf{v}$ to the Cartesian measurement error $\mathbf{v}_c$ is complicated. Also, this will run into the ambiguity in the choice of conditioning, as discussed before. More preferably, $E[\mathbf{z}_c|\mathbf{z}^{k-1}] \approx E[\mathbf{z}_c|\bar{\mathbf{z}}] \approx \bar{\mathbf{z}}_c$ and $\mathrm{cov}(\mathbf{z}_c|\mathbf{z}^{k-1}) \approx \mathrm{cov}(\mathbf{z}_c|\bar{\mathbf{z}}) \approx C_{\mathbf{z}_c}$ with $\mathbf{z}^{(0)} = \bar{\mathbf{z}}$ can be used as the predicted measurement and the residual covariance for measurement conversion, as discussed in Sec. 5.5.

More accurate measurement conversions can be obtained in a similar manner by using higher-order transformations (e.g., those match the skew and kurtosis of $\boldsymbol{\xi}$, as developed in [15, 66]) or more carefully designed sigma points [15, 66, 71]. Following the original development of the above symmetric second-order sigma points, several enhanced sets of sigma points

[8]It was recommended in [15] with theoretical justification that $\kappa = n - 3$ be used for a Gaussian distribution.

have been proposed in [15, 66, 71, 72, 73]. Briefly, [73] proposes an enhancement by scaling the pattern of sigma points; to match the mean and covariance of an n-dimensional vector $\boldsymbol{\xi}$, [72] proves $n+1$ to be the minimum number of required sigma points and derives a minimal set of asymmetric sigma points that minimizes the skew; a more robust implementation of the second-order transformation, utilizing more sigma points by augmenting $\boldsymbol{\xi}$ with its noise, was used in [69] and later more fully explored in [71], along with an associated square root.

A few general remarks are in order. First, as demonstrated in [15, 69, 66, 71, 72, 73], the accuracy of this transformation is surprisingly good[9] in view of the fact that a quite small number of sample points are used. Second, the *moments* are approximated directly in this approach, while in the linearization (or Taylor series expansion) based techniques, the transformation is approximated first and then moments are computed accordingly. Third, this approach is not limited to the case where $\boldsymbol{\xi}$ is Gaussian distributed, although the above symmetric second-order sigma points are probably best for a Gaussian distribution. Finally, a weakness of this approach is that, as for other sampling-based approaches, it does not provide enough insight into the problem.

6 Pseudomeasurement Models

6.1 Conventional Pseudolinear Models

The pseudomeasurement method, originated in [74], attempts to circumvent the bias problems of the EKF by avoiding explicit linearization of the nonlinear model (10) in the mixed coordinates. It relies on representing the nonlinear measurement model (10) in the following *pseudolinear* form

$$\mathbf{y}(\mathbf{z}) = H(\mathbf{z})\,\mathbf{x} + \mathbf{v}_y(\mathbf{x}, \mathbf{v}) \tag{67}$$

where the pseudomeasurement vector $\mathbf{y}(\mathbf{z})$ and matrix $H(\mathbf{z})$ are *known* functions of the actual measurement $\mathbf{z}$, and $\mathbf{v}_y(\mathbf{x}, \mathbf{v})$ is the corresponding pseudomeasurement error, now *state dependent*. The underlying idea of the approach is clear: Once a pseudolinear model (67) is available, a linear (Kalman) filter can be readily used with $\mathbf{y}(\mathbf{z})$, $H(\mathbf{z})$, and $R_y(\mathbf{x}^*) = \text{cov}[\mathbf{v}_y(\mathbf{x}^*, \mathbf{v})]$, where a common choice of $\mathbf{x}^*$ is the predicted state estimate $\bar{\mathbf{x}}$.

Developing a stochastic pseudomeasurement model (67) clearly depends on the particular measurement function $\mathbf{h}(\mathbf{x})$. As an example, for the mixed Cartesian-spherical measurement model (10)–(13), the following pseudolinear form can be obtained by algebraic and trigonometric manipulations:

$$\mathbf{y}(\mathsf{z}) = \begin{bmatrix} \mathsf{r} \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\mathsf{b}\cos\mathsf{e} & \sin\mathsf{b}\cos\mathsf{e} & \sin\mathsf{e} \\ -\sin\mathsf{b} & \cos\mathsf{b} & 0 \\ -\cos\mathsf{b}\sin\mathsf{e} & -\sin\mathsf{b}\sin\mathsf{e} & \cos\mathsf{e} \end{bmatrix}}_{H(\mathsf{z})} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{68}$$

