Averaging and near viability of singularly perturbed control systems

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The author dedicates this paper to Jean-Pierre Aubin

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Abstract

In this paper we consider issues related to averaging of singularly perturbed control systems (**SPCS**) in the viability context. We introduce a notion of near viability of SPCS and relate it to the viability of a specially constructed averaged differential inclusion.

1 Introduction

In this paper we consider issues related to averaging of singularly perturbed control systems (**SPCS**) in the viability context. Despite of the fact that the averaging techniques for SPCS have been studied very intensively (see [1]-[5], [8], [9], [13], [18], [19], [22], [23], [25], [29] for most recent developments and also for references to earlier results in the area), this topic, to the best of the author's knowledge, has not been considered in the literature.

The paper is organized as follows. In Section 2 we introduce a notion of near viability of SPCS and relate it to the viability of a specially constructed averaged differential inclusion (Theorems 2.1 and 2.2). In Section 3 we establish a result that can be interpreted as a generalization of Tichonov's theorem (Proposition 3.1), and we use this result to justify a relaxation of control systems with mixed control-state constraints proposed by J.-P.Aubin (Proposition 3.2). In Section 4 we consider some readily verifiable conditions which ensure the validity of the Assumptions of Theorems 2.1 and 2.2. In Section 5 the proofs of Theorems 2.1 and 2.2 are given.

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2 Averaging over viable fast motions and controls

Let us consider a singularly perturbed control system defined by the equations

$$\epsilon \dot{y}(t) = f(u(t), y(t), z(t)) \ , \quad (2.1)$$

$$\dot{z}(t) = g(u(t), y(t), z(t)) \ , \quad (2.2)$$

where $\epsilon > 0$ is a small parameter; $f : U \times R^m \times R^n \to R^m$, $g : U \times R^m \times R^n \to R^n$ are continuous vector functions satisfying Lipschitz conditions in z and y; U is a compact metric space and the controls are measurable functions satisfying the inclusion $u(t) \in U$.

Let $Z \subset R^n$ be a compact set and let $Y(z) : N(Z) \to 2^{R^m}$ be a point-to-set compact valued map defined in some (sufficiently large) neighborhood $N(Z)$ of Z. Assume that $Y(z)$ is uniformly continuous, that is, there exists a monotone decreasing function $\kappa(\cdot) : [0, \infty) \to [0, \infty)$, $\lim_{\theta \to \infty} \kappa(\theta) = 0$, such that

$$d_H(Y(z'), Y(z'')) \leq \kappa(||z' - z''||) \quad \forall z', z'' \in N(Z) \ , \quad (2.3)$$

where, here and in what follows, $d_H(\cdot, \cdot)$ is the Hausdorff metric defined on the bounded subsets of a finite dimensional space by the Euclidean norm.

Define the set $D \subset R^m \times R^n$ by the equation

$$D \stackrel{\text{def}}{=} \{(y, z) : y \in Y(z), \ z \in Z\} \ . \quad (2.4)$$

DEFINITION. A solution $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ of the SPCS (2.1)-(2.2) is called *near viable in D on the interval* $[0, T]$ if

$$\max_{t \in [0,T]} dist((y_\epsilon^{sp}(t), z_\epsilon^{sp}(t)), D) \leq \nu_T(\epsilon) \quad (2.5)$$

for some $\nu_T(\epsilon)$ tending to zero as ϵ tends to 0; it is called *near viable in D* if

$$\sup_{t \in [0,\infty)} dist((y_\epsilon^{sp}(t), z_\epsilon^{sp}(t)), D) \leq \nu(\epsilon) \ , \quad (2.6)$$

for some $\nu(\epsilon)$ tending to zero as ϵ tends to zero, where $dist(\cdot, \cdot)$ stands for the distance between a point and a set.

For the sake of brevity (and at the expense of some abuse of terminology) we refer to $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ as to a solution of the SPCS (2.1)-(2.2) instead of referring to it as to a parameterized by ϵ family of solutions.

Along with (2.1)-(2.2), let us consider the *associated system*

$$\dot{y}(\tau) = f(u(\tau), y(\tau), z), \ z = \text{ const} \ . \quad (2.7)$$

in which the controls are measurable functions satisfying the inclusion $u(\tau) \in U$. Note that (2.7) can be formally obtained from the "fast subsystem" of the SPCS (2.1) by changing the time scale: $\tau = \frac{t}{\epsilon}$ and replacing $z(\cdot)$ by a vector of constant parameters z.

ASSUMPTION I:

(i) For any $z \in N(Z)$ and any $y \in Y(z)$ there exists a control such that the solution $y(\tau)$ of the associated system (2.7), obtained with this control and the initial conditions $y(0) = y$, does not leave $Y(z)$; that is, the viability kernel of $Y(z)$ is equal to $Y(z)$ (sufficient and necessary conditions for this to be satisfied can be found in [6],[7]).

(ii) For any $\delta > 0$ small enough, any $z \in N(Z)$ and any $y \in Y(z) + \delta B$ (B is a closed unit ball in R^m), there exists a control such that the solution $y(\tau)$ of the associated system (2.7) obtained with this control and the initial conditions $y(0) = y$ reaches $Y(z)$ at some moment $\bar{\tau} \in [0, a]$ ($a > 0$ is a given constant), that is $y(\bar{\tau}) \in Y(z)$, and

$$dist(y(\tau), Y(z)) \leq \phi(\delta) \quad \forall \tau \in [0, \bar{\tau}] \ , \tag{2.8}$$

with $\lim_{\delta \to 0} \phi(\delta) = 0$.

DEFINITION. A pair $(u(\tau), y(\tau))$ will be referred to as *admissible* or δ-*admissible* for the associated system (2.7) on the interval $[0, S]$ if $u(\tau)$ is a control, $y(\tau)$ is the corresponding solution of the equation (2.7) and $y(\tau) \in Y(z)$ or, respectively, $y(\tau) \in Y(z) + \delta B$, $\forall \tau \in [0, S]$.

Define $V(z, S, y)$, $V(z, S)$ as the sets of the time averages

$$V(z,S,y) \stackrel{\text{def}}{=} \bigcup_{(u(\cdot),\ y(\cdot))} \left\{ \frac{1}{S} \int_0^S g(u(\tau), y(\tau), z) d\tau \right\} , \qquad V(z,S) \stackrel{\text{def}}{=} \bigcup_{y \in Y(z)} \left\{ V(z,S,y) \right\} , \tag{2.9}$$

where the first union is over all admissible pairs of the associated system (2.7) which satisfy the initial conditions

$$y(0) = y \ , \tag{2.10}$$

and the second union is over the initial conditions from $Y(z)$. Define also $V^\delta(z, S, y)$, $V^\delta(z, S)$ as

$$V^\delta(z,S,y) \stackrel{\text{def}}{=} \bigcup_{(u^\delta(\cdot),\ y^\delta(\cdot))} \left\{ \frac{1}{S} \int_0^S g(u^\delta(\tau), y^\delta(\tau), z) d\tau \right\} , \quad V^\delta(z,S) \stackrel{\text{def}}{=} \bigcup_{y \in Y(z) + \delta B} \left\{ V^\delta(z,S,y) \right\} , \tag{2.11}$$

where, in contrast to (2.9), the first union is over all δ-admissible pairs satisfying the initial conditions (2.10) and the second is over the initial conditions from $Y(z) + \delta B$. Note that, in accordance with these definitions,

$$V(z,S,y) \subset V^\delta(z,S,y) \ , \qquad V(z,S) \subset V^\delta(z,S) \ . \tag{2.12}$$

ASSUMPTION II:

(i) The following estimate is valid

$$d_H(V^\delta(z,S), V(z,S)) \leq \nu_1(S,\delta) \quad \forall z \in N(Z) \tag{2.13}$$

for some $\nu_1(S, \delta)$ such that $\lim_{S \to \infty, \delta \to 0} \nu_1(S, \delta) = 0$.

(ii) For any $z \in N(Z)$, there exists a convex and compact set $V(z)$ such that

$$d_H(V(z,S,y), V(z)) \leq \nu_2(S) \quad \forall y \in Y(z) \ , \forall z \in N(Z) \tag{2.14}$$

for some $\nu_2(S)$, $\lim_{S \to \infty} \nu_2(S) = 0$.

Note that (2.14) implies that $d_H(V(z,S),V(z)) \leq \nu_2(S)$ and, hence, from (2.13) it follows that

$$d_H(V^{\delta}(z,S),V(z)) \leq \nu_1(S,\delta) + \nu_2(S) \quad \forall z \in N(Z) \ . \tag{2.15}$$

In Section 4 below we discuss some readily verifiable assumptions that lead to the fulfillment of the Assumptions I and II.

Define the *averaged* differential inclusion (**ADI**) by the equation

$$\dot{z}(t) \in V(z(t)) \ . \tag{2.16}$$

REMARK. For any $z \in N(Z)$, let $P(U \times Y(z))$ be the space of probability measures defined on the Borel subsets of $U \times Y(z)$ and let $W(z)$ be the subset of $P(U \times Y(z))$ defined by the equation

$$W(z) = \{\gamma \ : \ \gamma \in P(U \times Y(z)); \ \int_{U\times Y(z)} (\phi_i'(y))^T f(u,y,z)\gamma(du,dy) = 0 \ , \ i = 1,2,... \ \} \ , \tag{2.17}$$

where $\phi_i'(\cdot)$ stands for the gradient of $\phi_i(\cdot)$, with $\{ \ \phi_i(\cdot) \ , \ i = 1,2,... \ \}$ being a sequence of continuously differentiable functions such that an arbitrary continuously differentiable function $\phi(\cdot)$ and its gradient $\phi'(\cdot)$ are simultaneously approximated on compact sets by linear combinations of functions from $\{ \ \phi_i(\cdot) \ , \ i = 1,2,... \ \}$ and their corresponding gradients (an example of such approximating sequence of functions is the sequence of monomials $y_1^{i_1}...y_m^{i_m}$, $i_1,...,i_m = 0,1,...$, where y_j stands for the jth component of y; see [21]). From results in [13] it follows that the limit set $V(z)$, the existence of which is postulated in Assumption II (ii), can be parameterized in the form

$$V(z) = \{\zeta : \zeta = \tilde{g}(\gamma,z) \ , \quad \gamma \in W(z)\} \ , \tag{2.18}$$

where

$$\tilde{g}(\gamma,z) \stackrel{\text{def}}{=} \int_{U\times Y} g(u,y,z)\gamma(du,dy) \quad \forall \gamma \in P(U \times Y) \ .$$

Hence, the ADI (2.16) is equivalent to the system

$$\dot{z}(t) = \tilde{g}(\gamma(t),z(t)) \ , \tag{2.19}$$

in which the controls are Lebesgue measurable functions $\gamma(\cdot) : [0,T] \to P(U\times Y)$ satisfying the inclusion

$$\gamma(t) \in W(z(t)) \ . \tag{2.20}$$

In this paper, we will be using the averaged system in the form of the differential inclusion (2.16) and not in the parameterized form (2.19)- (2.20).

Following [6] and [7], let us define the viability of the ADI (2.16) in Z.

DEFINITION. A solution $z(t)$ of the ADI (2.16) is called viable in Z on $[0,T]$ if $z(t) \in Z \ \forall t \in [0,T]$; it is called viable in Z if $z(t) \in Z \ \forall t \in [0,\infty)$.

