Verifications for Multiple Solutions of Triaxial Earth Rotation

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Abstract: In this study, we provide several verifications on existence for multiple solutions of rotation with triaxial Earth. In fact, the exact solution for Euler rotation equations of triaxial Earth case appears as elliptic functions of double periods in which the main term is Chandler wobble and the other is a periodic polar motion of 14.6 year. By quadratic polynomial approach, the triaxial condition of the principal moments of inertia satisfying inequality A<B<C simultaneously guarantees that two stable solutions exist. On the other hand, prime algorithm provides algebraic equations of two elliptic curves and a hyperbolic trajectory simultaneously so that shows the same consequences. As standard deduction for nonlinear differential equations, Jacobian matrix and its eigen values show that there are two stable and an unstable solutions for the rigid body rotation model with true solutions projection onto three principal axes. By observation data series of polar motion and LOD, we also provide filtering results illustrating explicit period of about 14 years oscillation. At last, phase portrait of the rotation equations without solving process is drawn with illustration of 2-torous graph in X—Y plane with spirals shows clearly existence of two periods and other two projections of phase portrait appear as explicit trajectory of single pendulum with hyperbolic trajectories in it illustrating complex permanent consequence in multiple solutions of tentative trajectories.

Key Words: Rigid body rotation, Earth rotation, polar motion, Length-of-Day (LOD)

1. Introduction

Rigid body theory has developed for long from Euler's era. Euler deduced the motion model for rigid body rotation but was astonished from the integration of the model and sighed for the strange as a sentence of "analysis forged us". Although the integration of Euler equations provided elliptic functions as exact solution, people yet could not understand the whole property of rotation for arbitrary rigid body. This situation has lengthened till Arnold [1989] pointed out from 1966 on that arbitrary triaxial rigid body might rotate with direct product of stable trajectory for rotation about the maximum moment of inertia and also stable for rotation about the minimum as well as unstable rotation about the medium. As all colleagues studying Earth rotation known, Earth or arbitrary rigid body rotates about a tentative axis. This axis may very much close to the maximum but never identical to the maximum axis. So, if we need true solution for the rotation, we had to make projections for the tentative spinning axis onto three principal axes of inertia momentum. Therefore, the three simple solutions of multiple trajectories exist. By this approach, we can get the true solution other than exact solution of elliptic functions, or other than assumption of the spinning axis very much close to the maximum axis of inertia momentum for obtaining the approximate result.

In this study, we provide several approach to the true solution of arbitrary rigid body with triaxial principal moments of inertia and point out. There are multiple solutions for triaxial Earth rotation with two stable periodic trajectories and an unstable hyperbolic. Thus Earth rotation possesses another free polar motion of period 14.6 year accompanied with Chandler wobble. Data series filtering approach also illustrates explicit period of about 14 year in polar motion as well as in LOD. First, we discuss the shape of the Earth especially concerning the principal axes of inertia momentum. In the main part of the paper, we provide several approaches to verify the true solution. At the last we make conclusion.

2. Earth's shape

Earth has anisotropic shape according to gravity or the geoid. In different era, people understand with Earth's shape in different precise. The earliest understanding is that Earth likes a global. Later by more precise observation, people know that Earth must be axi-symmetric flattening ellipsoid. The flattening parameter may be shown as

$$f = \frac{r_{\max} - r_{\min}}{r_{\max}} = \frac{1}{298.257} \tag{1}$$

Here $r_{\max}$ and $r_{\min}$ are the maximum and minimum radiuses of the Earth. In fact, the maximum and minimum radiuses of the Earth are observed as 6378147±5 m and 6356755±5 m with the difference about 21.385 km. Thus 1/300 becomes the alternative of Earth flattening.

However, as Earth spins with not too fast and not so slow a speed, the global property of the Earth can be conserved and the centrifugal force makes the equatorial part bulge in 21.385 km. The bulge affects the rotation of the Earth though the bulge is so tiny compared with the whole huge global. Hence people think of the oblate Earth as an orange or apple other than prolate sweet melon. This property is often called as biaxial. The biaxiality is described with the 2-order global harmonic coefficient J_2

$$J_2 = \frac{C - \bar{A}}{Ma^2} = (108263 \pm 0.2) \times 10^{-8} \tag{2}$$

Here M and a denote the total mass and gravity function with respect to the radius of the Earth. C and $\underline{A}$ as the maximum and the average minimum moment of inertia. For rotation, the dynamical ellipticity is often considered replacing the flattening. The Earth polar dynamical ellipticity for biaxial case is defined as

$$H_0 = \frac{C - \bar{A}}{\bar{A}} = \frac{C - (A+B)/2}{(A+B)/2} \tag{3}$$

For 1066A and 1066B Earth model, the polar dynamical ellipticity for biaxial case is obtained as

$$H_{0A} = 3.28475 \times 10^{-5} = 1/304.437 \tag{4}$$

For PREM Earth model, the polar dynamical ellipticity for biaxial case is specified as

$$H_{0P} = 3.27 \times 10^{-5} \tag{5}$$

with consideration of modification in visco-elasticity.

Furthermore, Earth is discovered with triaxial other than biaxial with respect to the principal dynamical ellipticity. In 1963, it was calculated that the geoid of the equator was not a circle but an ellipse from the perturbation of the satellite obit. In 1973 the principal momentum difference was noticed. The bulged part of the equator has a longer radius of 430 m than the shortest. In 1984 the principal momentum difference was noticed. Burša [1984] surveyed precisely that the difference of the equatorial principal moments as

$$B - A = 1.76 \times 10^{33}\,\mathrm{kgm}^2 \tag{6}$$

so that estimated the equatorial flattening as

$$(B - A)/A = 2 \times 10^{-5} \tag{7}$$

Later dynamical ellipticity is estimated in Groten [2000] as

$$H_1 = \frac{B - A}{C} = 2.196 \times 10^{-5} \tag{8}$$

with the minimum principal axis in the direction of angle

$$\Lambda = -14.93^\circ \approx -15^\circ \tag{9}$$

with respect to the Greenwich 0. Also see discussion in Marchenko & Schwintzer [2003].

If the difference of the polar and equatorial ellipticity for the whole Earth is so small that might be ignored, then the similar difference for the liquid core of the Earth might be great enough that could not be ignored. The flattening of the liquid core is determined as

$$f_c = \frac{C_c - B_c}{A_c} \approx 0.00256 \approx 1/391 \tag{10}$$

Van Hoolst & Dehant [2002] referred the polar and equatorial ellipticity for the liquid core as

$$H_{c0} = 2.65 \times 10^{-3} \tag{11}$$

$$H_{c1} = 5.8 \times 10^{-4} \tag{12}$$

The difference of the two is not larger than 5 times. How can it be ignored?

Souchay et al [2002] introduced the amount for comparison of the polar and equatorial ellipticity as

$$e = \frac{1}{2}\frac{\frac{1}{B} - \frac{1}{A}}{\frac{1}{C} - \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right)} \tag{13}$$

We provide here the comparison of the polar and equatorial ellipticity in the following table.

Table 1. Comparison of the polar and equatorial ellipticity for several planets

	Earth	Mars	Moon	Hyperion	Eros433
H_0	3.28475×10^{-3}	5.38×10^{-3}	5.2×10^{-4}	0.799	0.278?
H_1	2.1946×10^{-5}	5×10^{-4} or 6.896×10^{-4}	5.5669×10^{-5}	0.158	0.158?
e	0.00382	0.06463	0.214	0.978	0.723

However, the Earth has been discovered in shape of pear like other than apple. On the one hand, some place of the southern hemisphere is about 7.6 m higher than that of the northern hemisphere and the geoid of the southern is generally a little higher than the northern so that the southern hemisphere is a little fatter than the northern. On the other hand, the geoid of the northern pole is 15.2 m higher than the southern with the whole southern polar region a little shorter so that the Earth seems a little like a pear. Scientist of geophysics all over the world has confirmed this phenomenon. But the dynamical ellipticity surveying has not been worked out for the difference of the southern and the northern hemispheres.