for the noise-free case, where $H(\mathsf{z})$ is actually the transformation matrix from Cartesian to the *line-of-sight* (LOS) (more specifically, RHV) coordinates. As such, we have the following pseudolinear measurement model in the presence of measurement noise

$$\mathbf{y}(\mathbf{z}) = [H(\mathbf{z}), O]\mathbf{x} + \mathbf{v}_y(\mathbf{x}, \mathbf{v}) \tag{69}$$

with $\mathbf{x} = [x, y, z, \ldots]'$. The exact expression of $\mathbf{v}_y(\mathbf{x}, \mathbf{v})$ is rather involved [39]. Under the usual independence and Gaussian assumption for the spherical noise $\mathbf{v}$, it can be approximated (in the sense of a small mean-square error) by noise with zero mean and covariance given approximately by [39]

$$R_y(\bar{\mathbf{x}}) = \text{cov}[\mathbf{v}_y(\bar{\mathbf{x}}, \mathbf{v})] \approx \text{diag}[\sigma_r^2,\ (\bar{x}^2 + \bar{y}^2)\sigma_a^2,\ (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)\sigma_a^2] \tag{70}$$

where $\sigma_a = \sigma_b = \sigma_e$ is the standard deviation of angle measurement errors. The predicted state estimate $\bar{\mathbf{x}}$ is used in (70) since the true state $\mathbf{x}$ is not available.

The angular parts (i.e., excluding the first equation) of this pseudolinear measurement model are well known in angle-only tracking (see, e.g., [75, 76, 77, 11]). Albeit seemingly appealing, its straightforward implementation in the linear Kalman filter is known to exhibit a considerable bias (see, e.g., [78, 76, 77]). This arises from the strong state (measurement) dependency of $H(\mathbf{z})$ and the pseudomeasurement noise, which leads to, for example, a strong correlation between the filter gain and the measurement residual via the current measurement.

Apparently (69) can also be used by means of the standard pseudolinear technique in the Kalman filter, as explained above. Alternatively, a more subtle approach can be taken. It is based on the fact that the matrix $H(\mathbf{z})$ defined in the model (69)

[9] Especially for people who are familiar with the accuracy of Monte-Carlo-based methods.

provides the coordinate transformation from the sensor Cartesian frame to the LOS Cartesian (RHV) frame. This LOS frame is defined based on the sensor Cartesian frame by the two rotation angles b and e. It has x-axis pointing to the measured position of the target, y-axis lying in the horizontal plane, and z-axis determined by the right-hand rule. If the target state vector is transformed to LOS coordinates as $\mathbf{x}' = [H(\mathbf{z}), O]\mathbf{x}$, then the measurement model (69) is decoupled with the simplest possible measurement matrix: $\mathbf{y} = [I, O]\mathbf{x}' + \mathbf{v}_y$. The same equation also holds in the Cartesian coordinates except that the noise is now $H(\mathbf{z})^T\mathbf{v}_y$. This process in essence converts both pseudomeasurements and state to the same coordinates. This decoupled structure was exploited for synthesis of decoupled filters using pseudomeasurements [39, 40, 41, 79]. To reduce the bias, $H(\mathbf{z})$ is replaced by $H(\bar{\mathbf{z}})$ with the predicted measurement $\bar{\mathbf{z}}$ to decorrelate the filter gain and the current measurement residual [39]. The same idea was utilized in [40, 41, 79] (with [40] and [41] being largely duplicated), where the LOS frame is defined with reference to the prediction $\bar{\mathbf{z}}$ and thus $H(\bar{\mathbf{z}})$ was used. Note that using prediction-based (rather than measurement-based) pseudomeasurement matrix does not completely decorrelate filter gain and measurement residual; it also creates a dependence of the filter gain on the prediction.

Filter decoupling in LOS coordinates is an important and complex issue. Many publications have appeared. Component-wise formulas for each coordinate were presented in [80, 81, 82, 83, 84]. As pointed out in [85], they can be written much more elegantly and compactly in matrix notation. For more information concerning decoupling, the reader is referred to [8, 86, 11] and the references therein.

