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Gateways to Algebra at the Primary Level

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Abstract: This commentary presents an overview of the similarities and differences of the articles in this special issue, and utilizes the National Assessment of Educational Progress (NAEP) 2005 Mathematics Framework as a frame of reference for discussing the topics in each curriculum. Also included are discussions of differing perspectives and the scope and sequence of effective activities that develop algebraic thinking skills.

Introduction

Within the past decade, discussions and opinions pertaining to school algebra have changed dramatically. Professional organizations, policy makers, mathematicians, mathematics educators, administrators, and teachers who once considered algebra as a course just for university-bound students are now espousing the notion of algebra for all students. Overarching policy statements from these groups and individuals indicate a widening of the range of topics that constitute algebraic thinking to encompass now more than just the structural aspects of algebra. This broader description of thinking algebraically has led to the introduction of algebraic ideas into the curriculum at much earlier grade levels.

There is no doubt that teaching algebra and algebraic thinking is both complex and dynamic. Various professional organizations, researchers, and educators have varied perspectives on the exact nature of the topics to be included in an algebra curriculum (National Assessment Governing Board, 2003; National Council of Teachers of Mathematics, 2000; National Research Council 1998). With all these different frameworks and guiding principles, it is understandable that there are debates and disagreements among mathematicians and educators regarding the meaning of algebraic thinking and when it should be introduced into the curriculum. However, such frameworks provide venues for discussions and serve as focal points to help better comprehend components of algebraic thinking.

The theoretic algebra curriculum development framework elaborated by the National Assessment Governing Board (2003) pertaining to the National Assessment of Educational Progress (NAEP) 2005 Mathematics Framework provides a frame of reference for a commentary on the contributed articles in this issue. Specifically, this framework stipulates that algebra curriculum development can be analyzed on the basis of four criteria. These criteria are: 1) patterns, relations, and functions; 2) algebraic representations; 3) variables, expressions, and operations; and 4) equations and inequalities.

In this commentary I first present an overview of the similarities and differences of the curricula described in the articles in this issue. Second, I address each of the four primary grade level criteria from the NAEP framework as they appear in the contributions and then discuss the differing perspectives of each curriculum on when algebraic thinking skills should be introduced into the primary grades. I conclude with remarks about the place of the present contributions in the ongoing development of algebraic thinking in the primary grades.

Overview

Collectively, the articles in this issue illustrate how mathematics educators are using programs from various countries to develop materials, direct projects, and conduct research studies as a means of making mathematics curricula more effective. Of the five country-based articles in this special issue, three are case studies of the national mathematics curriculum in the pivotal countries of China, Korea, and Singapore. In the other two articles, Moyer, Huinker, and Cai report on a US study concerned with a single reform-based curriculum, *Investigations in Number, Data, and Space*, while Schmittau and Morris provide detail regarding implementation in the US of a curriculum based on the work of the Russian educator, Davydov. Although all five articles describe curriculum initiatives at the primary school level, it should be noted that the differences in the learning paradigms between the Davydov curriculum and the other four curricula are profound.

The notion of the mathematics of change is the main theme of the *Investigations* curriculum – a unifying idea that is also related to patterns and relationships, representations, and modeling. The Singapore curriculum applies three approaches in developing algebraic thinking: problem solving, generalizing, and functional analysis. The main focus of the Korean curriculum is on the development of six thinking skills that lead to improved performance in algebra: (1) generalization, (2) abstraction, (3) analysis, (4) dynamism, (5) modeling, and (6) organization. The Chinese primary school curriculum has an overarching goal of understanding quantitative relationships with a main focus on equations and equation solving, ratio and proportion, and variables and functions. In the Davydov curriculum, the chief goal is the students' cognitive development through the enhancement of theoretical thinking skills.

Although the *Investigations* program provides a curriculum for grades K-5, it differs from traditional programs in that it does not use student texts. Instead, teachers are encouraged to use resource books with reproducible student worksheets. The

Singapore curriculum employs a spiraling curriculum with teachers' guides and student editions of textbooks and workbooks, and like the Singapore curriculum, the Korean case study focuses on a national syllabus that includes both teachers' and students' editions of textbooks. The Chinese report also reveals a program that utilizes a national syllabus, teacher reference materials, and student texts.

Finally, the Davydov curriculum does not lend itself to a didactic presentation of traditional topics within primary school mathematics. Instead it can be described as a sequence of problems placed at the center of the program of study, and this central placement profoundly affects both the curriculum and related textual materials. Such a sequence aims to help students to analyze the problems theoretically rather than to form results empirically. Another notable distinction is that in the Davydov curriculum, the study of algebra precedes the study of arithmetic. This contrasts with the other curricula in the case studies wherein students have computational pre-algebraic experiences.

