A Triadic Approach to Formal Concept Analysis

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Abstract. *Formal Concept Analysis*, developed during the last fifteen years, has been based on the dyadic understanding of a concept constituted by its extension and its intension. The pragmatic philosophy of Charles S. Peirce with his three universal categories, and experiences in data analysis, have suggested a triadic approach to Formal Concept Analysis. This approach starts with the primitive notion of a *triadic context* defined as a quadruple (G, M, B, Y) where G, M, and B are sets and Y is a ternary relation between G, M, and B, i.e. $Y \subseteq G \times M \times B$; the elements of G, M, and B are called *objects*, *attributes*, and *conditions*, respectively, and $(g, m, b) \in Y$ is read: the object g has the attribute m *under* (or *according to*) the condition b. A *triadic concept* of a triadic context (G, M, B, Y) is defined as a triple (A_1, A_2, A_3) with $A_1 \times A_2 \times A_3 \subseteq Y$ which is maximal with respect to component-wise inclusion. The triadic concepts are structured by three quasiorders given by the inclusion order within each of the three components. In analogy to the dyadic case, we discuss how the ordinal structure of the triadic concepts of a triadic context can be analysed and graphically represented. A basic result is that those structures can be understood order-theoretically as *"complete trilattices"* up to isomorphism.

1 Triadic Contexts

Formal Concept Analysis views concepts as means of intersubjective understanding in situations of purpose-oriented action. The formalization of concepts and concept systems shall especially support the interpretation and communication of conceptual relationships in different situations. This requires that the formalization of concepts has to be transparent and simple, but also comprehensive so that all main aspects of a concept may have explicit references in the formal model.

The extensional and the intensional aspect, which were made explicit first by the "Port Royal Logic" [AN62] in the 17th century, are basic for the developed dyadic approach to Formal Concept Analysis. This approach starts with the primitive notion of a *formal context* as a triple (G, M, I) where G is a set of (*formal*) *objects* and M is a set of (*formal*) *attributes*, while I is a relation between G and M indicating when an object g of G has a certain attribute m of M. Formal contexts are often described by a two-dimensional cross table with a cross in position (g, m) if the relationship gIm holds, i.e., if the object g has the attribute m. A *formal concept* of a formal context (G, M, I) is defined as a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A = \{g \in G \mid gIm \text{ for all } m \in B\}$ and $B = \{m \in M \mid gIm \text{ for all } g \in G\}$, so the *extent* A contains exactly each object of G which has all the attributes of B, and the *intent* B contains exactly each attribute of M which is valid for all the objects of A. It turns out that the formal concepts of (G, M, I) are exactly the pairs (A, B) of subsets of G and M which are maximal with respect to the component-wise set inclusion in satisfying $A \times B \subseteq I$. If (G, M, I) is described by a cross table, this means that, under suitable permutations of the rows and the columns of the cross table, the concept (A, B) is represented by a maximal rectangle full of crosses. With respect to the *subconcept-superconcept-relation*, the concepts of a formal context (G, M, I) form a complete lattice called the *concept lattice* of (G, M, I).

Although the dyadic approach has been successful in many applications (see e.g. [Wi87],[Wi89a],[Wi89b],[Wi92a]), there have been situations suggesting an extension of formal concepts by a third component (cf. [Gi81],[Mar92], [KOG94]). Such a triadic approach to Formal Concept Analysis is also demanded by the pragmatic philosophy of Charles Sanders Peirce with his three universal categories (cf. [Wi94a], [Wi94b]) outlined further in the following.

Peirce developed a system of *"categories"* (as alternatives to Aristotle's and Kant's categories) and founded certain fundamental philosophical distinctions on this. In his lectures on pragmatism (1903) he gave the following description of his categories:

- Category the First is the Idea of that which is such as it is regardless of anything else. That is to say, it is a *quality* of feeling.
- Category the Second is the Idea of that which is such as it is as being Second to some First, regardless of anything else, and in particular regardless of any Law, although it may conform to a law. That is to say, it is *Reaction* as an element of the phenomenon.
- Category the Third is the Idea of that which is such as it is as being a Third, or Medium, between a Second and a First. That is to say, it is *Representation* as an element of the phenomenon. [Pe35;5.66]

Instead of the dyadic relation between objects and attributes used in Formal Concept Analysis up to now, the triadic approach is based on the triadic relation saying that *the object g has the attribute m under the condition b*. How should such a relationship be viewed through Peirce's three categories? The answer is not obvious because Peirce wrote about his categories over more than thirty years in a great variety of different explanations. The quoted description of the three categories allows us to interpret the triadic relationship as follows: *the object g is a First as some suchness to which the attribute m is a Second as some accident while the condition b is a Third as some medium between g and m.* In different situations, the Third as medium may be understood more specifically as relation, mediation, representation, interpretation, evidence, evaluation, modality, meaning, reason, purpose, condition etc. concerning a present connection between an object and an attribute.