Theorem 2.1. *Let Assumptions I and II be valid and let the map* $V(z) : N(Z) \to 2^{R^n}$ *satisfy Lipschitz conditions on* $N(Z)$*, that is,*

$$d_H(V(z'),V(z'')) \leq L||z' - z''|| \quad \forall z',z'' \in N(Z) \ , \tag{2.21}$$

where L is a constant. Let, also, the set of viable in Z on $[0,T]$ solutions of the ADI (2.16) be not empty. Then:

(i) For any solution $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ of the SPCS (2.1)-(2.2) which is near viable in D on the interval $[0,T]$, there exists a solution $z_\epsilon^a(t)$ of the ADI (2.16) which is viable in Z on $[0,T]$ and satisfies the inequality

$$\max_{t\in[0,T]} ||z_\epsilon^a(t) - z_\epsilon^{sp}(t)|| \leq \mu_T(\epsilon) \ , \tag{2.22}$$

where $\lim_{\epsilon\to 0}\mu_T(\epsilon) = 0$ ($\mu_T(\epsilon)$ is the same for all solutions of the SPCS (2.1)-(2.2) which satisfy (2.5) with the same function $\nu_T(\epsilon)$).

(ii) For any solution $z^a(t)$ of the ADI (2.16) which is viable in Z on $[0,T]$ and has the initial conditions

$$z^a(0) \stackrel{\text{def}}{=} \zeta_0 \tag{2.23}$$

and for any

$$y_0 \in Y(\zeta_0) \ , \tag{2.24}$$

there exists a solution $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ of the SPCS (2.1)-(2.2) which is near viable in D, has the initial conditions

$$(y_\epsilon^{sp}(0), z_\epsilon^{sp}(0)) = (y_0, \zeta_0) \ , \tag{2.25}$$

and satisfies the inequality

$$\max_{t\in[0,T]} ||z_\epsilon^{sp}(t) - z^a(t)|| \leq \mu_T(\epsilon) \ , \tag{2.26}$$

with $\mu_T(\epsilon)$ tending to zero as ϵ tends to zero (same for all $z^a(t)$ as above).

Proof of Theorem 2.1 is in Section 5.

To extend the statements of Theorem 2.1 to the infinite time horizon let us introduce the following assumption.

ASSUMPTION III: Corresponding to any solution $z_1^a(t)$ of the ADI (2.16) such that $z_1^a(t) \in N(Z)$ and corresponding to any $z \in N(Z)$, there exists a solution $z_2^a(t) \in N(Z)$ of the ADI (2.16) such that $z_2^a(0) = z$ and

$$||z_2^a(t) - z_1^a(t)|| \leq ae^{-bt}||z_2^a(0) - z_1^a(0)|| \quad \forall t > 0 \ , \tag{2.27}$$

where a and b are some positive constants.

Similarly to Lemma A.2 in [15], it can be shown that Assumption III is satisfied if there exist positive definite matrices C and D such that, for any $v' \in V(z')$, $z' \in N(Z)$ and any $z'' \in N(Z)$, there exists $v'' \in V(z'')$ such that

$$(v' - v'')C(z' - z'') \leq -(z' - z'')^T D(z' - z'') \tag{2.28}$$

For any two continuous functions $z'(\cdot), z'(\cdot) : [0,\infty) \to N(Z)$, let

$$\rho(z'(\cdot), z''(\cdot)) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty} 2^{-l} \max_{t\in[0,l]} ||z'(t) - z'(t)|| \ . \tag{2.29}$$

Note that $\rho(\cdot,\cdot)$ is a metric on the space of continuous bounded functions defined on the interval $[0,\infty)$ the convergence in which is equivalent to the uniform convergence on any finite time interval.

Theorem 2.2. *Let the conditions of Theorem 2.1 and Assumption III be satisfied. Let, also, the set of viable in Z solutions of the ADI (2.16) be not empty. Then:*

(i) For any solution $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ of the SPCS (2.1)-(2.2) which is near viable in D, there exists a solution $z_\epsilon^a(t)$ of the ADI (2.16) which is viable in Z and satisfies the inequality

$$\rho(z_\epsilon^a(\cdot), z_\epsilon^{sp}(\cdot)) \leq \mu(\epsilon) \ , \tag{2.30}$$

where $\lim_{\epsilon\to 0}\mu(\epsilon) = 0$ ($\mu(\epsilon)$ is the same for all solutions of the SPCS (2.1)-(2.2) which satisfy (2.6) with the same function $\nu(\epsilon)$).

(ii) For any solution $z^a(t)$ of the ADI (2.16) which is viable in Z and has the initial conditions (2.23) and for any y_0 satisfying (2.24), there exists a solution $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ of the SPCS (2.1)-(2.2), having the initial conditions (2.25), which is near viable in D and satisfies the inequality

$$\rho(z_\epsilon^{sp}(\cdot), z^a(\cdot)) \leq \mu(\epsilon) \ , \tag{2.31}$$

with $\mu(\epsilon)$ tending to zero as ϵ tends to zero (same for all $\zeta(t)$).

Proof of Theorem 2.2 is in Section 5.

3 A generalization of Tichonov's theorem and Aubin's relaxation

By formally taking $\epsilon = 0$ in (2.1)-(2.2), one obtains the system

$$0 = f(u(t), y(t), z(t)) \ , \tag{3.32}$$
$$\dot{z}(t) = g(u(t), y(t), z(t)) \ . \tag{3.33}$$

Under the additional assumption that, for any $z \in N(z)$, the equation $f(u, y, z) = 0$ has a unique root $y = \psi(u, z)$ on $U \times Y(z)$, that is,

$$(u, y) \in U \times Y(z) \ , \quad f(u, y, z) = 0 \qquad \Leftrightarrow \qquad y = \psi(u, z) \ , \tag{3.34}$$

the system (3.32)-(3.33) becomes equivalent to the system

$$\dot{z}(t) = g(u(t), \psi(u(t)z(t)), z(t)) \ , \tag{3.35}$$

in which the controls are measurable functions $u(t) \in U$. This is a so called reduced system. Results establishing a possibility to approximate the z-components of solutions of the SPCS by the solutions of the reduced system (3.35) are commonly referred to as generalizations of Tichonov's theorem (see, e.g., [10], [20], [26], [28]; the original Tichonov's theorem was established for the uncontrolled dynamics in [27]).

Below we introduce conditions, under which the ADI (2.16) becomes equivalent to the system (3.32)-(3.33) and, thus, the solutions of the latter approximate (in the sense of

Theorems 2.1-2.2) the near viable solutions of the SPCS (2.1)-(2.2). Note that the assumption that the equation $f(u, y, z) = 0$ has a unique root on $U \times Y(z)$ is not needed for the validity of this result.

Let $z \in N(Z)$ and $(\bar{u}, \bar{y}) \in U \times Y(z)$ be such that $f(\bar{u}, \bar{y}, z) = 0$. Then

$$\eta \stackrel{\text{def}}{=} g(\bar{u}, \bar{y}, z) \in V(z, S, \bar{y}) \quad \forall S > 0 \qquad \Rightarrow \qquad \eta \in V(z) \ ,$$

where $V(z)$ is introduced in Assumption II(ii). Following the terminology of [12], let us call the defined above η as a stationary regime point of $V(z)$. Denote by $V^{st}(z)$ the set of all stationary regime points of $V(z)$:

$$V^{st}(z) \stackrel{\text{def}}{=} \{\eta \mid \eta = g(u, y, z) \ , \ 0 = f(u, y, z) \ , \ (u, y) \in U \times Y(z)\} \ \subset V(z) \ . \tag{3.36}$$

Proposition 3.1. *Let Assumptions I and II be satisfied. Let also the set*

$$q(U, Y(z), z) \stackrel{\text{def}}{=} \{\eta \mid \eta = q(u, y, z) \ , \ (u, y) \in U \times Y(z)\} \tag{3.37}$$

be convex for any $z \in N(Z)$, where $q(u, y, z) \stackrel{\text{def}}{=} (f(u, y, z), g(u, y, z))$. Then

$$V(z) = V^{st}(z) \quad \forall z \in N(Z) \ . \tag{3.38}$$

Proof of the proposition is given in the end of this section.

From Proposition 3.1 it follows that the viable in D solutions of the ADI (2.16) coincide with the viable solutions of the differential inclusion

$$\dot{z}(t) \in V^{st}(z(t)) \ , \tag{3.39}$$

which is equivalent to the system (3.32)-(3.33).

Let us demonstrate one application of this result. Consider the system

$$\dot{z}(t) = g(y(t), z(t)) \ , \tag{3.40}$$

in which $y(\cdot)$ are controls (and not state variables as above). That is, $y(\cdot)$ in (3.40) are Lebesgue measurable functions that are assumed to satisfy the state constraint

$$y(t) \in Y(z(t)) \ . \tag{3.41}$$

J.-P. Aubin conjectured that this constraint can be relaxed in the sense that the viable solutions of (3.40)-(3.41) (that is, the solutions such that $z(t) \in Z$) can be approximated by the z-components of the near viable solutions of the SPCS

$$\epsilon \dot{y}(t) = u(t) \ , \tag{3.42}$$

$$\dot{z}(t) = g(y(t), z(t)) \ . \tag{3.43}$$

In this system, $y(\cdot)$ are state variables and controls are functions $u(\cdot)$ which are measurable and satisfy the inclusion $u(t) \in U \stackrel{\text{def}}{=} B$ (the closed unit ball in R^m). The following propositions can serve as a justification of such a relaxation.

Proposition 3.2. *Let $g(\cdot,\cdot) : R^m \times R^n \to R^m$ satisfy Lipschitz conditions and let the map $Y(\cdot) : N(Z) \to 2^{R^m}$ be convex and compact valued, and satisfy Lipschitz conditions (that is, (2.3) is valid with $\kappa(\theta) = L\theta,\ L =$ const). Assume that there exists $r > 0$ such that*

$$rB \subset Y(z) \qquad \forall z \in N(Z) \ , \tag{3.44}$$

where, as above, B is the closed unit ball in R^m and assume that the set

$$g(Y(z), z) \stackrel{\text{def}}{=} \{v \mid v = g(y, z) \ , y \in Y(z)\} \tag{3.45}$$

is convex for any $z \in N(Z)$. Then, corresponding to any near viable in D on $[0, T]$ solution $(y^{sp}_\epsilon(t), z^{sp}_\epsilon(t))$ of the SPCS (3.42)-(3.43), there exists a pair $(y^a_\epsilon(t), z^a_\epsilon(t))$ satisfying (3.40)-(3.41) such that $z^a_\epsilon(t)$ is viable in Z on $[0, T]$ and such that (2.22) is valid. Conversely, corresponding to any pair $(y^a(t), z^a(t))$ satisfying (3.40)-(3.41), with $z^a(t)$ being viable in Z on $[0, T]$, there exists a near viable solution of the SPCS (3.42)-(3.43) such that (2.26) is valid.