Grade Level Criteria

Patterns, relations, and functions

The next part of this commentary relates the four criteria that pertain to the primary level of the NAEP 2005 algebra framework to the contributions in this issue. The patterns, relations, and functions component of the NAEP framework includes the specific skills of recognizing, defining, and extending numerical patterns as well as being able to construct or explain a rule of a pattern or sequence. Also, the framework stipulates that students should be able to create a different representation of a pattern and to recognize or describe a relationship in which there is a proportional change in a quantity.

A major component in the *Investigations* program is its study of patterns that is designed to help children learn about change and the absence of change. Beginning at the first grade level, students focus on how changes in one pattern relate to changes in another pattern - a critical component of the study of functions. It is not the intent of the *Investigations* curriculum that students be able to formally write functions with algebraic symbols, rather the curriculum aims to teach students to describe and use the underlying properties of patterns and functions, namely the concept of change. As such, this curriculum shifts emphasis away from teaching how to use variables to represent procedures.

The Singapore curriculum lists understanding patterns, relations, and functions as a process component of its curriculum. The entire Singaporean program includes activities that require students to identify, understand, and extend challenging numerical and geometrical patterns. Notably, the Singapore program provides numerous activities in which students develop the concept of function and study how changes in input affect functions.

Lew's case study shows that initially Korean students intuitively learn the concept of function. The Korean curriculum also considers pattern recognition as an important learning process. The process of pattern discovery begins in Grade 1 and continues with additional activities throughout the primary school curriculum. Later, in grades 5-6, students are given opportunities to model problem situations that involve the use of proportional expressions.

Like the Korean children, Cai's study reveals that Chinese students at first learn the concept of function at an intuitive level. Although the Chinese curriculum permeates grades 1- 4 with problems dealing with functions and the arithmetic analysis of quantitative relationships, it is interesting to note that the term "function" is not defined in the Chinese primary school curriculum. In addition, Cai notes that Chinese students learn to solve application problems involving direct and indirect proportional relationships in Grade 6. Finally, it is remarkable that the Davydov curriculum includes no work with patterns at an intuitive level. However, starting in Grade 2 this curriculum does thoroughly develop the concept of proportion with complex multi-step problems involving part-whole relationships, proportional relationships, and other quantitative relationships.

Algebraic representations

The second criterion defined by the NAEP framework involves the translation between different algebraic representations such as verbal, pictorial, numerical, and symbolic. Also included in these criteria are graphing or interpreting points on grids and in the coordinate plane and verifying a conclusion using algebraic properties.

As Moyer et al.'s case study points out, the *Investigations* curriculum presents students with numerous opportunities to study graphical representations of change. In addition to using conventional Cartesian coordinate graphs and line graphs, students are also encouraged to devise their own graphs and to interpret others' graphs to gain exposure to graphical changes in time and direction. While Ng does not elaborate on these topics within her case study, there is evidence that students are given ample opportunities within the Singapore primary mathematics curriculum to deepen their knowledge of different algebraic representations and mathematical properties.

As with Ng's study of the Singapore curriculum, Lew's case study implicitly references representations and properties. However, Lew makes it clear that students are exposed to different mathematical representations at all grade levels. In fact, within the Korean primary mathematics curriculum, activities are suggested in which students identify mathematical properties such as the commutative law of addition as early as Grade 1. In contrast, Cai discusses translations between different algebraic representations and formal mathematical properties in the Chinese case study. For example, in the second half of Grade 4, students are taught to apply the commutative, associative, and distributive laws to specific problem situations. In Grade 6, extensive work involving multiple representations of functional relationships is presented along with ways to describe these relationships - verbally, numerically, symbolically, and graphically.

Algebraic representations and schematics are one of the cornerstones of the Davydov curriculum. The purpose of these representations is to help students abstract relationships between quantities, so they are given ample opportunities to work with algebraic representations within this curriculum. When students are first introduced to word problems, they are often presented with a story text without questions, are asked to analyze the relationships between the quantities, and to formulate their own questions. Later, students are asked to represent the quantitative information in an alternative form (such as in tables) and to analyze the relationships between the quantities.

Variables, expressions, and operations

Using letters and symbols to represent an unknown quantity in a simple mathematical expression is a first principle in the NAEP criteria concerning variables, expressions, and operations. A second principle indicates that students should express simple mathematical relationships using number sentences.

The concepts of variable and mathematical expressions are not given extensive treatment in the *Investigations* program. This curriculum does not introduce and teach algorithmic processes and representations before the students have achieved a solid foundation in investigative techniques and analysis. In fact, the *Investigations* curriculum stresses that introducing symbolism at too early an age is counterproductive to learning. In contrast, Ng's report on the Singapore program shows that there are numerous activities in which students study the concept of function, the notion of letters as variables, and how changes in input affect functions. It is notable that the rectangular regions within the Singapore model method serve as placeholders for the unknown quantities represented by letters in equations. Moreover, Ng describes the Singapore curriculum as one that emphasizes the use of letters as variables rather than as static unknown quantities.