6.2 Models Based on Universal Pseudo-Linearization

To reduce the bias associated with the EKF and the conventional pseudolinear measurement models, a second class of pseudolinear models has been developed, originated in [77]. It was given the impressive name "*universal linearization*" more recently in [39, 87]. A more precise name is "*universal pseudo-linearization*" because it is actually based on a *pseudolinear* representation of a nonlinear function, as is clear from below.

In this approach, it is assumed that the nonlinear measurement function $\mathbf{h}(\mathbf{x})$ in (10) can be rewritten in the following pseudolinear form

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(\bar{\mathbf{x}}) + G(\mathsf{z}, \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}), \quad \text{for all } \mathbf{x}, \bar{\mathbf{x}} \tag{71}$$

where $G(\mathsf{z}, \bar{\mathbf{x}})$ is some matrix function and $\mathsf{z} = \mathbf{h}(\mathbf{x})$ is the ideal measurement. The corresponding pseudolinear measurement model is

$$\mathbf{z} = G(\mathsf{z}, \bar{\mathbf{x}})\mathbf{x} + \mathbf{d}(\mathsf{z}, \bar{\mathbf{x}}) + \mathbf{v} \tag{72}$$

where $\mathbf{d}(\mathsf{z}, \bar{\mathbf{x}}) = \mathbf{h}(\bar{\mathbf{x}}) - G(\mathsf{z}, \bar{\mathbf{x}})\bar{\mathbf{x}}$. Consider a simple scalar example with $\mathsf{z} = h(x) = x^3$. Since $x^3 - \bar{x}^3 = (x^2 + x\bar{x} + \bar{x}^2)(x - \bar{x}) = (\mathsf{z}^{2/3} + \mathsf{z}^{1/3}\bar{x} + \bar{x}^2)(x - \bar{x})$, we have $G(\mathsf{z}, \bar{x}) = \mathsf{z}^{2/3} + \mathsf{z}^{1/3}\bar{x} + \bar{x}^2$. Clearly, not all nonlinear functions can be written in the form of (71); those that can, such as invertible functions, are called "modifiable" in [77]. This condition was in fact relaxed in [87] by writing $\mathbf{h}(\mathbf{x})$ in the following pseudolinear form

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(\bar{\mathbf{x}}) + H(\mathbf{x}, \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}), \quad \text{for all } \mathbf{x}, \bar{\mathbf{x}} \tag{73}$$

which can always be done. As in Sec. 4.2, assume $\mathbf{h} = [\mathbf{h}_1', \mathbf{h}_2']'$, where $\mathbf{h}_1$ is invertible, and let $\mathbf{x} = [\mathbf{x}_1', \mathbf{x}_2']' = [\mathbf{h}_1^{-1}(\mathsf{z}), \mathbf{x}_2']'$. Then, $G(\mathsf{z}, \bar{\mathbf{x}}) = H\left(\mathbf{h}_1^{-1}(\mathsf{z}), \bar{\mathbf{x}}_2, \bar{\mathbf{x}}\right)$ and (73) becomes $\mathbf{h}(\mathbf{x}) = \mathbf{h}(\bar{\mathbf{x}}) + H\left(\mathbf{h}_1^{-1}(\mathsf{z}), \bar{\mathbf{x}}_2, \bar{\mathbf{x}}\right)(\mathbf{x} - \bar{\mathbf{x}})$. Thus (71) is seen a special case of (73).

Unlike the conventional pseudolinear models, the above models have a better-defined structure of the measurement noise. However, the unknown z is simply replaced by the known measurement $\mathbf{z}$ in this approach, whereas the error of this replacement is accounted for at least partially in the conventional pseudolinear models.

In order to use this modeling technique, appropriate functions $G(\mathsf{z}, \bar{\mathbf{x}})$ are needed. Such functions for 3D angle-only tracking with bearing and elevation measurements were derived in the original paper [77]. The functions are exact for the bearing but approximate for the elevation. More handy but approximate functions for the bearing were also obtained in [88] and [77]. The original function for the elevation exhibits inadequacy for large elevation angles, which may occur in short-range tracking or homing missile applications. More precise functions for the elevation were obtained in [89]. Furthermore, functions for the two general 3D cases — passive (angle-only measurements) and active (range measurement included) tracking — were derived in [53]. Finally, [87] proposed a general procedure for obtaining such functions approximately. The idea is to approximate $\mathbf{h}$ by a polynomial $\tilde{\mathbf{h}}$ of an appropriate order (via the Maclaurin series expansion), manipulate each row of the difference $\tilde{\mathbf{h}}(\mathbf{x}) - \tilde{\mathbf{h}}(\bar{\mathbf{x}})$ into the form $\sum p_i(\mathbf{x}, \bar{\mathbf{x}})(x_i^n - \bar{x}_i^n)$, and via $x_i^n - \bar{x}_i^n = \left(x_i^{n-1} + x_i^{n-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{n-2} + \bar{x}_i^{n-1}\right)(x_i - \bar{x}_i)$ obtain $\mathbf{h}(\mathbf{x}) - \mathbf{h}(\bar{\mathbf{x}}) \approx \tilde{\mathbf{h}}(\mathbf{x}) - \tilde{\mathbf{h}}(\bar{\mathbf{x}}) = H(\mathbf{x}, \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})$.