Proof of Proposition 3.2. First note that (3.44) and the convexity of $Y(z)$ imply the validity of Assumptions I and II (see Corollary 4.4 below). Also, from the convexity of the set (3.45) it follows that the set (3.37), which, for the system (3.42)-(3.43), has the form

$$q(U, Y(z), z) = U \times g(Y(z), z) \ ,$$

is convex. Hence, the conditions of Proposition 3.1 are satisfied and

$$V(z) = V^{st}(z) = g(Y(z), z) \ . \tag{3.46}$$

That is, the ADI (2.16) is equivalent to the system (3.40)-(3.41). The validity of the proposition follows now from Theorem 2.1. □

In conclusion of this section, let us prove Proposition 3.1.

Proof of Proposition 3.1. By (3.36), it is enough to prove that $V(z) \subset V^{st}(z)$. Denote:

$$V_q(z, S, y) \stackrel{\text{def}}{=} \bigcup_{(u(\cdot),\ y(\cdot))} \left\{ \frac{1}{S} \int_0^S q(u(\tau), y(\tau), z) d\tau \right\} \ ,$$

where, as in (2.9), the union is over all admissible pairs of the associated system (2.7), which satisfy the initial conditions (2.10). Due to the convexity of $q(U, Y(z), z)$

$$V_q(z, S, y) \subset q(U, Y(z), z) \ , \quad \forall S > 0 \ .$$

Also, by (2.7),

$$||\frac{1}{S} \int_0^S f(u(\tau, y(\tau), z) d\tau|| = ||\frac{1}{S} \int_0^S \dot{y}(\tau) d\tau|| = \frac{1}{S} ||y(S) - y(0)|| \leq \frac{c(z)}{S} \ ,$$

where $c(z) \stackrel{\text{def}}{=} \max\{||y' - y''|| \mid y', y'' \in Y(z)\}$. Hence,

$$V_q(z, S, y) \subset \{(\eta_1, \eta_2) \in R^m \times R^n \ \mid \ ||\eta_1|| \leq \frac{c(z)}{S} \ , (\eta_1, \eta_2) \in q(U, Y(z), z)\}$$

$$\Rightarrow \quad \limsup_{S\to\infty} V_q(z,S,y) \subset \{(\eta_1,\eta_2) \in R^m \times R^n \ \mid \ \eta_1 = 0 \ , \ (\eta_1,\eta_2) \in q(U,Y(z),z)\}$$

It follows that

$$V(z) \subset \{\eta_2 \mid (\eta_1,\eta_2) \in \limsup_{S\to\infty} V_q(z,S,y)\} \subset \{\eta_2 \mid \eta_1 = 0 \ , \ (\eta_1,\eta_2) \in q(U,Y(z),z)\} \stackrel{\text{def}}{=} V^{st}(z) \ .$$

This completes the proof. □

Note that the proof above is similar to that of Proposition 3.2 in [12] that was proved under more restrictive conditions.

4 Verification of the assumptions.

In this section we consider some conditions, the fulfillment of which allows one to verify the validity of Assumptions I and II.

EXPONENTIAL STABILITY (**ES**) CONDITION:

Any two solutions, $y^1(\tau)$ and $y^2(\tau)$, of the associated system (2.7) obtained with the same control, possess the following convergence property:

$$||y^1(\tau) - y^2(\tau)|| \leq \alpha e^{-\beta\tau}||y^1(0) - y^2(0)|| \quad \forall \tau \geq 0 \ , \tag{4.47}$$

where α and β are positive constants (same for all $z \in N(Z)$).

If the ES condition is satisfied, then the system (2.7) has a forward invariant set which, also, is a global attractor for all of its solutions (Theorem 3.1(ii) in [12]). More specifically, there exists a compact set $Y^*(z)$ such that

$$Y(z,\tau,y) \subset Y^*(z) \quad \forall \ y \in Y^*(z) \ , \ \tau \geq 0 \tag{4.48}$$

$$d_H(Y(z,\tau,y),Y^*(z)) \leq \alpha_1 e^{-\beta\tau} dist(y,Y^*(z)) \quad \forall \ y \in R^m \ , \ \tau \geq 0 \ , \tag{4.49}$$

where $\alpha_1 = \ const$ and $Y(z,\tau,y) \subset R^m$ is the reachability set of (2.7), that is, the set of points which can be reached at the moment τ by the trajectories of (2.7) obtained with all controls and the initial conditions $y(0) = y$.

Proposition 4.1. *Let the ES condition be satisfied and there exists $r > 0$ such that*

$$y + rB \subset Y^*(z) \qquad \forall z \in N(Z) \tag{4.50}$$

for some $y \in Y^*(z)$, with B being the closed unit ball in R^m. Then Assumptions I and II will be satisfied if $Y(z)$ is such that: (i) It contains $Y^*(z)$ for any $z \in N(Z)$ and (ii) For every $y \in Y(z)$, there exists an admissible pair $(\bar{u}(\tau),\bar{y}(\tau))$ satisfying the relationships:

$$\bar{y}(0) = y \ , \qquad \bar{y}(\bar{\bar{\tau}}) \in Y^*(z) \tag{4.51}$$

for some $\bar{\bar{\tau}} \in [0,b]$ ($b > 0$ is a given constant).

Proof of this proposition is given in the end of this section.

REMARKS:

(i) Note that the set $V(z)$ will be the same for all $Y(z)$ containing $Y^*(z)$, and, thus, the same will be the ADI (2.16) the solutions of which approximate the z-components of the near viable solutions of the SPCS (2.1)-(2.2). Note also that the assumption about the validity of (4.50) is not required if $Y(z) = Y^*(z)$.

(ii) It is easy to verify (see e.g. [14]) that the ES condition will be satisfied if there exist *positive definite* matrices C and D such that, for any $y^1, y^2 \in R^m$ and any $u \in U$, $z \in N(z)$,

$$(f(u, y^1, z) - f(u, y^2, z))^T C(y^1 - y^2) \leq -(y^1 - y^2)^T D(y^1 - y^2) \ .$$

The fulfillment of the latter condition can be guaranteed if, e.g., $f(u, y, z)$ is linear in (u, y):

$$f(u, y, z) = F(z)y + G(z)u \ , \tag{4.52}$$

with $F(z)$, $G(z)$ being matrix functions of the corresponding dimensions and with the eigenvalues of $F(z)$ having negative real parts on the closure of $N(Z)$.

$Y(z)$-STRONG CONTROLLABILITY CONDITION ($Y(z)$-**SC**):

For any $y', y'' \in Y(z)$, there exists an admissible pair $(u(\tau), y(\tau))$ of the system (2.7) such that $y(0) = y'$ and $y(\bar{\tau}) = y''$ for some $\bar{\tau} \in [0, b]$ ($b > 0$ is a given constant). That is, any two points of $Y(z)$ can be connected by an admissible solution of (2.7) within a uniformly bounded interval of time.

It is obvious that the $Y(z)$-SC condition implies the validity of Assumption I(i). The following statement establishes that it also implies the validity of Assumption II(ii).

Proposition 4.2. *If the $Y(z)$-SC condition is satisfied, then Assumption II(ii) is valid*

Proof. Using the $Y(z)$-SC condition, one can verify that, for some constant $c > 0$,

$$d_H(V(z, S, y'), V(z, S, y'')) \leq \frac{c}{S} \quad \forall y', y'' \in Y(z) \ , \ S > b \ . \tag{4.53}$$

The validity of Assumption II(ii) can be proved on the basis of (4.53) by following exactly the same steps as those in the proof of the corresponding result in [11] or as those in the proof of Proposition 3.2 in [17], where a similar statement was established for the case when $Y(z)$ is forward invariant for the system (2.7). □

Let us now consider two sets of conditions leading to the fulfillment of Assumption II(i).

SET A:

(i) $Y(z)$ is convex and compact valued and there exists $r > 0$ such that

$$rB \subset Y(z) \ , \qquad \forall z \in N(Z) \ ;$$

(ii) U is a compact subset of a Banach space and there exists α_0, $1 > \alpha_0 \geq 0$, such that $\alpha U \subset U$, $\forall \ \alpha \in [\alpha_0, 1]$;

(iii) The function $f(u, y, z)$ satisfies the following "homogeneouity" condition in y and u:

$$\beta f(u, y, z) \ = \ f(\beta^{\kappa} u, \beta y, z) \ , \ \ \forall \ \beta > 0 \ ,$$

where κ is a positive constant.

SET B:

(i) $Y(z)$ is convex and compact valued;

(ii) U is a convex and compact subset of a Banach space;

(iii) The function $f(u,y,z)$ has the form (4.52) (that is, it is linear in u and y) and there exists $r > 0$ such that, for any $z \in N(Z)$,

$$\bar{y} + rB \in Y(z) \tag{4.54}$$

for some $\bar{y} \in Y(z)$ which, along with some $\bar{u} \in U$, satisfy the equation

$$F(z)\bar{y} + G(z)\bar{u} = 0 \ . \tag{4.55}$$

Proposition 4.3. *Assumption II(i) is valid if the conditions from the set A or conditions from the set B are satisfied.*

Proof. Assume, first, that the conditions from the set A are satisfied. From A(i) it follows that

$$(1-\frac{\delta}{r})(Y(z)+\delta B) \ = \ (1-\frac{\delta}{r})Y(z) + (1-\frac{\delta}{r})\delta B \ = \ (1-\frac{\delta}{r})Y(z) \tag{4.56}$$

$$+ \ (1-\frac{\delta}{r})(\frac{\delta}{r})(rB) \ \subset \ (1-\frac{\delta}{r})Y(z) \ + \ (\frac{\delta}{r})(rB) \ \subset \ (1-\frac{\delta}{r})Y(z) \ + \ (\frac{\delta}{r})Y(z) \ \subset Y(z) \ ,$$

where it is assumed that δ is such that $\frac{\delta}{r} \in (0,1)$.

Let $(u^{\delta}(\tau), y^{\delta}(\tau))$ be an arbitrary δ-admissible pair for (2.7) on the interval $[0,S]$ and let $u(\tau) \stackrel{\rm def}{=} (1-\frac{\delta}{r})^{\kappa}u^{\delta}(\tau)$ and $y(\tau) \stackrel{\rm def}{=} (1-\frac{\delta}{r})y^{\delta}(\tau)$. From the inclusions above and A(ii) it follows that $(u(\tau), y(\tau)) \ \in U \times Y(z)$ for δ small enough. Also, by A(iii),

$$\dot{y}(\tau) \ = \ (1-\frac{\delta}{r})\dot{y}^{\delta}(\tau) \ = \ (1-\frac{\delta}{r})f(u^{\delta}(\tau), y^{\delta}(\tau)) \ = \ f(u(\tau), y(\tau)) \ , \quad \tau \in [0,S].$$

Hence, the pair $(u(\tau), y(\tau))$ is admissible for the system (2.7).

