Unified Approach to KdV Modulations

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Abstract

We develop a unified approach to integrating the Whitham modulation equations. Our approach is based on the formulation of the initial-value problem for the zero-dispersion KdV as the steepest descent for the scalar Riemann-Hilbert problem [6] and on the method of generating differentials for the KdV-Whitham hierarchy [9]. By assuming the hyperbolicity of the zero-dispersion limit for the KdV with general initial data, we bypass the inverse scattering transform and produce the symmetric system of algebraic equations describing motion of the modulation parameters plus the system of inequalities determining the number the oscillating phases at any fixed point on the \((x, t)\)-plane. The resulting system effectively solves the zero-dispersion KdV with an arbitrary initial datum. © 2001 John Wiley & Sons, Inc.

1 Introduction

The initial-value problem for the Korteweg–de Vries equation

\begin{equation}
   u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \quad u(x, 0) = u_0(x),
\end{equation}

in the zero-dispersion limit \( \epsilon \to 0 \) has been the subject of intense study for more than 25 years. The physical interest of this limit is that it allows the modeling of dispersive shocking in dissipationless dispersive media. In contrast to the usual dissipative hydrodynamics, the regularization of a shock occurs here through the generation of small-scale nonlinear oscillations. Gurevich and Pitayevskiî in [16] proposed to describe these oscillations with the aid of the one-phase Whitham modulation equations [34]. The multiphase analogue of the Whitham equations was derived by Flaschka, Forest, and McLaughlin [10]. Later, Lax and Levermore [23] rigorously showed that the multiphased averaged equations appear in the zero-dispersion limit of the initial-value problem for the KdV equation with asymptotically reflectionless initial data. For reflecting potentials, the zero-dispersion theory was constructed by Venakides [31], who also identified the parameters describing...
the weak limit of the solution in the Lax-Levermore approach with the Riemann invariants of the modulation equations in [10]; see [30, 32].

The modulation system itself, however, has to be integrated to reveal the dependence of the Riemann invariants (modulation parameters) on $x$ and $t$. For this, the observation of Tsarëv [29] (see also [8]), who generalized the classical hodograph method to a multidimensional (in the space of dependent variables) case, was crucial. The result of applying this generalized hodograph transform to the averaged KdV is the overdetermined consistent system of linear PDEs. Tsarëv’s results were put into the algebro-geometrical setting by Krichever [18, 19].

Each solution to the Tsarëv system gives rise to some local solution of the nonlinear modulation system. These solutions generically exist only within definite regions of the $(x, t)$-plane. To obtain the global solution, one should supplement the constructed local solutions with information about the number $N$ of nonlinear phases in each region and provide the smooth matching of solutions with different $N$ on the phase transition boundaries. The simplest, yet very important, class of problems with $N \leq 1$ has been investigated in works of Tian [27, 28] and Gurevich, Krylov, and El and their collaborators [13, 14, 15, 20], who obtained a number of exact solutions for the initial-value problems with monotone and humplike data having the only break point. Tian [28] was also able to prove that the solutions to the Whitham system in this case do globally belong to $N = 1$.

More general local solutions to the modulation system in the case of arbitrary $N$ were constructed in a symmetric form by El [9] with the aid of the fundamental solution (an analogue of the Green function) to the Tsarëv equations. Some global solutions involving the case $N = 2$ were constructed recently by Grava [12] on the basis of Dubrovin’s variational approach to the Whitham equations [7].

The common feature of all pointed methods for constructing the global solutions to the Whitham equations is the need to follow all earlier times $t < t_0$ to obtain the solution at $t_0$. This difficulty has been bypassed recently by Deift, Venakides, and Zhou [6], who obtained the algebraic equations for the KdV modulations while avoiding the integration of the modulation equations themselves. The method of DVZ is based on reformulation of the initial-value problem for the zero-dispersion KdV with decaying solitonless analytic initial data as the steepest descent for the scalar Riemann-Hilbert problem for the complex phase function. As a result, they obtain not only the local solutions to the modulation equations, but also the system of inequalities enabling the number of phases to be determined locally at each point of the $(x, t)$-plane.

It is clear, however, that the requirements of decaying and pure reflection for the initial potential are not essential in the zero-dispersion limit. Assuming the finite speed of propagation (i.e., validity of the Whitham equations) for the zero-dispersion KdV with an arbitrary initial data, one can see that only the finite part of the initial data contributes to the solution at any finite $t$. In this case, considering the evolution of “multihump” potential and then tending the number of “humps” to infinity, one arrives at the solution for nondecaying initial data. Certainly, for
finite \( \epsilon \), the results obtained in this way are valid within a finite time interval until the semiclassical (Whitham) spectrum begins to compete with the fine spectrum of the potential. The assumption of hyperbolicity in the limit studied eliminates the requirement of analyticity for the initial data.

In this work, we combine methods of Deift, Venakides, and Zhou \([6]\) and El \([9]\) to produce the global solution to the KdV modulation equations with an arbitrary initial datum. Namely, by assuming validity of the modulation equations in the case of the zero-dispersion KdV with an arbitrary initial datum, we reformulate the initial-value problem for the Whitham system as the Riemann-Hilbert problem and produce the symmetric system of algebraic equations supplemented by the system of inequalities determining both the motion of the Riemann invariants and the change of the genus of the Riemann surface (number of oscillating phases). We represent the local part of the obtained general solution in a potential form with the generalized functional of the Peierls-Fröhlich type \([4, 5, 17]\) as a potential. Recently the functionals of this type in more particular form were shown to give rise by minimization to the global solutions for the Whitham equations with monotone initial data \([7]\).

We also establish exact correspondence between some results obtained in \([6]\) and the results obtained earlier in \([7, 9, 18]\), which unifies different approaches developed independently in this area.

## 2 Summary of the Riemann-Hilbert Steepest Descent for the Zero-Dispersion KdV

It is well-known that the inverse scattering transform can be reformulated as a matrix Riemann-Hilbert (RH) problem; see \([3, 26]\). Deift, Venakides, and Zhou \([6]\) showed by considering the quasi-classical asymptotics for the initial-value problem \((1.1)\) in the case of one-hump solitonless initial data \(0 \leq u_0(x) \leq 1\) that this problem can be asymptotically (as \(\epsilon \to 0\)) reduced to the scalar RH problem for the complex phase function \(g(\lambda)\). We present here the resulting formulae (see \([6, 33]\) for details).

Let the interval \((0, 1)\) of the real axis on the \(\lambda\)-plane be partitioned into a finite set of intervals

\[
\{G_k(k = 0, 1, \ldots, N) : (0, r_1), (r_2, r_3), \ldots, (r_{2N}, r_{2N+1})\},
\]

\[
\{B_n(n = 1, \ldots, N + 1) : (r_1, r_2), (r_3, r_4), \ldots, (r_{2N+1}, 1)\}.
\]

We introduce the combination

\[
\alpha(\lambda) = 4t\lambda^{3/2} + x\lambda^{1/2},
\]

which will be important hereafter, and the phases of the reflection (from the right) and the transmission coefficients for the scattering on the potential \(u_0(x)\),

\[
\rho_+(\lambda) = x^+\lambda^{1/2} + \int_{x^+}^{\infty} \left[\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}\right]dx',
\]
(2.3) \[ \tau(\lambda) = \int_{x^-}^{x^+} (u_0(x') - \lambda)^{1/2} dx'. \]

where \( x^\pm = x^\pm(\lambda) \) are the roots of the equation \( u_0(x) = \lambda \).

Now we introduce the complex phase function \( \varphi(\lambda) \), which plays the central role in what follows. This function is defined by the following scalar RH problem: On the intervals \( \{ G_k, (k = 0, \ldots, N) : (0, r_1), (r_2, r_3), \ldots, (r_{2N}, r_{2N+1}) \} \) the following relations hold:

(2.4) \[ \frac{g_+ + g_-}{2} = \rho_+ - \alpha', \]

(2.5) \[ -\tau < \frac{g_+ - g_-}{2i} < 0. \]

It follows from equality (2.4) that on each interval \( G_k \): \( g_+ g_- - 2\rho_+ + 2\alpha = -\Omega_k \), where \( \Omega_k \) is some constant of integration; without loss of generality, one can put \( \Omega_0 = 0 \).

On the intervals \( \{ B_k, (k = 1, \ldots, N) : (r_1, r_2), (r_3, r_4), \ldots, (r_{2N-1}, r_{2N}) \} \), we have

(2.6) \[ \frac{g_+ - g_-}{2i} = -\tau, \]

(2.7) \[ \frac{g_+ + g_-}{2} < \rho_+ - \alpha'. \]

The latter inequality is due to the further reduction of the RH problem with the aid of the steepest-descent method [6].