Although the given interpretation of the triadic relationship seems to be convincing, the description of the First as *quality* and the Second as *reaction* conflicts with this interpretation. Indeed, in Peirce's writings, the Firstness is connected and explained more often with the term 'quality' than with any other term (cf. [Kr60], p. 11) and the examples of First are mainly taken from the area of feelings as, for instance, the feeling of Red. The Secondness is explained by the reaction caused by facts which struggle with human feelings and thoughts. These explanations reflect Peirce's dominating interest in epistemology and phenomenology. Under this view the human subject is First and the objective world is Second. But the speech acts of intersubjective communication should be viewed differently. This becomes clear in the different understandings of a *predication* which combines a subject and a predicate: epistemologically, the subject is understood as an instance of the concept decribed by the predicate (or the subject concept is subsumed under the predicate concept); semantically, the predicate is understood to be assigned to the object denoted by the subject term (cf. [We89]). These two understandings of predication were extensively discussed in the Middle Ages as identity and inherence theory of predication with the copula 'is' (cf. [Ma79]). Already the etymology of the word 'attribute' indicates that the dyadic predication *'the object g has the attribute m'* should be understood semantically, i.e., m is accidental to g. This semantic interpretation is compatible with Peirce's general understanding of the First as *suchness* [Pe35;1.303] and the Second as *otherness* [Pe35;1.295].

The given interpretation of the dyadic predication also underlies the description of *concept* in [AN62] which is based on the distinction of extension and intension. The objects of the extension are considered as things which exist by themselves and the attributes of the intension are viewed as parts which determine things but cannot be without them. This object-dependent role of attributes is basic for concept theories until today which is witnessed, for instance, by the German standards DIN 2330 [DIN79]. The dependency suggests to interpret also the dyadic relationship between an object and an assigned attribute as instance of Peirce's second category. Especially in Peirce's investigations of logic, Secondness is the general character of dyadic relations and Thirdness is the general character of triadic relations. Thus, the triadic relationship between an object, an attribute and a condition may be interpreted as an instance of Peirce's third category too.

The triadic approach to Formal Concept Analysis is based on a formalization of the triadic relation connecting formal objects, attributes and conditions. Since real situations can only be analysed within restricted contexts, *Triadic Concept Analysis* is founded on a formal notion of triadic contexts which allows set-theoretical formalizations. A *triadic context* is defined as a quadruple (G, M, B, Y) where G, M, and B are sets and Y is a ternary relation between G, M, and B, i.e. $Y \subseteq G \times M \times B$; the elements of G, M, and B are called *(formal) objects*, *attributes*, and *conditions*, respectively, and $(g, m, b) \in Y$ is read: the object g has the attribute m *under* the condition b (the relational notation $b(g, m)$ might also be used for $(g, m, b) \in Y$). Just as dyadic contexts are often described by two-dimensional cross tables, triadic contexts may be represented by three-dimensional cross tables. Figures 1, 3, and 5 below show examples of such cross tables for triadic contexts.

Formal objects, attributes, and conditions may formalize entities in a wide range, but in the triadic context they are understood in the role of the corresponding Peircean category. In particular, the formal conditions may formalize - as mentioned above - relations, mediations, representations, interpretations, evidences, evaluations, modalities, meanings, reasons, purposes, conditions etc. If real data are described by a triadic context, the names of the formal objects, attributes, and conditions yield the elementary bridges to reality which are basic for interpretations (cf. [Wi92b]). For theoretical developments it is often convenient to use K_1, K_2, and K_3 instead of G, M, and B; the alternative symbols indicate that the elements of the component K_i are seen in the role of an instance of Peirce's i-th category.

