New Directions for Research on Mathematical Problem Solving

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The Tao that can be told is not the eternal Tao.
(Lau Tsu, *Tao Te Ching*)

The word *Tao* generally is translated into English as *way* or *path*; but, Taoists intend this term to refer to something that might better be described as *the deep structure of patterns and regularities beneath surface-level experiences.* Similarly, *naming* or *telling* refers to all types of language-based or symbol-based statements, propositions, or rules. Therefore, Lao Tsu's words of wisdom can be interpreted to say that: *The structure of things that can be reduced to a formal statements, propositions, or rules is not the true structure of things*.

Because mathematics is the study of structure, the Lao Tsu's observations are especially relevant to research on mathematics learning and problem solving — where, for more than fifty years, the field has experienced a series of 10-year pendulum swings between curriculum reform movements focusing on *basic facts and skills* (easy-to-test *behavioral objectives*) and *problem solving* (general *process objectives*). But, what both of these movements have in common is that: (a) they tend to reduce nearly everything that they judge to be most important to learn to relatively simple declarative statements or rules, and (b) neither makes it clear how the learning of facts, skills, or general problem solving processes is related to the development of powerful constructs (e.g., concepts, conceptual systems, models, or structural metaphors,) which people needed in order to interpret experiences mathematically.

This paper describes an agenda for research on mathematical problem solving abilities which is different than either of the two described above. It is based on *models & modeling perspectives* on problem solving, learning, and teaching (Lesh, 2003; Lesh & Doerr, 2003); and, it focuses on (a) expressive, interpretive, and structural/systemic aspects of mathematical thinking — without neglecting deductive and procedural aspects, (b) studies of the preceding abilities which observe them developmentally — similarly to the ways mathematics educators haveinvestigated children's emerging understandings of concepts such as fractions, rates, and proportions, and (b) tacit as well as formal/analytic understandings for relevant knowledge and abilities.

<u>Why Emphasize Interpretation Abilities?</u> In virtually every field where researchers have investigated differences between experts and novices, experts not only DO things differently, they also SEE (or interpret) things differently. They not only do things right, but they also do the right things — at the right time and for the right reasons.

Even in the popular press, the realization is beginning to emerge that some of the most important types of knowledge that people (and/or learning communities) develop are not reducible to simple procedural rules or declarative statements. For example, in his best selling book, *Don't Think of an Elephant*, Lakoff uses the term "frames" to refer to the conceptual systems and metaphors that people develop to interpret (describe, explain) their experiences; and, in particular, Lakoff use "frames" to explain why, in the last presidential election in the USA, so many citizens clearly voted against their own best interests.

> People think in frames. … To be accepted, the truth must fit people's frames. If the facts do not fit a frame, the frame stays and the facts bounce off. …Statements of belief generally have meanings that depend on systems of belief in which they function. (p.18)

Similarly, in their book, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being,* Lakoff and Nunez explain why the development of powerful conceptual metaphors is especially important in mathematics learning and problem solving. Just as in other domains of activity where expertise involves the development of complex systems of thinking or acting (e.g., sports, performing arts), it is clear that facts, skills, and procedures are relatively useless unless students also recognize when to use them. In fact, a large part of what it means to understand these facts, skills, and processes comes from the systems in which they function. This is why even successful coaches who emphasize the importance of drills on isolated fundamentals do not neglect scrimmages and other complex decision-making activities. It also is why reflection activities such as videotape analyses often are used to help students develop imagery and intuitions for making sense of what they are doing — without trying to reduce complex activities to lists of concatenated rules. Famous books attesting to these facts include *The Inner Game of Tennis* (Gallwey), to *Mental Warfare in Tennis* (Gilbert & Jamison), to *Zen in the Art of Archery* (Herrigel & Suzuki), to *Zen Golf: Mastering the Mental Game* (Parent), to *Zen and the Art of Motorcycle Maintenance* (Pirsig).

A Brief History of Mathematics Education Research on Problem Solving?

In the 1992 *NCTM Handbook for Research on Mathematics Teaching and Learning (Grouws*, 1992), Schoenfeld's chapter on problem solving expressed optimism about the continuation of a movement that he and many other people considered to be *"the decade of problem solving"* in mathematics education. Instead, what followed was a worldwide emphasis on high-stakes testing which ushered in another decade-long period in which basic facts and skills have been emphasized to the exclusion of nearly all else. Therefore, in a comparable chapter on problem solving for the next *NCTM Handbook* (Lester, in press), Zawojewski and I ask:*If the pendulum of curriculum change again swings back toward problem solving, have we learned anything new so that our next initiatives may succeed where past ones have failed?* ... To answer this question, consider the following brief overview of the history of problem solving research.

Polya-style problem solving heuristics — such as *draw a picture*, *work backwards*, *look for a similar problem*, or *identify the givens and goals* — have long histories of being advocated as important abilities for students to develop. But, what does it mean to "understand' them? Such strategies clearly have descriptive power. That is, experts often use such terms when they give after-the-fact explanations of their own problem solving behaviors — or those of other people that they observe. But, according to Begel's 1979 review of the literature on problem solving, there is little evidence that general processes that experts use to describe their past problem solving behaviors should also serve well as prescriptions to guide novices' next-steps during ongoing problem solving sessions. Begel concluded that *"(N)o clear cut directions for mathematics education are provided by the findings of these studies. In fact, there are enough indications that problem-solving strategies are both problem- and student-specific often enough to suggest that hopes of finding one (or a few strategies) which should be taught to all (or most students) are far too* *simplistic"* (p. 145).