On the remaining interval \( \{ B_{N+1} : (r_{2N+1}, 1) \} \), there exist two possibilities:

(2.8) \[ \text{Case A: } \frac{g_+ - g_-}{2i} = -\tau, \quad \frac{g_+ + g_-}{2} < \rho'_+(\lambda) - \alpha', \]

which coincides with conditions (2.6) and (2.7), or

(2.9) \[ \text{Case B: } g_+ - g_- = 0, \quad \frac{g_+ + g_-}{2} > \rho'_+(\lambda) - \alpha'. \]

We also have outside the interval \((0, 1)\)

(2.10) \[ g_+ + g_- = 0 \quad \text{if } \lambda < 0, \]

(2.11) \[ g_+ - g_- = 0 \quad \text{if } \lambda > 1. \]

In all the above formulae we denote

\[ g_\pm(\lambda) \equiv \lim_{\delta \to 0} g(\lambda \pm i\delta). \]

The additional requirement imposed on the function \( g(\lambda) \) is

(2.12) \[ \text{The functions } \sqrt{\lambda}g'(\lambda) \pm \text{ are continuous for real } \lambda. \]
It is also shown in [6] that the following asymptotic is true:

\[ g(\lambda) = g_1/\lambda^{1/2} + O\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to \infty. \]

One can see that the RH problem formulated above contains not only equations defining the function \( g(\lambda) \) on the intervals \( B_k \) and \( G_k \) but also the inequalities that define the overall number \( 2N \) of the intervals (if \( N \) is chosen wrongly, then at least one of the inequalities will be violated).

We observe that for each fixed \( x \) and \( t \), \( g' \) also satisfies a scalar RH problem on the real axis. Indeed, \( g'_+ + g'_- = 0 \) when \( \lambda < 0 \), and \( g'_+ - g'_- = 0 \) when \( \lambda > 1 \). When \( 0 < \lambda < 1 \), the equalities (2.4), (2.6), (2.8), and (2.9) specify \( g'_+ + g'_- = -2i\tau' \) outside these intervals. We only consider \( g'_+ - g'_- = -2i\tau' \) when \( \lambda \) lies between any two of the \( G_k \)'s, while on the remaining interval \( (r_{2N+1}^2, 1) \), we examine both possibilities, i.e., \( g'_+ - g'_- = -2i\tau' \) (case A) or \( g'_+ - g'_- = 0 \) (case B).

Necessarily \( g'(\lambda) \) has the following form:

\[
g'(\lambda) = \sqrt{R_{2N+1}^2(\lambda)} \left( \int_{\bigcup G_k} \frac{2\rho'(\mu) - 2\alpha'(\mu)}{\sqrt{R_{2N+1}^2(\mu)(\mu - \lambda)}} \frac{d\mu}{2\pi i} + \int_{(0, E)\setminus \bigcup G_k} \frac{-2i\tau'(\mu)}{\sqrt{R_{2N+1}^2(\mu)(\mu - \lambda)}} \frac{d\mu}{2\pi i} \right),
\]

where

\[
R_{2N+1}^2(\lambda) = \prod_{j=1}^{2N+1} (\lambda - r_j),
\]

and \( E = 1 \) in case A and \( E = r_{2N+1}^2 \) in case B. Here \( \sqrt{R_{2N+1}^2(\lambda)} \) is positive for \( \lambda > r_{2N+1} \). Also, \( \sqrt{R_{2N+1}^2(\lambda)} \) denotes the boundary value from above.

A sufficient number of conditions to determine the endpoints of the intervals \( G_j \) can now be written down. Indeed, the condition \( g(\lambda) = O(\lambda^{-1/2}) \) for large \( \lambda \) implies \( g'(\lambda) = O(\lambda^{-3/2}) \), which leads to the following moment conditions:

\[
\int_{\bigcup G_j} \frac{\rho'(\lambda) - \alpha'(\lambda)}{\sqrt{R_{2N+1}^2(\lambda)}} \lambda^k d\lambda + \int_{(0, E)\setminus \bigcup G_j} \frac{-i\tau'(\lambda)}{\sqrt{R_{2N+1}^2(\lambda)}} \lambda^k d\lambda = 0,
\]

\[ k = 0, \ldots, N. \]

A second set of conditions is obtained by integrating \( g' \) around each \( G_j \) and using (2.6). In case A we obtain

\[
\int_{G_j} (g'_+ - g'_-)(\lambda) d\lambda = -2i(\tau(r_{2j+1}) - \tau(r_{2j})), \quad j = 1, \ldots, N.
\]
In case B, \( \tau(r_{2N}) \) and \( \tau(r_{2N+1}) \) must be replaced by zero.

Conditions (2.16) and (2.17) represent a system of \((N + 1) + N = 2N + 1\) independent equations for the \(2N + 1\) unknowns (the branch points \( r_1, r_2, \ldots, r_{2N+1} \) of the Riemann surface (2.15)). Conversely, suppose that for given \( x, t, \) and some \( N, \) the quantities \( r_1, r_2, \ldots, r_{2N+1} \) satisfy conditions (2.16) and (2.17), giving rise to an explicit expression (2.14) for \( g' \), and hence for \( g \) by integration. Suppose further that the function \( g \) so constructed also satisfies inequalities (2.5)–(2.9). Then \( g \) is the desired solution of the scalar RH problem.

Deift, Venakides, and Zhou show that the parameters \( r_j \) represent the “semiclassical spectrum” of the problem; i.e., they are the branch points of the spectral Riemann surface defining the local finite-gap solution of the KdV. In Section 4 we shall directly identify the moment conditions (2.16) and (2.17) with the local solution of the KdV-Whitham system.

Now we present a number of important relations that will be very useful for establishing this correspondence.

We observe from relations (2.4)–(2.11) that \( g(\lambda) \) also satisfies an RH problem. Solving this RH problem in exactly the same way as the problem for \( g' \), we obtain, similarly to (2.14),

\[
g(\lambda) = \sqrt{R_{2N+1}(\lambda)} \sum_{j=0}^{N} \left( \int_{G_j} \frac{2\rho_+(\mu) - 2\alpha(\mu) - \Omega_j}{\sqrt{R^2_{2N+1}(\mu)(\mu - \lambda)}} d\mu \right) - \frac{2i \tau(\mu)}{2\pi i} \int_{(0,E) \cup G_j} \sqrt{R^2_{2N+1}(\mu)(\mu - \lambda)} d\mu.\]

(2.18)

The expression for the constant of integration \( \Omega_j \) is

\[
\Omega_j = -2x \oint_{a_{\infty}} \lambda^{1/2} \psi_j - 8t \oint_{a_{\infty}} \lambda^{3/2} \psi_j + 4 \oint_{\bigcup G_k} \rho_+ \psi_j - 4 \oint_{(0,E) \cup G_k} i \tau \psi_j
\]

\[= x \Omega_k + t \Omega_{k2} + \Omega_{k3}, \quad j = 1, \ldots, N, \quad \Omega_0 = 0.\]

(2.19)

We recall that

(2.20) \( E = 1 \) (case A) \quad or \quad \( E = r_{2N+1} \) (case B).

The basis of holomorphic differentials \( \psi_j \) is given by

\[
\psi_j = \sum_{k=0}^{N-1} c_{k,j} \frac{\lambda^k}{\sqrt{R_{2N+1}(r, \lambda)}} d\lambda, \quad k = 1, \ldots, N.
\]

(2.21)

\[
\oint_{a_k} \psi_j = \delta_{jk}, \quad k, j = 1, \ldots, N.
\]

(2.22)
The contours $\alpha_k (k = 0, 1, \ldots, N)$ surround the intervals $(r_2, r_3), (r_3, r_4), \ldots, (r_{2j}, r_{2j+1}), (r_{2N}, r_{2N+1})$ clockwise.

The following identities can be obtained for $\Omega_{j1}$ and $\Omega_{j2}$:

\begin{align}
\Omega_{j1} &= -\text{Res}_\infty \lambda^{1/2} \psi_j, & \Omega_{j2} &= -4 \text{Res}_\infty \lambda^{3/2} \psi_j, \\
\partial_x \Omega_j &= \Omega_{j1}, & \partial_t \Omega_j &= \Omega_{j2}.
\end{align}

One can observe from (2.23) that in normalization (2.22), $\Omega_{j1}$ and $\Omega_{j2}$ can be identified with the wave number $k_j$ and the frequency $\omega_j$, respectively [10], where $j$ is the number of the phase, $j = 1, 2, \ldots, N$. In fact (taking into account the change of the normalization in comparison with [10]), we have the expansion as $\lambda \to \infty$

\begin{equation}
\psi_j = \frac{1}{\lambda^{3/2}} \left( k_j + \frac{1}{4\lambda} \omega_j + \cdots \right),
\end{equation}

where

\begin{equation}
k_j = c_{N-1,j}, & \omega_j = 2c_{N-1,j} \sum_{m=1}^{2N+1} r_m + 4c_{N-2,j}.
\end{equation}

Taking the mixed derivatives of (2.24), we arrive at $N$ equations expressing the wave number conservation laws for each of $N$ phases

\begin{equation}
\partial_j k_j = \partial_x \omega_j, \quad j = 1, \ldots, N.
\end{equation}

### 3 Local Solutions to the Whitham Equations in a Symmetric Form

Now we propose an alternative way for deriving the equations for the branch points $r_j$, equivalent to the moment conditions (2.16) and (2.17), in a more symmetric and universal form.