Evidently, the triaxial rigid body has different rotation behavior compared with the biaxial rigid body. The pear-shaped Earth must have different rotation behavior compared with the triaxial apple-like Earth. We discuss the rotation of triaxial Earth in this paper starting from the triaxial rigid body and remain the discussion of pear-shaped Earth rotation in Wang Wen-Jun [2004b].

3. Exact solution

Euler dynamical equations for rigid body rotation write

$$\begin{aligned} A\dot{\omega}_1 + (C-B)\omega_2\omega_3 &= 0 \\ B\dot{\omega}_1 + (A-C)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_1 + (B-A)\omega_1\omega_2 &= 0 \end{aligned} \tag{14}$$

In order to obtain the analytical solution, we may get three first integral to determine the three variables and the integral constants. As the external torque is zero, the first integral gives the momentum as constant to conserve the momentum of the rotating rigid body.

$$\mathbf{H}\cdot\mathbf{H} = \mathbf{H}^2 = \text{const} \tag{15}$$

or
$$\begin{bmatrix} A\omega_1 & B\omega_2 & C\omega_3 \end{bmatrix} \begin{bmatrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{bmatrix} = \text{const} \tag{16}$$

so that

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = \mathbf{H}^2 \tag{17}$$

Also as the external torque vanishes the total motion energy should remain invariant for the second integral.

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H} = \text{const} \tag{18}$$

or
$$\begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \begin{bmatrix} A\omega_1 \\ B\omega_2 \\ C\omega_3 \end{bmatrix} = 2T \to \text{const} \tag{19}$$

so that

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T \tag{20}$$

In the first integral, the kinematical energy is zero, while in the second integral the potential energy is zero. Any of (14) can be selected to juxtapose simultaneously with the two integrals

$$\begin{aligned} B\dot{\omega}_2 + (A-C)\omega_1\omega_3 &= 0 \\ A\omega_1^2 + B\omega_2^2 + C\omega_3^2 &= 2T \\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 &= H^2 \end{aligned} \tag{21}$$

For $\omega_1, \omega_2, \omega_3$ integration, introduce transformation

$$2T = D\mu^2 \tag{22}$$

and
$$H = D\mu \tag{23}$$

so that

$$\mu = 2T / H \tag{24}$$

with $$D = H^2 /(2T) \tag{25}$$

here D and μ are positive constant. Thus two integrals in (21) write

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = D\mu^2 \tag{26}$$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = D^2\mu^2 \tag{27}$$

for eliminating the third variable, (26) multiplying C subtracts (27) and there holds

$$A\omega_1^2(C-A) + B\omega_2^2(C-B) = D\mu^2(C-D) \tag{28}$$

(27) subtracts (26) multiplying A and holds

$$C\omega_3^2(C-A) + B\omega_2^2(B-A) = D\mu^2(D-A) \tag{29}$$

using ω_2 to express ω_3 and using ω_2 to express ω_1, it holds

$$\omega_3^2 = \frac{1}{C(C-A)}[D(D-A)\mu^2 - A(B-A)\omega_1^2] \tag{30}$$

$$\omega_2^2 = \frac{1}{B(C-B)}[D(C-B)\mu^2 - A(C-A)\omega_1^2] \tag{31}$$

Hence

$$\omega_3^2 = \frac{A(B-A)}{C(C-A)}\left[\frac{D(D-A)}{A(B-A)}\mu^2 - \omega_2^2\right] \tag{32}$$

$$\omega_1^2 = \frac{B(C-B)}{A(C-B)}\left[\frac{D(C-B)}{A(C-B)}\mu^2 - \omega_2^2\right] \tag{33}$$

Notice $C-A>0, C-B>0, A-B<0$ it holds

$$D-A>0 \tag{34}$$

This must be carried out by initial condition $\left.\begin{matrix}\omega_{''0} \neq 0 \text{ or } \omega_{20} \neq 0 \\ \omega_{''0} \neq 0 \text{ and } \omega_{20} \neq 0\end{matrix}\right\}$

Similarly, there is

$$C-D>0 \tag{35}$$

This must be carried out by initial condition $\left.\begin{matrix}\omega_{10} \neq 0 \text{ or } \omega_{20} \neq 0 \\ \omega_{10} \neq 0 \text{ and } \omega_{20} \neq 0\end{matrix}\right\}$

From (3.2.21) and (3.2.22), with $C>A, C>D$, it holds

$$\frac{D(D-A)}{B(B-A)}\mu^2 > 0 \qquad \frac{D(C-D)}{A(C-A)}\mu^2 > 0$$

Set

$$m = \mu\sqrt{\frac{D(D-A)}{B(B-A)}} \tag{36}$$

$$n = \mu\sqrt{\frac{D(C-D)}{A(C-A)}} \tag{37}$$

so that (30) and (31) are expressed as

$$\omega_3^2 = \frac{B(B-A)}{C(C-A)}(m^2-\omega_2^2) \tag{38}$$

$$\omega_1^2 = \frac{B(C-B)}{A(C-A)}(n^2-\omega_2^2) \tag{39}$$

Substituting (38) and (39) into (21), it holds

$$B\dot{\omega}_2 \pm (C-A)\sqrt{\frac{B^2(C-A)(B-A)}{CA(C-A)^2}}\sqrt{(m^2-\omega_2^2)(n^2-\omega_2^2)} = 0$$

or $$\dot{\omega}_2 \pm mn\sqrt{\frac{(C-A)(B-A)}{BC}}\sqrt{(1-\frac{\omega_2^2}{m^2})(1-\frac{\omega_2^2}{n^2})} = 0 \tag{40}$$

Case 1. As $D > A$, $m^2 > n^2$. Introducing variable *y* such that

$$\omega_2 = ny \tag{41}$$

Set

$$k = \frac{n^2}{m^2} \qquad (k^2 < 1) \tag{42}$$

(40) changes to

$$\dot{y} \pm m\sqrt{\frac{(C-A)(B-A)}{BC}}\sqrt{(1-y^2)(1-k^2y^2)} = 0 \tag{43}$$

Introducing

$$\nu = \mu\sqrt{\frac{D(D-A)(D-B)}{ABC}} \tag{44}$$

Noticing (36), (43) changes to

$$\dot{y} \pm \nu\sqrt{(1-y^2)(1-k^2y^2)} = 0 \tag{45}$$

Determine condition before integration

$$y \in (0,\ y) \quad u \in (u_0, u) \quad t = \nu(u-u_0)$$

so that

$$u = \pm\int_0^y \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-k^2y^2)}} \tag{46}$$

It holds

$$y = \pm \mathrm{sn}u \tag{47}$$

Noticing (37) and (41), there is

$$\omega_2 = \pm\mu\sqrt{\frac{D(C-D)}{A(C-A)}}\mathrm{sn}u = \pm n\mathrm{sn}u \tag{48}$$

substituting (38) and (39), there are

$$\omega_1 = \frac{B(C-B)}{A(C-A)}n^2(1-\mathrm{sn}^2u) \tag{49}$$

$$\omega_3 = \frac{B(B-A)}{C(C-A)}m^2(1-\frac{n^2}{m^2}\mathrm{sn}u) \tag{50}$$

By the relations of elliptic functions, there are

$$\omega_1 = \pm\mu\sqrt{\frac{D(C-D)}{A(C-A)}}\mathrm{cn}u \tag{51}$$

$$\omega_2 = \pm\mu\sqrt{\frac{D(C-D)}{B(C-B)}}\mathrm{sn}u \tag{52}$$

$$\omega_3 = \pm\mu\sqrt{\frac{D(D-B)}{C(C-B)}}\mathrm{dn}u \tag{53}$$

To determine the sign, as $D > B, \mathrm{dn}u > 0$, this guarantees ω_3 never vanish so that the result is positive or negative for ever. The sign is determined from the initial value of ω_{30}. This is nothing but rotation velocity and direction. Set $\omega_{30} > 0$, rotation remains positive. In practice, the anti-clockwise is set as positive. Hence (53) should be set as positive only. If this has been determined from the initial condition, by the second equation in (14), ω_1 and $\dot{\omega}_2$ possess the same sign. Noticing the expressions of elliptic functions, there is