Similarly, Schoenfeld's 1992 review of the literature concluded that attempts to teach students to use general problem-solving strategies generally were not successful. He noted that Polya's descriptive processes are really more like names for large categories of processes rather than being well defined processes in themselves. Therefore, in an attempt to go beyond "descriptive power" to achieve "prescriptive power", he recommended: (a) developing and teaching more *specific problem-solving strategies* (that link more clearly to specific classes of problems), (b) studying how to teach *metacognitive strategies* (so that students learn to effectively deploy their problem-solving strategies and content knowledge), and (c) developing and studying ways to eliminate students' counter-productive beliefs while enhancing productive *beliefs* (to improve students' views of the nature of mathematics and problem solving). Unfortunately, this approach simply moved the basic problem to a higher level. That is, short lists of descriptive processes tend to be too general to be useful; yet, long lists of prescriptive processes tend to become so numerous that knowing when to use them becomes the heart of understanding them.

A decade later, at the end of another extensive review of the literature, Lester and Kehle (2003) again reported that little progress had been made in problem solving research — and that problem solving still had little to offer to school practice. In general, they agreed with Silver's 1985 assessment that, even in those few studies where some successful learning was reported, transfer of learning tended to be undocumented or unimpressive; and, successes generally occurred only when world-class teachers taught lengthy and complex courses where the complexity of the "treatment" makes it unclear *why* performance improved. Perhaps the students simply learned some mathematics concepts — rather than learning problem solving strategies, heuristics, or metacognitive processes!

Looking over the preceding history, Lesh & Zawojewski (in press) concluded that, when a field has experienced more than fifty years of pendulum swings between two ideologies, each of which have obvious and fundamental flaws, perhaps the time has come to consider the fact that these are not the only two options that are available. For example, one collection of alternatives which will be described in this paper are coming to be known as *models & modeling perspectives* on mathematics problem solving, learning and teaching. Unlike most of the problem solving studies referred to in the preceding reviews of the literature, research based on MMP developed out of research on concept development at least as much as research on problem solving. Consequently, results tend to focus on what it means to "understand" important concepts and processes - and on how these understandings develop — rather than rushing ahead to try to teach strategies, heuristics, processes, and beliefs whose nature we may not yet understand. For example, in the case of problem solving processes, research based on MMP investigates what develops when students learn to function proficiently in situations where they need to modify/adapt/extend/refine concepts and conceptual systems that ALREADY ARE AVAILABLE (at some level of development) rather than trying to teach the students to function better in situations where relevant ways of thinking are assumed to be LOST OR MISSING (i.e., What should they do when they're stuck?)

Lack of Accumulation is a Central Problem in Research on Problem Solving

Mathematics education research often is accused on failing to address teachers' problems. But, if we've failed to solve realistically complex problems, the reason surely is not because of a lack of trying or caring. More fundamental reasons may be because of the arrogance of governmental and professional organizations who believe that (a) they themselves already have clear understandings of the problems at hand, and (b) these problems should be solved though individual studies showing *what works.*

In contrast to the preceding points of view, in applied sciences and design sciences such as engineering or architecture, it is well known that: (a) at least half of the researcher's job typically involves helping "clients" understand their problems and ask better questions, (b) in order to deal with trade-offs involving competing goals (such as low costs and high quality) solutions usually need to draw on more than a single practical or theoretical perspective, (c) a large share of research effort needs to be aimed at the development of tools for researchers themselves to use for tasks such as measuring important constructs that cannot be observed directly, and (d) solutions to realistically complex problems usually need to involve the gradual accumulation of knowledge — and iterative efforts involving multiple people, projects, and projects which build on one another's work over long periods of time.

Lack of accumulation is a main reason why so little progress has been made in problem solving research. For example, in research on problem solving, each time the pendulum of curriculum reform has swung back toward problem solving, many previously discredited ideas have tended to be recycled — perhaps using new terminology. For instance, currently, past notions of metacognitive processes are being reincarnated using the language of *habits of mind* (Cuoco, Goldenberg, & Mark, 1996; Golderberg, Shteingold, Feurzeig, 2003).Yet, using this new jargon, the underlying ideas clearly involve nothing new; and, no new tools have been provided to document or assess most of the constructs that are claimed to be important.

One main flaw of research which is preoccupied with showing *"what works"* is that there seldom exists much clarity about what the "it" (i.e., the instructional treatment) really is that is claimed to work; nor is it clear what "working" (i.e., mastery) really means. Furthermore, there usually is an expectation that how well "it" works depends on how much and how well "it" is implemented. Therefore, because implementation is never complete, the results of such studies nearly always end up concluding that what is needed is to try more and harder. So, in response to repeated failures, old ways of thinking tend to be embellished rather than being re-examined or rejected. For example:

- When most students prove to be unable to use concepts and skills beyond the artificial situations in which they were introduced in school, problem solving strategies and heuristics are introduced.
- When instruction on strategies and heuristics proves to be ineffective, metacognitive functions and other habits of mind are introduced.
- When metacognitive functions are ineffective, beliefs and affective functions are introduced.
- When higher-order processes, beliefs, and higher-order habits of mind prove to be ineffective, the whole preceding sequence is currently being recycled again — with two minor embellishments focusing on: (i) situated learning and/or (ii) communities of practice.