Calculating $\partial_x \Omega_j$ directly from (2.19), we get

\begin{equation}
\partial_x \Omega_j = \Omega_{j1} + \sum_{k=1}^{2N+1} \frac{\partial \Omega_j}{\partial r_k} \partial_x r_k, \quad j = 1, \ldots, N,
\end{equation}

which, together with (2.24), implies

\[ \sum_{k=1}^{2N+1} \frac{\partial \Omega_j}{\partial r_k} \partial_x r_k = 0 \]

for any spatially nonuniform solution $r_k(x, t)$. Thus,

\begin{equation}
\partial_j \Omega_i = 0, \quad \partial_j = \frac{\partial}{\partial r_j}, \quad j = 1, \ldots, 2N+1, \quad i = 1, \ldots, N,
\end{equation}

provided $\partial_x r_j \neq 0, \quad j = 1, \ldots, 2N+1$.

System (3.2) is the system of $N(N+1)$ algebraic equations for $2N+1$ variables $r_j(x, t)$. Due to the uniqueness of the solution for the Riemann-Hilbert problem (2.4)–(2.11), it has to be equivalent to the moment conditions (2.16) and (2.17). To
show this, it is enough to prove the consistency of the system (3.2). In other words, one has to show that all $N$ closed systems for $r_j (j = 1, \ldots, 2N + 1)$, which form the overdetermined system (3.2), are equivalent. With this aim in mind, we present the following lemma:

**Lemma 3.1** The overdetermined system

$$
\frac{\partial}{\partial t} \Omega_i = 0, \quad j = 1, \ldots, 2N + 1, \quad i = 1, \ldots, N,
$$

where $\Omega_i$ is defined by (2.19), is consistent and equivalent to the symmetric system of $2N + 1$ algebraic equations with respect to $2N + 1$ variables $r_j$.

(3.3) \[ \oint_{\alpha_k} \{ x \lambda^{1/2} + 4t \lambda^{3/2} \} \Lambda_i = \oint_{\beta_n} \rho_n(\lambda) \Lambda_i - i \oint_{\beta_E} \tau(\lambda) \Lambda_i. \]

The contours $\alpha_k$, $k = 0, \ldots, N$, as mentioned above, surround the intervals $G_k$ clockwise. The contours $\beta_n$, $n = 1, \ldots, N + 1$, surround the intervals $B_n$ clockwise, and the contour $\beta_E$ surrounds the interval $(E, 1)$, where $E = 1$ (case A) or $E = r_{2N+1}$ (case B).

The differential $\Lambda_j$, $j = 1, \ldots, 2N + 1$, is defined by

(3.4) \[ \Lambda_j = \frac{\partial_j \psi_i}{\partial_j k_i} = \frac{\lambda^N + \sum_{k=1}^{N} \frac{\lambda^{N-k} p_{k,j}}{(\lambda - r_j) \sqrt{R_{2N+1}}}}{d\lambda}, \]

and the coefficients $p_{k,j}$ can be found unambiguously from the normalization conditions

(3.5) \[ \oint_{\alpha_m} \Lambda_j = 0, \quad m = 1, \ldots, N. \]

The proof of the lemma can be found in Appendix A.

**Theorem 3.2 (Identification)** The moment conditions (3.3) appearing in the RH problem for the zero-dispersion KdV equation are consistent with the $N$-phase averaged Whitham-KdV system

(3.6) \[ \partial_t r_j = V_j(r) \partial_r r_j, \quad j = 1, \ldots, 2N + 1, \]

where $V_j(r)$ are computed as certain combinations of complete hyperelliptic integrals [10, 23].

**Proof:** To identify the algebraic system (3.3) with the solution of the Whitham equations (3.6), we make use of the Tsarëv result [8, 29]: If $W_j(r_1, \ldots, r_{2N+1})$ is a solution of the linear overdetermined consistent system

(3.7) \[ \frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}, \quad i \neq j, \quad i, j = 1, \ldots, 2N + 1, \]

then the system of algebraic equations (generalized hodograph transform)

(3.8) \[ x + V_j t = W_j \]
gives implicitly the smooth solution \( r_j(x, t) \) to the Whitham system (3.6) provided \( \partial_x r_j \neq 0 \).

First, we observe that the following identities hold:

\[
\begin{align*}
-\frac{1}{2\pi i} \oint_{a_\infty} \lambda^{1/2} \Lambda_j = 1 & \quad \text{for all } j, \\
\frac{2}{\pi i} \oint_{a_\infty} \lambda^{3/2} \Lambda_j = V_j.
\end{align*}
\]

The first equality immediately follows from definition (3.4). The second one requires a bit more detailed consideration. We make use of the fact that system (3.6) implies existence of the wave-number conservation law (2.27). Then, introducing the Riemann invariants \( r_j \) into equation (2.27) explicitly, one easily gets the representation for the characteristic speeds of the Whitham system (3.6) (see [13, 14, 21] for the case \( N = 1 \) and [9] for arbitrary \( N \)).

\[
V_i = \frac{\partial_i \alpha_j}{\partial \kappa_j} \quad \text{for any } j.
\]

The expression (3.11) can be interpreted as a generalization of the group velocity notion to the case of nonlinear waves [9]. It should be noted that the relationships analogous to (3.11) arise in the classical theory of hyperbolic systems as the compatibility conditions providing existence of an additional conservation law [22].

Using (2.23), (2.25), and (3.4), we arrive at

\[
V_i = \frac{2}{\pi i} \frac{\partial_i}{\partial \kappa_j} \oint_{a_\infty} \psi_j \lambda^{3/2} d\lambda = \frac{2}{\pi i} \oint_{a_\infty} \lambda^{3/2} \Lambda_j = -4p_{1,i}.
\]

Solution (3.3) then takes the form (3.8) provided

\[
W_i = -\frac{1}{2\pi i} \left\{ \oint_{\cup \rho_k} \rho_+(\lambda) \Lambda_j - i \oint_{\cup \rho_k \setminus \rho_k} \tau(\lambda) \Lambda_j \right\},
\]

where the functions \( \rho_+(\lambda) \) and \( \tau(\lambda) \) are supposed to be analytic.

We have to prove, therefore, that (3.13) does solve the linear system (3.7), which implies proving that \( \Lambda_j \) does solve this system at any \( \lambda \). With this aim in mind, we consider the following combinations occurring in the left-hand part of (3.7):

\[
\partial_i \Lambda_j = \partial_i p_{1,j} \frac{\lambda^N + \sum_{k=1}^{N} \lambda^{N-k} a_{k,j}}{(\lambda - r_i)(\lambda - r_j)\sqrt{R_{2N+1}}} d\lambda,
\]

where all \( a_{k,j} \) are uniquely defined by the conditions following from the normalization (3.5)

\[
\oint_{a_m} \partial_i \Lambda_j = 0, \quad m = 1, \ldots, N.
\]
Another relevant combination is
\[(3.16)\quad \Lambda_i - \Lambda_j = (p_{1,i} - p_{1,j}) \frac{\lambda^N + \sum_{k=1}^{N} \lambda^{N-k} p_{k,i}}{(\lambda - r_i)(\lambda - r_j)\sqrt{R_{2N+1}}} d\lambda ,\]
where, again, the coefficients $b_{k,j}$ are given by the normalization
\[(3.17)\quad \oint a_m (\Lambda_i - \Lambda_j) = 0, \quad i \neq j ,\]
which implies $b_{k,i} = a_{k,i}$. Then
\[(3.18)\quad \frac{\partial_i \Lambda_j}{\Lambda_i - \Lambda_j} = \frac{\partial_i p_{1,j}}{p_{1,i} - p_{1,j}} .\]
Recalling that $p_{1,j} = -1/4V_j$ (see (3.12)), we prove the identification theorem.

We also present the equivalent form of the solution (3.3) parametrized by the phase of the reflection coefficient from the left $\rho_-$ (cf. (2.2)),
\[(3.19)\quad \rho_-(\lambda) = \int_{-\infty}^{-x^-} \left[ \lambda^{1/2} - (\lambda - u_0(x))^{1/2} \right] dx - \lambda^{1/2}x^- .\]
It can be shown that the following identity holds (see Appendix B for details):
\[(3.20)\quad \oint_{\alpha_k} \{ \rho_-(\lambda) + \rho_+(\lambda) \} \Lambda_j = i \oint_{\beta_E} \tau(\lambda) \Lambda_j = 0 .\]
Then solution (3.3) takes the form
\[(3.21)\quad \oint_{a_\infty} \{ x\lambda^{1/2} + 4t\lambda^{3/2} \} \Lambda_j = - \oint_{\alpha_k} \rho_-(\lambda) \Lambda_j + i \oint_{\beta_E} \tau(\lambda) \Lambda_j .\]
In particular, in case A we have the especially simple representation (we note that $\bigcup_{a_k} = a_\infty$)
\[(3.22)\quad \oint_{a_\infty} \{ x\lambda^{1/2} + 4t\lambda^{3/2} + \rho_-(\lambda) \} \Lambda_j = 0, \quad j = 1, \ldots, 2N + 1 .\]
This form of the solution coincides with the one obtained in [9].

