FAST POSITIONING USING THE LAMBDA-METHOD

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Abstract
Ambiguity resolution is essential for precise GPS diffe rential positioning. At the IAG General Meeting in Beijing 1993 [1], a new method for fast integer ambiguity estimation has been introduced. Some aspects of the theory have been reviewed at the DSNS 94 conference [2]. The LAMBDA-method, which stands for Least-squares AMBiguity Decorrelation Adj ustment, first decorrelates the ambiguities and next co mputes integer least-squares estimates for the ambiguities in a highly efficient way.

In this paper fast high precision positioning results are presented. For baselines up to 10 km the LAMBDA-method is able to come up with the correct integer estimates at a high success rate using only one epoch of data. These results show that the LAMBDA-method enables instantaneous ambiguity resolution and is therefore very well suited for real-time precise navigation.

Contents

1. Introduction
In relative positioning the unknowns of primary interest are the baseline coordinates. In high precision relative positioning the use of carrier phase observations is imperative, as their level of precision is some orders of magnitude better than that of the code observations. The carrier phase observations, however, are ambiguous and in order to obtain precise positioning results either a long observation time span or resolution of the ambiguities is required.

In this paper we will demonstrate the capabilities of the LAMBDA-method. This method is able to correctly estimate the integer ambiguities in a highly efficient way. For this estimation only a short observation time span is needed and therefore fast, precise GPS differential positioning is enabled.

The applications lie in precise navigation and surveying. In precise navigation the second receiver is in permanent motion and after a loss of lock the ambiguities should be fixed instantaneously as real time positioning results are required. In surveying, rapid static and kinematic techniques have emerged and in these applications ambiguity fixing should take place as soon as possible, from the viewpoint of surveying productivity.

2. Parameter estimation
In the parameter estimation, based on a model of double difference observation equations, two types of unknowns occur: baseline coordinates and double difference ambiguities. The *3* baseline coordinates are contained in vector b and the n double difference ambiguities in vector a. The m observations are collected in vector y. With e the vector of measurement noise, the full set of observation equations reads

$$y = \boldsymbol{A}a + \boldsymbol{B}b + e \quad (1)$$

First the **float solution** is computed, in which both the baseline coordinates and double difference ambiguities are treated as real valued parameters, thus $a \in R^n$ and $b \in R^3$. Based on the real valued estimates for the ambiguities $\hat{a}$ (and their variance covariance matrix $\sigma_o^2 Q_{\hat{a}}$), the **integer ambiguity estimation** is carried out, after which the ambiguities are fixed to the values of the integer least squares estimate $\check{a}$. With these additional constraints the **fixed solution** for the baseline coordinates $\check{b}$ is computed. It is important to note that fixing of the ambiguities is done for the reason of the large gain in precision of the baseline coordinates. For a discussion in more detail of the procedure for parameter estimation the reader is referred to [1] and [3].

3. Baseline-precision
In this section we will briefly consider the precision of the baseline coordinates in both the float and fixed solution. For the results in this paper the parameter

estimation is based on a simple mathematical model appropriate for short baselines. A standard troposphere model is used, providing a-priori tropospheric corrections for the observations, and no ionospheric model is used. The chosen a-priori standard deviation of unit weight σ_o and weighting of code and phase observables, correspond to σ=3mm for the phase observable and σ=60cm for the code. We assume that the observables are normally distributed and that no mutual- nor time-correlation exists. Dual frequency data are used.

In figure 1 the formal precision of the baseline coordinates is given. Only phase observations are used. In the graph the standard deviations [m] for the three WGS84 baseline coordinates of the float solution $\sigma_{\hat{x}}$ $\sigma_{\hat{y}}$ $\sigma_{\hat{z}}$ and of the fixed solution $\sigma_{\check{x}}$ $\sigma_{\check{y}}$ $\sigma_{\check{z}}$ are given as function of the observation time span, expressed in minutes. Note that both axes have logarithmic scale. For each solution only 2 epochs of data are used to clearly show only the effect of the length of the time span (i.e. the change in receiver satellite geometry) and not that of the growing redundancy.

