Proceedings of the International Congress of Mathematicians
Vancouver, 1974

# Some Systems of Second Order Arithmetic and Their Use

Harvey Friedman

The questions underlying the work presented here on subsystems of second order arithmetic are the following. What are the proper axioms to use in carrying out proofs of particular theorems, or bodies of theorems, in mathematics? What are those formal systems which isolate the essential principles needed to prove them?

Ultimately, answers to these questions will require use of systems that are not subsystems of second order arithmetic, but have variables ranging over objects such as sets of sets of natural numbers. Such systems would be needed in order to formalize directly theorems about continuous functions on the reals, or measurable sets of reals. But the language of second order arithmetic is sufficient to formalize directly several fundamental theorems, and is basic among the possible languages relevant to the formalization of mathematics. Furthermore, our preliminary investigations reveal that the most important systems not formalized in the language of second order arithmetic are conservative extensions of those that are. In this way, the systematic study reported here of subsystems of second order arithmetic is a necessary and important step in answering the underlying questions.

In our work, two principal themes emerge. The first is as follows.

I. When the theorem is proved from the right axioms, the axioms can be proved from the theorem.

When this theme applies, we have a unique formalization of the theorem, up to provable equivalence. I occurs surprisingly often, but not always.

The second is more technical.

II. Much more is needed to define explicitly a hard-to-define set of integers than merely to prove their existence.

An example of this theme which we consider is that the natural axioms needed to define explicitly nonrecursive sets of natural numbers prove the consistency of the natural axioms needed to prove the existence of nonrecursive sets of natural numbers.

The language $\mathscr{L}$ of second order arithmetic has numerical variables $n_i$ and set variables $x_i$, $0 \leq i$, the constant 0, the unary successor function symbol $N$, the binary function symbols $+$, $\cdot$, and the binary relation symbols $<$, $=$, $\in$.

The terms of $\mathscr{L}$ are given by (a) 0, and each numerical variable is a term, and (b) $s + t$, $s \cdot t$, and $N(s)$ are terms if $s$, $t$ are terms.

The atomic formulae of $\mathscr{L}$ are of the form $s = t$, $s < t$, or $s \in x$, for terms $s$, $t$, and set variables $x$.

The formulae of $\mathscr{L}$ are given by (a) atomic formulae are formulae, (b) if $A$, $B$ are formulae, so are $(\sim A)$, $(A \,\&\, B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$, and (c) if $A$ is a formula, $\alpha$ a variable of $\mathscr{L}$, then $(\forall\alpha)(A)$, $(\exists\alpha)(A)$ are formulae.

The language $\mathscr{L}$ has the following interpretation. An $\mathscr{L}$-structure is a system $(D, n, a, m, \mathbf{0}, \prec, K)$, where $D \neq \emptyset$, $n$, $a$, $m$ are unary, binary, and binary functions on $D$ interpreting $N$, $+$, $\cdot$, $\mathbf{0} \in D$ interpreting 0, $\prec$ is a binary relation on $D$ interpreting $<$, and $K \subset \mathscr{P}(D)$ is nonempty. We often write $\mathscr{A} = (M, K)$.

$\mathscr{A} \models A[f, g]$ is defined in the usual way, with $\mathscr{A} \models s = t[f, g]$ iff $\mathrm{Val}(\mathscr{A}, s, t) = \mathrm{Val}(\mathscr{A}, t, f)$, $\mathscr{A} \models s < t[f, g]$ iff $\mathrm{Val}(\mathscr{A}, s, f) \prec \mathrm{Val}(\mathscr{A}, t, f)$, $\mathscr{A} \models s \in x_i[f, g]$ iff $\mathrm{Val}(\mathscr{A}, s, f) \in g(i)$. Here $f(i) \in D$ is the interpretation of $n_i$, and $g(i) \in K$ is the interpretation of $x_i$.

We say that $\mathscr{A} = (M, K)$ is an $\omega$-structure just in case $M$ is the standard model of arithmetic. In this case, we identify $\mathscr{A}$ with $K \subset \mathscr{P}(\omega)$.

