MARGIN CALL STOCK LOANS

ERIK EKSTRÖM AND HENRIK WANNTORP

Abstract. We study margin call stock loans, i.e. loans in which a stock acts as collateral, and the borrower is obliged to pay back parts of the loan in case the value of the stock falls below the loan amount. We show that such a stock loan can be viewed as a down-and-out American barrier option with a rebate and negative interest rate, and we give explicit formulas for the value function and the optimal exercise time. Moreover, we provide a sensitivity analysis of the loan value with respect to the model parameters.

1. Introduction

Recently, Xia and Zhou studied perpetual non-recourse stock loans, see [8]. These are contracts in which the holder borrows money with a stock as collateral. The holder may at any instant choose to pay back the loan and when doing so, he retrieves the stock and the contract is terminated. The incentive for paying back would typically be a rise in the stock price. On the other hand, the holder of a non-recourse loan has no obligation to pay back the loan should the value of the collateral decrease. In [8] it is shown that the valuation of a non-recourse stock loan reduces to the valuation of an American call option with a negative interest rate, and the optimal exercise strategy is described. For related work, see also [7] in which an exchange option is studied in an economy with negative interest rate.

In reality, many stock loans are equipped with the additional feature that the lender may issue a margin call in case the stock value drops below the loan amount. The margin call forces the borrower to pay back a predetermined fraction of the loan and thus protects the lender from a large drop in value, or even default, of the collateral. In the current article we show that the valuation of a margin call stock loan reduces to the valuation of a certain barrier option with rebate and a possibly negative interest rate, and we describe the optimal repayment strategy. The analysis of this barrier option is also of independent interest.

The article is organised as follows. In Section 2 we review the results for non-recourse stock loans studied in [8], and we introduce margin call stock loans in detail. We show that the valuation of a margin call stock loan can be reduced to the valuation of a related down-and-out barrier option with rebate. This barrier option is studied in Section 3, in which explicit pricing formulas and optimal exercise times are given. Finally, in Section 4 we provide a sensitivity analysis with respect to the model parameters.

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2. Stock Loans

In this section we first review the valuation formulas for the non-recourse stock loan studied in [8], and we then introduce the margin call stock loan. A non-recourse stock loan is described as follows. At time 0, the borrower receives the loan amount $q > 0$ using one share of the stock as collateral. This amount grows at the rate γ, where γ is a constant loan interest rate stipulated in the contract, and the cost of repaying the loan at time t is thus given by $qe^{\gamma t}$. When paying back the loan, the borrower regains the stock and the contract is terminated. In the case of a non-recourse loan, the borrower has no obligation to ever pay back any part of the loan. However, if the contract is never terminated, then the stock will remain in the possession of the lender.

To model the stock loan, we assume that the market consists of a money market with deterministic price satisfying

$$dB_t = rB_t\,dt,$$

and a stock with price process S given under the pricing measure by

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t, \tag{1}$$

where the interest rate $r \geq 0$ and the volatility $\sigma > 0$ are constants and W is a standard Brownian motion. Note that we specify the stock dynamics directly under the pricing measure, so there is no need to change the measure when pricing options on S. Thus, according to standard arbitrage theory (see for example [2]), the value of a non-recourse stock loan is

$$V^0(x,q) = \sup_\tau E_x\left[e^{-r\tau}(S_\tau - qe^{\gamma\tau})^+\right], \tag{2}$$

where the index indicates that $S_0 = x$ and the supremum is taken over stopping times. Here and throughout the article we use the convention that $f(S_\tau) = 0$ on the event $\{\tau = \infty\}$ for any function f. It is clear from (2) that the value of a non-recourse stock loan coincides with the value of an American call option with time-dependent strike price $qe^{\gamma t}$. If we introduce the process $X_t = e^{-\gamma t}S_t$, i.e. the stock price discounted at the loan interest rate γ, then the value function can be written

$$V^0(x,q) = \sup_\tau E_x\left[e^{-(r-\gamma)\tau}(X_\tau - q)^+\right],$$

where

$$dX_t = (r-\gamma)X_t\,dt + \sigma X_t\,dW_t \tag{3}$$

and $X_0 = x$. The value can thus alternatively be described as the value of a perpetual American call option with strike price q and interest rate $r - \gamma$. It is well-known that if the interest rate is non-negative, then a call option should not be exercised early (see for example [3]), and the value of the perpetual option coincides with the value of the underlying. The main novelty of the following result in [8] is thus the formulas for $\gamma > r$ (i.e. for $\alpha > 0$ in the notation below).