Later in this paper, I will describe a variety of ways that *models & modeling perspectives* emphasize promising aspects associated with both sociocultural theories and theories of situated cognition. But for now, I'll simply state the claim that neither of these latter perspectives necessarily forces researchers to re-examine their own basic assumptions about the changing nature of elementary mathematics — nor the changing nature of "real life" problem solving situations in which some type of "mathematical thinking" is useful. In fact, just as in traditional research on problem solving, both of these perspectives tend to begin with the assumption that the researcher already possesses clear and accurate conceptions about the nature of elementary mathematics — and about what it means to "understand" relevant concepts and processes. So, neither necessarily avoids fatal flaws similar to those that have plagued traditional research on problem solving. For example, if theoretical perspectives associated with situated cognition emphasize learning in context, then difficulties tend to be highlighted which are associated with lack of transfer for what is learned. Or, if sociocultural perspectives emphasize learning which occurs though participation in relevant communities of practice, then the emphasis on building new ways of thinking tends to shift toward borrowing existing ways of thinking - so problem solving tends to be reduced to the little more than using cultural tools, beliefs, and behaviors which can be adopted from others with little adaptation.

Science makes Progress by Rejecting Ideas — Not Just by Endlessly Embellishing

Popper introduced the notion that falsifiability is the key characteristic that distinguishes science from ideology. However, Popper was thinking mostly about natural sciences, such as physics, where (a) the "things" being investigated were on the scene before the advent of humans, and they remain largely unchanged after of human intervention, and (b) the theories being developed can be thought of as being true or false in the sense that they accurately describe, predict, or explain events which occur naturally — rather than being created by humans.

Mathematics education, on the other hand, is a design science — more like engineering — where many of the most important things that we need to understand are systems that we ourselves help to create. Therefore, as soon as we understand these systems, we tend to change them. So, ideas about such systems are not so much true or false as they are useful or not useful. Furthermore, the development of such systems almost always involves trade-offs among competing factors such as high quality and low costs. So, their designs nearly always need to draw on multiple practical and theoretical perspectives; and, single grand theories seldom prove to be useful.

As Lau Tsu surely would remind us, and as I have argued in a number of different publications in the past (e.g., Lesh 2003), research is about knowledge development; and, not everything that we know is reducible to rules (tested hypotheses) and declarative statements (answered questions). For example, in both the design sciences and the natural sciences, much of what is known is embodied in models — which typically integrate ideas and procedures from multiple practical and theoretical perspectives, and which often are expressed using a variety of interacting representational media that tend to be purged from elegant theories.

Models are a type of knowledge that is useful for developing, describing, explaining, predicting, and controlling complex systems; and, "survival of the useful" is the main criteria which determines the acceptance or rejection of models (as well as the underlying conceptual systems that they embody). In particular, models which are not rejected generally need to be powerful (in specific situations), re-useable (in other situations), and sharable (with others). So: (i) scientists who are in the model development business design for these characteristics rather than simply testing for them, (ii) adaptability (e.g., modularity, revisability), rather than imperviousness to change, is a key attribute of models that survive, and (iii) every current product and model tends to be the nth step in process of continuous development. Therefore, the rationale for rejecting models (or components of models) are expected to come from auditable trails of documentation and explanation which are tested and revised over long periods of time — and at multiple sites where a variety of conditions and purposes prevail.

The preceding kinds of design processes should not seem mysterious. They are exactly the way scientists design space shuttles, stable ecological systems, and other systems where the whole is more than the sum of its parts. Knowledge development and artifact development occur simultaneously and interactively; and, this is especially true when the underlying design (i.e., the underlying conceptual system, or model) is considered to be an important part of the artifact being designed. In these latter cases, when the artifacts are tested, the conceptual systems that the artifacts embody tend to be tested.

Theory development should contribute to model development; model development should contribute to theory development; and, both should contribute to knowledge development as well as to the design of useful artifacts. Nonetheless, theory development and model development are somewhat different. For example, even though mono-theoretic models certainly do exist and sometimes are useful, in general, realistically complex problems are not likely to be solved by drawing on only a single grand theory.

In general, I believe that mathematics education researchers should consider themselves to be primarily in the model development business, as well as the theory development business; and, in the case of research on mathematical problem solving, the kind of models that we need are models of modeling (for problem solvers who range from students, to teachers, to curriculum designers, to policy makers). But, this has important implications for knowledge development — because, when model development is emphasized in mature fields such as engineering or architecture, it is well understood that all models are wrong but that some are useful! So, simpleminded notions of falsifiability are not appropriate. Nonetheless, Popper's basic notion is valid! That is, science makes progress mainly by rejecting ideas — not simply by embellishing them endlessly as we have done in mathematics education research on problem solving.

Problem Solving Research should Re-examine Foundation-level Assumptions

Models and modeling perspectives on mathematics, learning, and problem solving were born with the belief that the time has come to re-examine foundation-level assumptions about what it means to understand many of the most important concepts and processes in elementary mathematics — as well as basic assumptions about the nature of problem solving and mathematical thinking in real life situations beyond school.

One reason to re-examine assumptions about what it means to understand and use mathematics is because, in a technology-based *age of information*, regardless whether attention focuses on the lives of ordinary people or future-oriented professions, radical changes have been occurring in the kind of situations where some type mathematical thinking is useful beyond school (Lesh, Hamilton & Kaput, in press). Yet, if *usefulness beyond school* is considered to be a primary criteria for choosing content to be emphasized in the elementary mathematics curriculum, then the opinions of people who are heavy users of mathematics have seldom been heard in the development of standards for curriculum and assessment (Burkhardt & Pollak, 2006). The following trends are important to consider.

New ways of thinking about old situations:

Emerging new technologies are creating new multi-media, interactive, and dynamic ways to think about old situations. For example, we only need to look at daily newspapers such as *USA Today* to see ample evidence that modern mathematics is becoming multi-media mathematics. In sections of the newspaper ranging from editorials, to sports, to business, many of the articles are coming to resemble multi-media computer displays which are filled with graphs, tables, diagrams and hyperlinks to other resources. Consequently, even in situations that involve nothing more sophisticated than buying and selling groceries or automobiles, situations are created and described using computer-based, multi-media, computational models — which are creating completely new ways of thinking about problems that involve optimization, stabilization, and other goals that used to require algebra, calculus, or other topics in advanced mathematics. But today, such issues often are handled using computational, interactive, multi-media models which are based on extensions of basic ideas in measurement and arithmetic — rather than models which are based on single algebraic functions. Therefore, even though such models may be associated with topics such as discrete mathematics, systems theory, game theory, complexity theory, or mathematical modeling, they often are well within the limits of elementary mathematics.