A formula is called arithmetic if it has no bound set variables, and a sentence is a formula with no free variables.

The $\Delta_0$ formulae are given by (i) atomic formulae are $\Delta_0$, (ii) propositional combinations of $\Delta_0$ formulae are $\Delta_0$, and (iii) if $A$ is $\Delta_0$, $n$ is a numerical variable, $t$ a term, then $(\exists n)(n < t \,\&\, A)$, $(\forall n)(n < t \rightarrow A)$ are $\Delta_0$.

A formula is regular if it is of the form $(Q\alpha_1) \cdots (Q\alpha_n)(B)$, where $B$ is a $\Delta_0$ formula not beginning with a quantifier, $0 \leq n$. The quantifiers $(Q\alpha_1), \cdots, (Q\alpha_n)$ are called the leading quantifiers.

The $\Sigma_k^0$ ($\Pi_k^0$) formulae are the regular arithmetic formulae with at most $k$ leading quantifiers, beginning with an existential (universal) quantifier.

The $\Sigma_k$ ($\Pi_k$) formulae are those regular formulae whose leading quantifiers begin with a block of at most $k$ set quantifiers beginning with an existential (universal) one, followed by only arithmetic quantifiers.

The $\Sigma$ ($\Pi$) formulae are the regular formulae with no universal (existential) set quantifiers.

**I. Axioms for arithmetic sets.** RCA (recursive comprehension axiom system) consists of

1. (successor axioms) $N(n) \neq 0$, $N(n) = N(m) \rightarrow n = m$.
2. (recursion axioms) $n + 0 = n$, $n + N(m) = N(n + m)$, $n \cdot 0 = 0$, $n \cdot N(m) = (n \cdot m) + n$, $n < m \leftrightarrow (\exists r)(r \neq 0 \,\&\, n + r = m)$.

3. (induction axioms) $(A(0) \,\&\, (\forall n)(A(n) \to A(N(n)))) \to (\forall n)(A(n))$, where $A$ is arbitrary.

4. (recursive comprehension) $(\forall n)(A(n) \leftrightarrow B(n)) \to (\exists x)(\forall n)(n \in x \leftrightarrow A(n))$, where $A$ is $\Sigma_1^0$, $B$ is $\Pi_1^0$, $x$ not free in $A$.

Note that the $\omega$-models of RCA are just the collections of sets closed under join and relative recursivity. In RCA, we can define and prove the basic facts about coding. These include codes for finite sequences of natural numbers, for functions as sets, for finite and infinite sequences of sets and functions, for $\mathscr{L}$-structures, and for partial recursive functions and recursively enumerable sets (although not every index will provably define a p.r. function or an r.e. set, because of the weakness of the recursive comprehension axiom). In addition, the satisfaction relation for propositional calculus can be defined.

ACA (arithmetic comprehension axiom system) consists of RCA together with arithmetic comprehension: $(\exists x)(\forall n)(n \in x \leftrightarrow A(n))$, for arithmetic $A$ in which $x$ is not free. In ACA, we can define and prove the inductive clauses for the satisfaction relation for predicate calculus, which cannot be done in RCA. Note that the $\omega$-models of ACA are just the collections of sets closed under join and relative arithmeticity.

In formalizing model theory in RCA, we use the following conventions. Given a structure $\mathscr{A}$, and a sentence $A$, we let $A^{(\mathscr{A})}$ be the formula that asserts that $A$ holds in $\mathscr{A}$, obtained by relativizing the symbols in $A$ to $\mathscr{A}$. Thus $A^{(\mathscr{A})}$ and $A$ have the same complexity. For structures $\mathscr{A}$, set variables $y_1, \cdots, y_n$, set constants $c_1, \cdots, c_n$, we write $\operatorname{Rep}(\mathscr{A}, c_1, \cdots, c_n, y_1, \cdots, y_n)$ for the formula which asserts "$\mathscr{A}$ is a structure in the language $\mathscr{L}$ augmented with the set constants $c_1, \cdots, c_n$, and $\bar{n} \in c_i$ holds in $\mathscr{A}$ if and only if $n \in y_i$."