2 Triadic Concepts

Concepts are understood as units of thought. This means that a concept tends to be *homogeneous* and *closed.* If a concept is viewed through the triadic paradigm then, for the purpose of formalization, a concept should be seen as a combination of objects, attributes, and conditions which is homogeneous and closed. Homogeneity is attained if each object has each attribute under each condition within the concept. Closure is attained if the concept is maximal with respect to this property. Therefore, we define a *triadic concept* of a triadic context (G, M, B, Y) as a triple (A_1, A_2, A_3) with $A_1 \subseteq G$, $A_2 \subseteq M$, and $A_3 \subseteq B$ such that the triple (A_1, A_2, A_3) is maximal with repect to component-wise set inclusion in satisfying $A_1 \times A_2 \times A_3 \subseteq Y$, i.e., for $X_1 \subseteq G$, $X_2 \subseteq M$, and $X_3 \subseteq B$ with $X_1 \times X_2 \times X_3 \subseteq Y$, the containments $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, and $A_3 \subseteq X_3$ always imply $(A_1, A_2, A_3) = (X_1, X_2, X_3)$. If (G, M, B, Y) is described by a three-dimensional cross table, this means that, under suitable permutations of rows, columns, and layers of the cross table, the triadic concept (A_1, A_2, A_3) is represented by a maximal rectangular box full of crosses. For a particular triadic concept (A_1, A_2, A_3), the components A_1, A_2, and A_3 are called the *extent*, the *intent*, and the *modus* of (A_1, A_2, A_3), respectively.

As in the dyadic case, derivation operators are useful for the construction of triadic concepts within a triadic context. For the description of derivation operators, it is convenient to denote the underlying triadic context alternatively by $\mathbb{K} := (K_1, K_2, K_3, Y)$. For $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$ and for $X \subseteq K_i$ and $Z \subseteq K_j \times K_k$, the (i)*-derivation operators* are defined by

$$X \mapsto X^{(i)} := \{(a_j, a_k) \in K_j \times K_k \mid a_i, a_j, a_k \text{ are related by } Y \text{ for all } a_i \in X\},$$

$$Z \mapsto Z^{(i)} := \{a_i \in K_i \mid a_i, a_j, a_k \text{ are related by } Y \text{ for all } (a_j, a_k) \in Z\}.$$

This definition yields the derivation operators of the dyadic contexts defined by

$$\mathbb{K}^{(1)} := (K_1, K_2 \times K_3, Y^{(1)}),$$
$$\mathbb{K}^{(2)} := (K_2, K_1 \times K_3, Y^{(2)}),$$
$$\mathbb{K}^{(3)} := (K_3, K_1 \times K_2, Y^{(3)})$$

where $gY^{(1)}(m,b) :\Longleftrightarrow mY^{(2)}(g,b) :\Longleftrightarrow bY^{(3)}(g,m) :\Longleftrightarrow (g,m,b) \in Y$. For the construction of triadic concepts, further derivation operators are needed. For $\{i,j,k\} = \{1,2,3\}$ and for $X_i \subseteq K_i$, $X_j \subseteq K_j$, and $A_k \subseteq K_k$, the (i,j,A_k)-*derivation operators* are defined by

$$X_i \mapsto X_i^{(i,j,A_k)} := \{a_j \in K_j \mid a_i, a_j, a_k \text{ are related by } Y \text{ for all } (a_i,a_k) \in X_i \times A_k\},$$
$$X_j \mapsto X_j^{(i,j,A_k)} := \{a_i \in K_j \mid a_i, a_j, a_k \text{ are related by } Y \text{ for all } (a_j,a_k) \in X_j \times A_k\}.$$

This definition yields the derivation operators of the dyadic contexts defined by

$$\mathbb{K}^{ij}_{A_k} := (K_i, K_j, Y^{ij}_{A_k})$$

where $(a_i,a_j) \in Y^{ij}_{A_k}$ if and only if a_i, a_j, a_k are related by Y for all $a_k \in A_k$. In case of $(1,2,3) = (i,j,k)$, the relationship $(a_1,a_2) \in Y^{12}_{A_3}$ means that the object a_1 has the attribute a_2 under all conditions a_3 with $a_3 \in A_3$.

If one wants to get a triadic concept having a given object set X_1 in its extent, one may first generate a dyadic concept in $\mathbb{K}^{12}_{A_3}$ with X_1 in its extent and then extend it to a triadic concept using the corresponding (3)-derivation operator in $\mathbb{K}^{(3)}$. This is formally performed by forming the triadic concept

$$(X_1^{(1,2,A_3)(1,2,A_3)}, X_1^{(1,2,A_3)}, (X_1^{(1,2,A_3)(1,2,A_3)} \times X_1^{(1,2,A_3)})^{(3)}).$$