The differential $\Lambda_j$ was introduced for the first time by El [9] as a generating differential of the Whitham hierarchy and can be regarded as a nonlinear analogue of the Green function for the modulation equations. In fact, because $\Lambda_j$ depends on the free parameter $\lambda$ and satisfies identically the Tsarëv equations (3.7), we have automatically proven that it is a fundamental solution to the Tsarëv system. Expanding (3.4) in powers of $1/\lambda$ as $\lambda \to \infty$, we obtain the homogeneous solutions to Tsarëv’s equations. Those with odd indices of homogeneity $n = 3, 5, 7, \ldots$, are the characteristic speeds of the $N$-gap averaged $n^{th}$ KdV in the hierarchy [18]. Therefore, $\Lambda_j$ is the generating differential for the averaged KdV hierarchy.
effective formulae for the characteristic speeds of the hierarchy in terms of the generating differential are (up to a normalization constant)

\[ W_j^{(n)}(\tau) d\lambda = - \text{Res}_\infty \lambda^{n-3/2} \Lambda_j. \]

Corresponding solutions \( r_j(x, t) \) given by the system

\[ \oint_{\alpha_\infty} \left\{ x\lambda^{1/2} + 4i\lambda^{3/2} + c\lambda^{n-3/2} \right\} \Lambda_j = 0, \quad j = 1, \ldots, 2N + 1, \]

are self-similar [18];

\[ r_j(x, t) = t^{\gamma_j} \left( \frac{x}{t^{\gamma + 1}} \right), \quad \gamma = \frac{1}{n - 1}, \quad n = 3, 5, \ldots, \]

and solve the initial-value problem

\[ x = \left[ 2^n \frac{(2n - 1)!!}{n!} c \mu^n \right]_{t=0}. \]

Taking \( n = 3 \) we arrive at Potëmin’s solution [25] (see also [8]) describing the universal regime of formation of the collisionless shock in the vicinity of the break point [8, 13, 14, 16]. This solution was also obtained by Wright [35] applying the Lax-Levermore methods [23]. We also note that if

\[ \rho_-(\lambda) = - \sum_{k=2}^{2N+1} c_k \lambda^{k+1/2}, \]

then formula (3.22) gives the realization of the general Krichever prescription for constructing the algebraic-geometrical solutions to the Whitham equations [18].

### 4 Functionals of the Peierls-Fröhlich Type as the Potentials for the Local Solutions to the Whitham Equations

We point out the important relationship between the differential \( \Lambda_j \) and the standard meromorphic differential (quasi momentum)

\[ dp = \frac{\lambda^N + \sum_{k=1}^{N} \lambda^{N-k} q_k}{\sqrt{R_{2N+1}}} d\lambda \]

normalized by

\[ \oint_{\alpha_m} dp = 0, \quad m = 1, \ldots, N. \]

As can be easily seen,

\[ \Lambda_j = \frac{\partial_j dp}{\partial_j q_1 + \frac{1}{2}}. \]
In particular, for $N = 0$,

\[(4.4)\]

\[\Lambda = 2 \partial_r dp = \frac{d\lambda}{(\lambda - r)^{3/2}}.\]

It is well-known (see, for instance, [8]) that $\int dp$ is the generating function for the averaged Kruskal integrals $Q_k$:

\[(4.5)\]

\[P = \int dp = 2\sqrt{\lambda} + \sum_{k=0}^{\infty} \frac{2Q_k}{(2\sqrt{\lambda})^{2k+1}}.\]

Expanding (4.1) near infinity and comparing with (4.5), we find that

\[(4.6)\]

\[q_1 = Q_0 - \frac{1}{2} \sum_{j=1}^{2N+1} r_j,\]

where

\[(4.7)\]

\[Q_0 = \bar{u} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} u(x, r) dx.\]

Then it follows from (4.3) that

\[(4.8)\]

\[\Lambda_j = \frac{\partial_j dp}{\partial_j \bar{u}},\]

provided $\partial_j \bar{u} \neq 0$. Substituting the representation (4.8) into the solution (3.3), one arrives at the potential form of the local solution to the Whitham equations

\[(4.9)\]

\[\partial_j F_N(r; x, t) = 0, \quad j = 1, \ldots, 2N + 1,\]

where the potential

\[(4.10)\]

\[F_N(r; x, t) = \Phi[u_0(x); x, t, N] = \]

\[\frac{1}{2\pi i} \left[ \int_{d\infty} \left\{ x \lambda^{1/2} + 4t \lambda^{3/2} \right\} dp + 2 \int_{\bigcup G_k} \rho_-(\lambda) dp - 2i \int_{E} \tau(\lambda) dp \right].\]

In case $A (E = 1)$ we observe that $F_N$ represents the functional of the Peierls-Fröhlich type [4, 5, 17],

\[(4.11)\]

\[F_N(r; x, t) = \frac{1}{2\pi i} \left[ \int_{d\infty} \left\{ x \lambda^{1/2} + 4t \lambda^{3/2} \right\} dp + 2 \int_{\bigcup G_k} \rho_-(\lambda) dp \right].\]

The case of the Peierls-Fröhlich-type functional with (cf. (3.27))

\[(4.12)\]

\[\rho_-(\lambda) = -\sum_{k=2}^{2N+1} c_k \lambda^{k+1/2} + 2 \int_{\lambda}^{\infty} \frac{g(u)}{\sqrt{u - \lambda}} du,\]

where $g(u)$ is a sufficiently small smooth function, has been studied recently by Dubrovin [7], who found that the minimizer to (4.11) and (4.12) gives rise to the
solution of the Cauchy problem for the KdV-Whitham system with the monotone initial data

\begin{equation}
(4.13) \quad x = \left[ \sum_{k=2}^{2N+1} \frac{(2k+1)!!}{2^{k-1}k!} c_k u^k + g(u) \right]_{t=0}.
\end{equation}

One can suppose that the minimizer to the functional (4.10) gives rise to the solution of the Cauchy problem with hump-like initial data. We emphasize also that the solution in the form (4.9) and (4.10) does not require analyticity from the initial data.

5 Global Solutions to the Whitham System

5.1 General Formulation

As shown in Section 2, which gives an account of the results of [6], the steepest descent for the RH problem yields the global solution to the Whitham equations, bypassing the procedure of integration of the modulational system itself. The obtained solution, however, is restricted by the requirements of decaying at infinity and of analyticity for the initial data. In addition, in [6] the authors deal only with solitonless (pure reflective) initial data. On the other hand, the Whitham equations themselves admit more general formulation of the problem canceling these restrictions.

Our idea is to construct the global solutions to the general Cauchy problem for the Whitham equations by applying the results of the RH problem approach described above to a more general class of functions $\rho(\lambda)$ and $\tau(\lambda)$ (and therefore to a more general class of initial data) appearing in the solution (3.3). More specifically, the functions $\rho(\lambda)$ and $\tau(\lambda)$ can be multivalued (even infinitely valued) with a different number of branches for different $\lambda$. Actually, as we will show, such behavior for $\rho$ and $\tau$ corresponds to multihump (or infinite-hump nondecaying) initial data. Also, the resulting formulation of the RH problem does not require any analyticity from the functions $\rho(\lambda)$ and $\tau(\lambda)$, which eliminates the requirement of analyticity for the initial data and is consistent with the hyperbolic nature of the zero-dispersion KdV limit [24].

First, we define, following Dubrovin [7], the Whitham system as a sequence of the modulation systems (3.6) defined for $N = 1, 2, \ldots$. For $N = 0$ this coincides with the Riemann wave equation

\begin{equation}
(5.1) \quad \partial_t r - 6r \partial_x r = 0
\end{equation}

with the initial data $r(x, 0) = u_0(x)$ given. Solutions of the Whitham equations (3.6) for a given $N$ typically exist only within certain domains of the $(x, t)$-plane. The main problem in the theory of the Whitham equations is gluing together these solutions in order to produce a $C^1$-smooth multivalued function of $x$ that also depends $C^1$-smoothly on the parameters $r_1, r_2, \ldots, r_{2N+1}$.
Thus, to produce the global solution to the Whitham system with the aid of the obtained local solutions (3.3), one should

1. determine the right genus at every point \( x_0, t_0 \), and
2. provide \( C^1 \)-smooth matching of the solutions (3.3) for different genera on the phase transition boundaries [7, 8].

The properties of the phase transitions can be investigated inside the local Whitham theory.

### 5.2 Phase Transitions

We study what happens to the solution (3.3) when one of the Riemann invariants \( r_{2j} \) coalesces either with \( r_{2j-1} \) or with \( r_{2j+1} \). It follows from the solution (3.3) that its phase transition properties are completely determined by the properties of the differential \( \Lambda_j \) near the double points \( r_k = r_{k+1} \) and can be investigated directly using the representation (3.4) for \( \Lambda_j \). It is more convenient, however, to use the relationship (4.8) between \( \Lambda_j \) and the meromorphic differential \( dp \), properties of which are known well.