Since the concept of variable is also considered to be very important in the Korean curriculum, using symbols to represent unknown values is introduced as early as the first grade. At this level, the process of solving for an unknown value is intuitively introduced and its mastery is considered essential for understanding functions. Lew's case study also notes that students are expected to perform operational activities with abstract symbols and to use symbols related with concepts and principles.

Although the term "variable" is not defined in the Chinese curriculum, children do extensive work with symbolic and concrete representations of quantities. According to Cai, variables are permeated into the arithmetic analysis of quantitative relationships throughout the curriculum for grades 1 to 4. Particularly, Chinese students use variables as placeholders, pattern generalizers, and representations of relationships. The Davydov curriculum also explicitly requires that students develop an understanding of variable. For example, students learn to interpret letters as standing for different numerical values in grades 1 and 2.

Equations and inequalities

The equations and inequalities criterion in the NAEP framework specifies that students should be able to find the value of the unknown in a whole number sentence. As a result, this criterion involves properly using properties of number systems to represent ideas using equations and to manipulate equations.

The *Investigations* curriculum does not give extensive explanations of equations nor does it include formal equation solving within its program. Instead, intuitive equation-solving activities and problems are embedded within many of the curriculum units. Although there is an expectation that students will be able to write equations and model problems, there are no specific expectations that all students should learn how to solve them. By the same token, in the Singapore curriculum the emphasis is on constructing equations by using letters as unknowns and by drawing models that represent problem situations. In Singapore, students begin to learn the model method in Grade 2. Within this method, pictorial equations are formed and unknown quantities are replaced with rectangular regions. It is notable that even students with no formal training in algebra are able to solve challenging word problems with the help of this method and its pictorial equations.

A main focus in the Chinese primary school curriculum is on equations and equation solving. The development of these concepts is divided into three stages within the Chinese curriculum: intuitive, introduction, and application. The curriculum for grades 1-4 consists of the intuitive stage where students study the notion of variable, equations, equation solving, and function sense. The introduction stage, where students are formally introduced to equations and equation solving, begins in the first half of the Grade 5. The curriculum in the second half of Grade 5 and in Grade 6 further presents students with applications of equation solving involving fractions, percents, proportions, and statistics.

Equation solving is also an important theme in the Korean curriculum. Students are taught as early as Grade 1 in the Korean curriculum to use symbols to represent unknown values and to solve equations intuitively. In Grades 3-4 they are taught to solve equations by using inverse relations and operations. As a result, in the early primary grades, Korean children are taught to develop their analytic thinking abilities and by solving equations by finding any necessary conditions.

In the Davydov curriculum, students study the properties of equality and inequality relationships in-depth. Beginning in Grade 1, they learn to solve equations by thinking about the relationships between quantities. Consequently, in the early primary grades, children are encouraged to develop an understanding of equality and inequality relationships and the associated properties of these relationships.

Final Comments

The case studies also provide differing perspectives on when algebraic thinking skills should be introduced into the primary grades. The *Investigations* program starts the development of algebraic thinking skills at Grade 1. While algebraic thinking is not explicitly referenced in the Singapore curriculum, activities that contribute to its development are evident throughout the early grades in the curriculum. Since Korean students begin the formal study of algebra at Grade 7, many prerequisite activities are included within the early primary school grades to promote the development of algebraic thinking. From Cai’s study of the Chinese curriculum, it is evident that the facilitators of algebraic thinking permeate the first four grades and that the formal study of equations and equation solving begins at Grade 5. The case studies of these four curricula describe activities to develop algebraic thinking in the primary grades, while deferring the formal study of algebra, *per se*, until the early secondary grades. In contrast, the Davydov curriculum begins to develop the concepts of equality and other key algebraic notions as early as Grade 1.

Overall, a perusal of the contributions in this special issue leads one to conclude that the meaning of algebra as illustrated in curricula, textbooks, teacher resources, and notions of what constitutes algebraic thinking at the primary school level varies dramatically across cultures. Nonetheless, algebra for all students is a goal that enjoys consensus not only in these articles, but also one that continues to be supported by mathematics educators and policy makers worldwide. However, if students are to achieve success in algebra in the later school years, these case studies make it clear that primary school students need quality mathematical experiences to their develop algebraic readiness. Consequently, the articles in this issue contribute to the ongoing dialogue on the content of activities that develop algebraic thinking skills, and the scope and sequence of effective primary school curricula for promoting this crucial way of thinking.

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