# Minimizing angular backlash of a multistage gear train

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**Abstract:** An optimization model is constructed to select the optimum reduction ratios that minimize the total angular backlash of a gear train, under constraints on the total reduction ratio and available space. It is found that a proper layout of reduction ratios has a major effect on the total angular backlash of a gear train.

**Keywords:** gear train, angular backlash, optimization

## 1 INTRODUCTION

The prime function of gear transmission systems in many precision machines is to accurately transmit angular displacement. In such applications, gear backlash causes an unexpected angular position error when a rotating gear train reverses, and it also induces a transient impact force on the mating gear surface due to the moment of inertia of the system.

There is an enormous amount of research literature addressing precision gear design. Various design requirements have been considered; in particular, minimizing the transmission error has received much attention. Yoon and Rao [**1**] minimized the static transmission error using cubic splines for a gear tooth profile. Iwase and Miyasaka [**2**] modified the tooth profile to minimize a transmission error of helical gears, while Shibata *et al.* [**3**] tried to find the optimum tooth profile that minimizes transmission error for hypoid gears. Using quadrature factorial models, Yu and Ishii [**4**] modified the tooth profile to find a ‘robust optimum’ that has a minimum expected transmission error.

However, modifying the tooth profile in order to minimize the transmission error may not be a practical option for many precision machine designers. Typical design decisions faced by the designers are how to select proper reduction ratios, gear quality grades and tolerances, so that the required transmission precision of a gear transmission system can be obtained in the most cost effective way.

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In this research, it is found that a proper layout of reduction ratios of a multistage gear train has a major effect on its total angular backlash. An optimization model is constructed in order to find the optimum reduction ratios that minimize the total angular backlash of a gear train, under constraints on the total reduction ratio and available space.

## 2 ANGULAR BACKLASH OF A GEAR TRAIN

Factors contributing to angular backlash mainly come from two sources: (a) original manufacturing errors and (b) assembly errors, such as centre distance tolerance, bearing tolerance and shaft and bearing tolerance. In this study, the original manufacturing errors and the centre distance tolerance are considered.

The original manufacturing errors of a gear are specified by its quality grade. The quality grades of spur gears and helical gears range from 0 to 9 in JIS B 1703, and ranges from 0 to 12 in ISO 1328. These two standards also have slightly different definitions of error estimation for each quality grade. According to JIS B 1703 [**5**], the linear backlash at the pitch line $\delta$ is estimated by

$$\delta = B \times 10^{-3}(\sqrt[3]{2r} + 0.65m) \qquad (1)$$

where $r$ and $m$ are the radius and module of the gear respectively. The quality grade of the gear is reflected in the coefficient $B$ in equation (1). For example, the value $B$ ranges from 10 to 25 for grade 0 gears (the finest quality), from 10 to 28 for grade 1 gears and from 10 to 90 for grade 8 gears.

Figure 1 shows the layout of a typical three-stage gear train. The reduction ratios of the three gear pairs are

 

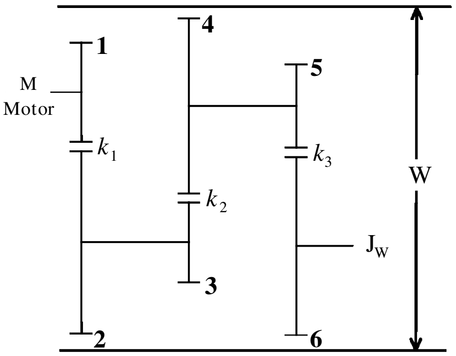


**Fig. 1** A three-stage gear transmission system

$k_1$, $k_2$ and $k_3$. The angular backlash of gear $i$ can be expressed as

$$\Delta\theta_i = \frac{\delta_i}{r_i} \tag{2}$$

where $\delta_i$ is the linear backlash and $r_i$ is the radius of gear $i$. The total angular backlash can be expressed as a linear sum of backlashes of individual gears, reflected at the output shaft of the gear train [**6**]. For the gear train in Fig. 1, the total angular backlash due to original manufacturing errors reflected at the shaft of gear 6 can be expressed as

$$\begin{aligned}\Delta\theta_{6m} &= \frac{\delta_6}{r_6} + \frac{\delta_5/r_5}{k_3} + \frac{\delta_4/r_4}{k_3} + \frac{\delta_3/r_3}{k_2k_3} + \frac{\delta_2/r_2}{k_2k_3} + \frac{\delta_1/r_1}{k_1k_2k_3} \\ &= \frac{\delta_6+\delta_5}{r_5k_3} + \frac{\delta_4+\delta_3}{r_3k_2k_3} + \frac{\delta_2+\delta_1}{r_1k_1k_2k_3}\end{aligned} \tag{3}$$

Note that $r_6 = r_5k_3$, $r_4 = r_3k_2$ and $r_2 = r_1k_1$.