$$\frac{\mathrm{dcn}u}{\mathrm{d}u} = -\mathrm{sn}u\mathrm{dn}u \tag{54}$$

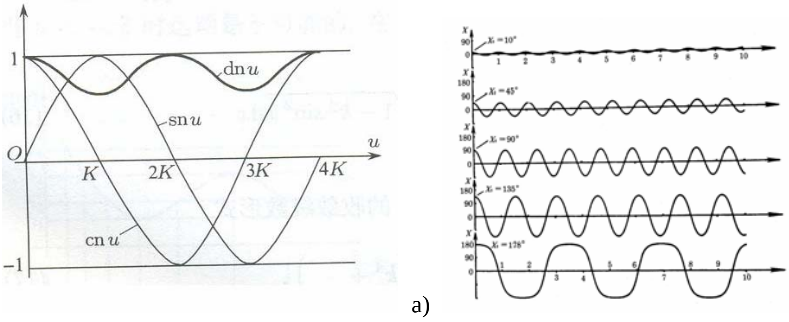
In the case of $\mathrm{dn}u > 0$, $\frac{\mathrm{dcn}u}{\mathrm{d}u}$ must possess different sign with respect to that of sn*u*. Consequently, ω_1, ω_2 may have different signs. ω_1 is often set as positive, so that ω_2 must set as negative. This is the essential consequence for Earth rotation theory and coordinate Y must be set with a minus sign. At last the solution of Euler equations write

$$\omega_1 = \mu\sqrt{\frac{D(C-D)}{A(C-A)}}\mathrm{cn}u \tag{55}$$

$$\omega_2 = -\mu\sqrt{\frac{D(C-D)}{B(C-B)}}\mathrm{sn}u \tag{56}$$

$$\omega_3 = \mu\sqrt{\frac{D(D-B)}{C(C-B)}}\mathrm{dn}u \tag{57}$$

As elliptic functions are periodic functions of *u* and double periodic functions, the analytic solution of Euler equations must be double periodic functions of time *t*. The property for elliptic functions to possess two periods in different directions can be seen in arbitrary textbook of elliptic functions. Wang Wen-Jun [2004a] verified that the main motion for the solution is Chandler wobble in Earth rotation case while the other wobble has a decadal period. The graph of elliptic functions can be seen in Fig 1.



a) b)

Fig 1. a) Graphs of elliptic functions b) cn*u* compared with trigonometric cos*u*

However, the exact solution for geodesy may not be easy to use and since the consideration of the elliptic function with respect to the sine function has difference in order 10^{-14}, people need not to treat with elliptic functions in practice. On the other hand, geodesy may be carried out as projection onto some defined coordinate frame. Elliptic functions are also not easy to be taken into account in the coordinate frame. Thus we suggest that the elliptic functions are not necessary to be applied in geodesy. But if we neglect the second free wobble together with elliptic functions, then the problem of polar motion will never reach the enough precision and the theory and the observation may never be made with both ends meet.

4. Momentum conservation

Conservative momentum approach to the components of rotation angular velocity ω_i, i=1, 2, 3, possess conservation for the angular momentum H and for the potential energy $T \propto m^2$.

$$\begin{aligned} &H = \tfrac{1}{2} M(\omega_1^2/A + \omega_2^2/B + \omega_3^2/C) \\ &\omega_1^2 + \omega_2^2 + \omega_3^2 = m^2 \end{aligned} \tag{58}$$

Let $x_i=\omega_i/m$, (i=1, 2, 3) and $\lambda_1 = 1/A, \lambda_2 = 1/B, \lambda_3 = 1/C$, $A<B<C$, $1/A>1/B>1/C$, here $1/C$ is the least. The first equation of (58) is changed to a quadratic polynomial of variables x_i in which the respective coefficients λ_i satisfy (+, +, +).

$$H = \tfrac{1}{2} m^2 M(\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2) = \tfrac{1}{2} m^2 M[(\lambda_1 - \lambda_3)x_1^2 + (\lambda_2 - \lambda_3)x_2^2 + \lambda_3] \tag{59}$$

Theory of quadratic polynomial tells that if and only if the determinant of the last formula of (9) is positively definite, then it has stable elliptic solution, otherwise it has instable saddle point solution. It is evident for $C>A$ and $C>B$ that (59) has a stable elliptic solution corresponding to the usual case of Chandler wobble. (59) has another form as $1/A$ the largest

$$H = \tfrac{1}{2} m^2 M[\lambda_1 + (\lambda_2 - \lambda_1)x_2^2 + (\lambda_3 - \lambda_1)x_3^2] \tag{60}$$

For the case of $A<B$ and $A<C$, the quadratic polynomial (60) also satisfies the positive definite condition coefficients satisfy (+, -, -). So there must be another stable elliptic solution for triaxial Earth.

This verifies that any triaxial rigid body rotation has two free wobbles simultaneously.

5. Algebraic curve equations

Equation (14) could be expressed as Euler dynamical equations without external torque

$$
\begin{aligned}
&A\dot{\omega}_1-(B-C)\omega_2\omega_3=0\\
&B\dot{\omega}_2-(C-A)\omega_3\omega_1=0\\
&C\dot{\omega}_3-(A-B)\omega_1\omega_2=0
\end{aligned}
\tag{61}
$$

Considering the free rotation of the rigid Earth, (61) is reduced as follows

$$
\begin{aligned}
&\dot{\omega}_1+\sigma_1\omega_2\omega_3=0\\
&\dot{\omega}_2-\sigma_2\omega_3\omega_1=0\\
&\dot{\omega}_3+\sigma_3\omega_1\omega_2=0
\end{aligned}
\tag{62}
$$

where σ_1, σ_2 and σ_3 are small quantities, defined by following equations

$$
\alpha=\sigma_1=(C-B)/A \qquad \beta=\sigma_2=(C-A)/B \qquad \gamma=\sigma_3=(B-A)/C \tag{63}
$$

Especially, σ_1 and σ_2 are in the order of 3×10^{-3}, and σ_3 is in the order of 2×10^{-5}. Various studies have provided the values of σ_1, σ_2 and σ_3, which are listed in Table 1.

In this paper, the values provided by Groten [2000] are applied, which possess little differences with the values provided by other authors (Table 2). From (63), one gets

$$
\begin{aligned}
&\sigma_2\omega_1^2+\sigma_1\omega_2^2=C_{12}\\
&\sigma_3\omega_2^2+\sigma_2\omega_3^2=C_{23}\\
&\sigma_1\omega_3^2-\sigma_3\omega_1^2=C_{31}
\end{aligned}
\tag{64}
$$

Where C_{ij} are constants to be determined. If $A = B$, then $\sigma_3 = 0$, $\sigma_1=\sigma_2$, and in this case, from (64) it holds that ω_3=constant, and $\omega_1^2+\omega_2^2=C_{12}/\sigma_1$ holds invariant. If $A<B$, the three equations express three cone curves. Especially, (64) determines three trajectories of two elliptic curves and a hyperbolic curve so that the three cone curves illustrate three solution components for the dynamical model of rotation. The three simple solution components exist simultaneously coupling as a total multiple solutions of elliptic functions.