New types of situations to understand and explain:

New kinds of situations also are emerging that need to be understood and explained. Again, this is because a distinguishing characteristics of a technology-based information age is that the same tools that provide new ways to think about existing worlds of experience also enable completely new worlds of experience to be designed. Consequently, complex systems — ranging from communication systems, to economic systems, to transportation systems, to ecological systems — are coming to be among the most powerful "things" that impact the lives of increasing numbers of people. And, such systems are especially significant in countries that are developing knowledge economies — where increasing globalization typically leads to feedback loops and second-order effects which often overpower local actions — especially when interactions involve multiple agents with partly conflicting goals.

New types of problem solvers and problem solving:

Modern jobs increasingly involve "learning organizations" which need to adapt rapidly in response to continually changing circumstances; and, the most important assets of these "learning organizations" often consist of knowledge and networks rather than large warehouses filled with physical goods and resources. Consequently, "problem solvers" often are not isolated individuals whose only tools consist of pencil and paper. Instead, they often are teams of diverse specialists representing a variety of different practical and theoretical perspectives, and having access to a wide range of rapidly evolving technical tools. Similarly, the knowledge and abilities that productive groups and individuals possess often do not reside within the minds of isolated individuals. Instead, knowledge and abilities tend to be distributed — and often are off-loaded to supporting networks of tools and colleagues. This is why job interviewers in future-oriented professions consistently emphasize the fact that the people who are in highest demand are not necessarily those who are skillful at scoring well on standardized tests. Instead, they tend to be people who are able to: (a) make sense of complex systems, (b) work within teams of diverse specialists, (c) adapt rapidly to a variety of rapidly evolving conceptual tools, (d) work on multi-staged projects that require planning and collaboration among many levels and types of participants, and (e) develop sharable and re-useable conceptual tools that usually need to draw on a variety of disciplines — and textbook topic areas (Lesh, Hamilton & Kaput, 2006). So, the mathematical abilities that set these people apart from their peers often have more to do with expression (e.g., interpretation, description, explanation, communication, argumentation, and construction) more than computation or deduction; and, they have as much to do with imposing structure on experience as they do with deriving or extracting meaning from information which is presumed to already be given in a mathematically meaningful form.

New kinds of products and design processes:

Today, when some kind of mathematical thinking is needed to solve real problems, the products that need to be produced often involve much more than short answers to pre-mathematized questions. For example, they often involve developing conceptual tools (or other types of complex artifacts) which are designed for some specific decision maker and for some specific decision-making purpose — but which seldom are worthwhile to develop unless they go beyond being powerful for a specific purpose to being sharable with others and re-useable beyond the immediate situations in which they were first needed. Consequently, solution processes often involve sequences of iterative development testing revising cycles in which a variety of different ways of thinking about givens, goals, and possible solution steps are iteratively expressed, tested, and revised (e.g., integrated, differentiated, or reorganized) or rejected. That is, the development cycles often involve a great deal more than simply progressing from pre-mathematized givens to goals when the path is not obvious. Instead, the heart of the problem often consists of conceptualizing givens and goals in productive ways.

Another reason for re-examining assumptions about what it means to understand and use mathematics is because, in research on concept development, mathematics educators have now developed relatively sophisticated mini-theories describing the development of whole number concepts, rational numbers concepts, early algebra concepts, and a wide range concepts related to other topic areas. Yet, these mini-theories often yield quite different descriptions of development — in spite of the fact that, in the minds of students, these concepts do not develop in isolation from one another, nor independently. For example, at the same time that whole number concepts are developing, early understandings also are beginning to develop about measurement, geometry, fractions, and even foundation-level ideas related to algebra and calculus. Furthermore, in recent years, researchers have produced a great deal of evidence showing that learning is far more piecemeal (diSessa, 2002), situated (Lave & Wenger, 1991), socially mediated (Wenger, 1998), and multi-dimensional (Lesh & Yoon, 2004) than most currently popular theories have us to believe. For example, in ethnographic comparisons of experts and novices in a variety of fields, results have consistently shown that expert knowledge is organized around experience at least as much as it is organized around the kinds of abstractions emphasized in schools. Furthermore, in an age when problem solvers have nearly continuous access to things like computer-based search engines, spell checkers, data processors, and multi-media tools for communication and collaboration, it is obsolete to think of intellectual capabilities (e.g., information storage, retrieval, or processing) as if they resided exclusively within the minds of isolated individuals. The whole point of using conceptual tools is that they enable people to off-load information and functions that once needed to be carried on in the mind. But, the nature of these tools is that, when they are introduced into a situation, the situations themselves tend to be transformed in a variety of fundamental ways.

What New Methodologies are Appropriate for Investigating Changes in the Kind of "Mathematical Thinking" that is Needed in for Success in the 21st Century?

Even though the observations in the preceding section describe several important ways that the nature of problem solving has been changing as we enter the 21st century, they do little to clarify answers to the following kinds of questions: (a) What kind of "mathematical thinking" is emphasized in these situations? (b) What does it mean to "understand" the most important of these ideas and abilities? (c) How do these competencies develop? (d) What can be done to facilitate development?