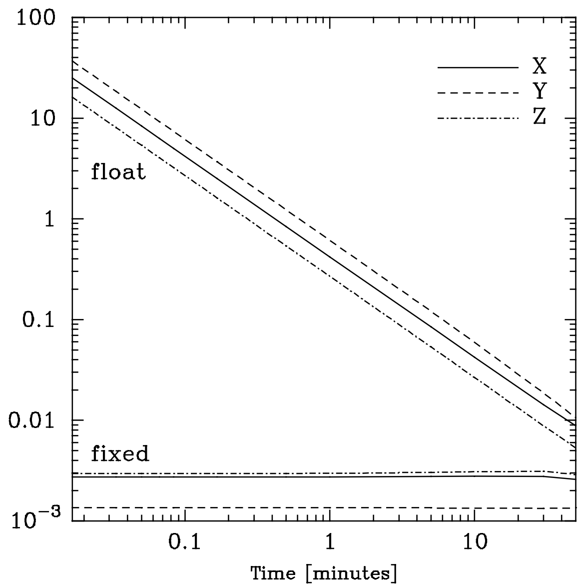


Figure 1: Standard deviation in meters of baseline coordinates of float and fixed solution, baseline 1, session 2

In figure 1 we see that the precision of the baseline coordinates of the fixed solution, which is typically at the mm-cm level, is nearly independent of the length of the observation time span. Similar conclusions are drawn in [6]. The precision of the float solution is much worse than the fixed solution, in particular for short observation time spans. In these cases the decrease in standard deviation for the baseline coordinates, by fixing the ambiguities, is tremendous.

4. Integer ambiguity estimation and LAMBDA-method

Integer ambiguity estimation is the key to the fixed solution, i.e. precise differential positioning results. With the least-squares principle we know that the most likely integer vector for the ambiguities is that vector that is nearest to the vector of real valued estimates of the float solution. An *n*-dimensional ambiguity hyper-ellipsoid needs to be searched through for the grid-point that is nearest to $\hat{a}$ in the metric of matrix $Q_{\hat{a}}^{-1}$. Previously, methods for ambiguity resolution lacked of efficiency and were therefore not able to estimate the integer ambiguities within a reasonable amount of computation time.

The LAMBDA-method has been introduced in [1] and reviewed in [2] and [3]. The two main features of the method are:

1. the decorrelation of the ambiguities
2. the integer ambiguity estimation

The efficiency of the method is explained by the decorrelation step. Using a short observation time span, the ambiguities (of the float solution) have large variances and are heavily correlated. The ambiguities are transformed such that the shape of the *n*-dimensional ambiguity hyper-ellipsoid is optimized as much as possible, for the search in the second step. The transformed ambiguities are largely decorrelated and their variances have been drastically reduced. This allows a fast and efficient integer estimation of the ambiguities. The efficiency of the method has been explained in detail by analysis of the precision and correlation of the GPS double difference ambiguities in [3] and [4].

5. Fast positioning results

In this section we will present the results of the integer ambiguity estimation. Two baselines are considered, which were measured on May the 5th in 1994 at the former Air Force Base Ypenburg (ϕ=52º02’ N; λ=4º21’ E). The characteristics of the experiment are:-
session 1: 16:50-17:40 UTC (s1)

- session 2: 19:55-20:45 UTC (s2)
- session 3: 23:15-00:05 UTC (s2)
- baseline 1: 2250 m
- baseline 2: 10440 m
- Trimble 4000 SSE receivers (cross correlation)
- 7 satellites in view (during the full session)
- elevation cutoff 10º
- sampling rate 1 second
- GPS in Initial Operational Capability (IOC) Anti Spoofing on
- dual frequency code and full wavelength phase observations
- data free of cycle slips

In tables 1 and 2 we give the rate of success in correctly estimating the integer ambiguities for several cases. The sessions, each of 50 minutes, are divided in batches of 30 seconds, giving 100 batches. Per batch both a float and fixed solution were computed for each time span within this batch: 0, 1, 5 and 15 seconds. For a time span of 0 seconds only one epoch of data is used, in all other cases two epochs are used. In the table we give the number of batches for which correct integer estimates for the ambiguities were computed. A deterministic validation took place: we checked whether the computed integer estimates correspond to the ones obtained using the full 50 minutes of data, which are considered to give the 'true' integer values for the ambiguities.

time span [s]	baseline 1		baseline 2	
	s1	s3	s2	s3
1	99	100	75	40
5	100	100	96	76
15	100	100	99	91

Table 1: Success rate of integer ambiguity estimation with dual frequency phase data

time span [s]	baseline 1		baseline 2	
	s1	s3	s2	s3
0	100	100	100	98
1	100	100	100	98
5	100	100	100	99
15	100	100	100	98

Table 2: Success rate of integer ambiguity estimation with dual frequency phase and code data

time span [s]	baseline 1, s1	
	4 svs	7 svs
0	98	100
1	98	100
5	100	100
15	100	100

Table 3: Success rate of integer ambiguity estimation with dual frequency phase and code data using 4 and 7 satellites.