Theorem 2.1. *Let* $\alpha = -\frac{2(r-\gamma)}{\sigma^2}$. *Then the value function* V^0 *satisfies*

$$V^0(x,q) = \begin{cases} x & \text{if } \alpha \le 1 \\ \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} q^{1-\alpha} x^\alpha & \text{if } \alpha > 1 \text{ and } x < \frac{\alpha q}{\alpha-1} \\ x - q & \text{if } \alpha > 1 \text{ and } x \ge \frac{\alpha q}{\alpha-1}. \end{cases} \quad (4)$$

Moreover, if $\alpha > 1$, *then the stopping time*

$$\tau^* = \inf\{t \ge 0 : S_t \ge \frac{\alpha q e^{\gamma t}}{\alpha - 1}\}$$

is optimal in the sense that

$$V^0(x,q) = E_x\left[e^{-r\tau^*}(S_{\tau^*} - qe^{\gamma\tau^*})^+\right].$$

If $\alpha \le 1$, *then there is no optimal stopping time in* (2).

Our main emphasis in the current article is on stock loans which allow for margin calls. These are described as follows. At time 0, the borrower receives the loan amount $q > 0$ using one share of the stock as collateral. At any later time t, he may regain the stock by paying $qe^{\gamma t}$, that is, the principal together with interest compounded at the loan interest rate. When doing so, the borrower retrieves the stock and the contract is terminated. However, if the stock price falls below the accrued loan amount at any time before the contract is terminated, then the borrower is forced to pay back a fraction $\delta \in (0, 1]$ of the loan. We assume that the bank is entitled to only one margin call (this could of course easily be generalised). Consequently, after the first instant that the stock price is below the accrued loan amount, the contract reduces to a non-recourse stock loan as described above.

Let the stock price S be modeled as in (1), and let τ_q be the first instant that the stock price falls below the accrued loan amount, i.e.

$$\tau_q = \inf\{t \ge 0 : S_t \le qe^{\gamma t}\}.$$

The value of a stock loan on the amount q and with a margin call fraction δ is

$$\begin{aligned} V(x) &= \sup_{\tau \le \tau_q} E_x\left[e^{-r\tau}(S_\tau - qe^{\gamma\tau})1_{\{\tau<\tau_q\}}\right. \\ &\qquad \left. + e^{-r\tau_q}(V^0(S_{\tau_q}, (1-\delta)qe^{\gamma\tau_q}) - \delta qe^{\gamma\tau_q})1_{\{\tau=\tau_q\}}\right]. \end{aligned}$$

Here the first term in the expected value corresponds to the borrower repaying the loan and thereby regaining the stock without the margin call being issued. The second term corresponds to the case when the lender issues a margin call and thus forces the borrower to pay back a fraction δ of the loan. Note that after paying the fraction δ, the borrower is left with a non-recourse stock loan with loan amount $(1-\delta)qe^{\gamma\tau_q}$ and value $V^0(S_{\tau_q}, (1-\delta)qe^{\gamma\tau_q})$. Using the equality $S_{\tau_q} = qe^{\gamma\tau_q}$ and the formula (4) we find that

$$V^0(S_{\tau_q}, (1-\delta)qe^{\gamma\tau_q}) = \begin{cases} qe^{\gamma\tau_q} & \text{if } \alpha \le 1 \\ \left(\frac{\alpha-1}{1-\delta}\right)^{\alpha-1} \frac{qe^{\gamma\tau_q}}{\alpha^\alpha} & \text{if } 1 < \alpha \le 1/\delta \\ \delta qe^{\gamma\tau_q} & \text{if } \alpha > 1/\delta. \end{cases}$$