Near the double points, the differential \( dp \) as well as the coefficients \( Q_k \) in the decomposition (4.5) has the following asymptotics [7]:

1. \( r_{2j+1} - r_{2j} \to 0 \) (small gap)

\[
f_N(r_1, \ldots, r_{2N+1}) = f_{N-1}(r_1, \ldots, \hat{r}_{2j}, \hat{r}_{2j+1}, \ldots, r_{2N+1})
+ v^2 f_{N,j}^1(r_1, \ldots, \hat{r}_{2j}, \hat{r}_{2j+1}, \ldots; \beta, v) + o(v^2).
\]

Here

\[
\beta = \frac{r_{2j} + r_{2j+1}}{2}, \quad v = \frac{r_{2j+1} - r_{2j}}{2}.
\]

Here and below the hat means that the corresponding coordinate is omitted.

2. \( r_{2j} - r_{2j-1} \to 0 \) (small band)

\[
f_N(r_1, \ldots, r_{2N+1}) = f_{N-1}(r_1, \ldots, \hat{r}_{2j-1}, \hat{r}_{2j}, \ldots, r_{2N+1})
+ \delta f_{N,j}^2(r_1, \ldots, \hat{r}_{2j-1}, \hat{r}_{2j}, \ldots; \eta, \delta) + o(\delta).
\]

Here

\[
\eta = \frac{r_{2j-1} + r_{2j}}{2}, \quad \delta = \left[ \log \frac{4}{(r_{2j} - r_{2j-1})^2} \right]^{-1}.
\]

One can see that in both cases, in the limit, the double points drop out of the function \( f_N \), and it turns into its analogue for the \( N - 1 \) genus. This fact follows from the normalization for the meromorphic differential

\[
\int_{r_{2j}}^{r_{2j+1}} dp = 0, \quad j = 1, \ldots, N,
\]
which implies that the polynomial
\[ \lambda^N + \sum_{k=1}^{N} q_k \lambda^{N-k} \]
in the numerator of \( dp \) has exactly one zero in each gap. Then, if one shrinks either gap or band, this zero inevitably coincides with the double point in the denominator.

Certainly, all the averaged Kruskal integrals \( Q_k(r_1, r_2, \ldots, r_{2N+1}) \) (see (4.5)) have the same properties near the double points. Then using the representation (4.8) for the generating differential, we arrive at the following asymptotics by differentiating (5.2) and (5.4):

1. \( r_{2j+1} - r_{2j} \to 0 \)

(5.7) \[
\Lambda_k^{[N]}(r_1, r_2, \ldots, r_{2N+1}; \lambda) = \\
\Lambda_k^{[N-1]}(r_1, r_2, \ldots, \hat{r}_{2j}, \hat{r}_{2j+1}, \ldots, r_{2N+1}; \lambda) + O(v^2) \quad \text{if} \ k \neq 2j, 2j + 1 ,
\]
and

2. \( r_{2j} - r_{2j-1} \to 0 \)

(5.8) \[
\Lambda_k^{[N]}(r_1, r_2, \ldots, r_{2N+1}; \lambda) = \Lambda_{2j+}^{[N]} + O(v) \quad \text{if} \ k = 2j \text{ or } k = 2j + 1 ,
\]
where \( \Lambda_{2j+}^{[N]} \) is the limiting value of the differential \( \Lambda_k^{[N]} \), which follows from (3.4) and (3.5) when one pinches the \( j \)th gap.

(5.9) \[
\Lambda_{2j+}^{[N]} \equiv \Lambda_k^{[N]}(r_1, r_2, \ldots, r_{2j-1}, r_{2j}, r_{2j+2}, \ldots, r_{2N+1}; \lambda) = \\
\lambda^N + \lambda^{N+1} + \cdots \\
(\lambda - r_{2j})^2 \sqrt{R_{2N-1}'} d\lambda ,
\]

\( R_{2N-1}' = (\lambda - r_1)(\lambda - r_2) \cdots (\lambda - r_{2j-1})(\lambda - r_{2j+2}) \cdots (\lambda - r_{2N+1}) \),
and \( \Lambda_{2j+}^{[N]} \) has a double pole at \( \lambda = r_{2j} \).

(5.10) \[
\Lambda_k^{[N]}(r_1, r_2, \ldots, r_{2N+1}; \lambda) = \\
\Lambda_k^{[N-1]}(r_1, r_2, \ldots, \hat{r}_{2j-1}, \hat{r}_{2j}, \ldots, r_{2N+1}; \lambda) + O(\delta) \quad \text{if} \ k \neq 2j - 1, 2j ,
\]
and

(5.11) \[
\Lambda_k^{[N]}(r_1, r_2, \ldots, r_{2N+1}; \lambda) = \Lambda_{2j-}^{[N]} + O(v/\delta) \quad \text{if} \ k = 2j - 1 \text{ or } k = 2j .
\]

Here \( \Lambda_{2j-}^{[N]} \) is the limiting value of the differential \( \Lambda_k^{[N]} \) when one pinches the \( j \)th band:

(5.12) \[
\Lambda_{2j-}^{[N]} \equiv \Lambda_k^{[N]}(r_1, r_2, \ldots, r_{2j-2}, r_{2j}, r_{2j+1}, \ldots, r_{2N+1}; \lambda) = \\
\lambda^{N-1} + \lambda^N + \cdots \\
(\lambda - r_{2j}) \sqrt{R_{2N-1}'} d\lambda ,
\]

\( R_{2N-1}' = (\lambda - r_1)(\lambda - r_2) \cdots (\lambda - r_{2j-2})(\lambda - r_{2j+1}) \cdots (\lambda - r_{2N+1}) \),
and \( \Lambda_{2j-}^{[N]} \) has a single pole at \( \lambda = r_{2j} \) (cancellation of one pole occurs due to the zero in the vanishing \( j \)th band).
One can see the substantial difference in the limiting behavior of the differential $\Lambda_j^{[N]}$ depending upon whether one shrinks the $j$th gap (5.9) or the $j$th band (5.12).

The asymptotics (5.7) and (5.10) provide natural $C^1$-smooth matching of the solution (3.3) for different genera on the phase transition boundaries. The limiting values in (5.8) and (5.11) determine the motion of those boundaries. Namely, the boundaries $x_j^\pm$ separating $N$-phase and $(N-1)$-phase regions correspond to closing either the $j$th gap (+) (linear wave degeneration) or the $j$th band (−) (soliton degeneration) and obey the ODEs

$$\frac{dx_j^\pm}{dt} = -4 \text{Res}_\infty \frac{\lambda^{3/2} \Lambda_j^{[N]}_j}{\lambda_j}.$$  

One should also check what happens to the solution (3.3) under the phase transition $(N = 1) \rightarrow (N = 0)$. For $N = 0$, $\Lambda_j^{[0]} = (\lambda - r)^{-3/2} d\lambda$ (see (4.4)), and, as a result, we arrive at two different equations corresponding to cases A and B, respectively. In case A we have

$$x - \frac{2t}{\pi i} \oint_{\infty} \frac{\lambda^{3/2}}{(\lambda - r)^{3/2}} d\lambda = \frac{2}{\pi} \frac{d}{dr} \int_0^r \frac{\tau(\lambda)}{\sqrt{\lambda - r}} d\lambda,$$

which is the solution of the Riemann wave equation (5.1) with the initial data in the form of the increasing part of the hump $u_0(x)$,

$$x + 6rt = x^−(r).$$

For case B we have a similar result for the decreasing part of the initial hump $u_0$,

$$x + 6rt = x^+(r).$$

It can also be easily seen that the decompositions of $\Lambda_j^{[1]}$ near $\Lambda_j^{[0]}$ have the form (5.7) and (5.10), which provides $C^1$-smoothness of the transition $N = 1 \rightarrow N = 0$. This transition was investigated in detail by Avilov, Krichever, and Novikov [2, 1]; see also [7, 8].

It is clear now that solution (3.3) in cases A and B corresponds to different parts of the global solution of the Whitham equations corresponding to the monotonic branches of the initial profile $u_0(x)$. We also note that formulae (5.14) and (5.16) give the solution to the Riemann equation in terms of the KdV scattering data and were obtained in this form by Geogjiev in [11].

5.3 Inequalities: Determination of Genus

As said before, the necessary part of constructing the solution to the Cauchy problem is determining of the genus of the Riemann surface at each point $x, t$. In fact, this problem cannot be resolved locally in the frame of the obtained solution (3.3) for the Whitham system. This point of view has not been established clearly in the literature so we will describe it in more detail.
The genus (the number of oscillating phases) $N$ is the “external” characteristics with respect to the local solutions to the Whitham system. To illustrate this, we present a simple example of ambiguity in the local determination of genus.

In Figure 5.1, there is a typical solution of the one-phase averaged Whitham equations for the initial data with the only break point (for example, the cubic breaking in the Gurevich-Pitayevskiǐ problem [16]) depicted by a solid line in the region $(x_-, x_+)$ while the three-valued solution to the Riemann wave equation (5.1) is depicted by a broken line in the region $(x_1, x_2)$, $x_1 < x_- < x_2 < x_+$. Outside those intervals, both solutions coincide.