In words, one first determines the set of all attributes which all objects of X_1 have under all conditions of A_3; secondly, X_1 is extended to the set of all objects having all those attributes under all conditions of A_3; thirdly, A_3 is extended to the set of all conditons under which each of the derived objects has each of the derived attributes. Of course, this construction of a triadic concept may analogously be applied to other choices of $X_i \subseteq K_i$ and $A_k \subseteq K_k$ for $i \neq k$ in $\{1,2,3\}$. It should also become clear that a triple (A_1,A_2,A_3) with $A_i \subseteq K_i$ for $i = 1,2,3$ is a triadic concept of $\mathbb{K}$ if and only if $A_i = (A_j \times A_k)^{(i)}$ for all $\{i,j,k\} = \{1,2,3\}$ with $j < k$. Note that there are always the extremal triadic concepts $\mathfrak{o}_1 := ((K_2 \times K_3)^{(1)}, K_2, K_3)$, $\mathfrak{o}_2 := (K_1, (K_1 \times K_3)^{(2)}, K_3)$, and $\mathfrak{o}_3 := (K_1, K_2, (K_1 \times K_2)^{(3)})$.

Next we discuss how the set $\mathfrak{T}(\mathbb{K})$ of all triadic concepts of the triadic context $\mathbb{K} := (K_1,K_2,K_3)$ is structured. The most natural structure is given by the set inclusion in each of the three components of the triadic concepts. For each $i \in \{1,2,3\}$, one obtains a quasiorder $\lesssim_i$ and its corresponding equivalence relations $\sim_i$ defined by

$$(A_1,A_2,A_3) \lesssim_i (B_1,B_2,B_3) :\Longleftrightarrow A_i \subseteq B_i \quad \textit{and}$$
$$(A_1,A_2,A_3) \sim_i (B_1,B_2,B_3) :\Longleftrightarrow A_i = B_i \quad (i = 1,2,3).$$

The three quasiorders satisfy the following *antiordinal dependencies* (cf. [Wi95]): For $\{i,j,k\} = \{1,2,3\}$, $(A_1,A_2,A_3) \lesssim_i (B_1,B_2,B_3)$ and $(A_1,A_2,A_3) \lesssim_j (B_1,B_2,B_3)$ imply $(A_1,A_2,A_3) \gtrsim_k (B_1,B_2,B_3)$ for all triadic concepts (A_1,A_2,A_3) and

(B_1, B_2, B_3) of $\mathbb{K}$. This becomes clear if one visualizes the two triadic concepts by maximal boxes within the three-dimensional cross table. It follows that, for $i \neq j$, the relation $\sim_i \cap \sim_j$ is the identity on $\mathfrak{T}(\mathbb{K})$, i.e., a triadic concept is uniquely determined by two of its components. Furthermore, $\lesssim_{ij} := \lesssim_i \cap \lesssim_j$ is an order on $\mathfrak{T}(\mathbb{K})$. Let us denote the equivalence class of $\sim_i$ which contains the triadic concept (A_1, A_2, A_3) by $[(A_1, A_2, A_3)]_i$. The quasiorder $\lesssim_i$ induces an order $\leq_i$ on the factor set $\mathfrak{T}(\mathbb{K})/\sim_i$ of all equivalence classes of $\sim_i$ which is characterized by $[(A_1, A_2, A_3)]_i \leq_i [(B_1, B_2, B_3)]_i \iff A_i \subseteq B_i$. Clearly, $(\mathfrak{T}(\mathbb{K})/\sim_1, \leq_1)$ can be identified with the ordered set of all extents of $\mathbb{K}$, $(\mathfrak{T}(\mathbb{K})/\sim_2, \leq_2)$ with to the ordered set of all intents of $\mathbb{K}$, and $(\mathfrak{T}(\mathbb{K})/\sim_3, \leq_3)$ with the ordered set of all modi of $\mathbb{K}$. For such an ordered set one cannot expect special properties in general because every ordered set with smallest and greatest element is isomorphic to the ordered set of all extents (resp. intents, modi) of some triadic context as shown in [Wi95]. Therefore the extents (resp. intents, modi) need not to form a *closure system* as in the dyadic case.

For analysing conceptual relationships within triadic contexts $\mathbb{K} := (K_1, K_2, K_3)$, it is basic to investigate the relational structures $\underline{\mathfrak{T}}(\mathbb{K}) := (\mathfrak{T}(\mathbb{K}), \lesssim_1, \lesssim_2, \lesssim_3)$. There is already a computer program which determines the triadic concepts and the relational structure $\underline{\mathfrak{T}}(\mathbb{K})$ for a given triadic context (see [GK94]). The program is based on a nested use of *Ganter's Algorithm* for dyadic contexts (see [Ga87]). The extented algorithm for determining the triadic concepts is explicitly discussed in [KOG94].