Consequently, the value function V can be written in terms of the process X in (3) as

$$V(x) = \sup_{\tau \leq \tau_q} E_x \left[e^{-(r-\gamma)\tau}(X_\tau - q) 1_{\{\tau < \tau_q\}} + e^{-(r-\gamma)\tau_q} \varepsilon 1_{\{\tau = \tau_q\}} \right], \tag{5}$$

where ε is defined as

$$\varepsilon = \begin{cases} q(1-\delta) & \text{if } \alpha \leq 1 \\ \left(\frac{\alpha-1}{1-\delta}\right)^{\alpha-1} \frac{q}{\alpha^\alpha} - \delta q & \text{if } 1 < \alpha \leq 1/\delta \\ 0 & \text{if } \alpha > 1/\delta. \end{cases} \tag{6}$$

Note that the hitting time τ_q is the hitting time of a constant barrier for the process X. More precisely,

$$\tau_q = \inf\{t \geq 0 : X_t \leq q\}. \tag{7}$$

From (5) and (7) it is clear that the value of the margin call stock loan reduces to the value of an American down-and-out barrier call option with strike price q, rebate ε at the barrier q, and with interest rate $r - \gamma$. Let $U(x, \mu, \varepsilon)$ denote the value function of such a barrier option in an economy with interest rate μ (see (8) below for a formal definition). With this notation, we thus have the following pricing formula for the margin call stock loan.

Theorem 2.2. *The value function V of a margin call stock loan satisfies*

$$V(x) = U(x, r - \gamma, \varepsilon),$$

where ε is defined in (6).

Remark If the loan interest rate γ is smaller than the interest rate r, then the loan is favourable to the holder and one might therefore anticipate that the holder should never exercise early. This intuition is indeed confirmed in Theorem 3.1 below. The economically more interesting case is perhaps when γ is larger than r. In this case, the stock loan is exercised optimally the first instant that the stock price discounted at the loan interest rate γ exceeds a level b, compare Theorems 3.2 and 3.3 below.

3. A Barrier Call Option with Rebate

In this section we study the American down-and-out barrier call option that was introduced above. We derive an explicit formula for its value function U, and we describe the optimal exercise time in cases when it exists. The analysis of this barrier option seems to be new in the literature, and may be of independent interest. For a study of a related problem, see [5].

To formulate the problem, let X solve

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t,$$

where $\mu \in \mathbb{R}$ is a constant and W is a Brownian motion. For $q > 0$, let

$$\tau_q = \inf\{t \geq 0 : X_t \leq q\}$$

be the first passage time of X over the level q. For $\varepsilon \in [0, q)$, consider the optimal stopping problem

$$U(x, \mu, \varepsilon) = \sup_{\tau \le \tau_q} E_x \left[e^{-\mu\tau} h(X_\tau) \right], \tag{8}$$

where

$$h(x) = \begin{cases} x - q & \text{for } x > q \\ \varepsilon & \text{for } x = q. \end{cases}$$

Depending on the sign of the parameter μ, the value function $U(x, \mu, \varepsilon)$ exhibits different qualitative behaviours, compare Figure 1 below. The different cases are treated in Theorem 3.1 and Theorems 3.2-3.3, respectively.

Theorem 3.1. *Assume that $\mu \ge 0$. Then the value of the barrier option is given by*

$$U(x, \mu, \varepsilon) = x - (q - \varepsilon) \left(\frac{q}{x} \right)^{2\mu/\sigma^2}$$

for $x \ge q$, and there is no optimal exercise time in (8).

Proof. Let

$$F(x) = x - (q - \varepsilon) \left(\frac{q}{x} \right)^{2\mu/\sigma^2}$$

denote the candidate value function. Then F'' exists and is continuous for all $x \ge q$, and straightforward calculations show that

$$\tfrac{1}{2}\sigma^2 x^2 F''(x) + \mu x F'(x) - \mu F(x) = 0. \tag{9}$$

Define the process $Y_t = e^{-\mu(t \wedge \tau_q)} F(X_{t \wedge \tau_q})$. The Itô formula and (9) give that

$$\begin{aligned} dY_t &= e^{-\mu t} \left(\tfrac{1}{2}\sigma^2 X_t^2 F''(X_t) + \mu X_t F'(X_t) - \mu F(X_t) \right) 1_{\{t \le \tau_q\}} \, dt \\ &\quad + e^{-\mu t} \sigma X_t F'(X_t) 1_{\{t \le \tau_q\}} \, dW_t \\ &= e^{-\mu t} \sigma X_t F'(X_t) 1_{\{t \le \tau_q\}} \, dW_t. \end{aligned}$$