We now consider two important combinatorial principles. König's lemma asserts that every infinite finitely branching tree of finite sequences of natural numbers has an infinite path. Weak König's lemma asserts that every infinite tree of finite sequences of 0's and 1's has an infinite path.

Take KL (WKL) to be the system consisting of RCA together with König's lemma (weak König's lemma).

Let SHB (sequential Heine-Borel system) be the system consisting of RCA together with the axiom which asserts that every sequence of open intervals which covers [0, 1] has a finite initial segment which covers [0, 1]. In the formulation of SHB, reals are identified with the set of rationals less than them, and open intervals are identified with appropriate pairs of reals.

Let SLUB (sequential least upper bound system) be the system consisting of RCA together with the axiom which asserts that every bounded infinite sequence of reals has a least upper bound.

Let MLUB (monotone least upper bound system) be the system consisting of RCA together with the axiom which asserts that every bounded monotone increasing sequence has a least upper bound.

Let SBW (sequential Bolzano-Weierstrass system) consist of RCA together with the axiom which asserts that to every bounded sequence of distinct real numbers, there is a real number every neighborhood of which contains at least two terms.

By formalizing familar recursion theoretic constructions, we have

THEOREM 1.1. *ACA is equivalent to* (a) *KL*, (b) *SLUB*, (c) *MLUB*, *and* (d) *SBW*.

Theorem 1.1 is an illustration of our theme I. The following theorem is another illustration of theme I.

THEOREM 1.2. *WKL is equivalent to* (a) *the compactness theorem for propositional calculus*, (b) *the completeness theorem for sets of sentences in propositional calculus*, (c) *SHB*, *and* (d)

$$A(x_1, \cdots, x_k) \rightarrow (\exists \mathscr{A})(\mathrm{Rep}(\mathscr{A}, c_1, \cdots, c_k, x_1, \cdots, x_k) \mathbin{\&} A(c_1, \cdots, c_k)^{(\mathscr{A})}),$$

*where $A$ has the free variables $x_1, \cdots, x_k$. Other equivalents are* (e) *every consistent theory in predicate calculus has a complete consistent extension in the same language*, *and* (f) *every consistent theory in predicate calculus has a Henkin complete extension* (*with new Henkin constants added*).

Observe that (d) above is a reflection principle, asserting that if a statement is true, there is a structure in which it holds.

The $\omega$-models of WKL have special significance. Let PA denote Peano arithmetic. A set $x \subset \omega$ is called binumerable in a complete consistent extension $K$ of PA just in case $x = \{n : A(\bar{n}) \in K\}$, for some formula $A$ with one free variable. A set $x \subset \omega$ is called representable in a model $\mathscr{A}$ of PA just in case $x = \{n : \mathscr{A} \models A(\bar{n})\}$, for some formula $A$ with one free variable. The first half of the following theorem is due to Scott [7]. Our proof of the second half uses the continuum hypothesis, but it most likely is eliminable.

THEOREM 1.3. *The countable $\omega$-models of WKL are precisely those collections of sets which, for some complete consistent extension $K$ of PA, are the sets binumerable in $K$. The $\omega$-models of WKL are precisely those collections of sets which, for some model $\mathscr{A}$ of PA, are the sets representable in $\mathscr{A}$.*

By taking a $\Delta_2^0$ complete consistent extension of PA, we have an $\omega$-model of WKL which is not an $\omega$-model of ACA. It is also clear that the recursive sets do not form an $\omega$-model of WKL.

Using formalized cut elimination, formalized recursion theory, and forcing, we obtain

THEOREM 1.4. *RCA and WKL prove the same $\Pi$ formulae. However, they do not prove the same $\Sigma_1$ sentences.*

We now consider what recursion theory can be proved in WKL. WKL proves the existence of a plethora of incomparable Turing degrees. The best theorem we know along these lines is

THEOREM 1.5. *WKL proves that for any $x_0$ there is a sequence $\{x_n\}$, $0 \leqq n$, such that each $x_n$ is nonrecursive, and the only sets recursive in more than one term are recursive.*

ACA would suffice to prove the existence of a perfect tree every two paths of which are of incomparable Turing degree, and we do not know if WKL is sufficient. ACA suffices to prove the existence of a set of minimal Turing degree, and again we do not know if WKL is sufficient.