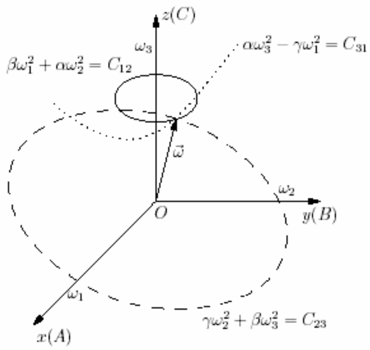


Fig 2: The orbit of the angular velocity vector: a combination of two ellipses and one hyperbola

Table 2. Values of σ_1, σ_2 and σ_3

Sources	σ_1	σ_2	σ_3
Burša (1984)	327341×10^{-8}	329588×10^{-8}	2.1888995×10^{-5}
Liu et al.(1991)	(not given)	(not given)	2.1946×10^{-5}
Yoder (1995)	328448×10^{-8}	330655×10^{-8}	2.20743409×10^{-5}
Groten (2000)	$(327353\pm 6)\times 10^{-8}$	$(329549\pm 6)\times 10^{-8}$	$(2196\pm 6)\times 10^{-8}$
WGS84 (2000)	327519×10^{-8}	329423×10^{-8}	1.90405380×10^{-5}

Wei (2005)	$(3273.5367 \pm 0.4905) \times 10^{-6}$	$(3295.4949 \pm 0.4938) \times 10^{-6}$	$(21.9584 \pm 0.0033) \times 10^{-6}$

6. Eigen value approach

Since axes *A*—*B*—*C* are not spinning, we can make the projection for the instantaneous spinning axis *Z* onto axes *A*—*B*—*C* obtaining projection components $\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$ on axes *A*—*B*—*C* respectively, with each component possessing vector feature of spinning angular velocity ($\mathbf{1}_1\Omega$ $\mathbf{1}_2\Omega$ $\mathbf{1}_3\Omega$). Suppose that the wobble about the axis *C* is extracted out of consideration or the free Chandler wobble is obtained out of the free wobble total motion. Then, the rotation remained must have the great angular momentum about the minimum inertia momentum axis so that the rotation must be selected close to the minimum inertia momentum axis. If it is treated as the same of the maximum inertia momentum, with rotation angular velocity as $\mathbf{1}_3\Omega$, then the result may not be identical to the fact of Earth that the minimum inertia momentum axis does rotate itself. The frame of axes *A*—*B*—*C* turns around the instantaneous spinning axis *Z* so that the axes *A* and *B* rotate while the axis *C* spinning. Although the rotation of axes *A* and *B* are smaller as 10^{-6} of that of axis *C*, spinning of axes *A* and *B* causes a plane and an inversed pendulum, respectively. Noticing that the minimum inertia momentum axis does not rotate but may wobble, the rotation angular velocity of the minimum inertia momentum axis may be treated with rotation angular velocity $\mathbf{1}_1\Omega$. Similarly the rotation angular velocity of the medium inertia momentum axis is set as with rotation angular velocity $\mathbf{1}_2\Omega$. Perhaps, this approach can emerge the rectangular relation to the axis rotating or inertia momentum axes *C*—*A*—*B* seem to possess no any coherent relation. As known, any of the inertial axes may not spin and there is only the instantaneous spinning frame spinning. It is reasonable that the spinning of the inertial axes expresses as ($\mathbf{1}_1\Omega$ $\mathbf{1}_2\Omega$ $\mathbf{1}_3\Omega$) determined after integration from initial condition of observation which of the axes in the instantaneous frame spins.

In theoretical physics and mechanics, it has been verified again and again that there are three solutions for rotation of triaxial body. But the multiple solutions have not transferred to Earth's polar motion till publication of Wang W-J [2004a]. According to elliptic function approach to the exact solution of Euler rotation equations for triaxial Earth case, we notice that the exact solution may have a real period and an image period. As Arnold [1989] specified this motion with decomposition to direct product of two plane pendulums and an inverted pendulum, we may solve the equations three times about one of the principal inertia momentums once.

For expressing the direct sum of solutions for two triaxial bodies, we define the spin for a frame as form of tensor matrix. Generally, a celestial body may spin about only one pole. The spinning axis is the instantaneous spinning but not axis *C* the maximum inertia momentum. The spinning axis has projecting onto each inertial axis. These projections may have spinning feature as well as that of the spinning axis. So these three projections must be seen as spinning with the same angular velocity of the spinning axis. If the rotation vector about a certain axis of the frame is expressed as $\boldsymbol{\omega} = (\omega_1\ \omega_2\ \omega_3)^T = (m_1\ m_2\ \mathbf{1}_3 + m_3)^T\Omega$, then the coherent feature of the three axes may not appear and some valuable integration results may be neglected. Thus the angular velocity of the rotation about a certain point of inertial frame must be defined as tensor matrix form.

$$\boldsymbol{\omega} = \begin{bmatrix} 1+w_1 \\ w_2 \\ w_3 \end{bmatrix} \Omega \otimes \begin{bmatrix} v_1 \\ 1+v_2 \\ v_3 \end{bmatrix} \Omega \otimes \begin{bmatrix} m_1 \\ m_2 \\ 1+m_3 \end{bmatrix} \Omega \tag{65}$$

Here symbol $\otimes$ represents the direct product of vectors. That is to say, axis A travels revolution around the instantaneous axis with positive sequence and axis B travels revolute with negative sequence while the rotating axis is only near axis C. The rotation tensor matrix describes that a body may rotate in any tensor space but especially spinning about the principal instantaneous axes but not the inertial axis C. However, the real spinning must be only about the instantaneous and if by asymmetry the two axes spin as well, then the rotating tensor matrix may be reformed on the first and second diagonal elements adding some real parts before the images.

Since the form of rotation equations from quaternion approach is similar to that of (61), we can solve the model with respect to the form of (61) with notice of three angular velocity components as three rotations in different complex planes. These three complex planes are coherent as a frame of inertial space. For a nonlinear system, former solving process of Euler rotation equations with linearization and obtains only Chandler wobble is unilateral.

It needs to confirm the existence of the image period of LOD. To specify the solution, (62) can be changed to matrix form as nonlinear function in homogeneous model with constant frequency components $\alpha = \sigma_1 = (C-B)/A$ $\beta = \sigma_2 = (C-A)/B$ $\gamma = \sigma_3 = (B-A)/C$.

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} -\sigma_1\omega_2\omega_3 \\ \sigma_2\omega_3\omega_1 \\ -\sigma_3\omega_1\omega_2 \end{pmatrix} \tag{66}$$

The equation can be written in compact form of vector with $\mathbf{F}$ a nonlinear column vector of three orders in the right hand side of (66).

$$\dot{\boldsymbol{\omega}} = \mathbf{F}(\boldsymbol{\omega}) \tag{67}$$

The equilibrium for the nonlinear vector equation takes place at the derivative operator case of Taylor expansion.

$$\dot{\boldsymbol{\omega}} = \mathbf{F}(\boldsymbol{\omega}_0) + \dot{\mathbf{F}}(\boldsymbol{\omega}_0)(\boldsymbol{\omega} - \boldsymbol{\omega}_0) + \mathbf{O}(\boldsymbol{\omega}^2) \tag{68}$$

The derivative operator of the vector $\mathbf{F}$ is obtained in Jacobian matrix.

$$\dot{\mathbf{F}}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\sigma_1\omega_3 & -\sigma_1\omega_2 \\ \sigma_2\omega_3 & 0 & \sigma_2\omega_1 \\ -\sigma_3\omega_2 & -\sigma_3\omega_1 & 0 \end{pmatrix} \tag{69}$$

It is interesting to decompose the Jacobian matrix (69) as quaternion with image units i_1, i_2, i_3 so that $\mathbf{1}_1$, $\mathbf{1}_2$, $\mathbf{1}_3$ may be seen as image units i_1, i_2, i_3 in the complex case and as unit vector components in the vector algorithm, respectively. Interestingly we can align quaternion approach to rotation with Jacobian matrix decomposed into three small rotation matrices in inertial planes. Here the three small rotation matrices in inertial planes are distorted without circular symmetry but can be seen as circular frequency rotation equivalently. It may be legal to ignore the higher order terms here in the expression. Interestingly here the components of decomposition show as three first order terms for rotations in each inertial plane.

$$\dot{\mathbf{F}}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_2 \\ 0 & -\sigma_3 & 0 \end{pmatrix} \omega_1 + \begin{pmatrix} 0 & 0 & -\sigma_1 \\ 0 & 0 & 0 \\ -\sigma_3 & 0 & 0 \end{pmatrix} \omega_2 + \begin{pmatrix} 0 & -\sigma_1 & 0 \\ \sigma_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \omega_3 \tag{70}$$

The physics meaning of (70) is that each rotation component may have small rotation acting on it.