3 Triadic Diagrams

In this section we discuss how the 'triadic' relational structure $\underline{\mathfrak{T}}(\mathbb{K})$ of a triadic context $\mathbb{K} := (K_1, K_2, K_3)$ may be represented graphically. For this, the relational structure can be understood as a combination of two types of structures: the *geometric structure* $(\mathfrak{T}(\mathbb{K}), \sim_1, \sim_2, \sim_3)$ and the *ordered structures* $(\mathfrak{T}(\mathbb{K})/\sim_i, \leq_i)$ for $i = 1, 2, 3$. The geometric structure gives rise to a 'partial 3-net' $(\mathfrak{T}(\mathbb{K}), \bigcup_{i=1}^{3}\{[(A_1, A_2, A_3)]_i \mid (A_1, A_2, A_3) \in \mathfrak{T}(\mathbb{K})\})$ in which the classes of different equivalence relations meet in at most one element. This suggests to represent the three equivalence relations $\sim_1$, $\sim_2$, and $\sim_3$ by three systems of parallel lines in the plane and to locate the elements of one equivalence class on one line of the corresponding parallel system. Unfortunately, such a representation cannot be performed by straight lines in general because that would never admit a violation of the so-called 'Thomsen Condition' (see [KLST71], pp.250 ff.) which might occur in triadic contexts (see [WZ95]). But it seems that the Thomsen Condition and its ordinal generalizations (cf. [WiU95]) is only seldom violated by real data wherefore we concentrate in this section on graphical representations of the geometric structure by straight lines. The ordered structures are represented as usual by line diagrams (Hasse diagrams). Some examples will show how the graphical representations of the geometric and the ordered structures may be combined into a *triadic diagram* of the whole relational structure.

Let us first consider an elementary type of 'triadic' relational structures whose

	1					2					3					4					5				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
1	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	
2	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×		×	×	×		
3	×	×	×	×	×	×	×	×	×	×	×	×	×	×		×	×	×			×	×			
4	×	×	×	×	×	×	×	×	×		×	×	×			×	×				×				
5	×	×	×	×		×	×	×			×	×				×									

Fig. 1. The triadic 5-chain context $\mathbb{K}_5^c$

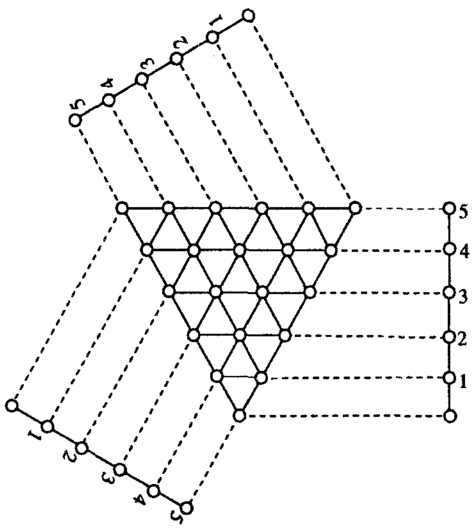

Fig. 2. A triadic diagram of the triadic 5-chain $\underline{\mathfrak{T}}(\mathbb{K}_5^c)$

ordered structures are only chains. The most regular subtype of those structures are given by the triadic *n-chain contexts* $\mathbb{K}_n^c := (\{1,\ldots,n\},\{1,\ldots,n\},\{1,\ldots,n\}, Y_n^c)$ and $(x_1,x_2,x_3) \in Y_n^c :\Longleftrightarrow x_1+x_2+x_3 \leq 2n$. The triadic concepts of those contexts form as geometric structure a regular triangle pattern as it is indicated by the following example. In Figure 1, the triadic 5-chain context $\mathbb{K}_5^c$ is described by a cross table in which the rows represent objects, the columns attributes, and the subtables conditions. A *triadic diagram* of $\underline{\mathfrak{T}}(\mathbb{K}_5^c)$ is shown in Figure 2. The geometric structure of the triadic concepts is represented by the triangular pattern in the center of the diagram. The circles represent the triadic concepts and the lines the equivalence classes, i.e., the horizontal lines those of $\sim_1$, the lines ascending to the right those of $\sim_2$, and the lines ascending to the left those of $\sim_3$. The perforated lines indicate the connection to the extent diagram on the right, to the intent diagram on the lower left, and to the modus diagram above. A circle of the line diagram on the right represents the extent consisting of those