Define the function $\psi(\theta)$ by the equation

$$\psi(\theta) \stackrel{\rm def}{=} \sup_{||u'-u''||+||u'-u''|| \leq \theta} \{ ||g(u',y',z) - g(u'',y'',z)|| \mid u',u'' \in U \ , \ y',y'' \in Y(z) + \delta_0 B \ , \ z \in N(Z) \} \ ,$$

where δ_0 is a fixed positive number. In accordance with this definition, for $\delta \in [0, \delta_0]$,

$$||g(u^{\delta}(\tau), y^{\delta}(\tau), z) \ - \ g((u(\tau), y(\tau), z)|| \ \leq \ \psi(\ ||u^{\delta}(\tau) - u(\tau)|| \ + \ ||y^{\delta}(\tau) - y(\tau)|| \ )$$

$$\leq \ \psi(\ (1 \ - \ (1-\frac{\delta}{r})^{\kappa}) \ \max_{u \in U} ||u|| + (\frac{\delta}{r}) \max_{y \in Y(z)+\delta B, z \in N(Z)} ||y||) \ \stackrel{\rm def}{=} \ \hat{\psi}(\delta) \ . \tag{4.57}$$

Note that, due to the continuity of $g(u,y,z)$, the function $\psi(\theta)$ tends to zero as θ tends to zero and, hence, the function $\hat{\psi}(\delta)$ (introduced above) tends to zero as δ tends to zero. Since $(u^{\delta}(\tau), y^{\delta}(\tau))$ is an arbitrary δ-admissible pair, the inequality obtained above implies that

$$V^{\delta}(z,S) \subset V(z,S) + \hat{\psi}(\delta)B' \ ,$$

where B' is the closed unit ball in R^n. This and (2.12) imply the validity of Assumption II(i) with $\nu_1(S,\delta) = \hat{\psi}(\delta)$ in (2.13).

Assume now that the conditions from the set B are satisfied. By (4.54), $rB \subset Y(z) - \bar{y}$. Hence, similarly to (4.56), one can obtain

$$(1-\frac{\delta}{r})(Y(z) - \bar{y} + \delta B) \subset Y(z) - \bar{y} \quad \Rightarrow \quad (1-\frac{\delta}{r})(Y(z) + \delta B) + \frac{\delta}{r}\bar{y} \subset Y(z) \qquad (4.58)$$

Let $(u^\delta(\tau), y^\delta(\tau))$ be an arbitrary δ-admissible pair for (2.7) on the interval $[0,S]$ and let

$$u(\tau) \stackrel{\text{def}}{=} (1-\frac{\delta}{r})u^\delta(\tau) + \frac{\delta}{r}\bar{u} \ , \qquad y(\tau) \stackrel{\text{def}}{=} (1-\frac{\delta}{r})y^\delta(\tau) + \frac{\delta}{r}\bar{y} \ .$$

By the condition B(ii) and (4.58), $(u(\tau), y(\tau)) \in U \times Y(z)$ for δ such that $\frac{\delta}{r} \in (0,1)$. Also, by (4.52) and (4.55),

$$\dot{y}(\tau) = (1-\frac{\delta}{r})\dot{y}^\delta(\tau) = (1-\frac{\delta}{r})(F(z)y^\delta(\tau)+G(z)u^\delta(\tau))+\frac{\delta}{r}(F(z)\bar{y}+G(z)\bar{u}) = F(z)y(\tau)+G(z)u(\tau) \ .$$

That is, the pair $(u(\tau), y(\tau))$ is admissible for the system (2.7). Similarly to (4.57),

$$||g(u^\delta(\tau), y^\delta(\tau), z) \ - \ g((u(\tau), y(\tau), z)|| \ \le \ \psi(\ ||u^\delta(\tau) - u(\tau)|| \ + \ ||y^\delta(\tau) - y(\tau)|| \ )$$

$$\le \ \psi(2\frac{\delta}{r}(\max_{u\in U}||u|| + \max_{y\in Y(z)+\delta B, z\in N(Z)}||y||)) \stackrel{\text{def}}{=} \phi(\delta) \quad \Rightarrow \quad V^\delta(z,S) \subset V(z,S) + \phi(\delta)B' \ .$$

This and (2.12) imply the validity of Assumption II(i) with $\nu_1(S,\delta) = \phi(\delta)$ in (2.13). □

Corollary 4.4. *Assume that the map $Y(z)$ is convex and compact valued and that (3.44) is satisfied. Assume also that $f(u,y) = u$, $g(u,y,z) = g(u,y)$, and that $U = B$ (as in the SPCS (3.42)-(3.43)). Then Assumptions I and II are valid*

Proof. Under the assumptions made, (4.52) is true with $F(z) = 0$, $G\ (z) = I$ (identity matrix) and the conditions from the set B are satisfied with $\bar{u} = 0$ and $\bar{y} = 0$. Also the $Y(z)$-SC condition is satisfied. Hence, by Propositions 4.2 and 4.3, Assumption II is valid. The verification of Assumption I is obvious. □

Proof of Proposition 4.1. The existence of the admissible pair satisfying (4.51) and the fact that $Y^*(z)$ is forward invariant imply that Assumption I (i) is satisfied. The validity of Assumptions I(ii) readily follows from (4.49),(4.50).

To establish the validity of Assumption II, let us introduce the following set of time averages

$$\hat{V}(z,S,y) \stackrel{\text{def}}{=} \bigcup_{\left(u(\cdot),\ y(\cdot)\right)} \left\{\frac{1}{S}\int_0^S g(u(\tau), y(\tau), z)d\tau\right\} \qquad (4.59)$$

where, in contrast to (2.9) or (2.11), the union is over all (not only over admissible or δ-admissible) controls and corresponding solutions of (2.7) which satisfy (2.10). Note that

$$V(z,S,y) \subset V^\delta(z,S,y) \subset \hat{V}(z,S,y) \quad \forall y \in Y(z) \qquad (4.60)$$

and that

$$V(z,S,y) = V^\delta(z,S,y) = \hat{V}(z,S,y) \quad \forall y \in Y^*(z) \ , \qquad (4.61)$$

the latter being valid since $Y^*(z)$ is forward invariant. It is easy to verify that from the validity of (4.47) and the Lipschitz continuity of $g(u,y,z)$ in y it follows that, for any compact set Q , there exists a constant c such that

$$d_H(\hat{V}(z,S,y'),\hat{V}(z,S,y'')) \leq \frac{c}{S} \quad \forall y',y'' \in Q \ , \ S>0 \ . \tag{4.62}$$

This, in turns, implies (see Theorem 3.1(i) in [12], Proposition 3.2 in [17] and earlier results in [11]) that there exists a convex and compact set $V(z)$ and a constant $\hat{c}$ such that

$$d_H(\hat{V}(z,S,y),V(z)) \leq \frac{\hat{c}}{S^{\frac{1}{2}}} \quad \forall y \in Q \ , \ S \geq 1 \ . \tag{4.63}$$

Hence, by (4.60),

$$V(z,S,y) \subset V(z) + \frac{\hat{\hat{c}}}{S^{\frac{1}{2}}}B' \quad \forall y \in Y(z) \ , \ S \geq 1 \ , \tag{4.64}$$

where B' is the closed unit ball in R^n and $\hat{\hat{c}} = const$. Note that, due to the uniformity of the estimate (4.62) with respect to $z \in N(Z)$, the constant $\hat{\hat{c}}$ can be chosen to be the same for all $z \in N(Z)$.

Let now $y \in Y(z)$ and let $(\bar{u}(\tau),\bar{y}(\tau))$ be an admissible pair which has the initial conditions $\bar{y}(0) = y$ and which satisfies the inclusion $\bar{y}(\bar{\bar{\tau}}) \in Y^*(z)$. Using the definition of the set $V(z,S,y)$, one can establish that the following inclusion is valid

$$\frac{1}{S}\int_0^{\bar{\bar{\tau}}} g(\bar{u}(\tau),\bar{y}(\tau),z) + \frac{S-\bar{\bar{\tau}}}{S}V(z,S-\bar{\bar{\tau}},\bar{y}(\bar{\bar{\tau}})) \subset V(z,S,y) \ . \tag{4.65}$$

Using (4.63), one can obtain that

$$d_H(\frac{1}{S}\int_0^{\bar{\bar{\tau}}} g(\bar{u}(\tau),\bar{y}(\tau),z) + \frac{S-\bar{\bar{\tau}}}{S}\hat{V}(z,S-\bar{\bar{\tau}},\bar{y}(\bar{\bar{\tau}})) \ , \ V(z)) \leq \frac{\bar{c}}{(S-\bar{\bar{\tau}})^{\frac{1}{2}}} = O(\frac{1}{S^{\frac{1}{2}}})$$

for some $\bar{c} = const$ and $S \geq \bar{\bar{\tau}} + 1$. Consequently,

$$V(z) \subset \frac{1}{S}\int_0^{\bar{\bar{\tau}}} g(\bar{u}(\tau),\bar{y}(\tau),z) + \frac{S-\bar{\bar{\tau}}}{S}\hat{V}(z,S-\bar{\bar{\tau}},\bar{y}(\bar{\bar{\tau}})) + O(\frac{1}{S^{\frac{1}{2}}})B' \ .$$

Since, by (4.61), $\hat{V}(z,S-\bar{\bar{\tau}},\bar{y}(\bar{\bar{\tau}})) = V(z,S-\bar{\bar{\tau}},\bar{y}(\bar{\bar{\tau}}))$, from (4.65) it follows that

$$V(z) \subset V(z,S,y) + O(\frac{1}{S^{\frac{1}{2}}})B' \quad \forall y \in Y(z) \ . \tag{4.66}$$

This, along with (4.64), lead to the validity of Assumption II(ii), with $\nu_2(S) = O(\frac{1}{S^{\frac{1}{2}}})$ in (2.14). Note that the latter implies, in particular, that

$$d_H(V(z,S),V(z)) = O(\frac{1}{S^{\frac{1}{2}}}) \tag{4.67}$$

$$\Rightarrow \quad V(z) \subset V(z,S) + O(\frac{1}{S^{\frac{1}{2}}})B' \subset V^{\delta}(z,S) + O(\frac{1}{S^{\frac{1}{2}}})B' \ .$$

From (4.60) and (4.63) it follows, on the other hand, that

$$V^{\delta}(z,S) \subset \bigcup_{y \in Y^{\delta}(z)} \left\{\hat{V}(z,S,y)\right\} \subset V(z) + O(\frac{1}{S^{\frac{1}{2}}})B' \ .$$

Hence, $d_H(V^{\delta}(z,S),V(z)) = O(\frac{1}{S^{\frac{1}{2}}})$. This and (4.67) establish the validity of Assumption II(i), with $\nu_1(S,\delta) = O(\frac{1}{S^{\frac{1}{2}}})$ in (2.13). The proof is completed. □

5 Proofs of Theorems 2.1, 2.2

Proof of Theorem 2.1(i). Let us rewrite the system (2.1)-(2.2) in the "stretched" time scale $\tau = \frac{t}{\epsilon}$

$$\dot{y}(\tau) = f(u(\tau), y(\tau), z(\tau)) \ , \tag{5.68}$$

$$\dot{z}(\tau) = \epsilon g(u(\tau), y(\tau), z(\tau)) \ , \tag{5.69}$$

where the controls are measurable functions satisfying the inclusion $u(\tau) \in U$.

In this time scale, the inequality (2.5) defining the near viability of the SPCS on the interval $[0, T]$ is replaced by

$$\max_{\tau\in[0,\frac{T}{\epsilon}]} dist((y^{sp}(\tau), z^{sp}(\tau)), D) \leq \nu_T(\epsilon) \ . \tag{5.70}$$

Note that here (and everywhere else in the proof of Theorem 2.1) we write $(y^{sp}(\tau), z^{sp}(\tau))$ instead of $(y^{sp}_\epsilon(\tau), z^{sp}_\epsilon(\tau))$ and, similarly, we will write $z^a(t)$ instead of $z^a_\epsilon(t)$ (thus omitting the subscript ϵ from our notations).

