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HETEROSKEDASTICITY-AUTOCORRELATION ROBUST STANDARD ERRORS USING THE BARTLETT KERNEL WITHOUT TRUNCATION

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In this note we show that the heteroskedasticity-autocorrelation (HAC) robust tests recently proposed by Kiefer, Vogelsang, and Bunzel (2000) are exactly equivalent to using Bartlett kernel HAC standard errors without truncation. This result suggests that valid tests (asymptotically pivotal) can be constructed using kernel based estimators with bandwidth equal to sample size. For clarity, we focus on the simple linear regression model $y_t = x_t'\beta + u_t$, $t = 1, 2, \ldots, T$, where β and x_t are $k \times 1$ vectors, u_t is autocorrelated and possibly conditionally heteroskedastic, and $E(u_t|x_t) = 0$. This last condition rules out lagged dependent variables but can be dropped by doing the analysis in the context of instrumental variable estimation. See Vogelsang (2000).

The focus is testing linear hypotheses about β. We consider the ordinary least squares (OLS) estimator, $\hat{\beta} = (\sum_{t=1}^{T} x_t x_t')^{-1} \sum_{t=1}^{T} x_t y_t$. Define $v_t = x_t u_t$. Using standard calculations we can write the normalized estimator as

$$\sqrt{T}(\hat{\beta} - \beta) = \left(T^{-1} \sum_{t=1}^{T} x_t x_t' \right)^{-1} T^{-1/2} \sum_{t=1}^{T} v_t.$$

Define $\Omega = \Lambda\Lambda' = \Gamma_0 + \sum_{j=1}^{\infty} (\Gamma_j + \Gamma_j')$ where $\Gamma_j = E(v_t v_{t-j}')$. Note that Ω is the asymptotic variance of $T^{-1/2} \sum_{t=1}^{T} v_t$. Define $\widehat{S}_t = \sum_{j=1}^{t} \hat{v}_j$ where $\hat{v}_t = x_t \hat{u}_t$, $\hat{u}_t = y_t - x_t'\hat{\beta}$.

Kiefer, Vogelsang, and Bunzel (2000) proposed using $\widehat{C} = T^{-2} \sum_{t=1}^{T} \widehat{S}_t \widehat{S}_t'$ in place of the usual consistent estimator of Ω when constructing standard errors. We now show that using $2\widehat{C}$ is exactly equivalent to using

$$\widehat{\Omega} = T^{-1} \sum_{i=1}^{T} \sum_{j=1}^{T} \hat{v}_i \left(1 - \frac{|i-j|}{T} \right) \hat{v}_j'$$

which is the Bartlett kernel estimator (see Newey and West (1987)) of Ω using bandwidth equal to the sample size. Making use of the identity

$$\sum_{j=1}^{T} a_j b_j = \sum_{j=1}^{T-1} \left((a_j - a_{j+1}) \sum_{i=1}^{j} b_i \right) + a_T \sum_{i=1}^{T} b_i$$

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and the fact that $\widehat{S}_T = \mathbf{0}$ by the normal equations for OLS, direct algebra gives

$$\begin{aligned}
(1)\qquad \widehat{\Omega} &= T^{-1}\sum_{i=1}^{T}\sum_{j=1}^{T}\hat{v}_i\left(1-\frac{|i-j|}{T}\right)\hat{v}_j' = T^{-1}\sum_{i=1}^{T}\hat{v}_i\sum_{j=1}^{T}\left(1-\frac{|i-j|}{T}\right)\hat{v}_j' \\
&= T^{-1}\sum_{i=1}^{T}\hat{v}_i\sum_{j=1}^{T}\frac{(-1)^{-(i-j)}}{T}\widehat{S}_j' = T^{-1}\sum_{j=1}^{T}\left(\sum_{i=1}^{T}\hat{v}_i\frac{(-1)^{-(i-j)}}{T}\right)\widehat{S}_j' \\
&= T^{-1}\sum_{j=1}^{T}\left(\sum_{i=1}^{T}\widehat{S}_i\mathbf{1}(i=j)\frac{2}{T}\right)\widehat{S}_j' = 2T^{-2}\sum_{j=1}^{T}\widehat{S}_j\widehat{S}_j' = 2\widehat{C}.
\end{aligned}$$

Consider testing the null hypothesis $H_0 : R\beta = r$ against the alternative hypothesis $H_1 : R\beta \neq r$ where R is a $q\times k$ matrix of constants with rank q and r is a $q\times 1$ vector of constants. Consider the F-type statistic

$$F^* = T(R\hat{\beta}-r)'[R\widehat{Q}^{-1}\widehat{\Omega}\widehat{Q}^{-1}R']^{-1}(R\hat{\beta}-r)/q$$

where $\widehat{Q} = T^{-1}\sum_{t=1}^{T}x_t x_t'$. In the case where $q=1$ one can construct a t-type statistic

$$t^* = \sqrt{T}(R\hat{\beta}-r)\Big/\sqrt{R\widehat{Q}^{-1}\widehat{\Omega}\widehat{Q}^{-1}R'}.$$

Under the regularity conditions assumed by Kiefer, Vogelsang, and Bunzel (2000), it directly follows from (1) that as $T\to\infty$,

$$F^* \Rightarrow W_q(1)'\left[2\int_0^1 B_q(r)B_q(r)'\,dr\right]^{-1}W_q(1)/q,$$

$$t^* \Rightarrow W_1(1)\Big/\sqrt{2\int_0^1 B_1(r)^2\,dr},$$

where $\Rightarrow$ denotes weak convergence, $B_q(r) = W_q(r) - rW_q(1)$, and $W_q(r)$ is a $q\times 1$ vector of independent standard Brownian motions. Critical values for the asymptotic distribution of t^* have been obtained analytically by Abadir and Paruolo (2002) and are tabulated in Table I for convenience. Asymptotic critical values for F^* for $q = 1, 2, 3, \ldots, 30$ can be obtained by multiplying by 0.5 the critical values tabulated by Kiefer, Vogelsang, and Bunzel (2000) in their Table II.

Given that the Bartlett kernel without truncation delivers valid HAC robust tests, it is logical to ask how tests using other kernels without truncation behave and perform. Kiefer and Vogelsang (2002) analyze the case of a general kernel, $k(x)$, with continuous second derivative, $k''(x)$, and show that,

$$\widehat{\Omega} = \widehat{\Gamma}_0 + \sum_{j=1}^{T-1}k(j/T)(\widehat{\Gamma}_j + \widehat{\Gamma}_j') \Rightarrow -\Lambda\int_0^1\int_0^1 k''(r-s)B_k(r)B_k(s)'\,dr\,ds\Lambda'.$$

TABLE I

ASYMPTOTIC CRITICAL VALUES OF t^*

1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95%	97.5%	99.0%
−6.090	−4.771	−3.764	−2.740	0.000	2.740	3.764	4.771	6.090

Source: Line 1 of Table I from Abadir and Paruolo (1997, p. 677) scaled by $1/\sqrt{2}$.

Because asymptotic proportionality to Ω is obtained for general kernels, valid asymptotic testing is possible using the class of kernel HAC estimators with bandwidth equal to sample size. A preliminary analysis on this class of tests is given in Kiefer and Vogelsang (2002).

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