ACA is obviously sufficient to explicitly define a nonrecursive set (e.g., the jump). WKL is not sufficient, and so the following theorem provides us with an illustration of our theme II. The proof uses forcing, symmetry arguments, and recursion theoretic diagonal arguments.

THEOREM 1.6. *There is an $\omega$-model of WKL + $(\exists! x)(A(x)) \rightarrow (\exists x)(A(x)$ & $x$ is recursive), where $A(x)$ is an arbitrary formula with $x$ as the only free set variable.*

The following concerns the corresponding rule.

THEOREM 1.7. *If WKL proves $(\exists x)(A(x)$ & $x$ is not recursive) then WKL proves $(\forall x)(\exists y)(A(y)$ & $y$ is not recursive and $(\forall n)((x)_n \neq y))$, where $A$ is a $\Sigma$ formula with $x$ as the only free set variable.*

**II. Axioms for hyperarithmetic sets.** HCA (hyperarithmetic comprehension axiom system) consists of RCA together with $(\forall n)(A(n) \leftrightarrow B(n)) \rightarrow (\exists x)(\forall n)(n \in x \leftrightarrow A(n))$, where $A$ is $\Sigma_1$, $B$ is $\Pi_1$, and $x$ is not free in $A$.

HAC (hyperarithmetic axiom of choice system) consists of RCA together with $(\forall n)(\exists x)(A(n, x)) \rightarrow (\exists y)(\forall n)(\exists x)(x = (y)_n$ & $A(n, x))$, where $A$ is arithmetic, $y$ not free in $A$.

HDC (hyperarithmetic axiom of dependent choice system) consists of RCA together with

$$(\forall x)(\exists y)(A(x, y)) \rightarrow (\forall w)(\exists z)(\forall n)(\exists x)(\exists y)$$
$$(x = (z)_n \mathbin{\&} y = (z)_{n+1} \mathbin{\&} A(x, y) \mathbin{\&} w = (z)_0),$$

where $A$ is arithmetic, $n$, $z$, $w$ not free in $A$.

ABW (arithmetic Bolzano-Weierstrass) consists of RCA together with the axioms which assert that to every bounded arithmetic predicate of reals there is either a finite sequence of reals which includes all solutions, or a real, every neighborhood of which contains at least two solutions.

It is easy to see that HAC implies ABW, but we know very little about the consequences of ABW.

SL (sequential limit system) consists of RCA together with the axioms which assert that, whenever every neighborhood of $x$ contains at least two solutions to an arithmetic predicate, $x$ is the limit of some sequence of solutions from that predicate.

The following is an illustration of theme I.

THEOREM 2.1. *HAC is equivalent to SL.*

The first half of the following is due to Kreisel [5], and the second half is due to Feferman.

THEOREM 2.2. *The $\omega$-models of HCA are closed under join and relative hyper-*

*arithmeticity. Not every collection closed under join and relative hyperarithmeticity obeys HCA.*

It is easy to see that HDC $\vdash$ HAC $\vdash$ HCA. The following is due to Friedman [1] and [3].

THEOREM 2.3. *HCA and HDC prove the same $\Pi_2$ formulae. There is a $\Sigma_2$ sentence provable in HDC but false in some $\omega$-model of HAC.*

J. Steel has recently proved that HCA and HAC are not equivalent (in fact, there is an $\omega$-model of HCA not satisfying HAC), solving a long outstanding problem. It is still open whether HCA proves each instance of HAC without parameters. Steel has also proved the independence of the relativized Kleene-Souslin theorem (every set $\Delta_1$ in $x$ is hyperarithmetic in $x$) from HDC. It is still open whether HDC (or HCA) proves the Kleene-Souslin theorem.