Traditional ethnographic approaches have been useful for investigating the preceding kinds of issues. But, such methodologies also tend to presuppose, to a far greater extend than we believe is acceptable in our own research, that the researcher already knows <u>where</u> to observe (e.g., in tailor shops, in grocery stores, in science laboratories), <u>whom</u> to observe (e.g., people who are still adapting, or experts who have reduced large classes of former problems to routine exercises), <u>when</u> to observe (e.g., when computations or deductions are being made, or when structure is being imposed on raw experiences), and <u>what</u> to observe (e.g., situations where the goal is to produce answers to pre-mathematized questions, or situations where the goal is to design tools that are sharable and reuseable).That is, ethnographic approaches tend to assume that answers already are known to the very questions that we want to investigate. Furthermore, such prejudices tend to be especially significant if the kind of thinking that we wish to emphasize focuses on students' interpretation abilities. This is because students' interpretations of situations are influenced by the structure of the situations in which observations are made — as well as by students' structuring abilities. Consequently, the thinking that researchers observe tends to be strongly influenced by the researcher's choice (or design) of the situations in which observations are made. Therefore, what occurs under such circumstances is like a psychological version of Heisenberg's *Indeterminancy Principle* in physics — where observations induce changes in the "thing" being observed.

Each time a person develops (extends, refines, revises) an interpretation of a situation, their interpretation abilities change. So, no two situations are ever exactly alike; there is no such thing as observing the same thing twice; and, the best that can be documented are trajectories of interactions as over time. Therefore, in order for our research to deal with such assumptions in developmental studies of problem solving abilities, MMP-based research based often uses combinations of the following research methodologies.

Thought-Revealing Activities:

Several past publications have described the most important principles for designing effective *thought-revealing activities* for students (Lesh, et al., 2000), for teachers (Jawojewski, et al., in press) or researchers (Lesh, Kelly & Yoon, in press).For any of the preceding "subjects", *thought-revealing activities* are situations which are both personally meaningful and structurally interesting — and which require problem solvers to express their current ways of thinking in forms that must go through several iterative cycles of testing and revising. Consequently, during *thought-revealing activities*, problem solvers typically make significant adaptations to their current ways of thinking; and, by going through a series of iterative expressing→testing→revising cycles, they automatically generate auditable trails of documentation which reveal important evidence about the nature of knowledge and abilities that are developed. In other words, *thought-revealing activities* are similar to Petrie dishes in high school chemistry or biology laboratories — because significant changes occur during sufficiently brief periods of time so that both problem solvers and researchers can directly observe the development of relevant concepts and abilities.

The specific kind of *thought-revealing activities* that we emphasize in our own research often have been called *model-eliciting activities* (Lesh & Doerr, 2003). They focus on subjects' interpretation abilities; and, they require subjects to express relevant ways of thinking in the form of models or conceptual tools which are designed to be powerful for some specific purpose — but which also need to be sharable and reuseable. Therefore, the design "specs" that are given to problem solvers describe products that are needed similarly to the way design "specs" are given to engineers or architects when they develop artifacts such as spacecraft to skyscrapers. These design "specs" typically include: (i) information about the resources that are expected to be use, (ii) issues that need to be addressed, and (iii) criteria that should be used to assess strengths and weaknesses of alternative products and ways of thinking. That is, they shape the products that students are supposed to create; and, they optimize the chances that significant developments will occur by making it possible for student to weed out inappropriate responses. But, they do not dictate the exact nature of what develops, nor do they dictate how development should occur. In other words, they are similar to photographic negatives of the concepts and conceptual systems whose development we want to observe. They filter out thinking that is of no interest. Nonetheless, because our goal is to study the nature of relevant models and conceptual systems, the design "specs" treat the exact nature of the model as something similar to an undefined term in a mathematical system. Similarly, the nature of relevant conceptual systems also is treated as an undefined term.

Evolving Expert Studies: In research based on models & modeling perspectives, evolving expert studies provide alternatives to methodologies in which researchers simply gather opinions from experts using observations, interviews, or questionnaires. For example, in research aimed at investigating the nature of new types of problems solving situations where new types of "mathematical thinking" is needed for success beyond school, we often enlist diverse teams of teachers, parents, policy makers, and professors or professionals in relevant fields to work together as co-researchers in semester-long projects whose goals are to develop *thought-revealing activities* for students which (the experts believe) are authentic simulations of new types of "real life" problem solving situations that: (i) will be especially important for students to master as preparation for success in a technology-based *age of information*, and (ii) emphasize important new types of elementary-but-powerful understandings and abilities that are likely to be needed for success in the preceding problem-solving situations. Participants in the preceding studies are considered to be experts because each has important views that should be considered to answer the relevant research questions. Yet, they are evolving experts because: (i) different experts often hold conflicting views, (ii) none have exclusive insights about truth, and (iii) all tend to evolve significantly when they repeatedly express their current ways of thinking in forms that go through multiple sequences iterative testing-and-revision cycles in which formative feedback and consensus building influence final conclusions that are reached. Thus, in "evolving expert" studies, the design of *thought-revealing activities* for students often provides an ideal context for equally *thought-revealing activities* for experts.

Principles for designing *thought-revealing activities* for experts have been described in several past publications (Zawojewski, et al., in press). For example, in "evolving experts" studies, other kinds of thought-revealing activities include: (i) designing *observation forms* that teachers or researchers can use to gathering information about the roles, processes, or concepts that contribute to students' success in the preceding activities, or (ii) designing *ways of thinking sheets* that teachers or researchers can use to identify strengths and weaknesses of alternative ways of thinking.

