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A CONVEXITY PRESERVING PEANO CURVE

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ABSTRACT. Let I denote the unit interval. It has been asked whether a function $T: I \to E^2$ exists such that T(I) is 2-dimensional and T([0,t]) is convex for all $t \in [0,1]$. In this paper such an example is provided.

1. Introduction. In 1890, G. Peano surprised the mathematical community by exhibiting a continuous function from the unit interval onto the closed unit square. Subsequently many such space filling curves were constructed; a recent example is given in [2]. D. Mauldin recently asked the following question (the origin of which is not clear): Does there exist a continuous function T from [0,1] onto a 2-dimensional subset of the Euclidean plane such that T([0,t]) is convex for all $t \in [0,1]$? Previous examples of space filling curves do not satisfy this convexity property. In Section 3 of this paper, such a convexity preserving map from the unit interval to the closed unit disk $D = \{(x,y) \in E^2 | x^2 + y^2 \leqslant 1\}$ will be constructed.

THEOREM. *There is a continuous surjection* $T: [0,1] \to D$ *with the property that* T([0,t]) *is convex for all* $t \in [0,1]$.

Our example does not have the property that T(J) is convex for every subinterval J of [0,1]. We conjecture that such an example cannot exist and comment further on this problem in Section 4.

2. Preliminaries. The *mesh* of a finite collection F of sets in a metric space is the maximum of the diameters of the members of F and will be denoted $\mu(F)$. For any collection F of sets, let F* denote the union of the members of F. Let F be a finite collection of compact subsets of a topological space X. If the interiors of the members of F are all non-empty and pairwise disjoint, then F will be called a *partition in* X. Note that F* = X is not assumed. The first proposition uses the notion of a defining sequence.

DEFINITION. Let X and Y be compact metric spaces and $X \subseteq E$. A sequence of triples $\{T_n\colon F_n \to G_n\}_{n=1}^{\infty}$ is a *defining sequence between* X *and* Y provided:

1. For each integer n the collections F_n and G_n are partitions in the spaces E and Y, respectively.
2. The set X is equal to $\cap_{n=1}^{\infty} F_n^*$.
3. For each n, $G_n^* = Y$.
4. The function $T_n\colon F_n \to G_n$ is a bijection with the property that if $f,f' \in F_n$ and $f \cap f' \neq \phi$, then $T_n(f) \cap T_n(f') \neq \phi$.
5. For each n, F_{n+1} refines F_n and G_{n+1} refines G_n.
6. Given $\epsilon > 0$, there is an integer n such that $\mu(G_n) < \epsilon$.
7. If $f_n \in F_n$, $f_{n+1} \in F_{n+1}$ and $f_{n+1} \subseteq f_n$, then $T_{n+1}(f_{n+1}) \subseteq T_n(f_n)$.

PROPOSITION 2.1. *If* $\{T_n\colon F_n \to G_n\}_{n=1}^{\infty}$ *is a defining sequence between* X *and* Y, *then there is a continuous surjection* $T\colon X \to Y$ *defined by the equation* $T(\cap_{n=1}^{\infty} f_n) = \cap_{n=1}^{\infty} T_n(f_n)$, *where the sequence* $\{f_n\}_{n=1}^{\infty}$ *is nested and* $f_n \in F_n$. *Moreover, the map* T *has the property that* $T(f_n) = T_n(f_n) = g_n$ *for all* $f_n \in F_n$.