Table 2 shows that with dual frequency phase and code data instantaneous correct integer estimation is possible at a nearly 100% rate of success. With dual frequency phase data, see table 1, correct integer estimation may require two epochs of data over a 15 seconds time span.

In table 3 the rates of success for baseline 1 are given using only 4 satellites. Phase and code data were used. The dataset of session 1 originally contained 7 satellites. The 4 satellites dataset was created by removing the three lowest elevation satellites, which corresponds to severe shadowing. Table 3 shows that at the 2.2 km baseline instantaneous ambiguity resolution is possible with only 4 satellites at a high rate of success!

6. Validation

To guarantee a certain quality of the results, the parameter estimation must be accompanied by a testing procedure, in which the validity of the model used in the estimation, is checked. In this section a first attempt is made to develop a viable procedure for the validation of the integer ambiguity estimate. The validation is carried out by means of statistical tests. Therefore we specify the following hypotheses:

$$(2) \quad \begin{array}{llll} H_1: & y = \boldsymbol{A}\boldsymbol{a} + \boldsymbol{B}\boldsymbol{b} + \boldsymbol{e} & \textit{with} & \boldsymbol{a} \in \boldsymbol{R}^n \\ & & \textit{and} & \boldsymbol{b} \in \boldsymbol{R}^3 \\ H_2: & y = \boldsymbol{A}\check{\boldsymbol{a}} + \boldsymbol{B}\boldsymbol{b} + \boldsymbol{e} & \textit{with} & \boldsymbol{b} \in \boldsymbol{R}^3 \\ H_3: & y \in \boldsymbol{R}^m & & \end{array}$$

The model of the float solution is given by H_1. In the hypothesis of the fixed solution H_2, we assume that we know a-priori the correct integer values of the ambiguities. In practice the ambiguities are fixed to the integer least squares estimates: $a := \check{a}$. The hypothesis H_3 is most relaxed; no restrictions are put on the vector of observations.

decision	reality	
	H true	*H* false
accept *H*	OK	β
reject *H*	α	OK

In deciding upon the acceptance of a certain hypothesis *H*, we can make two types of errors. The level of significance α, the type I error, is the probability of rejecting *H*, when in fact *H* is true. The type II error $\beta = 1 - \gamma$, is the probability of accepting *H*, when in fact *H* is false. Probability γ is called the power.

We prefer γ to be as large as possible: if the data does not correspond with the model and H is thus not true, we 'must' detect this and decide accordingly. The level of significance α will be allowed to be quite large, but β must be small at any price.

In this paper we assume that the model of the float solution holds true. Model H_1 can be checked on unspecified model errors by testing it against H_3 with a so-called Overall Model test [5]. We will concentrate on the validation of the integer ambiguity estimates. In this context we distinguish between **acceptance** and **discrimination** tests. The integer least squares estimate for the vector of ambiguities must be, **and** highly probable **and** clearly more probable than any other integer candidate vector.

Hypothesis H_2 will now be tested against the less relaxed hypothesis H_1. The teststatistic reads

(3) $$F(\check{a}) = \frac{(\hat{a}-\check{a})^T Q_{\hat{a}}^{-1}(\hat{a}-\check{a})}{n\ \hat{\sigma}^2}$$

which has an F-distribution under H_2 with n and m-n-3 degrees of freedom. The $\hat{\sigma}$ is the a-posteriori standard deviation of unit weight of the float solution. Teststatistic $F(\check{a})$ can be related to the ratio $\check{\sigma}/\hat{\sigma}$, where $\check{\sigma}$ is the a-posteriori standard deviation of unit weight of the fixed solution. If the test with statistic (3) is not passed, thus $F(\check{a}) \geq K_1$ where K_1 is the critical value, the confidence in the integer estimate $\check{a}$ is low and one should not fix the ambiguities. It is a test for **acceptance** of the model H_2. The critical value is set to K_1=15, except for the cases with code and phase data and two epochs of data in which K_1=31 is taken.