One can see that in the vicinity of any point in an open interval $(x_2, x_+)$, both cases $N = 0$ and $N = 1$ can be applied without violating the existence for the local solution. The missing part of the information that enables us to distinguish the unique solution in this interval is the break time $t_{\text{crit}}$ (if $t > t_{\text{crit}}$, then $N = 1$; otherwise $N = 0$), but this information is relevant to the global properties of the solution for the case $N = 1$. In other words, generically, to make the right decision about the genus at $t = t_0$, one should follow the solution at all times until $t_0$ and change genus after passing the critical points. So far rigorous global predictions of the genus have been made in only a few simple cases.

In the case of analytic initial data with the only break point, there are results by Tian [27, 28] that the solution beyond the break time globally belongs to $N = 1$.

Another result in this area is due to Grava [12]; it reads that the maximal genus in the Cauchy problem with polynomial initial data does not exceed the degree of the polynomial and $N \rightarrow 1$ asymptotically as $t \rightarrow \infty$ (the latter result can also be found in [9]).

But even in the simplest case of the polynomial data, determining the genus at a fixed point $x_0$, $t_0$ requires knowledge of the behavior of the solution at all earlier times.

The RH problem approach makes it possible to determine the right genus of the problem locally with the aid of the inequalities (2.7)–(2.11) for the complex phase function $g(\lambda)$. These original inequalities were derived for the decaying solitonless initial data in [6, 33]. It is clear, however, that if one assumes the finite speed of
Figure 5.2. Separatrix \( \{F(t) = \{(x, r) : r(x, t) = r_1, \ldots, r_{2N+1}\}\} \), one-hump case. (a) \( F(0) = \{(x, r) : r = u_0(x)\} \), (b) \( F(t), t > t_{\text{crit}} \).

propagation (hyperbolicity) in the zero dispersion limit (which has been rigorously proved for the case of decaying initial potential [23, 24, 31]), then one can consider an arbitrary initial profile and regard these inequalities as a complementary part to the general local solution (3.3) for finite \( t \). Actually, given the functions \( \rho_{\pm}(\lambda) \) and \( \tau(\lambda) \) as the simple integral (Abel) transforms (2.2), (2.3), and (3.19) of arbitrary initial data, one can construct the function \( g(\lambda) \) and its derivative \( g'(\lambda) \) and then, by trial and error, determine the genus with the aid of inequalities. Also, one should add the relations (2.8) and (2.9) separating cases A and B for the problem with nonmonotone initial data. Certainly, for the monotonically increasing data, we are always in case A.

Example: One-Hump Problem

Following [6], we present an example of a systematic procedure for obtaining \( N = N(x, t) \) for all \( x \) at any given \( t \). The result of this procedure is to construct for each \( t \) the separatrix \( F(t) = \{(x, r) : r = r(x, t), -\infty < x < \infty, r = r_1, r_2, \ldots, r_{2N+1}\} \) for the case of one-hump initial data with a finite number of break points; see Figure 5.2.

For times \( t \) less than some critical value \( t_{\text{crit}} \) one takes \( N = N(x, t) = 0 \) and solves (5.1) for \( r \); the time \( t_{\text{crit}} \) corresponds precisely to the time at which (1.1) with \( \epsilon = 0 \) breaks down. It turns out that the associated function \( g(\lambda) \) constructed as above indeed satisfies the auxiliary inequalities (2.5), (2.7), (2.8), and (2.9), and hence \( g \) is the desired solution. Thus, for \( t < t_{\text{crit}} \), we obtain a separatrix of the shape of Figure 5.2(a); moreover, we find that \( F(t) = (x, r) : r = u(x, t) \), where \( u(x, t) \) is the solution of (1.1) with \( \epsilon = 0 \). For \( x > x_0(t) \), we are in case B and for \( x < x_0(t) \), we are in case A. When \( t > t_{\text{crit}} \), one again sets \( N = N(x, t) = 0 \) for \( x \gg 1 \), computes \( r \) from (5.1), and verifies once again that the side conditions are satisfied for case B. However, as we move \( x \) to the left, we find that at least one of the inequalities (2.5), (2.7), (2.8), or (2.9) breaks down.

In Figure 5.2(b) the first inequality in (2.5), \( -\tau < (g_+ - g_-)/2i \), breaks down at \( x = x_1 \), and for \( x < x_1 \) the interval \( G_0 \) breaks up into two intervals \( G_0 \) and \( G_1 \). For such \( x \) one solves system (3.3) for one of the cases A or B to get \( r_1, r_2, \) and \( r_3 \) and verifies that indeed the associated \( N \) satisfies the above inequalities. In the scenario of Figure 5.2(b), as we move \( x \) further to the left, the same inequality
for $\tau$ again breaks down at some $x = x_2(t)$ for some $r \in G_0(x_2(t), t)$. Again $G_0$ splits up into two intervals yielding a total of three intervals $G_0, G_1,$ and $G_2$. As we continue to move towards $x_3$, the interval $G_1$ shrinks to a point and eventually disappears. For $x$ between $x_4$ and $x_3$, we are again in the two-interval case, and for $x < x_4$ we return to the one-interval case $\mathcal{N} = 0$. For $x > x_0$, we are always in case B, and for $x < x_0$ we are always in case A. The scenario of Figure 5.2(b) is representative of the generic case in which any finite number of intervals may be open at some $x$. For more complicated, indeed pathological initial data, infinitely many folds may appear.

6 Zero-Dispersion KdV with General Initial Data

The basic assumption that we make in the following consideration is the hyperbolicity of the zero-dispersion limit for the KdV with an arbitrary initial potential. By applying this assumption we extend the formulation of the RH problem made in Section 2 for the case of the solitonless analytic initial perturbation to the case of an arbitrary initial datum. As a result we construct the solution to the corresponding Cauchy problem for the KdV-Whitham system.

It is clear that due to the (assumed) hyperbolicity of the problem, only a finite number of humps in the initial perturbation will be involved in nonlinear interaction at any finite time at any particular point. Therefore, the solution to the problem for the Whitham equations with an arbitrary (nondecaying) initial data can be obtained by solving the multihump (decaying) problem in the zero-dispersion limit and then by tending the number of humps to infinity. Certainly, the solution obtained in this way will be valid during a limited time interval (presumably for $t \ll 1/\epsilon$) until the semiclassical spectrum (Riemann invariants of the Whitham equations) begins to compete with the fine spectrum of the initial potential and the problem loses its hyperbolic character.

The main technical obstacle to the direct extension of the formulation (2.4)–(2.11) to the case of a multihump initial data is the multivaluedness of the functions $\rho_+ (\lambda)$ and $\tau(\lambda)$ in (2.2) and (2.3) appearing in the RH problem. To manage this difficulty, it is convenient to first reformulate the RH problem for one-hump initial data by introducing the finite reference point instead of the reference point at infinity (scattering from the right).

6.1 Reformulation of the One-Hump Problem for the Finite Reference Point

We start with the function

$$g(\lambda, x, 0) = \int_x^{\infty} (\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}) dx',$$

which, as can be checked directly, solves the RH problem (2.4)–(2.11) for $t = 0, \mathcal{N} = 0$. 

We introduce a new function
\[ H(\lambda, x, x_0) = \int_{x_0}^{x} \left( \lambda^{1/2} - (\lambda - u_0(x'))^{1/2} \right) dx', \]
where \( x_0 \) is an arbitrary fixed reference point. Without loss of generality we put \( x_0 \) at the maximum of \( u_0(x) \); see Figure 5.2.

One can see that
\[ g_{\pm}(\lambda, x, 0) = H_{\pm} - \lambda^{1/2}x_0 + \rho_{\mp} \mp i\tau_{\pm}, \]
where
\[ \tau_{\pm}(\lambda) = \int_{x_0}^{x_+} (u_0(x) - \lambda)^{1/2} dx. \]

Then we have the relationships (recall that here \( t = 0 \))
\[ \frac{g_{+} + g_{-}}{2} = \frac{H_{+} + H_{-}}{2} - \lambda^{1/2}x_0 + \rho_{\mp}, \]
\[ \frac{g_{+} - g_{-}}{2i} = \frac{H_{+} - H_{-}}{2i} - \tau_{\pm}. \]

We also introduce
\[ \tau_-(\lambda) = \tau_+(\lambda) - \tau(\lambda) = \int_{x_0}^{x_-} (u_0(x) - \lambda)^{1/2} dx. \]

Then the function \( H(\lambda, x, x_0) \) is the solution of the RH problem that follows directly from (2.4)–(2.11): In the region \( x < x^- \) (see Figure 5.2) that lies to the left of the hump and contributes to the spectral band \( B_1 \), the correspondent relations follow from (2.8) (case A) taking into account (6.5), (6.6), and
\[ \frac{H_{+} - H_{-}}{2i} = \tau_-, \quad \frac{H_{+}' + H_{-}'}{2} < -\alpha'_0, \]
where
\[ \alpha_0(\lambda, x) = \lambda^{1/2}(x - x_0). \]

Analogously, for the region \( x^- < x < x^+ \) lying under the hump and contributing to the gap \( G_0 \), we have from (2.4) and (2.5),
\[ \frac{H_{+}' + H_{-}'}{2} = -\alpha'_0, \quad \tau_- < \frac{H_{+} - H_{-}}{2i} < \tau_+. \]

Finally, in the region \( x > x^+ \) that lies to the right of the hump and again contributes to the band \( B_1 \) in the original RH problem (case B) (cf. (2.9)), we have
\[ \frac{H_{+} - H_{-}}{2i} = \tau_+, \quad \frac{H_{+}' + H_{-}'}{2} > -\alpha'_0. \]

We also have outside the interval (0, 1) from (2.10) and (2.11)
\[ H_{+} + H_{-} = 0 \text{ if } \lambda < 0, \]
\[ H_{+} - H_{-} = 0 \text{ if } \lambda > 1. \]
FIGURE 6.1. Multihump initial data. (a) \(\rho\)- and \(\tau\)-domains. (b) Sequence \(\{x_k^\pm\}\).