Using that $|F'(x)| \le C$ for some constant C and all $x \ge q$ we find that Y_t is a martingale for $0 \le t < \infty$. Recalling our convention that $F(X_\infty) = 0$, an application of the Optional Sampling Theorem (see for example Problem 1.3.16 and Theorem 1.3.22 in [4]) shows that

$$F(x) = Y_0 \ge E_x [Y_\tau] = E_x \left[e^{-\mu\tau} F(X_\tau) \right]$$

for any stopping time $\tau \le \tau_q$. Since $F \ge h$, we find that

$$F(x) \ge \sup_{\tau \le \tau_q} E_x \left[e^{-\mu\tau} h(X_\tau) \right] = U(x, \mu, \varepsilon).$$

To derive the reverse inequality we consider stopping times of the form $\tau = t \wedge \tau_q$ for deterministic times t. For any t we have

$$\begin{aligned} U(x,\mu,\varepsilon) &\geq E_x\left[e^{-\mu(t\wedge\tau_q)}(X_{t\wedge\tau_q}-q)1_{\{t<\tau_q\}}+\varepsilon e^{-\mu\tau_q}1_{\{t\geq\tau_q\}}\right] \\ &= E_x\left[e^{-\mu(t\wedge\tau_q)}X_{t\wedge\tau_q}1_{\{t<\tau_q\}}\right]-qE_x\left[e^{-\mu(t\wedge\tau_q)}1_{\{t<\tau_q\}}\right] \\ &\quad + E_x\left[\varepsilon e^{-\mu\tau_q}1_{\{t\geq\tau_q\}}\right] \\ &= E_x\left[e^{-\mu(t\wedge\tau_q)}X_{t\wedge\tau_q}\right]-qE_x\left[e^{-\mu t}1_{\{t<\tau_q\}}\right] \\ &\quad + E_x\left[(\varepsilon-q)e^{-\mu\tau_q}1_{\{t\geq\tau_q\}}\right] \\ &= x-qe^{-\mu t}P_x(\tau_q>t)-(q-\varepsilon)E_x\left[e^{-\mu\tau_q}1_{\{t\geq\tau_q\}}\right], \end{aligned}$$

where we in the last equality used the martingale property of the process $e^{-\mu t}X_t$. Letting $t \to \infty$ and using the expression for the Laplace transform of τ_q, see for example page 197 in [4], we obtain

$$U(x,\mu,\varepsilon) \geq x-(q-\varepsilon)E_x\left[e^{-\mu\tau_q}\right] = x-(q-\varepsilon)(\frac{q}{x})^{2\mu/\sigma^2} = F(x).$$

Finally, for any stopping time $\tau \leq \tau_q$ we have

$$\begin{aligned} E_x\left[e^{-\mu\tau}h(X_\tau)\right] &= E_x\left[e^{-\mu\tau}(X_\tau-q)1_{\{\tau<\tau_q\}}+\varepsilon e^{-\mu\tau_q}1_{\{\tau=\tau_q\}}\right] \\ &\leq E_x\left[e^{-\mu\tau}X_\tau 1_{\{\tau<\tau_q\}}+\varepsilon e^{-\mu\tau_q}1_{\{\tau=\tau_q\}}\right] \\ &= E_x\left[e^{-\mu\tau}X_\tau\right]-E_x\left[e^{-\mu\tau}X_\tau 1_{\{\tau=\tau_q\}}\right]+\varepsilon E_x\left[e^{-\mu\tau_q}1_{\{\tau=\tau_q\}}\right] \\ &\leq x-(q-\varepsilon)(\frac{q}{x})^{2\mu/\sigma^2} = U(x). \end{aligned}$$

If $P(\tau < \tau_q) > 0$, then the first inequality is strict, and if $\tau = \tau_q$, then the second inequality is strict. Consequently, there is no optimal stopping time in (8). □