Let $(y^{sp}(\tau), z^{sp}(\tau))$ be a solution of (5.68)-(5.69) which satisfies (5.70). To prove Theorem 2.1(i), one needs to establish that there exists a solution $z^a(t)$ of the ADI (2.16) which is viable in Z on $[0, T]$ and satisfies the inequality

$$\max_{\tau\in[0,\frac{T}{\epsilon}]} ||z^{sp}(\tau) - z^a(\epsilon\tau)|| \leq \mu_T(\epsilon) \ . \tag{5.71}$$

We will do it in two stages. First, we will construct a solution $\tilde{z}^a(t)$ of the ADI (2.16) (not necessarily viable in Z) such that

$$\max_{\tau\in[0,\frac{T}{\epsilon}]} ||z^{sp}(\tau) - \tilde{z}^a(\epsilon\tau)|| \leq \mu'_T(\epsilon) \ , \tag{5.72}$$

where $\lim_{\epsilon\to 0} \mu'_T(\epsilon) = 0$. Secondly, we will show that there exists a solution $z^a(t)$ of the ADI (2.16) which is viable in Z on $[0, T]$ and satisfies the inequality

$$\max_{t\in[0,T]} ||z^a(t) - \tilde{z}^a(t)|| \leq \mu''_T(\epsilon) \ , \tag{5.73}$$

where $\lim_{\epsilon\to 0} \mu''_T(\epsilon) = 0$. This will establish (5.71) with $\mu_T(\epsilon) \stackrel{\text{def}}{=} \mu'_T(\epsilon) + \mu''_T(\epsilon)$.

Let us partition the interval $[0, \frac{T}{\epsilon}]$ by the points

$$\tau_l \stackrel{\text{def}}{=} lS_\epsilon \ , \quad l = 0, 1, ..., K_\epsilon \ , \quad \tau_{K_\epsilon+1} \stackrel{\text{def}}{=} \frac{T}{\epsilon} \ , \tag{5.74}$$

where S_ϵ is a function of ϵ such that

$$\lim_{\epsilon\to 0} S_\epsilon = \infty \ , \qquad \lim_{\epsilon\to 0} \epsilon S_\epsilon = 0$$

and K_ϵ is the integer part of $\frac{T}{\epsilon}$ ($\tau_{K_\epsilon} = \tau_{K_\epsilon+1}$ if $\frac{T}{\epsilon}$ is integer).

Denote by $(\bar{y}^{sp}(\tau), \bar{z}^{sp}(\tau)) \in D$ the projection of $(y^{sp}(\tau), z^{sp}(\tau))$ onto D. By (5.70),

$$\max_{\tau\in[0,\frac{T}{\epsilon}]} ||(y^{sp}(\tau), z^{sp}(\tau)) - (\bar{y}^{sp}(\tau), \bar{z}^{sp}(\tau))|| \leq \nu_T(\epsilon)$$

Hence,

$$||z^{sp}(\tau) - \bar{z}^{sp}(\tau)|| \leq \nu_T(\epsilon) \quad \forall \tau \in [0, \frac{T}{\epsilon}] \tag{5.75}$$

and

$$dist(y^{sp}(\tau), Y(\bar{z}^{sp}(\tau)) \leq ||y^{sp}(\tau) - \bar{y}^{sp}(\tau)|| \leq \nu_T(\epsilon) \quad \forall \tau \in [0, \frac{T}{\epsilon}] \ . \tag{5.76}$$

Note that from (5.75) it follows that, $\forall \tau \in [\tau_l, \tau_{l+1}]$,

$$||z^{sp}(\tau) - \bar{z}^{sp}(\tau_l)|| \leq ||z^{sp}(\tau) - z^{sp}(\tau_l)|| + ||z^{sp}(\tau_l) - \bar{z}^{sp}(\tau_l)|| \leq M\epsilon S_\epsilon + \nu_T(\epsilon) \tag{5.77}$$

for sufficiently small ϵ, with

$$M \stackrel{\text{def}}{=} \max\{||g(u, y, z)|| \mid u \in U, \ (y, z) \in \hat{D}\} \ , \tag{5.78}$$

and $\hat{D}$ being a sufficiently large compact subset of $R^m \times R^n$ which, in particular, contains D in its interior. Note also that (5.75)-(5.77) together with (2.3) imply that, $\forall \tau \in [\tau_l, \tau_{l+1}]$,

$$dist(y^{sp}(\tau), Y(\bar{z}^{sp}(\tau_l))) \leq dist(y^{sp}(\tau), Y(\bar{z}^{sp}(\tau))) + d_H(Y(\bar{z}^{sp}(\tau)), Y(z^{sp}(\tau)) \tag{5.79}$$

$$+ d_H(Y(z^{sp}(\tau)), Y(\bar{z}^{sp}(\tau_l))) \leq \nu_T(\epsilon) + \kappa(\nu_T(\epsilon)) + \kappa(M\epsilon S_\epsilon + \nu_T(\epsilon)) \stackrel{\text{def}}{=} \omega(\epsilon) \ .$$

Denote by $y_l(\tau)$ the solution of the associated system (2.7) considered on the interval $[\tau_l, \tau_{l+1}]$ with the control $u(\tau)$ (the same as the one used to obtain $(y^{sp}(\tau), z^{sp}(\tau))$), with the initial conditions $y_l(\tau_l) \stackrel{\text{def}}{=} y^{sp}(\tau_l)$, and with $z = \bar{z}^{sp}(\tau_l)$. One has

$$||y^{sp}(\tau) - y_l(\tau)|| \leq \int_{\tau_l}^{\tau} ||f(u(s), y^{sp}(s), z^{sp}(s)) - f(u(s), y_l(s), \bar{z}^{sp}(\tau_l))|| ds$$

$$\leq L \int_{\tau_l}^{\tau} (||y^{sp}(s) - y_l(s)|| + ||z^{sp}(s) - \bar{z}^{sp}(\tau_l)||) ds \tag{5.80}$$

where L is a Lipschitz constant. Using (5.77), one can obtain from here that

$$||y^{sp}(\tau) - y_l(\tau)|| \leq LS_\epsilon[M\epsilon S_\epsilon + \nu_T(\epsilon)] + L \int_{\tau_l}^{\tau} ||y^{sp}(s) - y_l(s)|| ds \ ,$$

whereas, by Gronwall-Bellman lemma, it follows that

$$||y^{sp}(\tau) - y_l(\tau)|| \leq LS_\epsilon[M\epsilon S_\epsilon + \nu_T(\epsilon)]e^{LS_\epsilon} \quad \forall \tau \in [\tau_l, \tau_{l+1}] \ .$$

Assuming (without loss of generality) that $\nu_T(\epsilon) \geq \epsilon$ and taking $S_\epsilon = \frac{1}{2L} ln \frac{1}{\nu_T(\epsilon)}$, one obtains now that, for sufficiently small ϵ,

$$||y^{sp}(\tau) - y_l(\tau)|| \leq [LMS_\epsilon^2 + 1]\nu_T(\epsilon) \frac{1}{\nu_T^{\frac{1}{2}}(\epsilon)} \leq \nu_T^{\frac{1}{4}}(\epsilon) \quad \forall \tau \in [\tau_l, \tau_{l+1}] \ . \tag{5.81}$$

The latter inequality and (5.79) imply that

$$y_l(\tau) \in Y(\bar{z}^{sp}(\tau_l)) + \delta(\epsilon)B \quad \forall \tau \in [\tau_l, \tau_{l+1}] \ , \tag{5.82}$$

with $\delta(\epsilon) \stackrel{\text{def}}{=} \omega(\epsilon) + \nu_T^{\frac{1}{4}}(\epsilon)$.

By definition (see(2.11)),

$$\frac{1}{S_\epsilon}\int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), \bar{z}^{sp}(\tau_l))d\tau \ \in \ V^{\delta(\epsilon)}(\bar{z}^{sp}(\tau_l), S_\epsilon) \ . \tag{5.83}$$

Hence, by (2.15), there exists $v_l \in V(\bar{z}^{sp}(\tau_l))$ such that

$$||\frac{1}{S_\epsilon}\int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), \bar{z}^{sp}(\tau_l))d\tau \ - \ v_l|| \leq \nu_1(S_\epsilon, \delta(\epsilon)) + \nu_2(S_\epsilon) \stackrel{\text{def}}{=} \omega_1(\epsilon) \ . \tag{5.84}$$

Let $\zeta_0 \stackrel{\text{def}}{=} z^{sp}(\tau_0)$ and

$$\zeta_{l+1} = \zeta_l + \epsilon S_\epsilon \tilde{v}_l \ , \quad l = 0, 1, ..., K_\epsilon - 1 \ , \tag{5.85}$$

where $\tilde{v}_l$ is the projection of v_l onto $V(\zeta_l)$. Note that from (2.21) and (5.75) it follows that

$$||v_l - \tilde{v}_l|| = dist(v_l, V(\zeta_l)) \leq d_H(V(\bar{z}^{sp}(\tau_l)), V(\zeta_l)) \leq L||\bar{z}^{sp}(\tau_l) - \zeta_l|| \tag{5.86}$$

$$\leq L||z^{sp}(\tau_l) - \zeta_l|| + L\nu_T(\epsilon) \ .$$

Subtracting (5.85) from the equation

$$z^{sp}(\tau_{l+1}) = z^{sp}(\tau_l) + \epsilon\int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y^{sp}(\tau), z^{sp}(\tau))d\tau \tag{5.87}$$

and taking into account (5.77), (5.84), (5.86), one obtains

$$||z^{sp}(\tau_{l+1}) - \zeta_{l+1}|| \leq ||z^{sp}(\tau_l) - \zeta_l|| + \epsilon\int_{\tau_l}^{\tau_{l+1}} ||g(u(\tau), y^{sp}(\tau), z^{sp}(\tau)) - g(u(\tau), y^{sp}(\tau), \bar{z}^{sp}(\tau_l))||d\tau$$

$$+\epsilon S_\epsilon||\frac{1}{S_\epsilon}\int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y^{sp}(\tau), \bar{z}^{sp}(\tau_l))d\tau - v_l|| + \epsilon S_\epsilon||v_l - \tilde{v}_l|| \leq ||z^{sp}(\tau_l) - \zeta_l||$$

$$+\epsilon S_\epsilon L(M\epsilon S_\epsilon + \nu_T(\epsilon)) + \epsilon S_\epsilon \omega_1(\epsilon) + \epsilon S_\epsilon L(||z^{sp}(\tau_l) - \zeta_l|| + \nu_T(\epsilon)) \ , \quad l = 0, 1, ...K_\epsilon \ ,$$

where L is a Lipschitz constant. Using the last inequality and applying Proposition 5.1 from [12], one can obtain that

$$||z^{sp}(\tau_l) - \zeta_l|| \leq \omega_2(\epsilon) \quad \forall l = 0, 1, ...K_\epsilon \tag{5.88}$$

for some $\omega_2(\epsilon)$ tending to zero as ϵ tends to zero.