**III. Axioms for arithmetic recursion.** ATR (arithmetic transfinite recursion) consists of ACA together with axioms that assert that arithmetic recursion can be performed on any well ordering of natural numbers. (The $H$-sets on recursive well orderings are examples of the result of such transfinite recursions.)

The weak $\Pi_1$-AC consists of RCA together with $(\forall n)(\exists m)(A(n, m)) \rightarrow (\exists f)(\forall n)(A(n, f(n)))$, where $A$ is $\Pi_1$, $f$ not free in $A$.

CWO (comparability of well orderings system) consists of RCA together with the axiom which asserts that to each pair of well orderings of natural numbers, there is an isomorphism of one onto an initial segment of the other.

PST (perfect subtree theorem system) consists of RCA together with the axiom that asserts that every tree of finite sequences such that no infinite sequence of functions includes all infinite paths has a perfect subtree.

CDS (countability of discrete sets system) consists of ACA together with axioms which assert that to every arithmetic predicate of reals, every two distinct solutions of which are at least one unit apart, there is a sequence which includes all its solutions.

The following is an illustration of theme I.

THEOREM 3.1. *ATR is equivalent to* (a) *weak $\Pi_1$-AC,* (b) *PST,* (c) *CWO,* (d) *CDS, and* (e) *ACA + "to each pair of well orderings there is an isomorphism from one into the other."*

As far as comparisons with the axioms for hyperarithmetic sets, we have

THEOREM 3.2. *ATR proves HAC, but not HDC. ATR proves the existence of an $\omega$-model of HDC. ATR + HDC proves the existence of an $\omega$-model of ATR.*

The first part of the following theorem is due to Kreisel [6], and the second essentially due to Simpson [8].

THEOREM 3.3. *ATR proves the relativized Kleene-Souslin theorem. ATR proves the existence of a perfect tree, the paths of which are of distinct nonzero minimal hyperdegree.*

In the next section we will state that TI, a system which proves ATR, does not suffice to define explicitly a nonhyperarithmetic set. The next theorem concerns the corresponding rule.

THEOREM 3.4. *If ATR proves* $(\exists x)(A(x)$ *& x not hyperarithmetic), then ATR proves* $(\forall x)(\exists y)(A(y)$ *& y is not hyperarithmetic &* $(\forall n)((x)_n \neq y))$*, where A is a* $\Sigma_2$ *formula with x as the only free set variable.*

**IV. Axioms for transfinite induction.** TI (transfinite induction system) consists of RCA together with axioms which assert that transfinite induction can be applied to any well ordering of natural numbers with respect to any formula.

RFN (reflection system) consists of ACA together with the axioms

$$A(x_1, \cdots, x_k) \rightarrow (\exists \mathscr{A})(\mathrm{Rep}(\mathscr{A}, c_1, \cdots, c_k, x_1, \cdots, x_k) \\ \& \ A(c_1, \cdots, c_k)^{(\mathscr{A})} \ \& \ \mathscr{A} \text{ is an } \omega \text{ structure}),$$

where $A$ has only the free variables $x_1, \cdots, x_k$.

By formalizing the proof of the completeness of cut free rules for $\omega$-logic, we obtain the following.

THEOREM 4.1. *TI and RFN are equivalent.*

Many questions arise in connection with systems obtained by restricting the complexity of the formulae to which the transfinite inductions are applied in TI. In the following theorem, which answers a few of the questions that arise, all systems are understood to include RCA.

THEOREM 4.2. *TI for* $\Sigma_1$ *formulae is equivalent to ATR. ATR does not prove TI for* $\Pi_1$ *formulae, but HDC does. TI for* $\Pi$ *formulae proves HAC. TI for* $\Pi_1$ *formulae proves HCA. TI for* $\Sigma_2$ *formulae proves HDC.*

A $\beta$-structure is a $K \subset \mathscr{P}(\omega)$ such that if $P(x_1, \cdots, x_k)$ is true then $P(x_1, \cdots, x_k)$ holds in $K$, where $x_1, \cdots, x_k \in K$, and $P$ is $\Sigma_1$ with only $x_1, \cdots, x_k$ free. Observe that any $\beta$-structure forms an $\omega$-model of TI.