An EKF-type filter was developed in [77, 90] based on (71), where $H(\bar{\mathbf{x}}) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\bar{\mathbf{x}}}$ is used for filter gain and residual covariance, but $G(\mathbf{z}, \bar{\mathbf{x}})$ is used for error covariance update:

$$
\begin{aligned}
K &= \bar{P}H' \left(H\bar{P}H' + R\right)^{-1} && (74) \\
\hat{\mathbf{x}} &= \bar{\mathbf{x}} + K(\mathbf{z} - \bar{\mathbf{z}}) && (75) \\
P &= (I - KG)\bar{P}(I - KG) + KRK' && (76)
\end{aligned}
$$

It was referred to as a "modified gain" EKF, which is a misnomer because it is the error covariance, not the filter gain, that is modified. In other words, it differs from the EKF only in that it uses $G(\mathbf{z}, \bar{\mathbf{x}})$ instead of $H(\bar{\mathbf{x}})$ for the covariance update. The use of $G(\mathbf{z}, \bar{\mathbf{x}})$ in the gain and residual covariance is avoided so as to remove the direct dependence of the gain on the current measurement $\mathbf{z}$.

Generally, this modified EKF appears to provide somewhat improved performance over the EKF. It was demonstrated in [77] to have a reduced bias as compared to that of the conventional pseudomeasurement based and the EKF (to a lesser degree). It was also shown to be theoretically stable under some conditions [77] and it alleviates the noncredibility [60] problems of the original EKF so typical for many bearing-only tracking applications [77, 88, 11]. [53] also showed that the modified EKF proposed therein for active tracking is more credible and provides marginally more accurate position and velocity estimates than the EKF. [87] considered simple 1D examples. Except for the uncontrollable cases with zero process noise resulting in divergence of the EKF, the universal pseudo-linearization based EKF presented therein and the EKF have virtually indistinguishable performance in all other scenarios. Also, the derivative-based linearized model for the sinusoidal example does not need any polynomial approximation to begin with.

In our opinion, however, all the models of this subsection always can be and should be replaced by the difference-based linearized model of Sec. 4.2, as explained next.

6.3 Superiority of Difference-Based Linearization Models

A major task in the universal pseudo-linearization based modeling technique is to find an appropriate universal function $G(\mathrm{z}, \bar{\mathbf{x}})$ for the problem at hand such that

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(\bar{\mathbf{x}}) + G(\mathrm{z}, \bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}}), \quad \text{for all } \mathbf{x}, \bar{\mathbf{x}} \tag{77}$$

It may involve extensive work and approximations. Once such a *function* is found, however, it is only used to compute the slope *value* of the straight line connecting $\mathbf{h}(\bar{\mathbf{x}})$ and $\mathbf{h}(\mathbf{x})$ approximately at $\mathrm{z} = \mathbf{z}$ as (by abusing the vector notation for brevity[10])

$$\frac{\mathbf{h}(\mathbf{x}) - \mathbf{h}(\bar{\mathbf{x}})}{\mathbf{x} - \bar{\mathbf{x}}} \approx G(\mathbf{z}, \bar{\mathbf{x}}), \qquad \mathbf{z} \approx \mathbf{h}(\mathbf{x}) \tag{78}$$

However, the sought-after slope can be found by (28) directly, easily, generally, and exactly without knowing the universal function $G(\mathrm{z}, \bar{\mathbf{x}})$! Since only the slope *values* are used in the filter, our effort on finding the universal function $G(\mathrm{z}, \bar{\mathbf{x}})$ is a waste.