Similarly to the proof of Lemma 2.1 in [12], define the piece-wise linear function $\zeta(t) : [0, T] \to R^n$ by the equations:

$$\zeta(t) \stackrel{\text{def}}{=} \zeta_l + (t - t_l)\tilde{v}_l \ \ \forall t \in [t_l, t_{l+1}) \ , \qquad l = 0, 1, ...K_\epsilon - 2 \ , \tag{5.89}$$

and $\zeta(t) \stackrel{\text{def}}{=} \zeta_{K_\epsilon - 1} + (t - t_{K_\epsilon - 1})\tilde{v}_{K_\epsilon - 1} \ \ \forall t \in [t_{K_\epsilon - 1}, T]$, where $t_l \stackrel{\text{def}}{=} \epsilon\tau_l$. By (2.21), for any $t \in (t_l, t_{l+1})$,

$$dist(\dot{\zeta}(t), V(\zeta(t))) = dist(\tilde{v}_l, V(\zeta(t))) \leq dist(\tilde{v}_l, V(\zeta_l)) + d_H(V(\zeta_l), V(\zeta(t)))$$

$$\leq L||\zeta_l - \zeta(t)|| \leq LM\epsilon S_\epsilon \ ,$$

where M is as in (5.78) and it is taken into account that

$$\sup\{||\eta|| \mid \eta \in V(z) \ , z \in N(z)\} \leq M \ . \tag{5.90}$$

Using Filippov's theorem (see e.g. [7] , p. 401), one can conclude now that there exists a solution $\tilde{z}^a(t)$ of the ADI (2.16) such that

$$\max_{t\in[0,T]} ||\tilde{z}^a(t) - \zeta(t)|| \leq c\epsilon S_\epsilon \ , \qquad c \ = \ const \ , \tag{5.91}$$

which leads to (see (5.88))

$$||z^{sp}(\tau_l) - \tilde{z}^a(\epsilon\tau_l)|| \leq ||z^{sp}(\tau_l) - \zeta_l|| + ||\zeta_l - \tilde{z}^a(t_l)|| \leq \omega_2(\epsilon) + c\epsilon S_\epsilon \quad \forall l = 0, 1, ...K_\epsilon \ .$$

The latter implies (5.72) with $\mu'_T(\epsilon) = \omega_2(\epsilon) + O(\epsilon S_\epsilon)$ since

$$||z^{sp}(\tau) - z^{sp}(\tau_l)|| \leq M\epsilon S_\epsilon \ , \qquad \forall \tau \in [\tau_l, \tau_{l+1}] \ , \tag{5.92}$$

$$||\tilde{z}^a(\epsilon\tau) - \tilde{z}^a(\epsilon\tau_l)|| \leq M\epsilon S_\epsilon \quad \ \forall \tau \in [\tau_l, \tau_{l+1}] \ . \tag{5.93}$$

Let us now establish that there exists a solution $z^a(t)$ of the ADI (2.16) which is viable in Z on $[0, T]$ and satisfies (5.73). Let $\mathcal{M}_\mathcal{T}$ stand for the set of all solutions of the ADI (2.16) which are viable in Z on $[0, T]$. Note that from the fact $V(z)$ is convex and compact valued (see Assumption II(ii)), it follows that $\mathcal{M}_\mathcal{T}$ is compact in the metric of uniform convergence $\rho_T(z'(\cdot), z''(\cdot)) \stackrel{\text{def}}{=} \max_{t\in[0,T]} ||z'(t) - z''(t)||$. Let $\mu''_T(\epsilon) \stackrel{\text{def}}{=} \min_{z'(\cdot)\in\mathcal{M}_\mathcal{T}} \rho_T(z'(\cdot), \tilde{z}^a(\cdot))$. The required statement will be established if one shows that $\lim_{\epsilon\to 0} \mu''_T(\epsilon) = 0$. Assume it is not true, then there exist a positive number $\beta > 0$ and a sequence $\epsilon_i \to 0$ such that

$$\min_{z'(\cdot)\in\mathcal{M}_\mathcal{T}} \rho_T(z'(\cdot), \tilde{z}^a_i(\cdot)) \geq \beta \quad \forall i = 1, 2, ... \ ,$$

where $\tilde{z}^a_i(\cdot)$ stands for $\tilde{z}^a(\cdot)$ with $\epsilon = \epsilon_i$. Again using the fact that $V(z)$ is convex and compact valued, one may assume (without loss of generality) that $\tilde{z}^a_i(\cdot)$ converges to a solution $\tilde{z}^a_*(\cdot)$ of the ADI (2.16) ($\lim_{i\to\infty} \rho_T(\tilde{z}^a_i(\cdot), \tilde{z}^a_*(\cdot)) = 0$). This "limit" solution will satisfy the inequality

$$\min_{z'(\cdot)\in\mathcal{M}_\mathcal{T}} \rho_T(z'(\cdot), \tilde{z}^a_*(\cdot)) \geq \beta \ . \tag{5.94}$$

On the other hand, by (5.70) and (5.72), one can write down

$$\max_{t\in[0,T]} dist(\tilde{z}^a_i(t), Z) \leq \nu_T(\epsilon_i) + \mu'_T(\epsilon_i) \quad \Rightarrow \quad \max_{t\in[0,T]} dist(\tilde{z}^a_*(t), Z) = 0 \quad \Rightarrow \quad \tilde{z}^a_*(\cdot) \in \mathcal{M}_\mathcal{Z} \ .$$

The latter inclusion contradicts (5.94) and, hence, proves (5.73). □

Proof of Theorem 2.1(ii). Let $z^a(t)$ be a solution of the ADI (2.16) which is viable in Z on $[0, T]$ and has the initial conditions (2.23). The required statement will be proved if one shows that there exists a solution $(y^{sp}(\tau), z^{sp}(\tau))$ of (5.68)-(5.69) having the initial conditions (2.25) and satisfying (5.70) such that the inequality (5.71) is valid.

Let us again partition the interval $[0, \frac{T}{\epsilon}]$ by the points (5.74), this time with

$$S_\epsilon = \frac{1}{2L} ln(\frac{1}{\epsilon}) \ , \tag{5.95}$$

where L is a Lipschitz constant of the function $f(u, y, z)$ (with respect to y and z). On the interval $[\tau_0, \tau_1)$, define a control $u(\tau)$ in such a way that the corresponding to this control solution $y_0(\tau)$ of the associated system (2.7), obtained with $z = \zeta_0$ and the initial conditions $y(0) = y_0$, satisfies the relationships:

$$y_0(\tau) \in Y(\zeta_0) \qquad \forall \tau \in [\tau_0, \tau_1] \ , \tag{5.96}$$

$$|| \frac{1}{S_\epsilon} \int_{\tau_0}^{\tau_1} g(u(\tau), y_0(\tau), \zeta_0) d\tau \ - \ v_0 || \leq \nu_2(S_\epsilon) \ , \tag{5.97}$$

where v_0 is the projection of $\frac{1}{\epsilon S_\epsilon} \int_{t_0}^{t_1} \dot{z}^a(t) dt$ onto $V(\zeta_0)$. As in the proof of Theorem 2.1(i), here and in what follows, $t_l \stackrel{\text{def}}{=} \epsilon \tau_l \ \forall l = 0, 1, ..., K_\epsilon + 1$. The fact that the control $u(\tau)$ ensuring the validity of (5.96)-(5.97) exists, follows from Assumption II(ii) (see (2.14)). Let $(y^{sp}(\tau), z^{sp}(\tau))$ be the solution of (5.68)-(5.69) having the initial conditions (2.23), which is obtained with the given control $u(\tau)$ on the interval $[\tau_0, \tau_1]$. Note that, similarly to (5.80), one obtains that

$$||y^{sp}(\tau) - y_0(\tau)|| \leq L \int_{\tau_0}^{\tau} (||y^{sp}(s) - y_0(s)|| + ||z^{sp}(s) - z_0||) ds \ . \tag{5.98}$$

Since $||z^{sp}(s) - z_0|| \leq M \epsilon S_\epsilon \ \forall s \in [\tau_0, \tau_1]$, one can now apply Gronwall-Bellman lemma to obtain

$$||y^{sp}(\tau) - y_0(\tau)|| \leq L S_\epsilon (M \epsilon S_\epsilon) e^{L S_\epsilon} \leq \epsilon^{\frac{1}{4}} \qquad \forall \tau \in [\tau_0, \tau_1] \ , \tag{5.99}$$

where the validity of the last inequality (for ϵ small enough) follows from (5.95).

Assume that the control $u(\tau)$ has been defined on the intervals $[\tau_0, \tau_1), ..., [\tau_{l-1}, \tau_l)$ ($l = 0, 1, ..., K_\epsilon - 1$) and that the solution $(y^{sp}(\tau), z^{sp}(\tau))$ of (5.68)-(5.69) obtained with this control satisfies the inequalities

$$dist(y^{sp}(\tau_i), Y(z^{sp}(\tau_i))) \leq \epsilon^{\frac{1}{4}} + \kappa(M \epsilon S_\epsilon) \ , \quad i = 0, 1, ..., l \tag{5.100}$$

and

$$\max_{\tau \in [\tau_{i-1}, \tau_i]} dist(y^{sp}(\tau), Y(z^{sp}(\tau))) \leq \epsilon^{\frac{1}{4}} + \kappa(M \epsilon S_\epsilon) + \phi(\epsilon^{\frac{1}{4}} + \kappa(M \epsilon S_\epsilon)) \ , \ \ i = 1, ..., l \tag{5.101}$$

where $\kappa(\cdot)$, $\phi(\cdot)$ and M are as in (2.3), (2.8) and (5.78) respectively. Let us extend the definition of the control to the interval $[\tau_l, \tau_{l+1})$.

Let $\tilde{v}_l$ be the projection of $\frac{1}{\epsilon S_\epsilon} \int_{t_l}^{t_{l+1}} \dot{z}^a(t) dt$ onto $V(z^a(t_l))$ and v_l be the projection of $\tilde{v}_l$ onto $V(z^{sp}(\tau_l))$. Denote by $y_l(\tau)$ the solution of the associated system (2.7) considered on the interval $[\tau_l, \tau_{l+1}]$ with $z = z^{sp}(\tau_l)$ and with the initial conditions $y_l(\tau_l) = y^{sp}(\tau_l)$. Define the control $u(\tau)$ on the interval $[\tau_l, \tau_{l+1})$ in such a way that $y_l(\tau)$ has the following properties:

$$y_l(\tau_l + a_l) \in Y(z^{sp}(\tau_l)) \ , \tag{5.102}$$

$$dist(y_l(\tau), Y(z^{sp}(\tau_l))) \leq \phi(\epsilon^{\frac{1}{4}} + \kappa(M \epsilon S_\epsilon)) \quad \forall \tau \in [\tau_l, \tau_l + a_l] \tag{5.103}$$

for some $a_l \in [0, a]$ and

$$y_l(\tau) \in Y(z^{sp}(\tau_l)) \quad \forall \tau \in [\tau_l + a_l, \tau_{l+1}] \ , \tag{5.104}$$

$$|| \frac{1}{S_\epsilon - a_l} \int_{\tau_l + a_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), z^{sp}(\tau_l)) d\tau - v_l || \leq \nu_2(S_\epsilon - a_l) \ . \tag{5.105}$$

The fact that a control $u(\tau)$ which ensures the validity of (5.102)-(5.103) and (5.104)-(5.105) exists follows from Assumption I (ii) and Assumption II (ii) respectively.