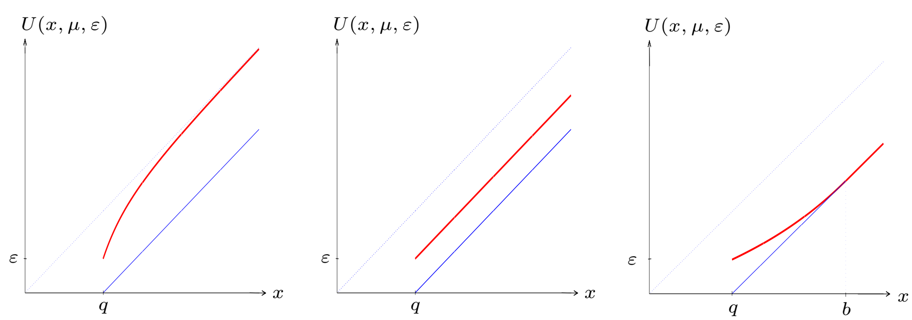

FIGURE 1. The three graphs show the value function $U(x,\mu,\varepsilon)$ in the cases $\mu > 0$, $\mu = 0$ and $\mu < 0$, respectively.

Theorem 3.2. *Assume that $\mu < 0$ and that $2\mu+\sigma^2 \neq 0$. Let b be the unique solution to the equation*

$$(1-\varepsilon/q)\frac{b}{q}-\frac{2\mu}{2\mu+\sigma^2}-\frac{\sigma^2}{2\mu+\sigma^2}\left(\frac{b}{q}\right)^{1+2\mu/\sigma^2} = 0 \tag{10}$$

satisfying $b \geq q$. *Then*

$$U(x,\mu,\varepsilon)=\begin{cases}(1-\frac{2\mu q}{b(2\mu+\sigma^2)})x-\frac{q\sigma^2}{2\mu+\sigma^2}\left(\frac{b}{x}\right)^{2\mu/\sigma^2} & \textit{if } q\leq x<b\\ x-q & \textit{if } x\geq b.\end{cases}$$

Moreover, the stopping time

$$\tau_{qb}=\inf\{t\geq 0: X_t\leq q \textit{ or } X_t\geq b\}$$

is optimal in the sense that

$$U(x,\mu,\varepsilon)=E_x\left[e^{-\mu\tau_{qb}}(X_{\tau_{qb}}-q)1_{\{\tau_{qb}<\tau_q\}}+\varepsilon e^{-\mu\tau_q}1_{\{\tau_{qb}=\tau_q\}}\right].$$

Remark It is straightforward to check that the value function and the optimal stopping boundary in Theorem 3.2 together constitute the unique solution (F,c) to the free boundary problem

$$\begin{cases}\frac{1}{2}\sigma^2x^2F''(x)+\mu xF'(x)-\mu F(x)=0 & \text{for } x\in(q,c)\\ F(x)=x-q & \text{for } x\geq c\\ F'(x)=1 & \text{for } x=c\\ F(q)=\varepsilon. & \end{cases}\tag{11}$$

The proof below verifies that the function F in (11) indeed coincides with the value function U. One can also check that if $\mu\geq 0$, then the free boundary problem (11) has no solution. This corresponds to the setting of Theorem 3.1 in which there is no optimal stopping time. For a general discussion on the connection between optimal stopping problems and free boundary problems we refer to the recent monograph [6].

Proof. First note that the function

$$f(y)=(1-\varepsilon/q)y-\frac{2\mu}{2\mu+\sigma^2}-\frac{\sigma^2}{2\mu+\sigma^2}y^{1+2\mu/\sigma^2}\tag{12}$$

grows linearly for large y, is convex, and satisfies $f(1)=-\varepsilon/q$ (and if $\varepsilon=0$, then $f'(1)=0$). This proves the existence of a unique solution $b\geq q$ to equation (10). Next, let

$$F(x)=\begin{cases}(1-\frac{2\mu q}{b(2\mu+\sigma^2)})x-\frac{q\sigma^2}{2\mu+\sigma^2}\left(\frac{b}{x}\right)^{2\mu/\sigma^2} & \text{if } q\leq x<b\\ x-q & \text{if } x\geq b\end{cases}$$

denote the candidate value function. Then F is continuously differentiable on $[q,\infty)$, and twice continuously differentiable on $[q,\infty)\setminus\{b\}$. Moreover, F satisfies