Due to the finite speed of propagation in the zero-dispersion KdV with decaying initial data, the picture (Figure 5.2(a)) does not change its topology under \(t\)-evolution (see Figure 5.2(b)). Finitely many folds that appear at some finite \(t\) determine the band-gap structure at any point \(x\) so that the region \(x < x^-\) on the \((u, x)\)-plane of the initial data will provide the bands \(B_n, n = 1, 2, \ldots, N + 1\) (\(B_{N+1}\) corresponds here to case A), the region \(x^- < x < x^+\) will provide the gaps \(G_k, k = 0, 1, \ldots, N\), and the region \(x > x^+\) will always correspond to the band \(B_{N+1}\) in case B. Then one can set the time-dependent RH problem for finite \(t\) by the substitution introducing the linear temporal evolution of the phase \(\alpha(\lambda)\) (2.1)

\[
\alpha_0(\lambda, x) \rightarrow \alpha_0(\lambda, x, t) = \lambda^{1/2}(x - x_0) + 4\lambda^{3/2}t
\]  

(6.14)

into the formulae (6.8)–(6.13).

Thus, we have effectively removed the infinite reference point from the formulation of the RH problem describing evolution of the one-hump perturbation.

6.2 Evolution of an Arbitrary Profile

The strategy of investigating the evolution for an arbitrary initial perturbation \(u_0(x)\) is essentially the same as in the previous subsection: We formulate the RH problem for the function \(H(\lambda, x, x_0)\) (6.2) at \(t = 0\) starting with the fixed reference point \(x_0\) and then by assuming the finite speed of propagation (hyperbolicity of the zero-dispersion KdV), we arrive at the time-dependent \(H\)-function by substitution (6.14) for the phase \(\alpha_0\). As was mentioned, the situation is complicated by the multivaluedness of the functions \(\rho(\lambda)\) and \(\tau(\lambda)\) appearing in such a problem.

We begin with the multihump initial potential and split up the strip \(0 < u < 1\) in the \((u, x)\)-plane of the initial data into two types of domains (see Figure 6.1):

- \(\tau\)-domains correspond to humps in the initial perturbation and contribute to the gaps in the semiclassical spectrum, and
- \(\rho\)-domains correspond to wells in the initial perturbation and contribute to the bands in the semiclassical spectrum.

We fix the reference point \(x_0\) at some maximum of the function \(u_0(x)\) and enumerate the monotonic parts of \(u_0(x)\) for \(x > x_0\), labeling them by the roots of the equation \(\lambda = u_0(x)\) (to avoid unnecessary complexity in notation, we put the
number of humps on the right from the point \( x_0 \) equal to the number of humps on the left):

\[
\text{(6.15)} \quad x_{-M}^- < x_{-M+1}^+ < \ldots < x_{-1}^+ < x_1^- \leq x_0 \leq x_1^+ < x_2^- < \ldots < x_{M-1}^- < x_M^+ ,
\]

\[
M \in \mathbb{N}, \quad \frac{dx_m^-}{d\lambda} > 0, \quad \frac{dx_m^+}{d\lambda} < 0.
\]

We denote the domains of definition for each \( x_n^\pm(\lambda) \) as \( D_m^\pm \). Then for each \( \lambda = \lambda^* \) one can define two subsequences, \( \{x_{m_j}^- (\lambda^*)\} \) and \( \{x_{m_j}^+ (\lambda^*)\} \),

\[
m_j = \pm 1, \pm 2, \ldots, \pm M_j, \quad j = j(\lambda) = 1, 2, \ldots, 2M - 1, \quad x_{kj}^+ > x_{kj-1}^- ,
\]

such that only those \( x_k^\pm \) participate in \( \{x_{m_j}^\pm\} \) for which \( \lambda^* \in D_k^\pm \) (Figure 6.1).

Then, for each \( j = j(\lambda) \) one can define the sequences \( \{\rho_{m_j}\} \) and \( \{\tau_{m_j}\} \),

\[
\rho_{m_j}(\lambda) = \text{Re} \int_{x_0}^{x_{m_j}^-} (\lambda - u_0(x'))^{1/2} dx',
\]

\[
\tau_{m_j}(\lambda) = \text{Im} \int_{x_0}^{x_{m_j}^+} (\lambda - u_0(x'))^{1/2} dx'.
\]

Then the function \( H(\lambda, x, x_0, 0) \) defined by (6.2) corresponds to a multihump analytic initial datum \( u_0(x) \). This function is related to the original function \( g(\lambda, x, 0) \) (cf. (6.3)),

\[
\text{(6.17)} \quad \text{if } x_{m_j-1}^- < x < x_{m_j}^+, \quad m_j > 0, \quad g_\pm = H_\pm - \lambda^{1/2}x_0 + \rho_\pm - \rho_{m_j-1} \mp i\tau_{m_j} ,
\]

where \( \rho_\pm(\lambda) \) is redefined for the multihump perturbation in the following way:

\[
\text{(6.18)} \quad \rho_\pm(\lambda) = x_{m_j}^\pm \lambda^{1/2} + \int_{x_{m_j}^-}^{x_{m_j}^+} [\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}] dx' .
\]

Here one should put \( \rho_0 \equiv 0 \) and \( x_0^\pm \equiv x_0 \). If \( m_j < 0 \), then the relationship (6.17) is valid within the interval \( x_{m_j}^- < x < x_{m_j+1}^+ \). Then, at \( t = 0 \) on the entire axis, we have instead of (6.5) and (6.6),

\[
\text{(6.19)} \quad \frac{g_+ + g_-}{2} = \frac{H_+ + H_-}{2} - \lambda^{1/2}x_0 + \rho_+ - \rho_{m_j-1} ,
\]

\[
\text{(6.20)} \quad \frac{g_+ - g_-}{2i} = \frac{H_+ - H_-}{2i} - \tau_{m_j} .
\]

Now we can formulate the RH problem that is satisfied by the function \( H \). We substitute (6.19) and (6.20) into the original RH problem (2.4)–(2.11), taking into account the new definition (6.18) of \( \rho_\pm(\lambda) \): In each \( \tau \)-domain \( (x_{m_j-1}^- < x < x_{m_j}^+ \) for \( m_j > 0 \) and \( x_{m_j}^- < x < x_{m_j+1}^+ \) for \( m_j < 0 \), one has from (2.4) and (2.5)

\[
\text{(6.21)} \quad \frac{H'_+ + H'_-}{2} = -\alpha_0' + \rho_{m_j}^{m_j-1} , \quad \tau_{m_j-1} < -\frac{H_+ - H_-}{2i} < \tau_{m_j} .
\]
In each $\rho$-domain $x_{m_j}^- < x < x_{m_j}^+$, we have from (2.6)–(2.9),

$$
(6.22) \quad \frac{H_+ - H_-}{2i} = \tau_{m_j}, \quad -\alpha'_0 + \rho'_{m_j} < \frac{H'_+ + H'_-}{2} < -\alpha'_0 + \rho'_{m_j}.
$$

We note that inequalities (2.7)–(2.9) have converted into one double inequality. One should also put $\tau_0 \equiv 0$.

Outside the interval $(0, 1)$, one has analogously to (6.12) and (6.13),

$$
(6.23) \quad H_+ + H_- = 0 \quad \text{if } \lambda < 0,
$$

$$
(6.24) \quad H_+ - H_- = 0 \quad \text{if } \lambda > 1.
$$

As well as in the previous subsection, the time-dependent RH problem is obtained from (6.21)–(6.24) by adding the term $4\lambda^{3/2}t$ to $\alpha_0(\lambda, x)$; see (6.14). However, to retain validity of the formulated RH problem after this substitution, one should assume hyperbolicity of the zero-dispersion limit for the KdV with general initial data. In this case the topology of the separatrix would not change under the evolution.