Considering the centre distance tolerance, as shown in Fig. 2, the effect of a radial separation on the linear backlash is

$$\delta = 2C \times \tan\phi \tag{4}$$

where $\phi$ and $C$ are the pressure angle and the centre distance tolerance respectively. Similar to the derivation of equation (3), the total angular backlash due to the centre distance tolerance reflected at the shaft of gear 6 can be expressed as

$$\Delta\theta_{6c} = 2C\tan\phi\left(\frac{1}{r_5k_3} + \frac{1}{r_3k_2k_3} + \frac{1}{r_1k_1k_2k_3}\right) \tag{5}$$

It will be cumbersome to consider all possible combinations of signs (plus or minus) of individual tolerances. Therefore the total angular backlash at the output shaft of a gear train $\Delta\theta_{\text{out}}$ is expressed as the geometric sum of individual tolerances $\Delta\theta_{6m}$ and $\Delta\theta_{6c}$:

$$\begin{aligned}\Delta\theta_{\text{out}} &= (\Delta\theta_{6m}^2 + \Delta\theta_{6c}^2)^{1/2} \\ &= \left[\left(\frac{\delta_6+\delta_5}{r_5k_3} + \frac{\delta_4+\delta_3}{r_3k_2k_3} + \frac{\delta_2+\delta_1}{r_1k_1k_2k_3}\right)^2 \right. \\ &\quad \left. + 4C^2\tan^2\phi\left(\frac{1}{r_5k_3} + \frac{1}{r_3k_2k_3} + \frac{1}{r_1k_1k_2k_3}\right)^2\right]^{1/2}\end{aligned} \tag{6}$$

## 3 THE OPTIMUM DESIGN MODEL

An optimum design model for a multistage gear train is constructed in this section. The objective function is to minimize the total angular backlash of the gear train [equation (6)]. There are two constraints in this model: the total available space and the total reduction ratio of the gear train. Referring to Fig. 1, the allowable space occupied by the gear train is limited within a given value $W$:

$$\begin{aligned}&2(r_2 + r_1) \leqslant W \\ &2r_4 + r_3 + r_2 \leqslant W \\ &2r_6 + r_5 + r_4 \leqslant W\end{aligned} \tag{7}$$

The total reduction ratio of the gear train has to be larger than $K_t$; therefore

$$k_1k_2k_3 \geqslant K_t \tag{8}$$

Finally, the optimum design model can be written as

Minimize

$$\Delta\theta_{\text{out}} = \left[\left(\frac{\delta_6+\delta_5}{r_5k_3} + \frac{\delta_4+\delta_3}{r_3k_2k_3} + \frac{\delta_2+\delta_1}{r_1k_1k_2k_3}\right)^2 + 4C^2\tan^2\phi\left(\frac{1}{r_5k_3} + \frac{1}{r_3k_2k_3} + \frac{1}{r_1k_1k_2k_3}\right)^2\right]^{1/2}$$