PROOF. To prove this proposition it is sufficient to check that the seven properties of a defining sequence ensure that T is well-defined on X, continuous, and surjective. If $\{f_n\}_{n=1}^{\infty}$ is a nested sequence, where $x \in f_n \in F_n$ for all n, then define $T(x) = \cap_{n=1}^{\infty} T_n(f_n)$. By property 7, the sequence $\{T_n(f_n)\}_{n=1}^{\infty}$ is also nested. By property 6, T(x) is a single point. If $\{f_n'\}_{n=1}^{\infty}$ is any other nested sequence such that $x \in f_n' \in F_n$ for all n, then $f_n \cap f_n' \neq \phi$, so that $T_n(f_n) \cap T_n(f_n') \neq \phi$, by property 4. Thus $\cap_{n=1}^{\infty} T_n(f_n) = \cap_{n=1}^{\infty} T_n(f_n')$ and T is well-defined. Properties 4 and 6 imply that T is continuous. Since the interiors of the members of G_n and G_{n+1} are pairwise disjoint and since G_{n+1} refines G_n, each member of G_{n+1} lies in exactly one member of G_n. Thus, if $f \in F_n$, the union of the members of F_{n+1} in F_n is equal to f. Since T_{n+1} is a bijection, the set $\cup_{f' \subseteq f} T_{n+1}(f')$ contains all the members of G_{n+1} in $T_n(f)$. Thus $T(f) = T_n(f)$. In particular, T is a surjection.■

PROPOSITION 2.2. *There exists a sequence of partitions* $\{G_n\}_{n=1}^{\infty}$ *in* D *with the following properties:*

1. *For each* n, $G_n^* = D$.
2. *The collection* G_{n+1} *refines* G_n.

3. *The collection* G_n *can be ordered* $G_n = \{G_{n,1},\ldots,G_{n,m_n}\}$ *so that* $\cup_{i=1}^{k} G_{n,k}$ *is convex for each* $k \leqslant m_n$.

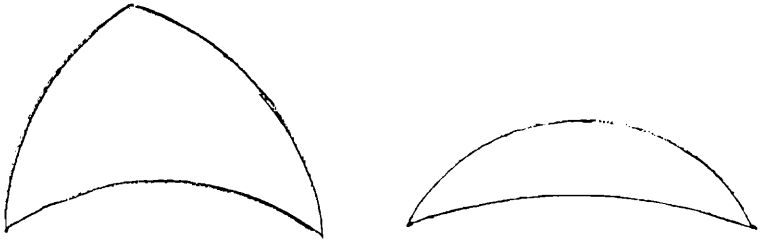
4. *For each* n, $\mu(G_{n+1}) \leqslant (\sqrt{3}/2)\mu(G_n)$.

PROOF. Let the first partition G_1 be the collection whose only member is the disk D. Assume inductively that a partition G_n has been found which satisfies properties 1 - 4. In addition, assume G_n satisfies the hypotheses:

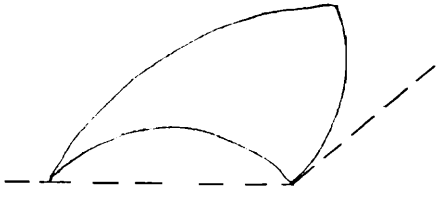
5. For each n, $G_{n,1}$ is a disk.

6. For each $n > 1$ and each $i > 1$, $G_{n,i}$ is a region bounded by either 2 or 3 circular arcs, as in Figure 1a.

7. The intersection $G_{n,i} \cap \cup_{j=1}^{i-1} G_{n,j}$ is equal to one of the circular arcs making up the boundary $G_{n,i}$. This arc will be called the *base* of $G_{n,i}$. It is also assumed that the base angles of $G_{n,i}$ are acute to avoid the possible pathology of Figure 1b.



a. The regions $G_{n,i}$.



b. A non-acute angle.

Figure 1.