$$\tfrac{1}{2}\sigma^2x^2F''(x)+\mu xF'(x)-\mu F(x)=0$$

for $q\leq x<b$, and

$$\tfrac{1}{2}\sigma^2x^2F''(x)+\mu xF'(x)-\mu F(x)=\mu q$$

for $x\geq b$. Let $Y_t=e^{-\mu(t\wedge\tau_q)}F(X_{t\wedge\tau_q})$. Since F is convex, an extension of the Itô formula (see for example Theorem 3.6.22 in [4]) yields

$$dY_t=e^{-\mu t}\mu q1_{\{X_{t\wedge\tau_q}\geq b\}}\,dt+e^{-\mu t}\sigma X_tF'(X_t)1_{\{t\leq\tau_q\}}\,dW,$$

so Y is a supermartingale. An application of the Optional Sampling Theorem gives

$$F(x)=Y_0\geq E_xY_\tau=E_x\left[e^{-\mu\tau}F(X_\tau)\right],$$

for any stopping time $\tau \leq \tau_q$, and using that

$$F(x) \geq h(x) = \begin{cases} x - q & \text{for } x > q \\ \varepsilon & \text{for } x = q \end{cases}$$

for all x, we obtain

$$F(x) \geq E_x\left[e^{-\mu\tau} h(X_\tau)\right].$$

Since this holds for any stopping time τ we conclude that $F(x) \geq U(x)$.

To prove the reverse inequality we note that $Y_{t\wedge\tau_{qb}}$ is in fact a martingale, so

$$F(x) = E_x\left[e^{-\mu\tau_{qb}} F(X_{\tau_{qb}})\right] = E_x\left[e^{-\mu\tau_{qb}} h(X_{\tau_{qb}})\right]$$

since $F(q) = h(q)$ and $F(b) = h(b)$. Consequently,

$$F(x) \leq \sup_{\tau \leq \tau_q} E_x\left[e^{-\mu\tau} h(X_\tau)\right] = U(x),$$

and τ_{qb} is optimal in (8). □

Remark If $\mu < 0$ and $\varepsilon = 0$, then $b = q$. Consequently, immediate exercise is optimal and the problem degenerates. In view of (6), a margin call stock loan is thus optimally exercised immediately if $\alpha \geq 1/\delta$.

Remark One may note that if $\varepsilon > 0$, then

$$f\left(\frac{q}{q-\varepsilon}\right) = \frac{1 - (q/(q-\varepsilon))^{1+2\mu/\sigma^2}}{1-\alpha} < 0,$$

where f is as in (12). Since b/q is the unique zero of f satisfying $b/q \geq 1$ it follows that the optimal stopping boundary b satisfies $b > q/(1 - \varepsilon/q)$.

Our next result can be proved similarly to Theorem 3.2, and we omit the details.

Theorem 3.3. *Assume that $\mu < 0$ and that $2\mu + \sigma^2 = 0$. Let b be the unique solution to the equation*

$$(1 - \varepsilon/q)(b/q) - \ln(b/q) - 1 = 0$$

satisfying $b \geq q$. Then

$$U(x, \mu, \varepsilon) = \begin{cases} (1 - \frac{q(1+\ln b)}{b})x + \frac{q}{b} x \ln x & \text{if } q \leq x < b \\ x - q & \text{if } x \geq b. \end{cases}$$

Moreover, the stopping time

$$\tau_{qb} = \inf\{t \geq 0 : X_t \leq q \text{ or } X_t \geq b\}$$

is optimal in (8).

4. Robustness Properties

In this section we study robustness properties of the stock loan value. Note that the model parameters that need to be estimated from market data are the interest rate r and the volatility σ; the remaining parameters are specified in the contract. We have the following monotonicity result.

Theorem 4.1. *The value function $V = V(x, \alpha)$ of a margin call stock loan is decreasing with respect to the parameter $\alpha = -2(r - \gamma)/\sigma^2$.*

Proof. If $\alpha \le 0$, then it follows from Theorems 2.2 and 3.1 that

$$V(x,\alpha) = x - \delta q (x/q)^\alpha,$$

which is clearly decreasing in α for $x \ge q$. For $\alpha > 0$, it is straightforward to check that ε given by (6) is decreasing in α. Moreover, the function

$$f(y) = (1-\varepsilon/q)y - \frac{\alpha}{\alpha - 1} - \frac{1}{1-\alpha} y^{1-\alpha},$$

compare (10), satisfies

$$f(1) = -\varepsilon/q,$$

which is independent of α, and

$$f'(y) = (1-\varepsilon/q) - y^{-\alpha},$$

which is increasing in α for $y \ge 1$. Since b/q is the unique zero of f satisfying $b/q \ge 1$, it follows that the optimal stopping boundary b is decreasing as a function of α.