Multi-tier Design Experiments:

Multi-tier design experiments have been described in several past publications (Lesh, 2003; Kelly & Lesh, in press). They can be thought of as longitudinal development studies in conceptually rich environments which were developed especially for the purposes of:(i) using multi-disciplinary perspectives to focus on the interacting development of students, teachers, and other "evolving experts" including researchers, and (ii) producing auditable trails of documentation focusing on the development of the participants' ways of thinking about mathematical problem solving situations — or about the nature of mathematics, learning, problem solvers, or problem solving beyond school. Such studies are called "multi-tier" because thought-revealing activities for students typically provide the context for thought-revealing activities for teachers (or other evolving experts) which in turn provide the context for thought-revealing activities for researchers. For example, in cases where *model-eliciting activities* are used, students develop models of mathematical problem solving situations; teachers develop models of students modeling abilities; and, researchers develop models of interactions among teachers and students. Thus, everybody is considered to be in the model development business; and, similar principles are expected to govern scientific inquiry at each level. In particular, at each level, *thought-revealing activities* optimize the chances that development will occur without dictating its direction; and, as participants at all levels repeatedly express, test, and revise important aspects of their current ways of thinking, they automatically generate auditable trails of documentation which reveal important aspects about the nature of developments that occur.

Alternatives to Expert-Novice Studies:

According to *models & modeling perspectives*, we often are interested in studying the development of conceptual systems, or the development of models and other conceptual tools — without regard to issues about whether relevant conceptual systems are developing in individuals, or by small groups, or by larger learning communities. Consequently, in somewhat the same way the other researchers have found it useful to compare experts with novices (or gifted students with average ability students), we often have found it useful to compare problem solvers who are groups to problem solvers who are isolated individuals. This does not mean that we believe that there are no significant differences between groups and individuals — any more than expert-novice studies presuppose that there are no significant differences between groups and individuals. But, it does mean that studies about one type of "problem solver" often help researchers understand other types of problem solvers (Lesh, 1985).

Results and Directions for Problem Solving Research Based on Models & Modeling Perspectives

Note: A longer version of this paper will be posted on the author's web site — where more extensive references are given. The following section, in particular, consists mainly of bullet points that are addressed in greater detail is the longer paper.

Are average ability students able to develop powerful mathematical models?

Traditional research on mathematical problem solving gives the overwhelming impression that only exceptionally bright students are capable of solving significant problems - or developing significant mathematical concepts — unless step-by-step guidance is provided by a teacher. But, research based on *models & modeling perspectives* is filled with transcripts, as well as results from large scale studies (e.g., Schorr & Gearhart, 2006, Zawojewski, Deifes-dux & Bowman, in press) documenting the following facts.

Average ability students routinely develop impressively sophisticated and deeply mathematical ways of thinking in response to *model-eliciting activities* in which solutions can be developed by going through several iterative expressing→testing→revising cycles. Last-draft responses that emerge tend to be much better indicators of ability than the kind of first-draft responses that are emphasized on standardized tests.

Students who emerge as being especially productive on such tasks often include a significant percentage who do not have histories of achieving the highest scores on standardized tests. Conversely, many students who have excellent records of doing well in school do not do well in simulations of "real life" problem solving situations. These facts appear to be true for many of the same reasons why, in future-oriented professions which are heavy users of mathematics, science, and technology, expert job interviewers generally look for a great deal more than A's in entry level courses and high scores on standardized tests. These interviewers know that such indicators reflect only a narrow, shallow, and non-central band of competencies — and, that when a broader range of deeper understandings are emphasized, a broader range of students naturally emerge as having extraordinary potential.

Transfer of learning often is impressive — especially if the conceptual tools that problem solvers develop are designed from the beginning to be useful (for specific purposes and specific people) — but also sharable (with others) and re-useable (beyond the immediate situation). In fact, in follow-up interviews with students who have worked on *model-eliciting activities,* it is not uncommon to hear extraordinarily detailed descriptions of activities that were completed many months earlier; and, over-generalization often appears to be more of a problem than lack of generalization. One is reminded here of the saying that: *When you have a hammer in your hand, lots of things begin to look like nails!* Whereas rule-based knowledge is inert (with condition-action rules sitting dormant until some stimulus causes them to be executed), interpretation systems are far more dynamic and alive. Once developed, interpretation systems seem to be like living organisms that go around seeking out situations in which they can function.

Do model-eliciting activities work?

Model-eliciting activities were not created to teach problem solving. They were created to help researchers understand and assess problem solving. Therefore, it is a serendipitous byproduct of MMP research that MEAs have proven to be powerful from the point of view of instruction. In fact, in large scale curriculum reform studies, or college levels, the following are two of the main reasons why MEAs appear to have been effective - and, both involve second-order effects on learning.

MEAs tend to contribute to teacher development. This is because: (i) MEAs are designed to be *thought-revealing activities*, and (ii) research on *cognitively guided instruction* has shown that one of the most effective ways to help teachers improve their teaching is to help them become more insightful about the nature of their students' thinking.

MEAs often encourage students to spend more time thinking about mathematics — both in class and outside of school. For example, during 60-minute model-eliciting activities, teachers typically report extraordinarily high levels of engagement throughout high percentages of the problem solving sessions; and, because MEAs are simulations of "real life" situations in which some type of mathematical thinking is useful beyond school, students often continue to work outside of class.

For nearly every *model-eliciting activity* for which transcripts have been reported in the literature, we make the bold claim that the results not only apply to large samples of students who are similar to those involved in the relevant study; but, in general, they are even highly likely to apply to colleagues who are reading the reports — or to their children or students. …What other genre of research is bold enough to make such claims? On the other hand, no problem is model-eliciting unless it actually succeeds in eliciting a model! In fact, for some students, such activities may not even be problematic or meaningful. So, it is nearly a tautology to say that *model-eliciting activities* work. They cannot not work and still be MEAs. Consequently, the main goal of MMP research is to show how and why MEAs work — not just to show that they work.