The partition G_{n+1} can be formed by refining each of the sets $G_{n,i}$ in the following manner. The region $G_{n,1}$ is partitioned into 7 sets as indicated in Figure 2. This partition is formed by joining three equally spaced points A, B and C on the boundary of G by three circular arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CA}$ with equal radii large enough so that each of the seven regions has diameter not greater than $\sqrt{3}/2$ diam $G_{n,1}$. The set $G_{n+1,1}$ will be the disk whose center is the origin and which is tangent to each of the circular arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CA}$. For $i > 1$, let P be the center of the circle forming the base $\overset{\frown}{AB}$ of $G_{n,i}$ (see Figure 3), and let r be the distance from P to the point Q in $G_{n,i}$ which is the farthest from P. Take R to be any number larger than r. Choose points A_1 and B_1 on the arcs $\overset{\frown}{AQ}$ and $\overset{\frown}{BQ}$, respectively, so that $d(A_1,P) = d(B_1,P)$ and so that the diameter of the region bounded by circular arcs $\overset{\frown}{A_1Q}$, $\overset{\frown}{B_1Q}$ and $\overset{\frown}{A_1B_1}$ has diameter less than $\sqrt{3}/2$ diam $G_{n,i}$. Let $r_1 = d(A_1,P)$ and let C_1 be the midpoint of the arc $\overset{\frown}{A_1B_1}$ with center P through A_1 and B_1. Let $\overset{\frown}{A_1C_1}$ denote the subarc of $\overset{\frown}{A_1B_1}$ with endpoints A_1 and C_1, and let $\overset{\frown\frown}{A_1C_1}$ denote the subarc of the circle containing the points A_1 and C_1 with center P but with radius R. (See Figure 4.) Define $\overset{\frown\frown}{C_1B_1}$ similarly. Let r_2 be the distance from P to the midpoint of $\overset{\frown\frown}{A_1C_1}$ (or equivalently $\overset{\frown\frown}{C_1B_1}$). The points A_2 and B_2 are the intersection of $\overset{\frown}{AQ}$ and $\overset{\frown}{BQ}$, respectively, with the circle whose center is P and radius r_2. The partition of the region $\{x \in G_{n,i} | r_2 \leqslant d(x,P) \leqslant r_1\}$ into 5 spherical regions is shown in Figure 4. Continue the process with $\overset{\frown}{A_2B_2}$, $\overset{\frown}{A_3B_3}$, etc. Note that $r_3 - r_2 \geqslant r_2 - r_1$, so that after a finite number of steps the arc $\overset{\frown}{AB}$ will be reached. While the argument that this last region can be partitioned as before is a little different from the one given above (the last A_k and B_k may not agree with A and B exactly and the value of R must be changed slightly), it is similar and will be left to the interested reader. Now enumerate the members of G_{n+1} in $G_{n,i}$ as $G_{n+1,m+1}, G_{n+1,m+2}, \cdots$, where G_{n+1} already has m members in $H = \cup_{j=1}^{i-1} G_{n,j}$. The members closest to H will have the lower indices. Property 3 in the induction will be satisfied because $\cup_{j=1}^{k} G_{n+1,j}$ is the intersection of convex sets. For example, the region $\cup_{j=1}^{5} G_{n,j}$ in Figure 2 is the intersection of 3 large disks; the region $\cup_{j=1}^{2} G_{n,j}$ in the same figure is the intersection of the two convex sets in Figure 5. The remainder of the properties 1 - 7 can be easily checked. ■

3. Main theorems. The decomposition of the unit disk of Proposition 2.2 is now

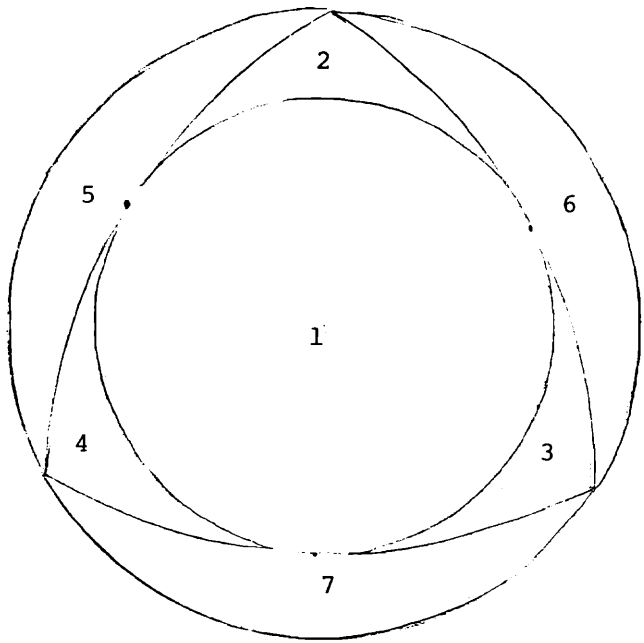


Figure 2. The partition G_{n+1}.