The extension of the solution $(y^{sp}(\tau), z^{sp}(\tau))$ of (5.68)-(5.69) to the interval $[\tau_l, \tau_{l+1}]$ obtained with this control satisfies the inequalities (5.92) and

$$||y^{sp}(\tau) - y_l(\tau)|| \leq \ \epsilon^{\frac{1}{4}} \qquad \forall \tau \in [\tau_l, \tau_{l+1}] \ , \tag{5.106}$$

the validity of the latter being established similarly to (5.99). Using (5.106), (5.104), (5.92), as well as (2.3), one obtains

$$dist(y^{sp}(\tau_{l+1}), Y(z^{sp}(\tau_{l+1}))) \leq ||y^{sp}(\tau_{l+1}) - y_l(\tau_{l+1})|| + dist(y_l(\tau_{l+1}), Y(z^{sp}(\tau_l))) \tag{5.107}$$

$$+d_H(Y(z^{sp}(\tau_l)), Y(z^{sp}(\tau_{l+1}))) \leq \ \epsilon^{\frac{1}{4}} + \kappa(M\epsilon S_\epsilon) \ ,$$

which extends the validity of (5.100) to $i = l+1$. Similarly, using (5.103),

$$dist(y^{sp}(\tau), Y(z^{sp}(\tau))) \leq ||y^{sp}(\tau) - y_l(\tau)|| + dist(y_l(\tau), Y(z^{sp}(\tau_l))) \tag{5.108}$$

$$+d_H(Y(z^{sp}(\tau_l)), Y(z^{sp}(\tau))) \leq \ \epsilon^{\frac{1}{4}} + \kappa(M\epsilon S_\epsilon) + \phi(\epsilon^{\frac{1}{4}} + \kappa(M\epsilon S_\epsilon)) \quad \forall \tau \in [\tau_l, \tau_{l+1}] \ .$$

This implies the validity of (5.101) for $i = l+1$.

Proceeding as above, one defines the control $u(\tau)$ and the corresponding solution $(y^{sp}(\tau), z^{sp}(\tau))$ of (5.68)-(5.69) so that (5.105), (5.108) are satisfied on each interval $[\tau_l, \tau_{l+1})$, $l = 0, 1, ...K_\epsilon - 1$ (with (5.97) being interpreted as (5.105) with $l = 0$ and $a_0 \stackrel{\text{def}}{=} 0$). On the "last" interval $[\tau_{K_\epsilon}, \tau_{K_\epsilon+1}]$ the controls is chosen so that just (5.108) with $l = K_\epsilon$ is satisfied.

Subtract now the equation

$$z^a(\epsilon\tau_{l+1}) = z^a(\epsilon\tau_l) + \int_{\epsilon\tau_l}^{\epsilon\tau_{l+1}} \dot{z}^a(t)dt \tag{5.109}$$

from the equation (5.87) and obtain the inequality

$$||z^{sp}(\tau_{l+1}) - z^a(\epsilon\tau_{l+1})|| \leq ||z^{sp}(\tau_l) - z^a(\epsilon\tau_l)|| \tag{5.110}$$

$$+\epsilon \int_{\tau_l}^{\tau_{l+1}} ||g(u(\tau), y^{sp}(\tau), z^{sp}(\tau)) - g(u(\tau), y_l(\tau), z^{sp}(\tau_l))||d\tau$$

$$+\epsilon S_\epsilon || \frac{1}{S_\epsilon} \int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), z^{sp}(\tau_l))d\tau - v_l|| + \epsilon S_\epsilon ||v_l - \tilde{v}_l|| + \epsilon S_\epsilon ||\tilde{v}_l - \frac{1}{\epsilon S_\epsilon} \int_{\epsilon\tau_l}^{\epsilon\tau_{l+1}} \dot{z}^a(t)dt|| \ .$$

By (5.92) and (5.106),

$$\epsilon \int_{\tau_l}^{\tau_{l+1}} ||g(u(\tau), y^{sp}(\tau), z^{sp}(\tau)) - g(u(\tau), y_l(\tau), z^{sp}(\tau_l))||d\tau \leq \epsilon S_\epsilon L(M\epsilon S_\epsilon + \epsilon^{\frac{1}{4}}) \ . \tag{5.111}$$

By (5.105),

$$|| \frac{1}{S_\epsilon} \int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), z^{sp}(\tau_l))d\tau - v_l||$$

$$\leq || \frac{1}{S_\epsilon} \int_{\tau_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), z^{sp}(\tau_l))d\tau - \frac{1}{S_\epsilon - a_l} \int_{\tau_l + a_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), z^{sp}(\tau_l))d\tau||$$

$$+||\frac{1}{S_\epsilon - a_l}\int_{\tau_l+a_l}^{\tau_{l+1}} g(u(\tau), y_l(\tau), z^{sp}(\tau_l))d\tau - v_l|| \leq \frac{2Ma}{S_\epsilon - a} + \nu_2(S_\epsilon - a) \ , \tag{5.112}$$

where it is taken into account that $a_l \leq a$ and it is assumed (without loss of generality) that $\nu_2(S_\epsilon - a_l) \leq \nu_2(S_\epsilon - a)$. Taking into account (2.21) and the fact that, by definition, v_l is the projection of $\tilde{v}_l \in V(z^a(\epsilon\tau_l))$ onto $V(z^{sp}(\tau_l))$, one can obtain that

$$||v_l - \tilde{v}_l|| = dist(\tilde{v}_l, V(z^{sp}(\tau_l)) \leq d_H(V(z^a(\epsilon\tau_l), V(z^{sp}(\tau_l)) \leq L||z^{sp}(\tau_l) - z^a(\epsilon\tau_l)|| \ . \tag{5.113}$$

To evaluate the last term in the right hand side of (5.110), let us note that, by (2.21) and (5.93), for almost all $\tau \in [\tau_l, \tau_{l+1}]$,

$$\dot{z}^a(t) \in V(z^a(t)) \subset V(z^a(\epsilon\tau_l)) + (LM\epsilon S_\epsilon)B' \ \Rightarrow \ \frac{1}{\epsilon S_\epsilon}\int_{\epsilon\tau_l}^{\epsilon\tau_{l+1}} \dot{z}^a(t)dt \in V(z^a(\epsilon\tau_l)) + (LM\epsilon S_\epsilon)B' \ ,$$

where B' is the closed unit ball in R^n. Since $\tilde{v}_l$ is the projection of $\frac{1}{\epsilon S_\epsilon}\int_{\epsilon\tau_l}^{\epsilon\tau_{l+1}} \dot{z}^a(t)dt$ onto $V(z^a(\epsilon\tau_l))$, it follows that

$$||\tilde{v}_l - \frac{1}{\epsilon S_\epsilon}\int_{\epsilon\tau_l}^{\epsilon\tau_{l+1}} \dot{z}^a(t)dt|| = dist(\frac{1}{\epsilon S_\epsilon}\int_{\epsilon\tau_l}^{\epsilon\tau_{l+1}} \dot{z}^a(t)dt, V(z^a(\epsilon\tau_l))) \leq LM\epsilon S_\epsilon \ . \tag{5.114}$$

Substituting (5.111)-(5.114) into (5.110), one can obtain that, for $l = 0, 1, ...K_\epsilon - 1$,

$$||z^{sp}(\tau_{l+1}) - z^a(\epsilon\tau_{l+1})|| \leq ||z^{sp}(\tau_l) - z^a(\epsilon\tau_l)|| + L\epsilon S_\epsilon||z^{sp}(\tau_l) - z^a(\epsilon\tau_l)|| + \epsilon S_\epsilon\gamma(\epsilon) \ , \tag{5.115}$$

where $\gamma(\epsilon) \stackrel{\text{def}}{=} L(M\epsilon S_\epsilon + \epsilon^{\frac{1}{4}}) + \frac{2Ma}{S_\epsilon - a} + \nu_2(S_\epsilon - a) + LM\epsilon S_\epsilon$ tends to zero as ϵ tends to zero. By virtue of Proposition 5.1 from [12], the validity of (5.115) implies that

$$||z^{sp}(\tau_l) - z^a(\epsilon\tau_l)|| \leq c_1\gamma(\epsilon) \, l = 0, 1, ..., K_\epsilon \ , \quad c_1 = \ const \ , \tag{5.116}$$

which, in turn, by (5.92)-(5.93),implies the validity of (5.71) with $\mu_T(\epsilon) = c_1\gamma(\epsilon) + O(\epsilon S_\epsilon)$. To complete the proof, one needs to verify that $(y^{sp}(\tau), z^{sp}(\tau))$ satisfies (5.70). Note that from the fact that (5.108) is valid for $l = 0, 1, ..., K_\epsilon$, it follows that

$$dist(y^{sp}(\tau), Y(z^{sp}(\tau))) \leq \ \epsilon^{\frac{1}{4}} + \kappa(M\epsilon S_\epsilon) + \phi(\epsilon^{\frac{1}{4}} + \kappa(M\epsilon S_\epsilon)) \stackrel{\text{def}}{=} \omega_3(\epsilon) \quad \forall \tau \in [0, \frac{T}{\epsilon}] \ . \tag{5.117}$$

Using this and (5.71), one obtains

$$dist((y^{sp}(\tau), z^{sp}(\tau)), D) \leq \min\{||y^{sp}(\tau) - y|| + ||z^{sp}(\tau) - z|| \mid y \in Y(z), z \in Z\}$$

$$\leq \min\{||y^{sp}(\tau) - y|| + ||z^{sp}(\tau) - z^a(\epsilon\tau)|| \mid y \in Y(z^a(\epsilon\tau))\} = dist(y^{sp}(\tau), Y(z^a(\epsilon\tau)))$$

$$+||z^{sp}(\tau) - z^a(\epsilon\tau)|| \leq \omega_3(\epsilon) + \mu_T(\epsilon) \quad \forall \in [0, \frac{T}{\epsilon}] \ ,$$

which establishes (5.70) with $\nu_T(\epsilon) \stackrel{\text{def}}{=} \omega_3(\epsilon) + \mu_T(\epsilon)$ and, thus, completes the proof. □

Proof of Theorem 2.2(i). Let us choose T_0 in such a way that

$$ae^{-bT_0} \stackrel{\text{def}}{=} \delta_0 < 1 \tag{5.118}$$

a and let $(y^{sp}_\epsilon(t), z^{sp}_\epsilon(t))$ be a solution of (2.1)-(2.2) such that the estimate (2.6) is valid. By Theorem 2.1, there exists a viable in Z on $[0, T]$ solution $\bar{z}^a_\epsilon(t)$ of the ADI (2.16) such that

$$||\bar{z}^a_\epsilon(t) - z^{sp}_\epsilon(t)|| \leq \mu_{T_0}(\epsilon) \quad \forall t \in [0, T_0]. \tag{5.119}$$

Using Theorem 2.1 again, one can establish that there exists a solution $\tilde{z}^a_\epsilon(t)$ of the ADI (2.16) viable in Z on the interval $[T_0, 2T_0]$ such that

$$||\tilde{z}^a_\epsilon(t) - z^{sp}_\epsilon(t)|| \leq \mu_{T_0}(\epsilon) \quad \forall t \in [T_0, 2T_0]. \tag{5.120}$$

By Assumption III, the solution $\bar{z}^a_\epsilon(t)$ of the ADI (2.16) used in (5.119) can be extended to the interval $[T_0, 2T_0]$ in such a way that, for any $t \in [T_0, 2T_0]$,

$$||\bar{z}^a_\epsilon(t) - \tilde{z}^a_\epsilon(t)|| \leq ae^{-b(t-T_0)}||\bar{z}^a_\epsilon(T_0) - \tilde{z}^a_\epsilon(T_0)|| \tag{5.121}$$