If the rotation vector about the certain axis C of the inertial frame is expressed as $\boldsymbol{\omega} = (\omega_1\ \omega_2\ \omega_3)^T = (m_1\ m_2\ 1_3 + m_3)^T \Omega$, then the coherent feature of the three axes may not appear and some valuable integration results may be neglected. So it is only truncated process near axis C considered for the former treatment to obtain the rotation about the spinning axis C neglecting the possible wobbles and inverted pendulum motion about other two axes, respectively. Any of the three equilibriums of the inertia momentum ellipsoid of the rigid body considered can be treated with separately. For axis C, there is eigen equation

$$\begin{vmatrix} \lambda & \sigma_1\Omega & 0 \\ -\sigma_2\Omega & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0 \tag{71}$$

Equivalently, we have $\lambda(\lambda^2 + \sigma_1\sigma_2\Omega^2) = 0$ with eigenvalues $\lambda_{1,2} = \pm i_3\sqrt{\sigma_1\sigma_2}\Omega,\ \lambda_3 = 0 i_3$. Thus, we obtain a stable elliptic trajectory of free normal mode for rotation of rigid triaxial body.

For the equilibrium $\mathbf{1}_2\Omega = (0\ 1\ 0)^T \Omega$ or near axis C, the eigen equation may be

$$\begin{vmatrix} \lambda & 0 & -\sigma_1\Omega \\ 0 & \lambda & 0 \\ -\sigma_3\Omega & 0 & \lambda \end{vmatrix} = 0 \tag{72}$$

There is eigen equation $\lambda(\lambda^2 - \sigma_1\sigma_3\Omega^2) = 0$ and eigenvalues $\lambda_2 = 0 i_2$, $\lambda_{1,3} = \pm\sqrt{\sigma_1\sigma_3}\Omega$. Here is no image i_2 unit in the eigenvalue. Thus the motion behavior of the medium inertia momentum pole may be unstable saddle point. For the equilibrium $\mathbf{1}_1\Omega = (1\ 0\ 0)^T \Omega$, it is evident that there is another stable ellipse trajectory as the eigenvalue may be image.

$$\begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \sigma_2\Omega \\ 0 & -\sigma_3\Omega & \lambda \end{vmatrix} = 0 \tag{73}$$

There is eigen equation $\lambda(\lambda^2 + \sigma_2\sigma_3\Omega^2) = 0$, so that eigenvalues $\lambda_1 = 0 i_1,\ \lambda_{2,3} = \pm i_1\sqrt{\sigma_2\sigma_3}\Omega$. This shows that there must be another stable trajectory about the axis of the minimum inertia momentum similar to the result in Wang W-J [2004a]. The wobble about the axis is obtained with periodic motion rectangular to the stable polar motion about the axis of the maximum inertia momentum. The angular momentum about this axis may not be less as the second order magnitude so that should not be neglected. If the triaxial case of Earth rotation is seen as the second order of approximation and the biaxial case as the first order, then this must be the second order approximation for Earth rotation.

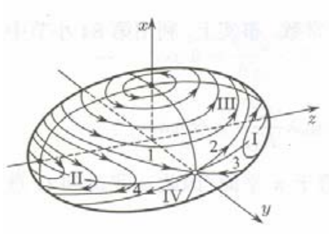


Fig 4. The three simple solutions on the surface of a triaxial rigid body

In order to obtain the rotation about axis *A*, the rotation pole may be set near axis *A* and the rotation angular velocity components should be set as $\omega_1^{(1)} = (\mathbf{1}_1 + w_1)\Omega$, $\omega_2^{(1)} = w_2\Omega$, $\omega_3^{(1)} = w_3\Omega$. In mind, $\mathbf{1}_1$=(1 0 0)^T and with w_1, w_2, w_3 dimensionless amounts. Notice that the rotation about axis *C* is obtained and has been separated from the total rotation. This time ω_1 is much more intense in module than the other two components. Then Euler rotation equations near the equilibrium of axis *A* become into

$$\begin{aligned} &\dot{w}_1 = 0 \\ &\ddot{w}_2 + \omega_a^2 w_2 = 0 \\ &\ddot{w}_3 + \omega_a^2 w_3 = 0 \end{aligned} \tag{74}$$

Here rotation velocity variation about axis *A* is separated from motion of axis *X* determined by the second and the third equation of (74). Evidently, the last two variables are separated as single pendulums and combine as plane pendulum positively hung. There exists a stable motion about axis *A* without rotating. Hence the decomposition theorem for triaxial rotation is complete.

Thus the three direction cosines in (62) are alternately symmetry but the principal inertia momentums *A*, *B*, *C* may dominate the rotation behavior about a certain axis. As σ_3 =2.196$\times 10^{-5}\,\Omega$ =1/45566.4 day^{-1}=1/124.76 yr^{-1} about 1/150 of σ_c, so is often ignored. However, σ_3 is not zero or higher order decimal as 1/124.76 cpy for the rigid Earth case. rotation about axis *A* may be comprehended. Similar to (74), (73) can be solved easily.

$$\begin{aligned} &w_1 = 0 \\ &w_2 = \alpha_1 \cos(\omega_a t + \varphi) \\ &w_3 = \alpha_2 \sin(\omega_a t + \varphi) \end{aligned} \tag{75}$$

Here α_1 and α_2 are amplitudes of elliptic wobble orbit, and φ the initial phase to be specified. The constant for w_1 also may be determined as approximate to 0 so that the rotation angular velocity may have $\omega_1 = i_1\Omega$. This illustrates that axis *A* may not rotate but the rotating pole is the axis cross with axis *A* in direction –90° .

There is an inherent wobble of period 10.2 yr for axis *A* of rigid case. We can obtain the inherent frequency specified as $\omega_a = \mathbf{1}_1\sqrt{\sigma_2\sigma_3}\Omega = \mathbf{1}_1\sigma_a\Omega = \mathbf{1}_1/\sqrt{0.832\times 124.76} = \mathbf{1}_1/10.2$ cpy and the solution for (73). Here the frequency may have an image unit as introduced in elliptic function approach the period of this solution must be image and rectangular to the period about axis *C*. If the parameter σ_3 =(*B*–*A*)/*C* were zero or regarded as zero, then there would be no wobble for axis *A*. But now σ_3 is not zero so that there is a wobble of period about 124.76 yr for axis *A*. As known, the effects of mantle elasticity, liquid core action and ocean excitation contribute the free wobble of rigid Earth as 304.5 d to 435.2 d or lengthen the period of 10.2 yr to more than 1.424

times according to van Hoolst & Dehant [2002]. So the inherent quasi-period for wobble of axis *A* may be specified as about 14.6 yr according to the following equation and reformation of Smith & Dahlen [1981].

$$\omega_A = (\sigma_1\sigma_3)^{1/2}\Omega(1-\frac{k_2}{\kappa})\frac{A}{A^m} \tag{76}$$

Here k^2 and k are Love number and the secular Love number. A^m is the minimum moment of inertia of the outer core. Three solutions for rotation of triaxial Earth may be seen in sketch of Fig 4 with stable trajectories about the maximum and the minimum inertia momentum axes while about the medium an unstable trajectory. In Vondrák [1999] period spectral series of decadal polar motion including 14 yr period and in Schuh et al [2001] decadal spectrum including about period of 17 yr for polar motion data series. Sidorenkov [2000] conjectured that the modulation period for Chandler wobble might be of 18.6 yr. But since no theoretical verification for decadal free polar motion has been established, people do not know what periods really exist and what period may be forged yielding in the filtering processes. In fact, any filter approach may cause forged peak of periods so that there can be seen a series of periods in the spectra of free polar motion time series. We provide here the phase difference time series from IERS data C01 in Fig 3 and show clearly a quasi-period of about 11~19 yr. For elliptic function solution illustrates that the decadal free polar motion period may be an image respect to the Chandler period, the appearance of the decadal free period may be rectangular to the transverse wave of the Chandler wobble so that modulates the phase. In mathematical solution it appears as image period and in physics this must be a vertical waveform respect to the transverse waveform.

Indeed, Chandler wobble about axis *C* has no effect on rotation velocity for the axis itself as the spinning pole. As seen in Fig 4, the free wobble about axis *A* independent to that about axis *C* may affect LOD with half period acceleration and the other half period deceleration.