subject to

$$g_1 : 2(r_2 + r_1) - W \leqslant 0$$

$$g_2 : 2r_4 + r_3 + r_2 - W \leqslant 0$$

$$g_3 : 2r_6 + r_5 + r_4 - W \leqslant 0$$

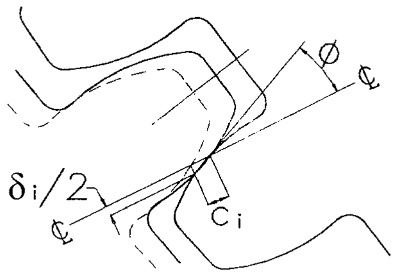


**Fig. 2** The influence of centre distance tolerance on gear backlash

$$h_1 : K_t - k_1 k_2 k_3 = 0$$

$$h_2 : r_2/r_1 - k_1 = 0$$

$$h_3 : r_4/r_3 - k_2 = 0$$

$$h_4 : r_6/r_5 - k_3 = 0$$

$$h_5 : \delta_1 - B \times 10^{-3}(\sqrt[3]{2r_1} + 0.65M) = 0$$

$$h_6 : \delta_2 - B \times 10^{-3}(\sqrt[3]{2r_2} + 0.65M) = 0$$

$$h_7 : \delta_3 - B \times 10^{-3}(\sqrt[3]{2r_3} + 0.65M) = 0$$

$$h_8 : \delta_4 - B \times 10^{-3}(\sqrt[3]{2r_4} + 0.65M) = 0$$

$$h_9 : \delta_5 - B \times 10^{-3}(\sqrt[3]{2r_5} + 0.65M) = 0$$

$$h_{10} : \delta_6 - B \times 10^{-3}(\sqrt[3]{2r_6} + 0.65M) = 0$$

$$r_i \geqslant N_{min} M, \qquad i = 1, \ldots, 6$$

$$1 \leqslant k_i \leqslant K_{max}, \qquad i = 1, \ldots, 3 \qquad (9)$$

The last two constraints in equations (9) post upper and lower bounds to the design variables $r_i$ and $k_i$, where $N_{min}$ is the minimum number of teeth of the gears and $K_{max}$ is the maximum allowable reduction ratio. Note that W, $K_t$, $B$, $C$, $M$, $N_{min}$ and $K_{max}$ are written in capital letters to represent parameters that are given for a certain case, but may vary from case to case.

## 4 SOLVING THE OPTIMUM DESIGN MODEL

To solve numerical values of this optimization model, a set of parameters is first assumed: $W = 100$ mm, $K_t = 120$, $B = 30$, $C = 0.020$ mm, $N_{min} = 18$, $K_{max} = 7$ and $M = 0.5$ mm. In this study, numerical optimization software GAMS [7] was used to solve for the optimum reduction ratios that minimize the total angular backlash of the gear train (Table 1). Since the number of teeth has to be an integer, the radii of the gears are rounded up to the closest integers multiplied by $M/2$. Both continuous and integer solutions are presented in the table. Note that with the integer solution, some of the constraints in equations (9) may be slightly violated.

Table 2 shows the reduction ratios that 'maximize' the total angular backlash of the gear train under the same requirements of space and total reduction ratio. Comparing Tables 1 and 2, it can be seen that by simply varying the reduction ratios, the maximum total angular backlash is 2.52 times the minimum total angular backlash in this example. Note that both cases have the same quality grade gears and the same centre distance tolerances; therefore their manufacturing costs should be about the same.

Also note that $k_1 \leqslant k_2 \leqslant k_3$ for the reduction ratios that minimize the total angular backlash. On the contrary, $k_1 \geqslant k_2 \geqslant k_3$ for the reduction ratios that maximize the total angular backlash. This can easily be explained by observing the objective function in equation (10). The reduction ratio $k_3$ appears in the denominators of all six terms of the objective function, while $k_2$ appears in the denominators of four terms and $k_1$ appears in the denominators of only two terms. To minimize the total angular backlash, obviously $k_3$ has the top priority to be as large as possible. The reduction ratio $k_2$ has the second priority, followed by $k_1$.

At the optimum solution shown in Table 1, space constraints $g_1$ and $g_3$ are active. The total angular backlash can be further reduced if the space constraints are relaxed. As shown in Fig. 3, for the same gear train discussed above, the minimum total angular backlash decreases as $W$ increases.

Figures 4 and 5 show the change in minimum total angular backlash with respect to the change in parameters $B$ and $C$. Comparing Figs 4 and 5, it can be seen that the total angular backlash is more sensitive to the parameter $B$ in this case. From this kind of parameter analysis, designers can evaluate the cost of changing into higher quality gears and the cost of tightening the centre distance tolerance, to decide how to reduce the total angular backlash in the most cost effective way.

**Table 1** Optimum reduction ratios that minimize total angular backlash

| | $\Delta\theta_{out}$ | $k_1$ | $k_2$ | $k_3$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Continuous solution | $7.18 \times 10^{-3}$ | 3.53 | 4.85 | 7.00 | 11.03 | 38.97 | 4.50 | 21.83 | 5.21 | 36.47 |
| Integer solution | $7.20 \times 10^{-3}$ | 3.55 | 4.83 | 6.95 | 11.00 | 39.00 | 4.50 | 21.75 | 5.25 | 36.50 |

**Table 2** Reduction ratios that maximize total angular backlash

| | $\Delta\theta_{out}$ | $k_1$ | $k_2$ | $k_3$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Continuous solution | $18.08 \times 10^{-3}$ | 7.00 | 7.00 | 2.45 | 4.50 | 31.50 | 4.50 | 31.50 | 4.50 | 11.02 |
| Integer solution | $18.12 \times 10^{-3}$ | 7.00 | 7.00 | 2.44 | 4.50 | 31.50 | 4.50 | 31.50 | 4.50 | 11.00 |