An additional test is needed to compare the likelihoods of integer candidates. A clear **discrimination** between the integer least squares estimate and any other candidate is required in order to proceed with the ambiguity fixing. The integer estimate $\check{a}$ should be clearly more probable than any other integer vector $\check{a}'$, e.g. the second best candidate. The teststatistic reads

(4) $$\Delta T(\check{a}',\check{a}) = T(\check{a}') - T(\check{a})$$

where

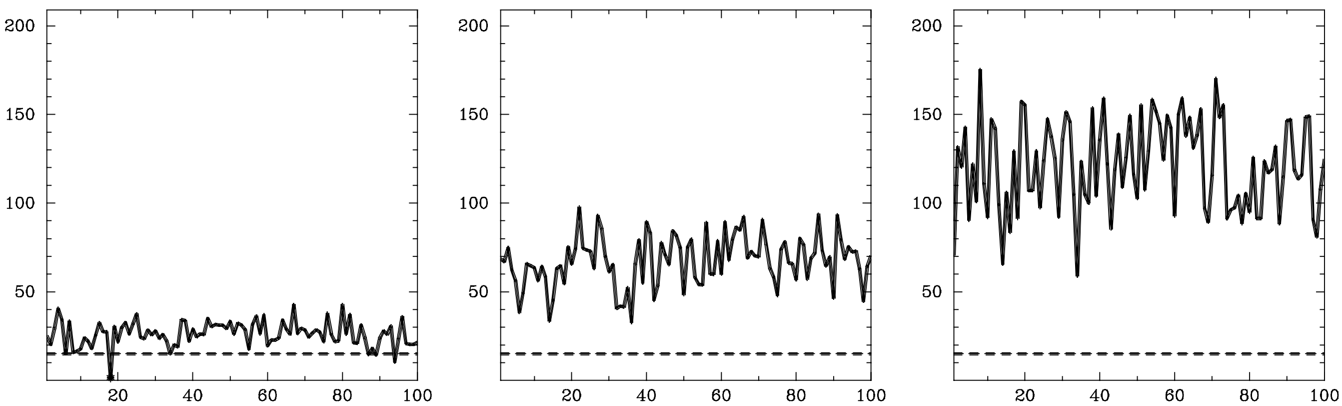
(5) $$T(\check{a}) = (\hat{a}-\check{a})^T \sigma_o^{-2} Q_{\hat{a}}^{-1}(\hat{a}-\check{a})$$

The value of (5) for the second best candidate, is found by replacing $\check{a}$ by $\check{a}'$. The difference in distance from the real valued estimate $\hat{a}$ to the second best candidate $\check{a}'$ must be sufficiently larger than the distance to the integer estimate $\check{a}$. The distances are measured in the metric of matrix $Q_{\hat{a}}^{-1}$ and note that the a-priori σ_o is common to both $T(\check{a})$ and $T(\check{a}')$. In the sequel the critical value K_2 is set to 15, which follows from preliminary experience with 7 satellite cases. If the test with statistic (4) is not passed, thus $\Delta T(\check{a}',\check{a}) \leq K_2$, the discrimination between $\check{a}$ and $\check{a}'$ is not positive.

In summary: if the following criterion is satisfied

(6) $$F(\check{a}) < K_1 \ \wedge \ \Delta T(\check{a}',\check{a}) > K_2$$

then the integer estimate is considered to be sufficiently likely and the likelihood that $\check{a}$ is the correct integer ambiguity vector is clearly larger than the likelihood of the second best vector $\check{a}'$. Condition (6) should be fulfilled in order to fix the vector of ambiguities to the integer estimate and compute the fixed solution. We are then confident to a (high) level about the results obtained with the model H_2.



(a) (b) (c)

Figure 2: Discrimination test, baseline 1, session 1, phase only, time span (a) 1 second, (b) 5 seconds and (c) 15 seconds.

The values of teststatistic (4) as function of the batch number are given in figure 2. For positive discrimination the value of the teststatistic should be larger than the critical value, which is indicated by the dashed line. An asterisk represents a case in which the integer ambiguity estimate is incorrect. Figure 2 concerns baseline 1 in session 1.

Enlarging the time span greatly enhances the discrimination. The mean value of teststatistic $\Delta T(\check{a}^{\prime},\check{a})$ in figure 2c is clearly larger than in figure 2a. The only one incorrect estimate in figure 2a can not be distinguished from the second best candidate and discrimination is negative indeed.