Perhaps the most important way that MEAs have proven to work is that, for at least the past ten years, research based on *models & modeling perspectives* has been able to begin with the assumption that the following facts can be counted on to occur. If the principles for MEAs are followed, then the activities can be expected to reliably elicit significant forms of development within 60-90 minute problem solving episodes. Consequently, recent research has been aimed at investigating: (i) the nature of new types of problem solving situations where some type of "mathematical thinking" is needed for success beyond school, (ii) the nature mathematical abilities that distinguish people who are extraordinarily successful in simulations of the preceding "real life" situations beyond school, and (iii) the nature of contexts that contribute to long-term memory and transfer.

How has research on MEAs influenced the way we think about Problem Solving?

In traditional word problems, where solution paths lead from mathematical givens (which are pre-mathematized and computation-ready) to mathematical goals (whose practical purposes are seldom sufficiently clear to provide criteria for judging the quality of trial solutions), there seldom is any need to leave the world of mathematics. But, for *model-eliciting activities*, where relevant processes are aimed at imposing structure on the situation rather than deriving information from the situations, solution processes generally involve involve: (i) mathematization — systematization, quantification, dimensionalization, coordinatization, etc. (ii) derivation, (iii) interpretation, and (iv) verification. So, only one-fourth of such cycles occurs exclusively within the world of mathematics.

We no longer define problem solving to be *a process of getting from givens to goals when the steps are not immediately obvious*. Instead, *problematic situations* are defined to be *goal directed activities in which adaptations need to be made in existing ways of thinking about givens, goals, and possible solution steps*. In other words, problem solving and concept development are expected to be highly interdependent.

We no longer think of *problem solving strategies* as if their main purposes were to provide answers to questions that focus on selecting and executing procedures: *What should I do when I'm stuck (i.e. when I am not aware of any productive ways of thinking about the problem at hand)?* Instead, useful *problem solving strategies* are expected to help problem solvers develop more productive interpretations — which are based on adaptations to existing conceptual systems (which are expected to be at intermediate stages of development - not fully developed nor completely undeveloped).

We no longer expect problem solving strategies, or metacognitive functions, or beliefs to be reducible of explicit rules which problem solvers should learn; and, in general, we no longer expect attributes that have descriptive power (for describing past behaviors of experts) to also have prescriptive power (for guiding the next steps of novice problem solvers). Instead, we recognize the possibility that the meanings of most rules of thought may depend on systems of thought, and that students may be able to think <u>with</u> these systems without necessarily needing to think <u>about</u> them (by making them explicit objects of thought).In other words, such systems are expected to function implicitly rather than explicitly; and, the relevant strategies, functions, and beliefs whose meanings depend on them are expected to often function tacitly rather than formally and analytically.

What results have emerged from MMP-based research on *model-eliciting activities*?

Because *model-eliciting activities* require students make significant adaptations to existing ways of thinking during single 60–90 minute episodes, and because important aspects of these ways of thinking need to be expressed in forms that are observable by both researchers and problem solvers themselves, it often is possible to directly observe not only the stages through with development occurs but also the processes that contribute to development.

Problem solvers typically go through 3–8 modeling cycles during a single 60–90 minute *model-eliciting activity*. One way to identify different models is because they tend to involve somewhat different mathematical "objects" (e.g., quantities), relations, operations, and patterns or other systemic properties of the relevant conceptual system-as-a-whole. This approach delimits, but does not define, the nature of the models (or underlying conceptual systems) that we want to investigate. That is, the principles for designing *model-eliciting activities* specify how models (and underlying conceptual systems) can be elicited in forms that can be observed; and, research tools that accompany *model-eliciting activities* often provide ways to classify and assess important aspects of what is observed. But, the main purpose of *model-eliciting activities* is to study the nature of relevant constructs and processes. Therefore, the research methodologies that we use leave a great deal of latitude about the nature of what can be observed.

Model-eliciting activities can be thought of a *local concept development activities*. The sequences of modeling cycles that problem solvers go through during a single model-eliciting activity often appear to be local or situated versions of the stages of concept development that Piaget-inspired mathematics educators have observed over periods of several months or years (Lesh & Carmona, 2003; Lesh & Harel, 2003). Consequently, it is possible to use developmental theories to help describe the kind of processes that contribute to problem solving effectiveness during single *model-eliciting activities*.

Note: The preceding developmental sequences sometimes are referred to as *learning trajectories*. But, even though is to possible to think of a thinking as being at some specific stage of development at some given moment in a specific problem solving session, it is misleading to think of students themselves as being at a given stage of thinking across all tasks which the researcher might consider to be characterized by the same basic structure. This is because: (i) during a given *model-eliciting activity,* a problem solver may evolve from *stage #N thinking* to *stage #N+n thinking* during a single problem solving episode, and (ii) if the problem solver later encounters another *model-eliciting activity* which the researcher considers to be characterized by the same structure, then it is highly likely that thinking on the second task might begin at some intermediate stage between (or before or after) #N and #N+n.

Research based on *models & modeling perspectives* generally supports Piaget's emphasis on the holistic/systemic nature of mathematics construct. That is, the most important characteristic that distinguishes mathematics concepts from other kinds of concepts is that their meanings depend on reasoning that is based on conceptual systems-as-a-whole. Nonetheless, MMP research does not support Piaget's notion that thinking is organized primarily around abstract structures. Instead, our results are more consistent with modern theories of situated cognition (Lave & Wenger, 1991) where knowledge and abilities generally are organized around experience at least as much as around abstractions. However, unlike most current theories of situated cognition, where learning is portrayed as being exceedingly task specific (with minimum transfer being expected to occur among tasks which researchers might consider to be characterized by the same underlying abstract structure), *models & modeling perspectives* also emphasize the fact that models generally are designed to be sharable and re-useable. So, they represent transportable forms of knowledge — even though they also represent situated forms of knowledge (which are strongly shaped by the situations in which they were developed).