	1			2			3		
	1	2	3	1	2	3	1	2	3
1		×	×	×	×	×	×	×	×
2	×	×	×	×		×	×	×	×
3	×	×	×	×	×	×	×	×	

Fig. 3. The triadic power set context $\mathbb{K}^b_{\{1,2,3\}}$

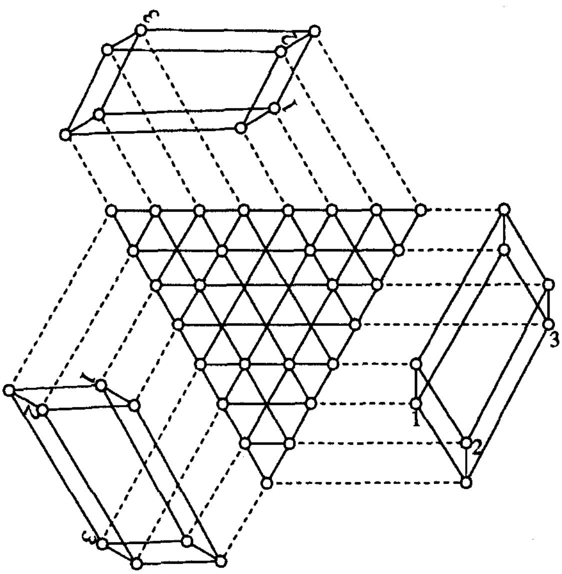

Fig. 4. The power set trilattice $\underline{\mathfrak{T}}(\mathbb{K}^b_{\{1,2,3\}})$

objects whose signs are attached to this circle or a circle below. The intents and modi can analogously be read from the diagram where the intents get larger from the upper left to the lower right and the modi get larger from the upper right to to the lower left. Using the given information, all triadic concepts can be completely determined by the triadic diagram. For instance, the next circle vertically above the lowest circle connects horizontally with the extent $\{1,2\}$, to the lower left with the intent $\{1,2,3,4\}$, and to the upper left with the modus $\{1,2,3,4\}$; hence it represents the triadic concept $(\{1,2\},\{1,2,3,4\},\{1,2,3,4\})$.

Another elementary type of triadic contexts are the triadic *power set contexts* defined for arbitrary sets S by $\mathbb{K}^b_S := (S,S,S,Y^b_S)$ with $Y^b_S := S^3 \setminus \{(x,x,x) \mid x \in S\}$. The triadic concepts of $\mathbb{K}^b_S$ are exactly the triples $(X_1,X_2,X_3) \in \mathfrak{P}(S)^3$ with $X_1 \cap X_2 \cap X_3 = \emptyset$ and $X_i \cup X_j = S$ for $i \neq j$ in $\{1,2,3\}$. In Figure 3, the triadic power set context $\mathbb{K}^b_{\{1,2,3\}}$ is described by a cross table. A triadic diagram

	a						b						c						d					
	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
A	×	×					×	×					×	×				×	×	×		×		
B	×											×	×				×		×				×	
C	×	×	×	×	×	×	×	×	×	×	×	×	×	×				×	×	×				×

A: Superordinate stereotypes
B: Unspecified
C: Subtypes
a: Recall
b: Impression Formation
c: Behavior Prediction
d: Evaluation

1. Physical appearance
2. Political beliefs
3. Attitudes
4. Behavior
5. Traits
6. Situations

Fig. 5. A triadic context of experimental data

of $\underline{\mathfrak{T}}(\mathbb{K}^b_{\{1,2,3\}})$ is shown in Figure 4. The triadic diagram is analogously read as the diagram of Figure 2; the only difference is that not every intersection point in the triangular pattern represents a triadic concept.

The last example, based on experimental data, is a triadic context taken from [KOG94] and desribed by the cross table in Figure 5. The objects of the triadic context are person categories named by 'A: Superordinate stereotypes', 'B: Unspecified', and 'C: Subtypes', the attributes are attribute types named by '1: Physical appearance', '2: Political beliefs', '3: Attitudes', '4: Behavior', '5: Traits', and '6: Situations', and the conditions are goals named by 'a: Recall', 'b: Impression Formation', 'c: Behavior Prediction', and 'd: Evaluation'. A cross in the table indicates that the object/attribute/condition-triple was rated above the median in the considered study. A triadic diagram of the triadic data context is shown in Figure 6. It seems to be quite typical for real data that the eleven triadic concepts spread rather loosely in the relatively large triangular pattern. In the modus diagram above the name 'a: Recall' occurs twice since there is not a smallest modus containing the condition 'a: Recall'. This might happen because modi (resp. intents, extents) need not to form a closure system which was already mentioned in Section 2. As the diagram shows, there are two triadic concepts with minimal modus containing the condition 'a: Recall', namely $(\{C\}, \{1,2,3,4,5,6\}, \{a,b\})$ and $(\{A,B,C\}, \{1\}, \{a,c,d\})$.