Let $\alpha_0 > 0$ be given and fixed, and let ε_0 be the rebate and b_0 be optimal stopping boundary corresponding to α_0. For $\alpha \in (0, \alpha_0]$, let $F(x,\alpha)$ be the solution of

$$\begin{cases} x^2 F_{xx} - \alpha x F_x + \alpha F = 0 & \text{if } x \in (q, b_0) \\ F(q) = \varepsilon_0 \\ F(b_0) = b_0 - q. \end{cases} \tag{13}$$

Since the above ODE is increasing in its boundary values and the stock loan value V satisfies

$$\begin{cases} x^2 V_{xx} - \alpha x V_x + \alpha V = 0 & \text{if } x \in (q, b_0) \\ V(q) = \varepsilon \ge \varepsilon_0 \\ V(b_0) \ge b_0 - q, \end{cases}$$

we have $F(x,\alpha) \le V(x,\alpha)$. For $\alpha \ne 1$, equation (13) has the unique solution

$$F(x,\alpha) = (\varepsilon_0 - D)x/q + D(x/q)^\alpha,$$

where

$$D = \frac{(q-\varepsilon_0)(b_0/q) - q}{(b_0/q)^\alpha - b_0/q}.$$

Note that it follows from the inequality $b \ge q/(1-\varepsilon/q)$ in the second remark following the proof of Theorem 3.2 that D has the same sign as $\alpha - 1$. A straightforward differentiation gives

$$F_\alpha(x,\alpha) = D\left((x/q)^\alpha \ln(x/q) - \frac{(x/q)^\alpha - x/q}{(b_0/q)^\alpha - b_0/q} (b_0/q)^\alpha \ln(b_0/q) \right).$$

Since the function

$$g(y) = \frac{y^\alpha \ln y}{y^\alpha - y}$$

is increasing in y for $y \ge 1$, it follows that $F_\alpha(x,\alpha) \le 0$ for $\alpha \ne 1$. Together with the fact that $F(x,\alpha)$ is continuous in α for all $\alpha > 0$, this finishes the proof. □

Corollary 4.2. *The value V of the margin call stock loan is decreasing in the volatility σ if $r > \gamma$ and increasing if $r < \gamma$. Moreover, it is increasing in the interest rate r, and decreasing in the loan interest rate γ.*

Remark The stock loan value is increasing in the volatility for parameter values that give a convex price ($\alpha > 0$), and decreasing for parameter values with a concave price ($\alpha < 0$). This is in accordance with general convexity theory for option pricing, see for example [1] or [3].

References

[1] Alvarez, L.H.R. (2003). On the properties of r-excessive mappings for a class of diffusions. *Ann. Appl. Probab.* **13**, 1517–1533.

[2] Björk, T. (2004). *Arbitrage Theory in Continuous Time.* Second edition. Oxford University Press.

[3] El Karoui, N., Jeanblanc-Picqué, M. and Shreve, S. (1998). Robustness of the Black and Scholes formula. *Math. Finance* **8,** no. 2, 93–126.

[4] Karatzas, I. and Shreve, S. (1991). *Brownian Motion and Stochastic Calculus.* Second edition. Graduate Texts in Mathematics, 113. Springer-Verlag, New York.

[5] Karatzas, I. and Wang, H. (2000). A barrier option of American type. *Appl. Math. Optim.* **42**, 259–279.

[6] Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-boundary Problems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel.

[7] Wong, S.K.T. (2008). The generalized perpetual American exchange-option problem. *Adv. Appl. Probab.* **40**, 163–182.

[8] Xia, J. and Zhou, X.Y. (2007). Stock loans. *Math. Finance* **17**, 307–317.

Department of Mathematics, Uppsala University, Box 480, SE-751 06 Uppsala, Sweden