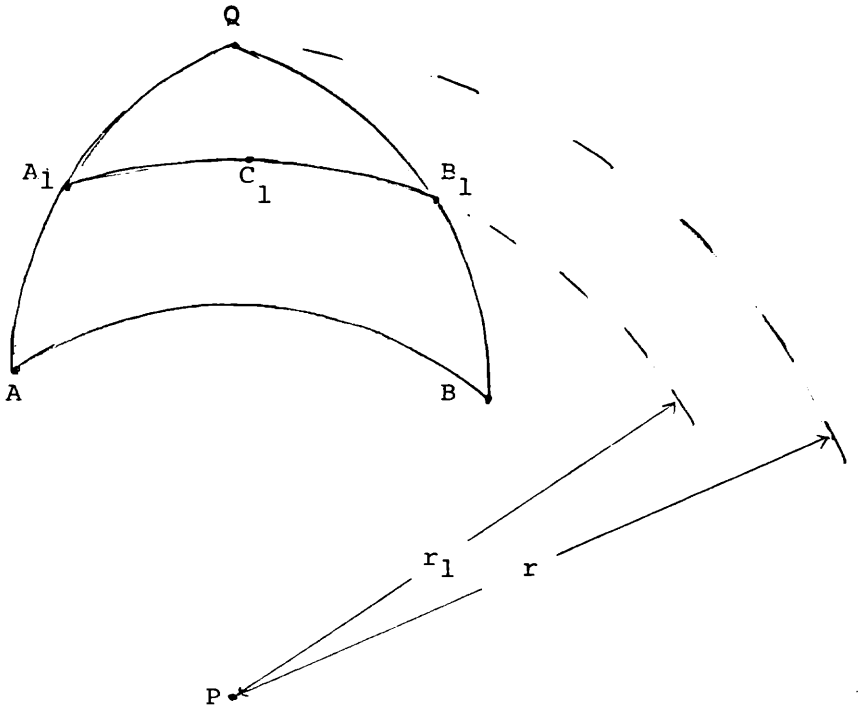


Figure 3.

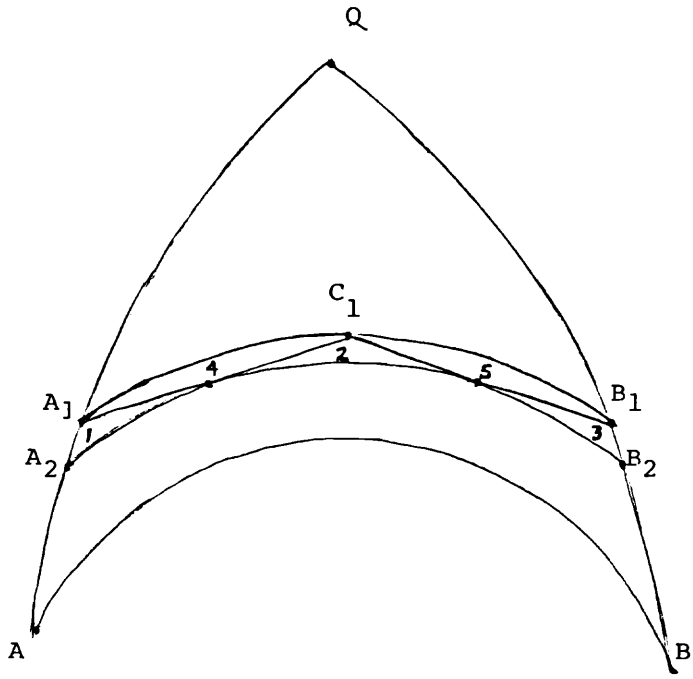


Figure 4. A step in the refinement of $G_{n,i}$.