Model development seldom occurs along a single developmental path. In spite of the preceding observations about similarities between the sequences of modeling cycles that occur in *model-eliciting activities* and the developmental trajectories that have been observed by Piaget and others, model development seldom occurs along a single developmental path when students learn by expressing→testing→revising their own ways of thinking. In fact, if thinking is not guided along artificially narrow conceptual paths toward teachers' or textbooks' ways of thinking, then: (i) models tend to develop by sorting out, integrating, and reorganizing concepts which are associated with a variety of textbook topic areas, (ii) models tend to be expressed using a variety of interacting representational media — each of which emphasize somewhat different characteristics of the systems being modeled or the conceptual system being used to describe them, and (iii) development usually involve several interacting dimensions such as concrete-abstract, external-internal, simple-complex, tacit-explicit, situated-decontextualized, or global-analytic — where the model that is most useful is not necessarily the one that is most abstract, or most complex, or most de-contextualized. Consequently, model development tends to resemble a genetic inheritance tree where grandchildren inherit characteristics from all of their grandparents and ancestors — not just from those representing a single line of descent.

Very different kinds of understanding and abilities tend to develop depending on whether learning activities focus on *mathematizing reality* or *realizing mathematics*. Recently, Lesh, Yoon & Zawojewski (in press) compared two forms of instruction which both professed to emphasize "real life" problem solving experiences. In the treatment that focused on *realizing mathematics*, the goal was to teach first and then apply what was taught. Whereas, in the treatment that focused on *mathematizing reality*, the goal was for students to express, test, and revise their own ways of thinking — before teachers helped them "clean up", decontextualize, and in other ways empower their results. The basic difference between these two approaches is similar to what John Dewey described as *making science practical* and *making practice scientific*. Different kinds of understanding and abilities develop depending on which of these two approaches is emphasized. For example, when students develop models by expressing→testing→revising their own ways of thinking, then (i) the concepts and conceptual systems that evolve tend to be multi-topic chunks of knowledge, and (ii) they tend to be expressed using a variety of interacting media — many of which tend to be purged from textbook treatises which try to collapse everything into a small number of textbook languages, symbols and diagrams.

Note: According to *teach-first-then-apply perspectives*, learning to solve "real life" problems is assumed to be more difficult than solving their decontextualized counterparts in text-books and tests — because realistically "messy" problems require students to know context-specific information in addition to knowledge about relevant concepts and processes. Whereas, according to *develop-first-then-harvest perspectives*, learning to solve meaningful (naturally occurring) "real life" problems is assumed to be easier than solving their decontextualized counterparts in textbooks or tests — because the latter requires students to make meaning out of symbolically described situations before sensible steps can be taken to generate solutions.

Models & modeling perspectives (and *model-eliciting activities*) naturally emphasize *sociocultural* influences on learning — and the nature of what is learned. First, because *model-eliciting activities* are intended to be simulations of "real life" problem solving situations, problem solvers often are not isolated individuals, but instead are groups or learning organizations. Second, because the preceding problem solvers often have tools and other cultural capital to use as "cognitive amplifiers", their capabilities tend to be strongly influenced by these resources. Third, the nature of models and tools that are developed are shaped not only by the structure of the problem solving context, but also are by the continually evolving goals and purposes that problem solving communities impose on the situations. Fourth, because problem solvers typically have access to tools such as Google and other technologies for that radically enhance problem solvers' abilities for construction, communication, and conceptualization, it is important for theories of learning and problem solving to advance beyond the notion most cognitive functions are carried out within the minds of isolated individuals. It is important to recognize that knowledge and abilities often are distributed across specialists, across tools, and across representational media. Both individual humans and learning communities continually off-load information and capabilities that once functioned in the minds of isolated individuals.

Whereas, sociocultural theories emphasize the mind in society, *models & modeling perspectives* also emphasize societies in the mind. That is, we emphasize societies of conceptual systems in the mind. For example, in cases where the "problem solver" is not simply an isolated individual, the burden of proof clearly lies on anybody who would claim that model development consists of gradually refining and extending a single monolithic conceptual system. In *model-eliciting activities*, regardless whether we look at the thinking of isolated individuals or learning communities, it is reasonable to assume that early interpretations usually involve a collections of unstable, poorly differentiated, and poorly integrated ways of thinking — each of which tend to be at intermediate stages of development. Consequently, the most important "things" that problem solvers transform during model-eliciting activities are their own ways of thinking; and, it is only secondarily that they transform data within any given interpretation. (Note: During a typical 60–90 minute *model-eliciting activity*, it is rare for more than 15 minutes to focus on data processing. The remaining time focuses on activities that emphasize interpretation processing.)

When problem solvers interpret realistically complex problem solving situations, the conceptual systems that they engage are not purely logical or mathematical in nature. They also involve beliefs, attitudes, dispositions, values, problem solving processes, and other attributes of a *mathematics learning and problem solving personae*. Consequently, which attributes are engaged in a given situations depends on how the situation is interpreted; and, rather than thinking of beliefs, attitudes, values, dispositions, or processes as being explicitly learned prescriptive rules or declarative statements which are learned separately and then used for specific purposes, it is useful to think of them as being parts of the models that problem solvers develop for interpreting situations. That is, it is useful to think of them as parts of a productive problem solving personae; and, it is useful to study them developmentally — by investigating the nature of early understandings and how they develop.

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