4 Concept Trilattices

The relational structures $\underline{\mathfrak{T}}(\mathbb{K}) := (\mathfrak{T}(\mathbb{K}), \lesssim_1, \lesssim_2, \lesssim_3)$ play an analogous role in triadic concept analysis as the *concept lattices* in the dyadic case. Therefore the investigation of $\underline{\mathfrak{T}}(\mathbb{K})$ should identify the algebraic operations which are the triadic analogues to infima and suprema. Since those operations have the aim of producing triadic concepts from others, it seems natural to reach the operations via the construction of triadic concepts described in Section 2. This

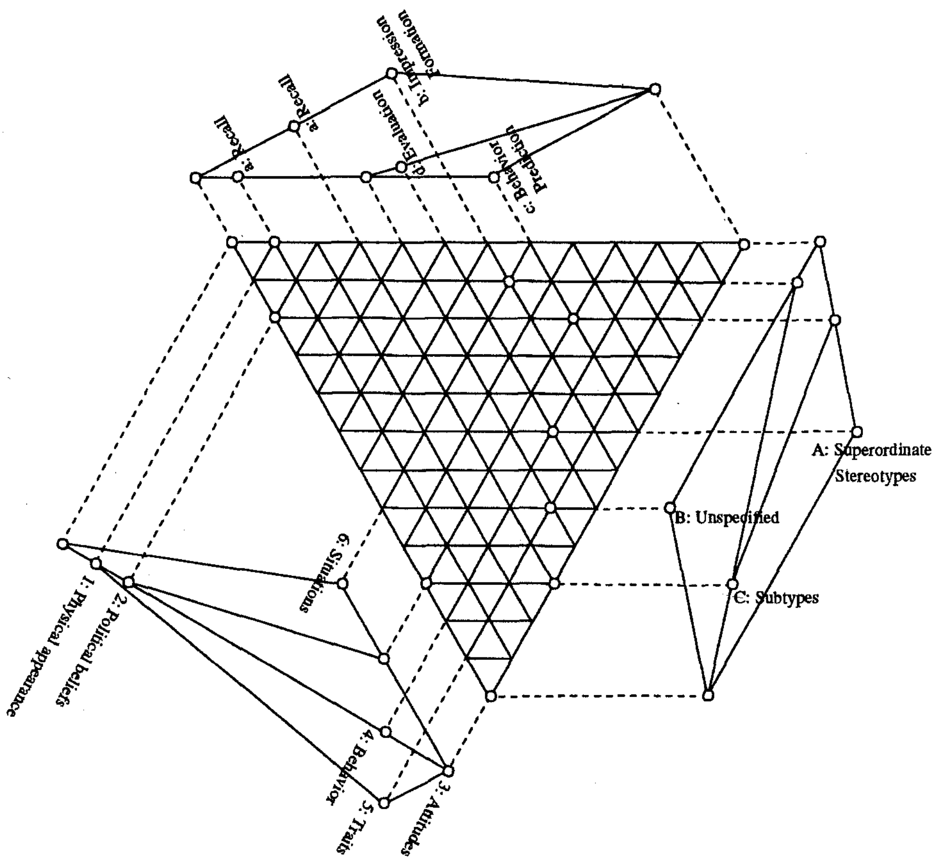

Fig. 6. A triadic diagram of the 'triadic' relational structure of the context in Figure 5

construction indicates that one needs two subsets of triadic concepts to produce another triadic concept. Let $\{i,j,k\} = \{1,2,3\}$ and let $\mathfrak{X}_i$ and $\mathfrak{X}_k$ be sets of triadic concepts of $\mathbb{K}$; furthermore, let $X_i := \bigcup\{A_i \mid (A_1,A_2,A_3) \in \mathfrak{X}_i\}$ and $X_k := \bigcup\{A_k \mid (A_1,A_2,A_3) \in \mathfrak{X}_k\}$. Then the *ik-join* of the pair $(\mathfrak{X}_i,\mathfrak{X}_k)$ is defined to be the triadic concept

$$\begin{aligned}\nabla_{ik}(\mathfrak{X}_i,\mathfrak{X}_k) &:= (B_1,B_2,B_3) \text{ with}\\ B_i &:= X_i^{(i,j,X_k)(i,j,X_k)},\\ B_j &:= X_i^{(i,j,X_k)},\\ B_k &:= (X_i^{(i,j,X_k)(i,j,X_k)} \times X_i^{(i,j,X_k)})^{(k)}.\end{aligned}$$