In figure 3 we give the values of both teststatistics (3) and (4). Figure 3 concerns baseline 2 in session 2. When simultaneously the teststatistic at left is below the critical value (dashed line) and the teststatistic on the right above, the validation of the integer ambiguity estimate is positive. In that case one is allowed to proceed with the fixed solution.

The results for the acceptance test in figure 3a are bad. Almost all integer estimates are rejected. This is not surprising as it concerns a 10 km baseline with 2 epochs of phase only data. When code and phase data are used, see figure 3d, correct instantaneous integer ambiguity estimation and positive validation are possible at a nearly 90% success rate.

time span [s]		correct		not correct	
		OK	α	β	OK
phase only	1	14	85	0	1
	5	65	35	0	0
	15	86	14	0	0
phase and code	0	100	0	0	0
	1	100	0	0	0
	5	100	0	0	0
	15	100	0	0	0

Table 4: Validation of baseline 1, session 1

time span [s]		correct		not correct	
		OK	α	β	OK
phase only	1	5	95	0	0
	5	31	69	0	0
	15	70	30	0	0
phase and code	0	96	4	0	0
	1	96	4	0	0
	5	99	1	0	0
	15	100	0	0	0

Table 5: Validation of baseline 1, session 3

time span [s]		correct		not correct	
		OK	α	β	OK
phase only	1	1	74	0	25
	5	29	67	0	4
	15	59	40	0	1
phase and code	0	89	11	0	0
	1	88	12	0	0
	5	93	7	0	0
	15	97	3	0	0

Table 6: Validation of baseline 2, session 2

time span [s]		correct		not correct	
		OK	α	β	OK
phase only	1	0	40	0	60
	5	4	72	0	24
	15	29	62	0	9
phase and code	0	61	37	0	2
	1	50	48	0	2
	5	59	40	0	1
	15	73	25	0	2