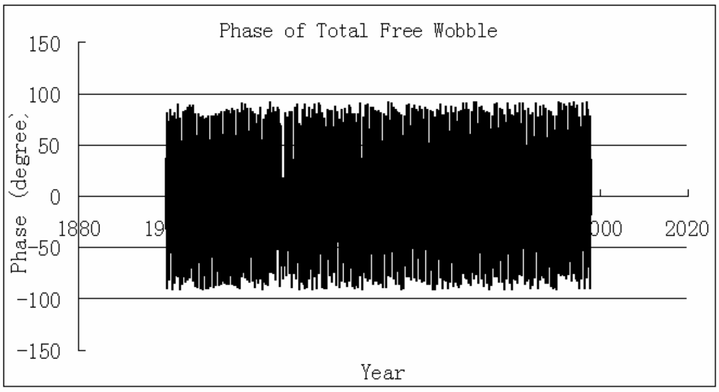
7. Filtering from observations

The longitudinal wave modulation is a phenomenon of phase modulation. In the period of 14.6 years, the main periodic motion may vary the phase tiny on the way transversely waving. To estimate the phase modulation magnitude, it may be seen that the 14.6 yr period possesses frequency of 0.0685 cpy. The frequency is distributed by half period of 14.6 years. Then, there is variation of 0.00938 cpy/cycle in the period of 360°. Thus the peak-to-peak variation of the average phase modulation is about 3.38°. In other words, the observed free wobble should have average phase modulation of 3.38° with 14.6 yr period. Fig 3 shows observed amplitude modulation in polar motion from C01 data series. The amplitude is identical to the analysis above.

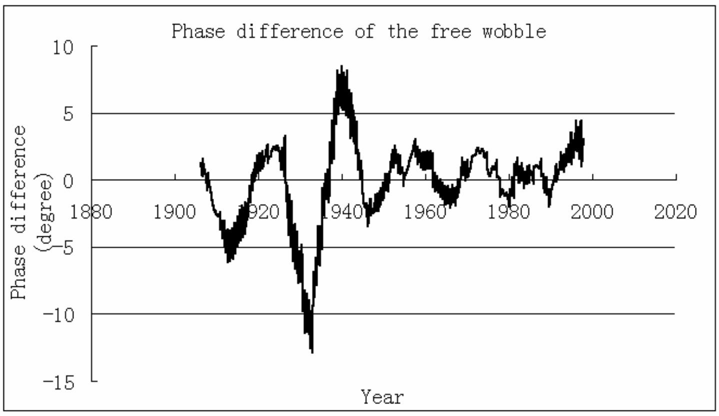
There are many oscillation terms seen in the LOD observation. Formerly, these terms are always considered as forced nutations since the Earth is seen as biaxial body with spinning. The decadal oscillations have been found early in 1950's. For about half a century, since no other source of sufficient angular momentum has been discovered, it has been generally assumed that the decadal fluctuations in LOD reflect core-mantle coupling torques. Geodetic observations thus provide information about fluid flows in the core and important physical constraints on dynamical core models derived from geomagnetic data. This is only a conjecture as there is no other sources of sufficient decadal angular momentum have been discovered. But now the decadal fluctuations in LOD have been found as free motion according to rotation of triaxial Earth. Hence the guessed cause of core-mantle coupling of shape and electromagnetic dynamo could be alternated.

Generally, we always consider polar motion and LOD as two distinct problems though they are nonlinearly coupling. The LOD fluctuations may affect polar motion with the average rotation rate $\Omega = \Omega(t) = \Omega_0 + \delta_0(t)$, here Ω_0 is the inherent frequency of Earth rotation and $|\delta_0(t)|$ the free oscillation magnitude of the rotation velocity. If the sign of $\delta_0(t)$ is positive then there exists acceleration and vice versa. This can be specified easily from linear treatment. On the other hand, nonlinear process causes nonlinear coupling for polar motion upon LOD. As there are free polar motions in the case of rotation for triaxial Earth, free fluctuations may actually be coupled in LOD. The variations of LOD in fact introduce accelerations. So we call these LOD fluctuations as free accelerations.

Firstly, we consider the free fluctuations appear in elliptic function solution. Evidently, the third equation in (32) emerges the LOD and fluctuations. The minimum periods of the third class elliptic sine function dn are $2K/\omega$ for the real and $4K'/\omega$ for the image. The two periods have rectangular coherent relation with the frame that may be understood as a transverse oscillation and the other vertical. Both two emerge rotating velocity variations as common free periodic accelerations. By initial condition of the third component, the integral ought to provide the tiny magnitudes but we concern the periods of the free accelerations. Two periods of accelerations should be inherent variations always called as free motion determined by internal differences of the triaxial Earth.



(a)



(b)

Fig 3. The phase time series in (a) computed from IERS C01 data series of the free wobble position coordinates. In fact, the series represent the total free wobbles including mainly Chandler wobble and the decadal free wobble. The result in (b) illustrates phase difference relating to the referred phase of Chandler wobble with roughly smooth average filtering.

In rotation velocity of triaxial Earth, there are free accelerations. Earth rotating velocity may have a free acceleration variation of period about half Chandler wobble period as 217 d and another free acceleration variation of period as decadal polar motion period 14.6 yr. So the velocity acceleration and deceleration periods may have 108.5 d and 7.3 yr. According to the property of elliptic functions, the third elliptic function dn may have oscillation under constant 1. The variation range can be specified from co-mode k' or the function dn varies with amplitude $\frac{1}{2}(1-k')$. This means that the average rotation angular velocity should be $\Omega[1-\frac{1}{2}(1-k')]$ and the velocity may have acceleration magnitude $\frac{1}{2}(1-k')$ in period of 7 months. As k' is determined the acceleration magnitude is specified as about 1.8 μs. In fact, the wobble of the rotation axis has Chandler period but the wobble amplitude may not contribute to the rotation variation (LOD). This is the effect of nonlinear coupling of rotation and polar motion.

Secondly, we treat with the result of the direct product decomposition approach.

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}=\begin{pmatrix}j\Omega\\ \alpha_1\cos\omega_a t\\ \alpha_2\sin\omega_a t\end{pmatrix}\otimes\begin{pmatrix}R_1e^{\omega_b t}+R_1'e^{-\omega_b t}\\ -j\Omega\\ R_2e^{\omega_b t}+R_2'e^{-\omega_b t}\end{pmatrix}\otimes\begin{pmatrix}\beta_1\cos\omega_c t\\ \beta_2\sin\omega_c t\\ \Omega\end{pmatrix} \tag{77}$$

Seen the square in (32), the third component of the solution is compounded from direct product of three parts. Evidently, the main part is the rotation angular rate Ω with the middle part as a hyperbolic function with respect to time or often called as one-way pendulum. The other part of the rotation rate component possesses a magnitude α_2 and a sine function of frequency ω_a. As 1-dimension component, the direct product may be easily obtained as simple superposition. Hence, the rotation component ω_3 possesses an inherent average frequency Ω and a free fluctuation of period 14.6 yr from the contribution of the unstable solution as one-way pendulum.

The free wobble about axis *A* may have contribution upon rotation fluctuations seen in Fig 3. We develop an approach to estimate the magnitude of the free fluctuation for LOD. Former studies may not consider fluctuation magnitude and leave the amount for observation. However, if the magnitude has not been estimated from the theoretical and empirical analysis, then the observation results cannot be connected with the theoretical analysis. So we estimate the amplitude of the decadal free wobble and translate it to the understanding of the free fluctuation of LOD. Since the wobble of axis *A* has amplitude about 13 mas and 1 ms equivalent to 15 mas in unit of radian (Eubanks 1993), the 14.6 yr period acceleration magnitude may be about twice of 0.87 ms. While the wobble carries through anticlockwise or common with Earth rotation direction, the rotation velocity gains acceleration in period of 7.3 years and vice versa. It can be seen that there is half period of acceleration from –0.87 ms to 0.87 ms so that the acceleration magnitude may be 1.74 ms. In the other half period, there appears as deceleration with magnitude 1.74 ms too. This result can be seen clearly in observation of LOD that there is a term of variation in decadal period with magnitude about 1.5 ms. Lambeck [1980] and Eubanks [1993] attach importance with the decadal oscillations of LOD. But till now the variation of LOD with decadal period has not acknowledged whether caused by core-mantle boundary coupling or electromagnetic coupling. Those conjectures have not been verified quantitatively. Here we conclude that there are free acceleration terms in LOD for rotation of triaxial Earth. Seen in (77) the third component of the solution of rotation for triaxial Earth appears as direct product of a rotating motion with angular velocity Ω and a sine wave with another motion of hyperbolic trajectory.