From the above relationship, it is also clear that $G(\mathbf{h}(\bar{\mathbf{x}}), \bar{\mathbf{x}}) = H(\bar{\mathbf{x}}, \bar{\mathbf{x}}) = H(\bar{\mathbf{x}}) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\bar{\mathbf{x}}}$ and thus this pseudolinear model reduces to the derivative-based linearized model of the EKF if the predicted state estimate is used. In general, they are different. An analysis of their differences made in [53] indicates that the difference may become critical for range estimation in long-range tracking.

Since replacing $\mathbf{x}_p$ by $\mathbf{h}^{-1}(\mathbf{z})$ can be viewed as a crude measurement conversion, which is known to be biased (see Sec. 5.2), care must be exercised. For example, the bias can be removed by the debiasing techniques discussed in Sec. 5, which is most likely superior to ad hoc remedies, such as passing the raw measurements through a low pass filter prior to their use in the above replacement, as mentioned in [87].

6.4 Pseudomeasurement Models for Kinematic Constraints

It is possible in practice to have additional information about the target in terms of constraints on its motion that is not accounted for by the target model. *Kinematic constraints* on the state vector $\mathbf{x}$ in a form of $c(\mathbf{x}) = 0$ are the most common class. Use of such constraints can improve state estimation. The pseudomeasurement models provide a convenient framework for incorporating such constraints. This can be done by augmenting the original measurement model with a pseudomeasurement

[10] The exact meaning of this ratio of vectors is given by (28).

model $0 = h_c(\mathbf{x}) + v_c$, where $h_c(\mathbf{x}) = c(\mathbf{x})$ and v_c is a zero-mean random error, introduced to relax the rigidity of the constraint. Accordingly, the measurement vector is augmented by a zero-value pseudomeasurement as $\mathbf{z}_a = [\mathbf{z}', 0]'$.

This idea was proposed in [91] for tracking targets in a coordinate turn [11] and implemented within the framework of the pseudolinear measurement model (69). The target is expected to have a nearly constant speed motion. This is a fact that is not explicitly embedded into the target motion model — a modified first-order Markov acceleration process (Singer model) with adaptive mean jerk (see Part I [1]). Under such a circumstance, the kinematic constraint arising from the constant speed assumption is that the acceleration vector is nearly orthogonal to the velocity vector: $\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} \approx 0$. Thus, the corresponding pseudomeasurement model is

$$0 = H_c(\mathbf{x})\,\mathbf{x} + v_c \tag{79}$$

where $\mathbf{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}]'$ and $H_c(\mathbf{x}) = [0, 0, 0, \ddot{x}, \ddot{y}, \ddot{z}, \; \dot{x}, \dot{y}, \dot{z}]$, which is replaced by $H_c(\bar{\mathbf{x}})$ with predicted state estimate $\bar{\mathbf{x}}$ in the implementation. The noise v_c is assumed to have zero mean and a variance R_c, which is a design parameter. An extended Kalman filter was implemented using this model and model (69) in [91].

It was proposed in [93, 94, 95] to modify the above scheme by replacing $H_c(\bar{\mathbf{x}})$ with $H_c^*(\hat{\mathbf{x}}) = \left[0, 0, 0, 0, 0, 0, \hat{\dot{x}}, \hat{\dot{y}}, \hat{\dot{z}}\right]/\hat{s}$, where $\hat{\mathbf{x}}$ is the best state estimate before using the kinematic constraint at the time and $\hat{s}$ is the corresponding target speed. Specifically, the state update is done in two steps sequentially [49]: the first step is conventional without the kinematic constraint; its estimate $\hat{\mathbf{x}}$ is used in the second step to provide a pseudomeasurement model more accurate than if the predicted state estimate $\bar{\mathbf{x}}$ is used. The estimate $\hat{\mathbf{x}}$ is then corrected by re-filtering using the pseudomeasurement alone. The replacement of $H_c(\mathbf{x}) = [0, 0, 0, \ddot{x}, \ddot{y}, \ddot{z}, \; \dot{x}, \dot{y}, \dot{z}]$ by $H_c^*(\mathbf{x}) = [0, 0, 0, 0, 0, 0, \dot{x}, \dot{y}, \dot{z}]/s$ suggested in [95] is based on two considerations: the acceleration estimates are often less accurate than the velocity estimates and the normalization by the speed s makes v_c less dependent on the velocity[12] and thus the design of R_c is easier. By the same token, an additional normalization by (the estimate of) the acceleration magnitude can simplify the design of R_c even further. Such normalization, however, makes the pseudomeasurement model more nonlinear and may introduce additional errors. An ad hoc design formula for R_c was used in [93, 94, 96, 95]. An performance improvement by the use of the kinematic constraints over the Kalman filter was demonstrated in [95], along with an analysis of stability and unbiasedness of the resulting filter.