We now state the theorem mentioned previously about the failure of TI in explicitly defining a nonhyperarithmetic set. This is an illustration of theme II.

THEOREM 4.3. *There is an* $\omega$*-model of TI (in fact, a* $\beta$*-structure) which satisfies* $(\exists ! x)(A(x)) \rightarrow (\exists x)(A(x)$ *& x is hyperarithmetic), for arbitrary A whose only set variable is x.*

There is the corresponding rule:

THEOREM 4.4. *If TI proves* $(\exists x)(A(x)$ *& x is not hyperarithmetic), then TI proves* $(\forall x)(\exists y)(A(y)$ *& y is not hyperarithmetic &* $(\forall n)((x)_n \neq y))$*, for* $\Sigma_2$ *formulae A with x as its only free set variable.*

**V. Axioms for the hyperjump.** $\Pi_1$-CA consists of RCA together with $(\exists x)(\forall n)(n \in x \leftrightarrow A(n))$, for $\Pi_1$ formulae $A$ without $x$ free.

PKT (perfect kernel theorem system) consists of RCA together with the axiom which asserts that to every tree $T$ of finite sequences with no infinite sequence of functions including all infinite paths, there is a perfect subtree $S$ and a sequence of functions such that every infinite path through $T$ is a path through $S$ or a term in the sequence.

ALUB (arithmetic least upper bound system) consists of RCA together with the axioms which assert that if the solutions to a nonempty arithmetic predicate of real numbers have an upper bound, they have a least upper bound.

The following is an illustration of theme I.

THEOREM 5.1. *$\Pi_1$-CA, PKT, and ALUB are equivalent.*

The last part of the following theorem is proved in Friedman [**2**].

THEOREM 5.2. *$\Pi_1$-CA proves ATR + HDC. There is an $\omega$-model of $\Pi_1$-CA that does not satisfy TI. $\Pi_1$-CA proves the existence of an $\omega$-model of TI (in fact, the existence of a $\beta$-structure).*

The second clause in Theorem 5.2 can be generalized. Let $T$ be any finite extension of RCA. Clearly RFN + $T$ proves the existence of an $\omega$-model of $T$. By the incompleteness theorem for $\omega$-logic, not every $\omega$-model of $T$ satisfies RFN, or equivalently TI.

We have considered stronger systems of second order arithmetic, but our results to date do not provide significant illustrations of our themes. We have also considered systems with restricted induction (see Friedman [**4**]).

## Bibliography


**1.** H. Friedman, *Subsystems of set theory and analysis*, Dissertation, M.I.T., Cambridge, Mass., 1967.

**2.** ———, *Bar induction and $\Pi^1_1$-CA*, J. Symbolic Logic **34** (1969), 353–362. MR **40** #4109.

**3.** ———, *Iterated inductive definitions and $\Sigma^1_2$-AC*, Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968), North-Holland, Amsterdam, 1970, pp. 435–442. MR **44** #1555.

**4.** ———, *Systems of second order arithmetic with restricted induction*. I, II (abstract presented by-title, April 1975 meeting of the Association for Symbolic Logic), J. Symbolic Logic (to appear).

**5.** G. Kreisel, *The axiom of choice and the class of hyperarithmetic functions*, Nederl. Akad. Wetensch. Proc. Ser. A **65** = Indag. Math. **24** (1962), 307–319. MR **25** #3838.

**6.** ———, *A survey of proof theory*, J. Symbolic Logic **33** (1968), 321–388. MR **43** #7295.

**7.** D. S. Scott, *Algebras of sets binumerable in complete extensions of arithmetic*, Proc. Sympos. Pure. Math., vol. 5, Amer. Math. Soc., Providence, R.I., 1962, pp. 117–121. MR **25** #4993.

**8.** S. Simpson, *Minimal covers and hyperdegrees*, Trans. Amer. Math. Soc. **209** (1975), 45–64.


STATE UNIVERSITY OF NEW YORK AT BUFFALO
AMHERST, NEW YORK 14226, U.S.A.