The acceleration amplitude may be estimated as 1.8 μs for period 217 d and 1.74 ms for period 14.6 yr. If the bifurcation process is noticed for the free acceleration of LOD, then a series of half periods may be seen occur as 7.3 yr, 3.65 yr and 1.825 yr with magnitude about to emerge in observation.

Since the free wobble about axis *A* has magnitude of about 13 mas, it can be estimated that the free fluctuations for LOD of period 14.6 yr should have amplitude about 1.74 ms from –0.87 ms to 0.87 ms. This shows that LOD of triaxial Earth may have acceleration in 7.3 yr period while in the other half period deceleration. The acceleration amount may be average 16.3 μs /yr^2 or 7.55 mm/yr^2. This anomaly acceleration may make a great centrifugal force upon the ocean water especially near the Equator of the Earth.

Interestingly, the El Nino events are aware of reason for Earth accelerations while La Nina events for Earth decelerations. El Nino events have discovered of quasi-period of 2~7 yr. Notice that bifurcation may take place in the process of Earth's free acceleration for LOD, maybe the El Nino events are caused by the free accelerations of periods 7.3 yr with bifurcations. In 7.3 years, the ocean water near the equator may be translated about 0.402 m as huge matter redistribution for water fluid persisting in period of intra annual to one side of the ocean. So the water flows to pile up in one side of Pacific Ocean eastward or reflux westward. The pile-up of the equatorial water brings warmth eastward or westward and changes the inherent equilibrium of the equatorial atmosphere so that causes the weather to change yielding El Nino or La Nina. Hence this may be the dynamical force causing ENSO. Evidently, there are some terms of forced fluctuations in LOD. But as known, forced fluctuations may not cause accelerations. Forced fluctuations are made from the variations of atmosphere and ocean flood while free fluctuations yield centrifugal accelerations. Only free fluctuations may cause accelerations and further centrifugal forces that yield water flowing periodically one way of a direction.

However, the periods are difficult to explain the ENSO occurring near Equator in Pacific Ocean. We deduce here the free fluctuations for LOD may have inherent period of 14.6 yr from the triaxiality of real Earth. As known, the ENSO events have no inherent periods but from 1.5 to 7 yr as. We make some discussions here for the phenomenon. As discussed in the text, the model of rotation equations has been seen evidently as nonlinear and we solve the model with linearized approach of direct product decomposition. The real result of the solution must be treated in product formulation. So the linearized solution of period 14.6 yr must be affected by product of other fluctuations in the process. Thus the inherent period of 14.6 may become instable and may yield bifurcation to split into series of semi-periods such as 7.3 yr, 3.65 yr, 1.825 yr and such on. This may be the different periods for ENSO to occur as pseudo-periods 7 yr, 4 yr and 2 yr. On the other hand, the centrifugal force of acceleration amount 7.55 mm/yr^2 also may not stable in the nonlinear process. Some times the centrifugal force is strengthened and other times decreased so that the ENSO events may burst strong some times and weak other times. Generally, the ENSO events occur in this nonlinear process unstably caused by the free fluctuations of LOD. Here we discover the dynamical force of free accelerations of rotation velocity. This model solves the resources of dynamical force for ENSO but is to study further and deeper. In order to test the two free acceleration variations deduced in theoretically in this study of data series for El Nino and Southern Oscillation (ENSO) events, subharmonic beating and synchronic need to be noticed. The appearance quasi-period of ENSO may not be identical to the prediction here. The main quasi-period of ENSO may be correlated to the free oscillation periods but with variations of subharmonic beating and nonlinear coupling. Anyway, the free acceleration of rotation velocity may be clue to discuss quasi-period of ENSO.

We provide here the numerical result of LOD by wavelet filtering and obtain evident free fluctuations of period as theoretically predicted. Data series is from Gross [2001] called LUNAR97 spanning 1832.5-1997.5 to extract decadal periods. The series has been combined excess from other authors' results but possesses evident appearance with decadal variation feature for LOD. We provide the wavelet band-pass filtering results constructed from Daubitches wavelet seen in Hu et al [2005]. Fig 5 shows the data series of LUNAR97. Fig 6 illustrates the wavelet spectrum in the band of period from 8 to 16 yr. It can be seen clearly there is a peak at frequency about 0.07196 cpy respect to period 13.89 yr or approximate to 14 yr. The difference to the theoretical result may be the effect of visco-elastic deformation of Earth's mantle. Since the effect of visco-elastic deformation makes the period decrease, the free fluctuations of period 14.6 yr for LOD may be conjectured as retrograde. That is to say, if the observed period is longer than the theoretical result, then the periodic fluctuation may be possibly prograde. Fig 7 shows the filtered time series of the period fluctuations with evident average amplitude about less than 0.5 ms. Hence the numerical results illustrate that the data series possesses fluctuations of period and magnitude almost identical to the theoretical analysis in this study.

Fig 5. The time series of data LUNAR97

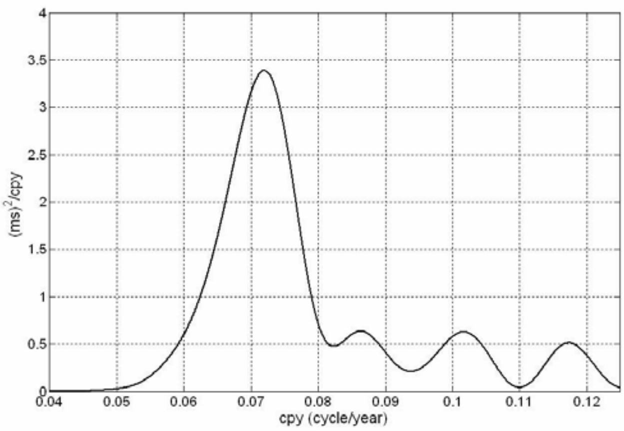


Fig 6. Spectrum of wavelet for data series LUNAR97 with evident peak at period 14 yr

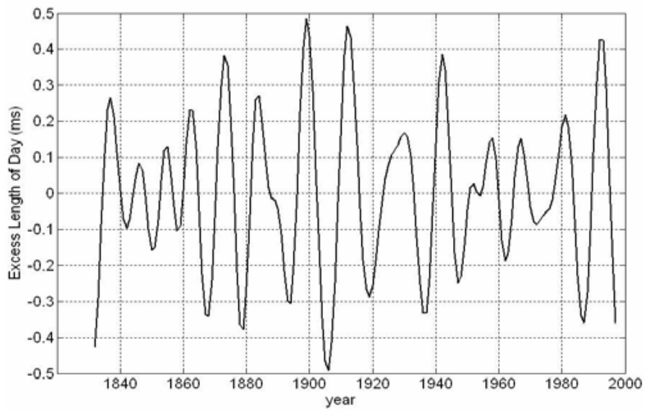


Fig 7. The filtered time series of data LUNAR97 with evident period 14 yr and average amplitude less than 0.5 ms

8. Phase portrait illustration

We state here briefly the process for solving nonlinear rotation equations, for the process may not be introduced in general course and most of colleagues are not familiar to the approach. In modern solving method of nonlinear equations the direct product decomposition may be carried out from eigen value approach for Jacobian derivative matrices and from illustration of phase portrait and section surface like that of Poincaré section.

Decades ago, scientists developed an approach using computer to illustrate the phase portrait and section surface of 3-dimensional differential equations without solving the nonlinear equations. In the process, chaotic trajectory was discovered with complex behaviors such as initial condition sensation and strange phase portrait and section surface. Lorenz model of dissipation was shown as an example of chaotic system in 1960s. Later, conservative system also was discovered of chaotic behavior in higher order dynamical model. A foundational fact came from planet rotation with triaxial moments of inertia. This is the discussion of tumbling for triaxial sub-planet Hyperion of the seventh satellite of Saturn. Wisdom et al [1984] raised a lower dimensional model to consider the variation of the spinning axis of the sub-planet and deduced that the axis might travel to a certain direction with respect to time so that Hyperion must be tumbling at last. Especially, they applied computer illustration to draw the phase portrait and section surface of the rotation for Hyperion and showed that there were quasi-periodic trajectories in the phase portrait as well as chaotic trajectories.