This along with (5.119)-(5.120) allow us to establish that, for any $t \in [T_0, 2T_0]$,

$$||\bar{z}^a_\epsilon(t) - z^{sp}_\epsilon(t)|| \leq ||\bar{z}^a_\epsilon(t) - \tilde{z}^a_\epsilon(t)|| + ||\tilde{z}^a_\epsilon(t) - z^{sp}_\epsilon(t)||$$
$$\leq ae^{-b(t-T_0)}[\ ||\bar{z}^a_\epsilon(T_0) - z^{sp}_\epsilon(T_0)|| + ||z^{sp}_\epsilon(T_0) - \tilde{z}^a_\epsilon(T_0)||\ ] + \mu_{T_0}(\epsilon)$$
$$\leq ae^{-b(t-T_0)}[\ ||\bar{z}^a_\epsilon(T_0) - z^{sp}_\epsilon(T_0)|| + \mu_{T_0}(\epsilon)\ ] + \mu_{T_0}(\epsilon).$$

Continuing in a similar fashion, one can define a solution $\bar{z}^a_\epsilon(t)$ of the ADI (2.16) on the interval $[0, \infty)$ (not necessarily viable in Z) such that the inequalities

$$||\bar{z}^a_\epsilon(t) - z^{sp}_\epsilon(t)|| \leq ae^{-b(t-lT_0)}[\ ||\bar{z}^a_\epsilon(lT_0) - z^{sp}_\epsilon(lT_0))|| + \mu_{T_0}(\epsilon)\ ] + \mu_{T_0}(\epsilon) \tag{5.122}$$

are satisfied for all $t \in [lT_0, (l+1)T_0],\ l = 0, 1, ...$. It follows (see (5.118)) that

$$||\bar{z}^a_\epsilon((l+1)T_0) - z^{sp}_\epsilon((l+1)T_0)|| \leq \delta_0[\ ||\bar{z}^a_\epsilon(lT_0) - z^{sp}_\epsilon(lT_0)|| + \mu_{T_0}(\epsilon)\ ] + \mu_{T_0}(\epsilon)$$
$$\Rightarrow \quad ||\bar{z}^a_\epsilon((l+1)T_0) - z^{sp}_\epsilon((l+1)T_0)|| \leq \frac{1+\delta_0}{1-\delta_0}\mu_{T_0}(\epsilon)\ .$$

These and (5.122) imply that

$$\sup_{t\in[0,\infty)} ||\bar{z}^a_\epsilon(t) - z^{sp}_\epsilon(t)|| \leq \mu_{T_0}(\epsilon)[\ a\frac{1+\delta_0}{1-\delta_0} + a\ ] + \mu_{T_0}(\epsilon) = \mu_{T_0}(\epsilon)(\frac{2a}{1-\delta_0} + 1) \stackrel{\text{def}}{=} \bar{\mu}(\epsilon)\ . \tag{5.123}$$

Denote by $\mathcal{M}$ the set of all solutions of the ADI (2.16) which are viable in Z. According to the conditions of the theorem, $\mathcal{M}$ is not empty and it can be also shown that it is compact in the metric $\rho(\cdot,\cdot)$ defined in (2.29). Let $\bar{\bar{\mu}}(\epsilon) \stackrel{\text{def}}{=} \min_{z(\cdot)\in\mathcal{M}} \rho(\bar{z}^a_\epsilon(\cdot), z(\cdot))$. Theorem 2.2(i) will be established if one shows that $\lim_{\epsilon\to 0} \bar{\bar{\mu}}(\epsilon) = 0$. Assume it is not true. Then there exist a positive number β and a sequence $\epsilon_i \to 0$ such that

$$\inf_{z^a(\cdot)\in\mathcal{M}} \rho(\bar{z}^a_{\epsilon_i}(\cdot), z^a(\cdot)) \geq \beta\ , \qquad i = 1, 2, ...\ .$$

Using the fact that the map $V(z)$ is convex and compact valued, one can show that there exist subsequences $\{\epsilon_{i_l}\} \subset \{\epsilon_{i_{l-1}}\}$, $l = 1, 2, ...$, with $\{\epsilon_{i_0}\} \stackrel{\text{def}}{=} \{\epsilon_i\}$, such that, for each l, $\bar{z}^a_{\epsilon_{i_l}}(\cdot)$ converges (in the uniform metric on $[0, l]$) to a solution the ADI (2.16). That is,

$$\lim_{\epsilon_{i_l}\to 0} \max_{t\in[0,l]} ||\bar{z}^a_\epsilon(\cdot) - \bar{\bar{z}}^a(\cdot))|| = 0\ .$$

Applying now the diagonalization argument, one can come to the conclusion that there exists a subsequence $\{\epsilon_{i'}\} \subset \{\epsilon_i\}$ and a solution $\bar{\bar{z}}^a(\cdot)$ of the ADI (2.16) defined on $[0, \infty)$ such that

$$\lim_{\epsilon_{i'}\to 0} \rho(\bar{z}^{\epsilon_{i'}}(\cdot), \bar{\bar{z}}^a(\cdot)) = 0 \qquad \Rightarrow \qquad \inf_{z^a(\cdot)\in\mathcal{M}} \rho(\bar{\bar{z}}^a(\cdot), z^a(\cdot)) \geq \beta\ . \tag{5.124}$$

From (2.6) and (5.123) it follows, however, that

$$\sup_{t\in[0,\infty)} dist(\bar{z}^a_{\epsilon_{i'}}(t), Z) \leq \nu(\epsilon_{i'}) + \bar{\mu}(\epsilon_{i'}) \quad \Rightarrow \quad \bar{\bar{z}}^a(t) \in Z \ \forall t \in [0,\infty) \quad \Rightarrow \quad \bar{\bar{z}}^a(\cdot) \in \mathcal{M}$$

The latter contradicts (5.124) and thus proves the required statement. □

Proof of Theorem 2.2(ii). Let $z^a(t)$ be a viable in D solution of the ADI (2.16). Choose T_0 to satisfy (5.118). By Theorem 2.1(ii), there exists a solution $(y^{sp}_\epsilon(t), z^{sp}_\epsilon(t))$ of the SPCS (2.1)-(2.2) which is near viable in D on $[0, T_0]$, has the initial conditions $z^{sp}_\epsilon(0) = z^a(0)$ and $y^{sp}_\epsilon(0) \in Y(z^a(0))$, and which satisfies the inequality

$$||z^{sp}_\epsilon(t) - z^a(t)|| \leq \mu_{T_0}(\epsilon) \quad \forall t \in [0, T_0] \ . \tag{5.125}$$

Construct a solution $\tilde{z}^a_\epsilon(t)$ of the ADI (2.16) such that

$$||\tilde{z}^a_\epsilon(t) - z^a(t)|| \leq ae^{-b(t-T_0)}||\tilde{z}^a_\epsilon(T_0) - z^a(T_0)|| \ \ \forall t \in [T_0, 2T_0] \ , \quad \tilde{z}^a_\epsilon(T_0) = z^{sp}_\epsilon(T_0) \ . \tag{5.126}$$

The existence of such a solution is implied by Assumption III. Note that $\tilde{z}^a_\epsilon(t)$ is not necessarily viable in D, but, still, using reasoning similar to that in the proof of Theorem 2.1(ii), one can extend the solution $(y^{sp}_\epsilon(t), z^{sp}_\epsilon(t))$ of the SPCS (2.1)-(2.2) to the interval $[T_0, 2T_0]$ in such a way that

$$||z^{sp}_\epsilon(t) - \tilde{z}^a_\epsilon(t)|| \leq \mu_{T_0}(\epsilon) \quad \forall t \in [T_0, 2T_0] \tag{5.127}$$

and

$$dist(y^{sp}_\epsilon(t), Y(z^{sp}_\epsilon(t)) \leq \mu_{T_0}(\epsilon) \quad \forall t \in [T_0, 2T_0] \tag{5.128}$$

By (5.126) and (5.127),

$$||z^{sp}_\epsilon(t) - z^a(t)|| \leq ||z^{sp}_\epsilon(t) - \tilde{z}^a_\epsilon(t)|| + ||\tilde{z}^a_\epsilon(t) - z^a(t)||$$

$$\leq \mu_{T_0}(\epsilon) + ae^{-b(t-T_0)}||z^{sp}_\epsilon(T_0) - z^a(T_0)|| \quad \forall t \in [T_0, 2T_0] \ .$$

Continuing in a similar way, one constructs a solution $(y^{sp}_\epsilon(t), z^{sp}_\epsilon(t))$ of the SPCS (2.1)-(2.2) on the interval $[0,\infty)$ in such a way that the relationships

$$||z^{sp}_\epsilon(t) - z^a(t)|| \leq \mu_{T_0}(\epsilon) + ae^{-b(t-lT_0)}||z^{sp}_\epsilon(lT_0) - z^a(lT_0)|| \tag{5.129}$$

and

$$dist(y^{sp}_\epsilon(t), Y(z^{sp}_\epsilon(t)) \leq \mu_{T_0}(\epsilon) \tag{5.130}$$

are satisfied for $t \in [lT_0, (l+1)T_0]$, $l = 0, 1, ...$. From (5.118) and (5.129) it follows that

$$||z^{sp}_\epsilon((l+1)T_0) - z^a((l+1)T_0)|| \leq \mu_{T_0}(\epsilon) + \delta_0||z^{sp}_\epsilon(lT_0) - z^a(lT_0)||$$

$$\Rightarrow \quad ||z^{sp}_\epsilon(lT_0) - z^a(lT_0)|| \leq \frac{\mu_{T_0}(\epsilon)}{1-\delta_0} \ , \quad l = 0, 1, ... \ .$$

The latter and (5.129) implies that

$$\sup_{t\in[0,\infty)} ||z^{sp}_\epsilon(t) - z^a(t)|| \leq (1 + \frac{a}{1-\delta_0})\mu_{T_0}(\epsilon) \stackrel{\text{def}}{=} \mu(\epsilon)$$

which, in turn, implies the validity of (2.31). Taking into account (2.3) and (5.130), one obtains now that

$$dist((y_\epsilon^{sp}(t), z_\epsilon^{sp}(t)), D) \leq \min\{||y_\epsilon^{sp}(t) - y|| + ||z_\epsilon^{sp}(t) - z^a(t)|| \mid y \in Y(z^a(t))\} \leq dist(y_\epsilon^{sp}(t), Y(z^a(t))$$

$$+\mu(\epsilon) \ \leq \ dist(y_\epsilon^{sp}(t), Y(z_\epsilon^{sp}(t)) + \kappa(\mu(\epsilon)) + \mu(\epsilon) \ \leq \ \mu_{T_0}(\epsilon) + \kappa(\mu(\epsilon)) + \mu(\epsilon) \quad \forall t \in [0, \infty) \ .$$

Hence, $(y_\epsilon^{sp}(t), z_\epsilon^{sp}(t))$ satisfies (2.6) with $\nu(\epsilon) = \mu_{T_0}(\epsilon) + \kappa(\mu(\epsilon)) + \mu(\epsilon)$, that is, it is near viable in D. This completes the proof. □

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