If the k-th-components of the triadic concepts in $\mathfrak{X}_i$ are all equal to X_k, then $\nabla_{ik}(\mathfrak{X}_i,\mathfrak{X}_k)$ is just the supremum of $\mathfrak{X}_i$ in the ordered set $(\{(B_1,B_2,B_3) \in \mathfrak{T}(\mathbb{K}) \mid$

$B_k = X_k\}, \lesssim_i)$ and, because of the antiordinal dependency, the infimum of $\mathfrak{X}_i$ in the dually ordered set $(\{(B_1, B_2, B_3) \in \mathfrak{T}(\mathbb{K}) \mid B_k = X_k\}, \lesssim_j)$.

For the further development of Triadic Concept Analysis it is useful to understand the operations ∇_{ik} on the purely order-theoretic level. Here we can only sketch the order-theoretic treatment elaborated in [Wi95]. First a *triordered set* is defined as a relational structure $(S, \lesssim_1, \lesssim_2, \lesssim_3)$ for which the relations $\lesssim_i$ are quasiorders on S such that $\lesssim_i \cap \lesssim_j \subseteq \gtrsim_k$ for $\{i,j,k\} = \{1,2,3\}$ and $\sim_1 \cap \sim_2 \cap \sim_3 = id_S$ where $\sim_i := \lesssim_i \cap \gtrsim_i$ $(i = 1,2,3)$. It immediately follows that $\sim_i \cap \sim_j = id_S$ for $i \neq j$. The triadic analogues of suprema and infima can order-theoretically be defined as follows: For $\{i,j,k\} = \{1,2,3\}$ and $X_i, X_k \subseteq S$, an element u of S is called an *ik-bound* of (X_i, X_k) if $u \gtrsim_i x$ for all $x \in X_i$ and $u \gtrsim_k x$ for all $x \in X_k$; an ik-bound u of (X_i, X_k) is called an *ik-limit* of (X_i, X_k) if $u \gtrsim_j v$ for all ik-bounds v of (X_i, X_k). There exists at most one ik-limit u of (X_i, X_k) with $u \lesssim_k v$ for all ik-limits v of (X_i, X_k) in $\underline{S}$; this element u is called the *ik-join* of (X_i, X_k) and denoted by $\nabla_{ik}(X_i, X_k)$. Now, a *complete trilattice* is defined to be a triordered set $\underline{L} := (L, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-joins exist for all $i \neq k$ in $\{1,2,3\}$ and all pairs of subsets of L. In a complete trilattice $\underline{L}$, the extremal element $0_i := \nabla_{jk}(L, L)$ $(= \nabla_{kj}(L, L))$ is uniquely determined by $0_i \lesssim_i x$ for all $x \in L$ (note that also $0_i = \nabla_{ij}(\emptyset, L) = \nabla_{ik}(\emptyset, L) = \nabla_{ji}(\emptyset, \emptyset) = \nabla_{ki}(\emptyset, \emptyset)$).

By the *Basic Theorem* of Triadic Concept Analysis (proved in [Wi95]), the relational structure $\underline{\mathfrak{T}}(\mathbb{K})$ of a triadic context $\mathbb{K}$ is a complete trilattice whose order-theoretical ik-joins are exactly the ik-joins defined by using the derivation operators. Since $\underline{\mathfrak{T}}(\mathbb{K})$ is a complete trilattice, it is called the *concept trilattice* of the triadic context $\mathbb{K}$. Conversely, every complete trilattice $\underline{L} := (L, \lesssim_1, \lesssim_2, \lesssim_3)$ is isomorphic to a concept trilattice of a suitable triadic context for which one can choose $(L, L, L, Y_{\underline{L}})$ with $Y_{\underline{L}} := \{(x_1, x_2, x_3) \in L^3 \mid$ there exists an $u \in L$ with $u \gtrsim_i x_i$ for $i = 1,2,3\}$. Thus, the concept trilattices are up to isomorphism the complete trilattices (as the concept lattices are up to isomorphism the complete lattices in the dyadic case). Hence, on the purely order-theoretic level, the concept trilattices are generally understood, but the detailed study of those structures is just at the beginning.

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