Another kinematic constraint based pseudomeasurement model, arising from bounds on the target speed and/or on the along-track acceleration, was considered in [97] and some of the references therein.

7 Concluding Remarks

Target motion models are best described by target state in Cartesian coordinates while measurements of the target state are directly available in the original sensor coordinates (usually in spherical coordinates or in terms of range and direction cosines). As a result, measurement models in a variety of coordinate systems have been developed.

The most natural and widely used measurement models are in the Cartesian-sensor mixed coordinates, where Cartesian target state is measured in sensor coordinates. They are highly nonlinear due to the nonlinear relationship between the two coordinate systems. Effective tracking with these models relies on nonlinear filtering. The most popular approach here is EKF-based, which relies on derivative-based linearization of the nonlinear models. Many enhancement techniques exist. If simple models are desirable, it appears that the newly-developed difference-based linearized models have a better potential than the derivative-based models. More important, these nonlinear models provide a framework particular suitable for most more sophisticated nonlinear filtering techniques, covered in the subsequent parts of this survey. Consequently, most tracking applications of advances in nonlinear filtering have appeared and will continue to appear in this framework without hidden difficulty.

Most measurement models in Cartesian coordinates have an attractive "linear" structure. They rely on a proper conversion of the measurement models in the original sensor coordinates to the Cartesian coordinate. Linear filters can be applied to this model (but nonoptimally because its measurement noise is actually state dependent and highly non-Gaussian). The emphasis here has been on finding the first two moments of the converted measurement noise with appropriate conditioning. Several different debiasing techniques have been proposed based on a variety of conditional moments of the noise. The state dependence of the converted measurement noise has been accounted for only through the *conditional* moments of the noise. In fact it can be more effectively taken into account in an explicit manner. As pointed out in Sections 5.4 and 5.5, it appears to be more fundamental and appealing to convert the measurement (residual) and the associated covariance directly, not the noise moments, and all we need in the Cartesian coordinates is a measurement residual equivalent to the one in the

[11] Similar ideas have been used in other areas, such as power system state estimation [92], under the name "virtual measurements."

[12] R_c of [91] would be proportional to s^2, similarly as for the converted range rate measurement d of Sec. 3.2.

sensor coordinates. This approach circumvents the ambiguity in the conditioning for noise moments. Although incapable of providing good insight, the quasi-Monte-Carlo method based transformations are attractive for its simplicity and accuracy within this framework.

The pseudomeasurement approach goes one step further. To take advantage of a linear model, it builds a pseudolinear model by constructing appropriate pseudomeasurements or finding a universally applicable pseudolinear representation of a nonlinear function. The price is that the "linear" measurement matrices (and possibly noise) are actually state dependent. Blind applications of linear filters to such disguised nonlinear problems have proven unsatisfactory. Numerous heuristic techniques have been proposed for performance improvement. Few of them are, in our opinion, promising in terms of accuracy and applicability due to their lack of theoretical support. As explained in Sec. 6.3, the universal pseudo-linearization based models are substantially inferior to and should be replaced by the difference-based linearized models.

The approach to convert/express the target state in the sensor coordinates, where the measurement models are the simplest possible, leads to highly nonlinear motion models with significant pseudoaccelerations that must be accounted for effectively. From the nonlinear filtering viewpoint, this appears to be a much more difficult problem than the one with Cartesian state measured in sensor coordinates: In order to (approximately) summarize past information the state in the sensor coordinates must have a high dimension and, probably worse, the process noise is highly state dependent, although the measurement models are linear. To our knowledge, few theoretical results are available for such problems and most techniques developed here are engineering oriented without a solid theoretical foundation.

Models in other coordinates are more application dependent and should be considered given a particular tracking problem. They typically utilize a combination of the sensor and Cartesian coordinates, such as line-of-sight Cartesian coordinates constructed along the range vector in the original sensor coordinates. Here partial measurement and/or state conversions are usually needed. However, the associated bias and credibility issues have been largely ignored or overlooked and should be addressed.

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