Illustration of phase portrait and section surface for higher order dynamical systems is an approach to draw system points in the higher order space. If the space dimension is too higher to show the phase portrait we may draw projection graphs in planes to decrease the dimension. Thus there may be two approaches to illustrate the attractor portrait of Lorenz model—cubic Lorenz attractor or graphs of projections in phase planes. As seen Lorenz attractor was shown like a butterfly with couple wings representing two stable attracting points in the system and chaos took place in the process of trajectories bouncing stochastically from one of the attracting point to the other. But for conservative system, the chaotic trajectories present in almost period but no period such that in graphs of so-called Poincaré sectioning with interception points of the section with trajectories not contacting but only neighbored. Hence continuous curves in the graph express periodic trajectories but in-continuous points express chaos in the Poincaré section of the phase portrait of fixed point.

Euler equations may be seen as conservative system without attenuation from dissipation. As conservative system it has fixed point as the exact solution but has attractor for in-conservative system if the damping terms in the equations were taken into account. We consider phase portrait of triaxial rotation for Euler rotation equations.

$$\begin{aligned}
\dot{X} &= -\sigma_1 YZ - \lambda X + \psi_1 \\
\dot{Y} &= \sigma_2 ZX - \lambda Y + \psi_2 \\
\dot{Z} &= -\sigma_3 XY - \lambda Z + \psi_3
\end{aligned} \tag{78}$$

Here (*X*, *Y*, *Z*) are angular velocity components of the rotation in inertial frame. Parameters σ_1, σ_2 and σ_3 are specified from principal inertia momentums $A<B<C$. Attenuation parameter λ determined from visco-elastic deformation of the mantle. ψ_1, ψ_2, ψ_3 are external torque components often seen as excitation function. In Euler case, ψ_1, ψ_2, ψ_3 and λ are seen as zeros. In damped free rotation, λ is selected as tiny amount according to assumption of visco-elastic deformation of the mantle.

We provide source program of Visual C++ to show the phase portrait and section surface for Euler rotation equations in three projection planes. In fact, the motion of axis *C* for the maximum moment of inertia may be obtained in inertial plane *X*—*Y* as direct product decomposition from the triaxial rotation. The motion of axis *A* for the minimum moment of inertia may be obtained in inertia plane *Y*—*Z* as direct product decomposition from the triaxial rotation. The motion of axis *B* for the minimum moment of inertia may be obtained in inertia plane *Z*—*X* as direct product decomposition from the triaxial rotation. Multiple solutions of the above three compose the real complex rotation phase portrait and section surface in the inertial space but can be seen in three projection planes. The cubic stereo phase portrait and section surface of the solution for Euler rotation equations may be obtained from the phase portrait and section surfaces.

Fig 8 sketches the inside of the phase portrait and section surface for triaxial rotation. Fig 9 shows the projection of the phase portrait onto the inertial plane *X*—*Y* illustrates that the phase portrait and section surface may be of cubic torus like a tread tire. The top and bottom of the phase portrait are thicker than the left and right sides so that represent the cubic torus tilted in both top and bottom. There are great many fingerprints like loops and whorls on the surface of the phase portrait. Especially, there are hatchings of windmill arms wrapped on the cubic torus. Fig 10 appears as a tilted section of the tread tire with also loops and whorls of fingerprints on the surface. As known, the Poincaré section is defined of rectangular interception. Here it is a section for the cubic torus not cross in rectangle but tilted so that the section surface appears as that of a wedge. However, the surface of the wedge illustrates interesting hatchings essentially represent similar effects of the Poincaré section.

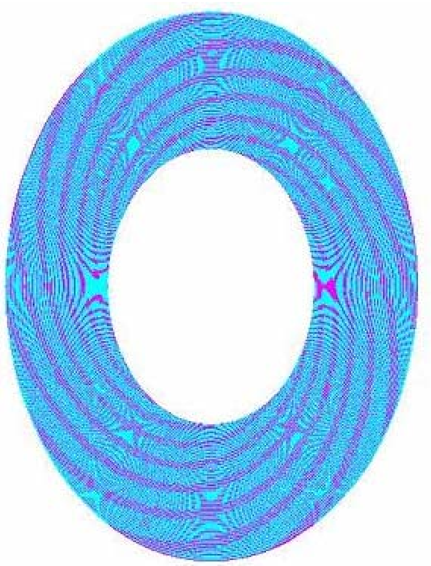


Fig 8

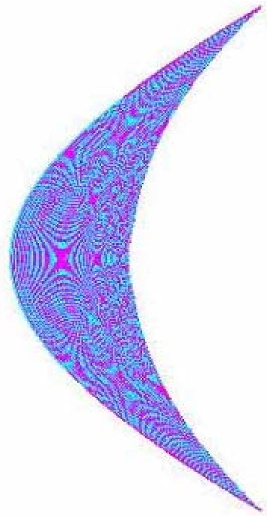


Fig 9

Fig 8 Phase portrait of rigid body rotation in X—Y plane Fig 9 Phase portrait of rotation in Z—Y plane

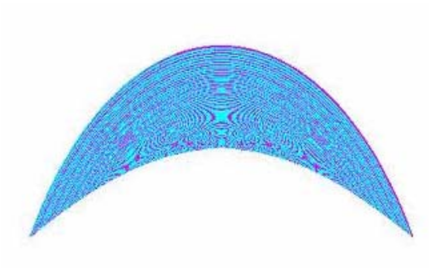


Fig 10a

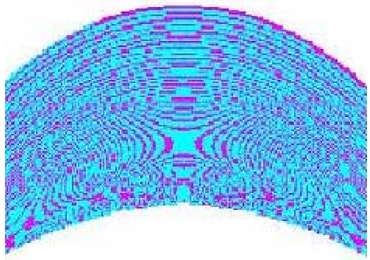


Fig 10b

Fig 10. a) Phase portrait of rotation in X—Z plane. b) Phase portrait of the amplified part in X—Z plane

9. Conclusion

Rotation of triaxial rigid body possesses exact solution of elliptic functions. By definite consideration, several algebraic and geometric approaches illustrate that the solution of the rotation appears as two stable and an elliptic trajectories. The routine solving method for nonlinear equations should be Jacobian matrix and its eigen values. For rotation of triaxial rigid body, the solution can be decomposed onto three principal axes or three main coordinate planes. The multiple solutions in the three main planes may be two stable elliptic trajectories respective to the Chandler wobble and another free wobble with period of 14.6 yr for real Earth rotation. The 14.6 yr periodic solution also can be obtained from observation data series. The phase difference of the observed free polar motion position illustrates evident decadal oscillations. On the other hand, the decadal free polar motion yet appears in variations of LOD while the LUNAR 97 data series filtered with wavelet treatment. Another approach to show the multiple solution of triaxial rotation is phase portrait illustrating the graph of the solution without solving the model. The 2-torus with two evident periodic trajectories and hyperbolic fingerprints show the analysis above may be quite positive. Rotation of a triaxial rigid body has been verified with three true solutions of two single plane pendulums in stable state and an unstable as inversed plane pendulum. Either classical mechanics or modern nonlinear dynamics shows the free wobble or polhody for a triaxial rigid body must be of two accompanied terms. The main term is caused from the polar flattening called as Chandler wobble while the other is caused from the equatorial flattening discovered in Wang W-J [2004]. Earth is not only triaxial other than biaxial with axi-symmetric equatorial main moments of inertia but can be treated with as a pear-shaped planet.

Nonlinear dynamics makes the Earth rotation breaking a revolution so that updated the whole polar motion science. We must humor the conformance of the advancing history without against the tidal current. Professor E Grafarend [2005] published a course about nonlinearity in geodesy as well as in geophysics. Therefore, Earth rotation theory must be reformed to a new Convention in which the two free wobbles should be stated clearly and the similar accompanied consequences as those of the Chandler wobble. Simultaneously, the whole theory of Earth rotation must face to new challenges of two stable and an unstable component solutions with chaotic final state.

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