Recalling that the $\tau$-regions contribute to the gaps $G_k$, $k = 0, 1, \ldots, N$, and the $\rho$-regions contribute to the bands $B_n$, $n = 1, 2, \ldots, N + 1$, one immediately obtains the function $H(\lambda, x, x_0, t)$ and its derivative; cf. (2.14) and (2.18). For simplicity we write down the result assuming the initial data to be such that each $\rho$-domain contributes to only one band (analogously, each $\tau$-domain contributes to only one gap). Consideration of more general data would only complicate notation and add nothing of substance.

$$
(6.25) \quad H'(\lambda, x, x_0, t) =
$$

\[
\sqrt{R_{2N+1}(\lambda)} \sum_{m=0}^{N} \left( \int_{G_m} \frac{2\rho_m^*(\mu) - 2\alpha_0(\mu, x, t)}{\sqrt{R_{2N+1}^+(\mu - \lambda)}} d\mu \right) 2\pi i
\]

\[
+ \int_{B_{m+1}} \frac{-2i\tau_m^*(\mu)}{\sqrt{R_{2N+1}^+(\mu - \lambda)}} d\mu 2\pi i,
\]

$$
(6.26) \quad H(\lambda, x, x_0, t) =
$$

\[
\sqrt{R_{2N+1}(\lambda)} \sum_{m=0}^{N} \left( \int_{G_m} \frac{2\rho_m^*(\mu) - 2\alpha_0(\mu, x, t) - \Omega_m}{\sqrt{R_{2N+1}^+(\mu - \lambda)}} d\mu \right) 2\pi i
\]

\[
+ \int_{B_{m+1}} \frac{-2i\tau_m^*(\mu)}{\sqrt{R_{2N+1}^+(\mu - \lambda)}} d\mu 2\pi i,
\]
\[
\Omega_m = -2(x - x_0) \oint_{\partial_{x_0}} \chi^{1/2} \psi_j - 8t \oint_{\partial_{x_0}} \chi^{3/2} \psi_j \\
+ 4 \sum_{m=0}^{N} \left( \int_{G_m} \rho_m^* \psi_j - i \int_{B_{m+1}} \tau_m^* \psi_j \right), \quad \Omega_0 = 0, \quad \rho_0^* = 0.
\]

Here
\[
\{ \rho_m^*(\lambda) \} \subset \{ \rho_{kj}(\lambda) \}, \quad \{ \tau_m^*(\lambda) \} \subset \{ \tau_{kj}(\lambda) \}, \quad m = 0, \ldots, N.
\]

The algebraic equations determining dependence of the band-gap structure on \(x\) and \(t\) are obtained from the condition (3.2) that \(\partial_j \Omega_i = 0\) and can eventually be represented in a potential form (4.9) with the generalized Peierls-Fröhlich-type functional as a potential (see Sections 3 and 4)
\[
\partial_j F_N(\mathbf{r}; x, t) = 0, \quad j = 1, \ldots, 2N + 1,
\]
\[
F_N(\mathbf{r}; x, t) = \frac{1}{2\pi i} \left[ \oint_{\partial_{x_0}} \{(x - x_0)\lambda^{1/2} + 4t\lambda^{3/2}\} \, dp \right.
\]
\[
- \left. 2 \sum_{m=0}^{N} \left( \int_{G_m} \rho_m^* \, dp - i \int_{B_{m+1}} \tau_m^* \, dp \right) \right].
\]

The system (6.29) and (6.30) represents the general local solution to the Whitham equations. Due to the properties of the meromorphic differential \(dp\) discussed in Section 5.2, this solution provides \(C^1\)-smooth matching on the phase transition boundaries where genus \(N\) of the hyperelliptic surface changes. The right choice of the local genus \(N\) and the subsets \(\{ \tau_m^* \}\) and \(\{ \rho_m^* \}\) of (6.28) can be verified by checking that inequalities (6.21) and (6.22) are fulfilled for the function \(H(\lambda, x_0, x, t)\).

As an example of an effective choice of the subsets \(\{ \tau_m^* \}\) and \(\{ \rho_m^* \}\), one can indicate the case of periodic initial data \(u(x)\) where, for \(T\) the period and \(m \in \mathbb{N}\),
\[
\tau_m^* = \left( m + \frac{1}{2} \right) \text{Im} \int_{x_0}^{x_0 + T} (\lambda - u(x'))^{1/2} \, dx',
\]
\[
\rho_m^* = \frac{m}{2} \text{Re} \int_{x_0}^{x_0 + T} (\lambda - u(x'))^{1/2} \, dx'.
\]

Thus, we have bypassed the inverse scattering transform for the zero-dispersion KdV with general initial data by assuming hyperbolicity of the zero-dispersion limit. One can see that the resulting construction requires neither decaying nor analyticity for the initial data.
Appendix: Proof of Lemma 3.1

Consider the function
\( \Phi^{(j)}_{i} = \frac{\partial_{i} \Omega_{j}}{\partial_{j} k_{j}}. \)

We rewrite the expression (2.19) in the form containing contour integrals. For analytic functions \( \rho_{+}(\lambda) \) and \( \tau(\lambda) \), we have
\[
\Omega_{j} = \oint_{\alpha_{k}} \left\{ -2x \lambda^{1/2} - 8t \lambda^{3/2} \right\} \psi_{j} + \oint_{\beta_{n} \backslash \beta_{E}} 2 \rho_{+}(\lambda) \psi_{j} - 2i \oint_{\beta_{n} \backslash \beta_{E}} \tau(\lambda) \psi_{j}.
\]

Here we have used the fact that the functions \( \rho_{+}(\lambda) \) and \( \tau(\lambda) \) have their own different Riemann surfaces with the branch points
\[
0, \infty \quad \text{for} \quad \rho_{+}(\lambda), \quad 1, \infty \quad \text{for} \quad \tau(\lambda).
\]

Then we obtain from (A.1)
\[
\Phi^{(j)}_{i} = -2 \oint_{\alpha_{k}} \frac{x \lambda^{1/2} + 4t \lambda^{3/2}}{\partial_{j} k_{j}} \partial_{i} \psi_{j}
\]
\[
+ 2 \oint_{\alpha_{k}} \rho_{+}(\lambda) \frac{\partial_{j} \psi_{j}}{\partial_{i} k_{j}} - 2i \oint_{\beta_{n} \backslash \beta_{E}} \tau(\lambda) \frac{\partial_{j} \psi_{j}}{\partial_{i} k_{j}}.
\]

We consider the differential
\( \Lambda^{(j)}_{i} = \frac{\partial_{i} \psi_{j}}{\partial_{j} k_{j}}. \)

Substituting (2.21) into (A.4), we immediately obtain
\[
\Lambda^{(j)}_{i} = \frac{\lambda^{N} + \sum_{k=1}^{N} \lambda^{N-k} p_{k,i}^{(j)}}{(\lambda - r_{i}) \sqrt{R_{2N+1}}} d\lambda.
\]

We integrate \( \Lambda^{(j)}_{i} \) over the \( \alpha \)-cycles. Then, taking into account (2.22) and (A.4), we get
\[
\oint_{\alpha_{m}} \Lambda^{(j)}_{i} = 0, \quad m, j = 1, \ldots, N, \quad i = 1, \ldots, 2N + 1.
\]

The normalization (A.6) uniquely defines the coefficients \( p_{k,i} \) independently on \( j \), i.e.,
\[
\Lambda^{(1)}_{i} = \Lambda^{(2)}_{i} = \cdots = \Lambda^{(N)}_{i} \equiv \Lambda_{i}.
\]

Therefore, the differential (A.4) does not depend on the index \( j \), and, according to (A.3), the function \( \Phi^{(j)}_{i} \) does not depend on \( j \) either. That means that system (3.2)
is consistent and takes the form

\[
\oint_{\alpha_{+}} \left\{ x \lambda^{1/2} + 4t \lambda^{3/2} \right\} \Lambda_{j} = \oint_{\alpha_{k}} \rho_{+}(\lambda) \Lambda_{j} - i \oint_{\beta_{E}} \tau(\lambda) \Lambda_{j},
\]

provided \( \partial_{x}k_{j} \neq 0 \).

**Appendix: Derivation of the Identity (3.20)**

We introduce the analytic function

\[
f(\lambda) = \int_{-\infty}^{\infty} \left[ \lambda^{1/2} - (\lambda - u_{0}(x'))^{1/2} \right] dx' = \rho_{-}(\lambda) + \rho_{+}(\lambda) - i \tau(\lambda).
\]

Then one can observe that the following identity holds:

\[
\oint_{\alpha_{+}} f(\lambda) \Lambda_{j} = 0.
\]

In fact, \( f(\lambda) \sim \lambda^{-1/2} \) and \( \Lambda_{j} \sim \lambda^{-3/2}d\lambda \) as \( \lambda \to \infty \). Considering (B.2) as the integral over the contour surrounding the interval \((0, 1)\), we arrive at the identity for analytic \( \rho_{-}(\lambda) \), \( \rho_{+}(\lambda) \), and \( \tau(\lambda) \).

\[
\oint_{\alpha_{k}} \left\{ \rho_{-}(\lambda) + \rho_{+}(\lambda) \right\} \Lambda_{j} - i \oint_{\beta_{E}} \tau(\lambda) \Lambda_{j} = 0.
\]

**Acknowledgments.** The research described in this paper was made possible by the financial support of the Civilian Research and Development Foundation for the Former Soviet Union and USA (CRDF) under Grant RM1-145. G. A. E. and A. L. K. also thank the Russian Foundation for Basic Research (RFBR) for partial financial support under Grant 00-01-00210.

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Received October 2000.