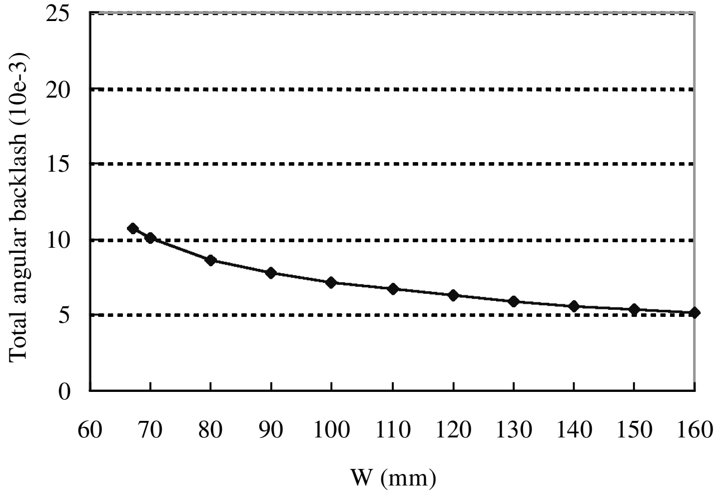


**Fig. 3** Minimum total angular backlash versus the change in available space

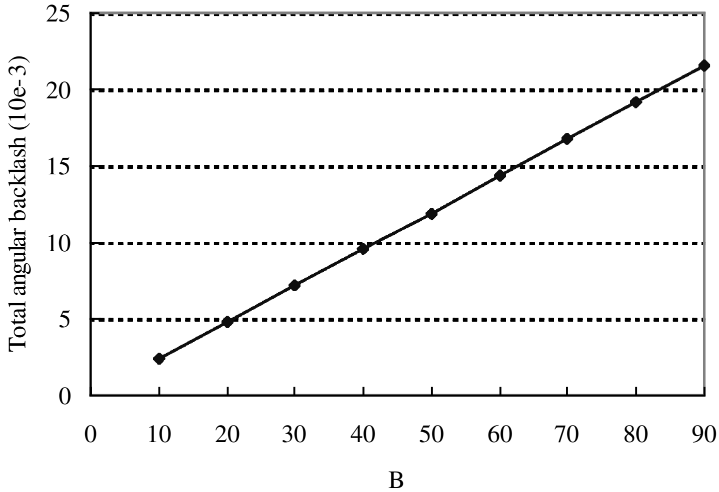


**Fig. 4** Minimum total angular backlash versus the effect of the change in parameter *B* on gear quality

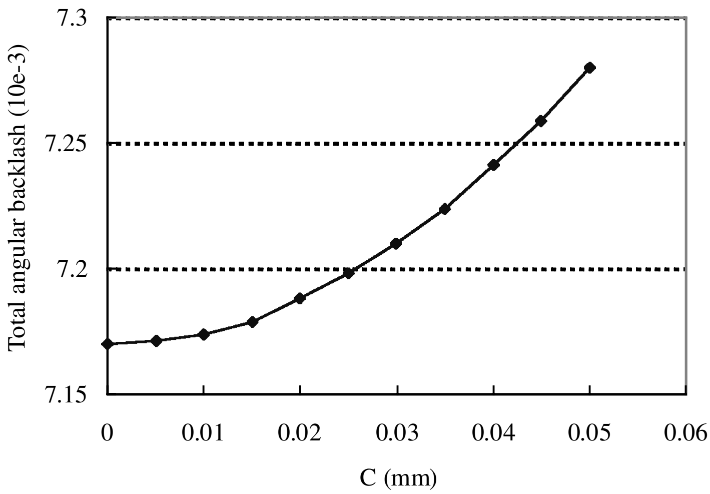


**Fig. 5** Minimum total angular backlash versus the change in centre distance tolerance

## 5 CONCLUSIONS

This paper presents an optimization model for finding the optimum reduction ratios that minimize the total angular backlash of a gear train. While the estimation of total angular backlash in the optimization model may not be quantitatively exact, several qualitative conclusions can be drawn:

1. Under the same design requirements on space and total reduction ratio, simply varying the reduction ratios can reduce the total angular backlash of a gear train, while the manufacturing cost stays the same.
2. To minimize the total angular backlash of a gear train, the reduction ratio closest to the output shaft has the highest priority to be made as large as possible.
3. The total angular backlash can be further reduced if the constraints on available space are relaxed.

This optimization model and the parameter analysis also provide a means to evaluate how to reduce the total angular backlash of a gear train in the most cost effective way.

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