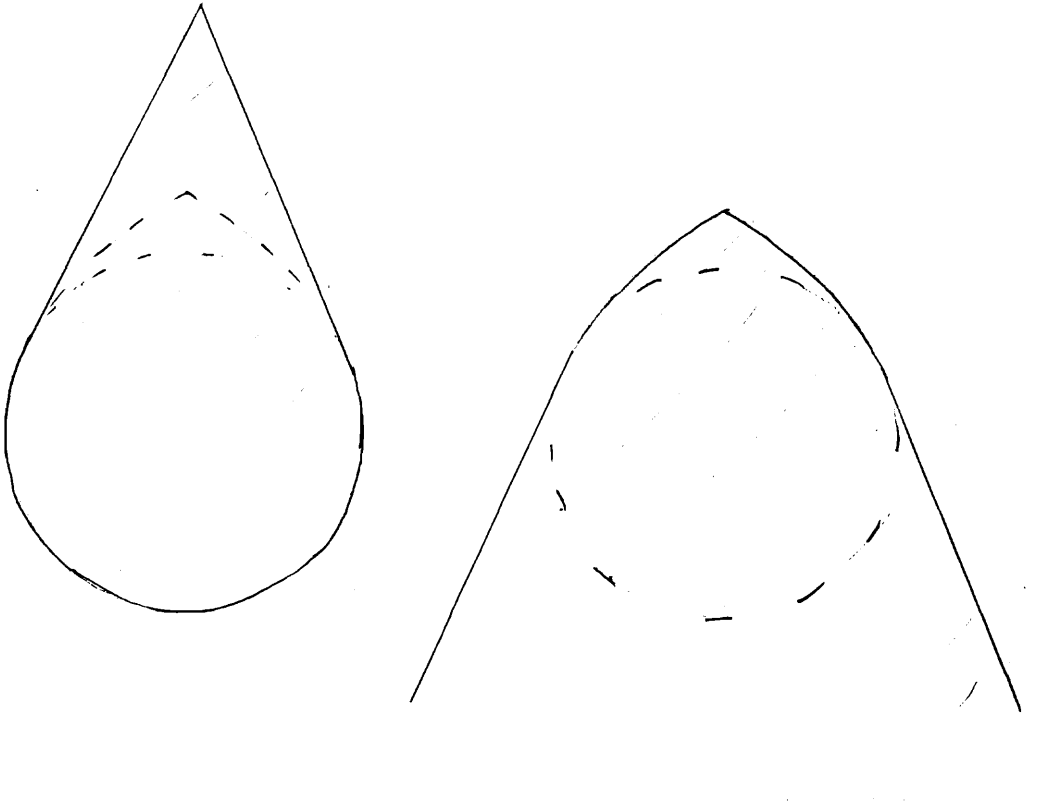


Figure 5. The convexity of $G_{n,1} \cup G_{n,2}$.

used to define continuous surjections onto D, first from the Cantor set and then from the unit interval. These maps are convexity preserving.

THEOREM 3.1. *If* X *is the standard Cantor set in the unit interval, then there is a continuous surjection* $T: X \to D$ *with the property that* $T([0,t] \cap X)$ *is convex for all* $t \in [0,1]$.