Table 7: Validation of baseline 2, session 3

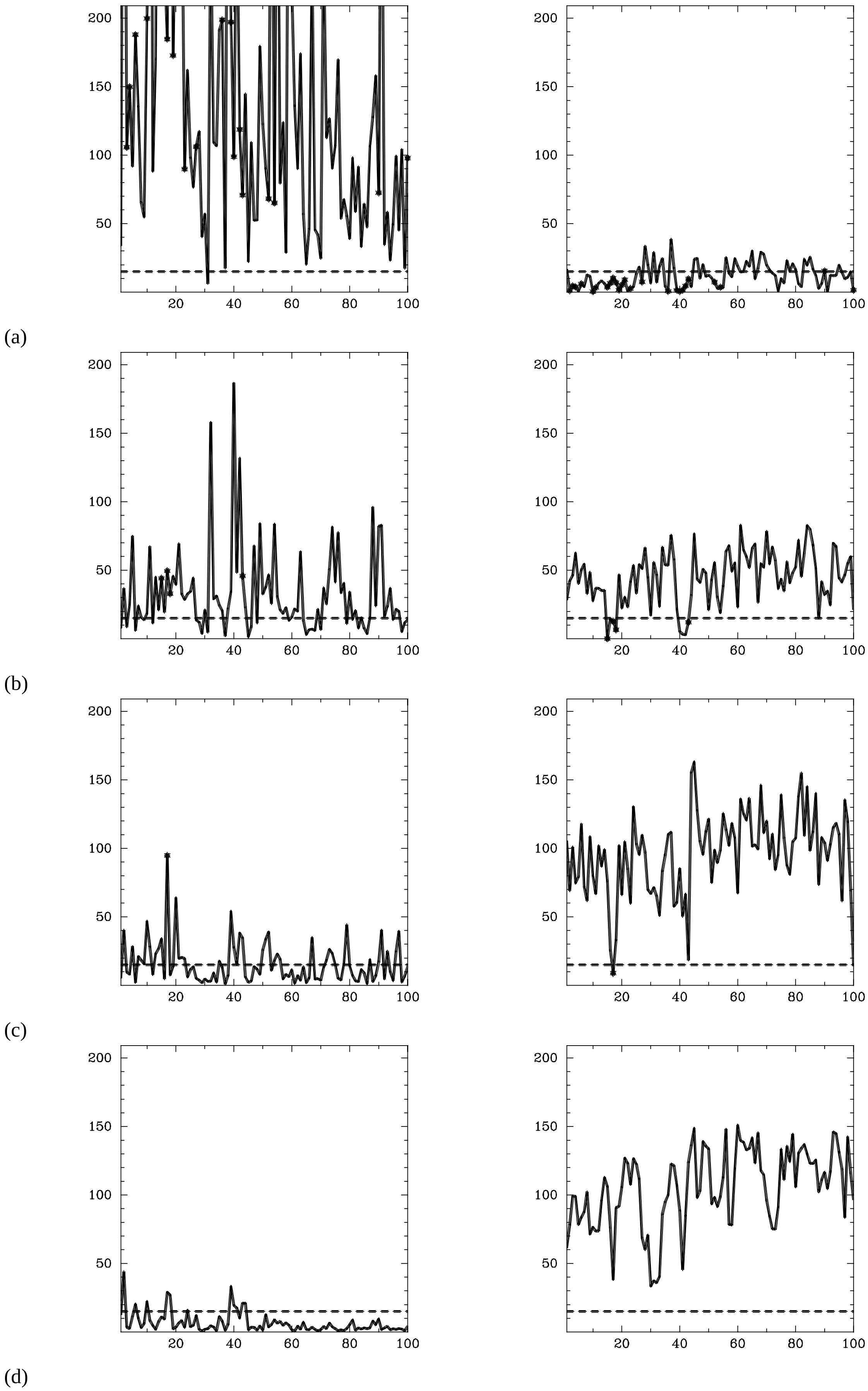


Figure 3: Acceptance (left) and discrimination test (right), baseline 2, session 2,
phase only, time span (a) 1 second, (b) 5 seconds (c) 15 seconds and (d) 0 seconds with code and phase.

In the tables 4 through 7 we give the number of batches in which correct integer estimates for the ambiguities were found and in which not; they are indicated by correct and not correct respectively. Both cases are subdivided into correct decision, OK, and wrong decision, the type I (α) and type II errors (β).

The integer ambiguities are correctly estimated at a high rate of success, cf. tables 1 and 2. In these cases we are also able to validate the integer estimate by means of statistical tests and using conservative testing parameters. With code and phase data, the validation rightly allows us to fix the ambiguities in nearly 100% of the cases for the 2.2 km baseline and in more than 50% of the cases for the 10 km baseline. Enlarging the time span generally increases these percentages. From the tables 4 through 7 it can also be concluded that the inclusion of code data greatly facilitates the validation of the integer ambiguity estimate. Once more it should be noted that per solution only two epochs of data are used. Increasing the sampling rate, will make that the validation criterion (6) is passed sooner as more data generally increases the reliability of the results.

These preliminary results indicate that the validation of the integer ambiguity estimate based on (6) is a viable procedure. No type II errors were made at all. In comparing the results of section 5, in which we carried out a deterministic validation, with those of section 6, we must conclude that the performance of the statistical validation by means of (6) slightly lags behind. We will make two remarks.

In order to increase the performance, the critical values of the testing procedure can be set looser. This will make you accept the result at an earlier stage, but will in general also increase the risk of accepting incorrect solutions. With the current setting we realized a very safe no type II errors or β=0%.

The second remark is of more importance and concerns the deficiencies of the mathematical model currently used in precise differential GPS positioning. The data collected in practice will not exactly obey the model. Several questions arise concerning the neglect of differential atmospheric delays in the functional model and the mutual and time correlation of the observables in the stochastic model. These topics deserve extensive research.

Another modelling aspect of serious concern is the probability density function of the integer estimator. Usually, once the integer ambiguity estimation has been carried out, the double difference ambiguities are treated as deterministic quantities. In this way the probability of vectors other than $\check{a}$, which hopefully is very small, is neglected.

7. Conclusions

The LAMBDA-method is capable of correctly estimating the integer ambiguities very fast and efficiently. As reported in [2] the integer estimation takes less than 0.1 second on a 486-PC. The results in this paper show that in order to successfully estimate the integer ambiguities, data of only a short time span are required. The method therefore enables instantaneous precise navigation and very rapid static surveying. We have also shown that the confidence in the integer estimate can be assessed.

Based on these preliminary results it is thought that especially the rapid static surveying productivity can be increased very much using the LAMBDA-method : the traditional static survey requires an occupation time on the order of 30 to 60 minutes and current rapid static surveys use 5-10 minutes occupation times. Time spans of below 1 minute are thought to be sufficient.

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