PROOF. Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of partitions guaranteed by Proposition 2.2. Recall that $G_1 = \{D\}$. Choose $F_1 = \{[0,1]\}$. Define $T_1: F_1 \to G_1$ by $T_1([0,1]) = D$. Assume inductively that there is a sequence $\{T_i: F_i \to G_i\}_{i=1}^{n}$ which satisfies the 7 properties of a defining sequence. In addition, assume that the members of F_i are pairwise disjoint closed intervals in $[0,1]$ which have been ordered so that $F_i = \{F_{i,1},\ldots,F_{i,m_i}\}$ implies that $\cup_{j=1}^{k} T_i(F_{i,j})$ is convex for all $k \leqslant m_i$. For each $G_{n,j} \in G_n$, count the number of members of G_{n+1} in $G_{n,j}$ and call this number α_{nj}. Now find α_{nj} pairwise disjoint closed intervals in $F_{n,j}$ of diameter $< 1/(n+1)$. The union of these collections will be the collection F_{n+1}. Order the members of F_{n+1} by the rule that $F_{n+1,i} < F_{n+1,j}$ if and only if $i < j$. The bijection $T_{n+1}: F_{n+1} \to G_{n+1}$ will be defined by $T_{n+1}(F_{n+1,i}) = G_{n+1,i}$. By the definition of T_{n+1} and property 3 of Proposition 2.2, $\cup_{j=1}^{n} T_{n+1}(F_{n+1,j})$ will be convex. By Proposition 2.1, there is a continuous surjection $T: X \to D$ with the property that $T(\cup_{i=1}^{k} F_{n,i})$ is convex for all $k \leqslant m_n$. If $t \in X$, then $[0,t] = \cap_{n=1}^{\infty} [0,a_n]$, where a_n is the right hand endpoint of some member of F_n. We can assume that $a_{n+1} \leqslant a_n$ for all n. Since $T(X \cap [0,a_n])$ is convex for all n, $T(X \cap [0,t]) = T(\cap_{n=1}^{\infty} X \cap [0,a_n]) = \cap_{n=1}^{\infty} T(X \cap [0,a_n])$ is convex. The last equality follows from the definition of T and its relation with the functions T_n.■

THEOREM 3.2. *There is a continuous surjection* $T: [0,1] \to D$ *with the property that* $T([0,t])$ *is convex for all* $t \in [0,1]$.

PROOF. As before, X denotes the standard Cantor set. Let $\hat{T}: X \to D$ be a continuous surjection with the property that $\hat{T}(X \cap [0,t])$ is convex for all $t \in [0,1]$. Let $T: [0,1] \to D$ be a linear extension of $\hat{T}$. More precisely, if a and b are endpoints of an open interval in $[0,1] - X$, then define $\hat{T}(s) = T(a) + (s-a/b-a)(\hat{T}(b) - \hat{T}(a))$ for all $s \in [a,b]$. The map T will have the desired properties.■

4. Related problems. The main theorem of this paper probably generalizes to

higher dimensions. We conjecture that there is a continuous map T from the unit interval onto the unit ball in R^n such that $T([0,1])$ is convex for all $t \in [0,1]$. It is the technical difficulties in extending Proposition 2.2 to higher dimensions that makes a proof difficult. Perhaps a simpler construction can be carried out that will allow such a generalization.

Our example of a space filling curve does not have the property that T(J) is convex for every subinterval J of [0,1]. We conjectuve that such a function cannot exist.

CONJECTURE 4.1. *There does not exist a continuous surjection* $T: [0,1] \to D$ *such that* T(J) *is convex for every closed subinterval* J *of* [0,1].

In connection with this conjecture, note that the following result is a consequence of Proposition 2.2. The convex sets H_i in this corollary can be taken to be the convex hulls of the $G_{n,i}$ in the statement of Proposition 2.2.

COROLLARY 4.2. *Given* $\epsilon > 0$, *there are convex sets* $\{H_1, H_2, \ldots, H_m\}$ *in the plane such that*

1. $\cup_{i=1}^{m} H_i = D$.
2. $\cup_{i=1}^{k} H_i$ *is convex for all* $k \leqslant m$.
3. diam $H_i < \epsilon$ *for all* i.

If Conjecture 4.1 is false, then, for any $\epsilon > 0$, there exist sets $\{H_1, H_2, \ldots, H_m\}$ such that

1. $\cup_{i=1}^{m} H_i = D$.
2. $\cup_{i=j}^{k} H_i$ is convex for all $1 \leqslant j \leqslant k \leqslant m$.
3. diam $H_i < \epsilon$ for all i.

We conjecture that this is not possible.

CONJECTURE 4.3. *There is an* $\epsilon > 0$ *such that the unit disk in the plane cannot be covered by the union of any sequence* $\{H_1, \ldots, H_m\}$ *of convex sets of diameter at most* ϵ *such that* $\cup_{i=j}^{k} H_i$ *is convex for all* $1 \leqslant j \leqslant k \leqslant m$.

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