# TRIANGULAR WITT GROUPS.
# PART II: FROM USUAL TO DERIVED

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June 1999

ABSTRACT. We establish that the *derived* Witt group is isomorphic to the *usual* Witt group when 2 is invertible. This key result opens the Ali Baba's cave of triangular Witt groups, linking the abstract results of Part I to classical questions for the usual Witt group. For commercial purposes, we survey the future applications of triangular Witt groups in the introduction. We also establish a connection between odd-indexed Witt groups and *formations*. Finally, we prove that over a commutative local ring in which 2 is a unit, the shifted derived Witt groups are all zero but the usual one.

## INTRODUCTION

The *usual* Witt group is the one defined by Knebusch (see [6]) for algebraic varieties. We present the obvious generalization of his definition to exact categories (see § 1) without any feeling of paternity. The *derived* Witt group is the Witt group of the derived category (see § 2), following the general definition for triangulated categories introduced in [1] and [2].

Let $\mathcal{E}$ be an exact category with a duality $* : \mathcal{E} \to \mathcal{E}$, *i.e.* $*$ is a contravariant exact functor and $*^2 \simeq \mathrm{Id}$. This includes the case of schemes (even with twisted dualities) and therefore rings and fields but also the case of abelian categories with duality, like the category of finite length modules over a regular local ring, for instance. Given such an exact category, its derived bounded category $\mathrm{D^b}(\mathcal{E})$ is equipped with an induced duality $\# : \mathrm{D^b}(\mathcal{E}) \to \mathrm{D^b}(\mathcal{E})$ which is contravariant, satisfies $\#^2 \simeq \mathrm{Id}$ and is exact, this last notion of exactness requiring some care. Our main objective here is to establish that the natural homomorphism from the usual Witt group to the derived Witt group:

$$\omega_{\mathcal{E}} : \mathrm{W}(\mathcal{E}) \longrightarrow \mathrm{W}\big(\mathrm{D^b}(\mathcal{E})\big)$$

is an isomorphism when $\frac{1}{2} \in \mathcal{E}$. This is theorem 4.3. The homomorphism $\omega_{\mathcal{E}}$ is described in 2.8.

Even if it might be of some metaphysical interest to know that a construction depending on an exact category is actually an invariant of its derived category, the reader will rapidly note that symmetric forms over complexes are much "heavier" (to keep this text within the boundaries of politeness) than the classical symmetric forms. Technical and calligraphic obstacles arise, like the size of the complexes and of the morphisms, like homotopies, like cone and cylinder constructions, like (fractions of) quasi-isomorphisms and are moreover of considerable unaesthetic weight.

Therefore, if the main question remains of course:

*What shall we use derived Witt groups for?*

a collateral question is:

*How can we avoid the technical toughness of the symmetric forms over complexes?*

The answer to the second question is obvious: we try to formalize everything in the framework of triangulated categories. This was successfully used in [2] to establish a long exact sequence for localization. If, in some cases, the triangular approach appears to be too ethereal, one can also have recourse to the more detailed formalism of bi-Waldhausen categories.

1991 *Mathematics Subject Classification.* 11E81, 18E30, 19G12.
*Key words and phrases.* Usual and derived Witt groups, surgery on symmetric complexes.

Let us advertise derived Witt groups in answering the first question : usefulness. The following ideas are not developed herein. They will appear in forthcoming articles in collaboration with Charles Walter.

First of all, we already mentioned the localization exact sequence constructed in part I of this series. Making all due allowance, this gives for Witt groups the analogue of Thomason's result in K-theory, namely a long exact sequence relating the Witt groups of a scheme, the Witt groups of an open subscheme and the Witt groups on those complexes of the scheme which are acyclic on the open subscheme (*confer* [13]). This is discussed in remark 4.9 below.

Secondly, among the triangulated categories with duality associated to a given scheme, there is also the derived category of quasi-coherent sheaves with coherent homology, considered by Hartshorne in [4] and equipped with Grothendieck's duality. This will lead to the definition of what could be considered as the *Witt group of coherent modules*, by analogy with G-theory (or K′-theory). This will also allow us to define, in this very flexible framework, a direct image homomorphism (a *transfer*) for a large collection of morphisms of schemes, introducing twists and shifts, all things which make full sense in the derived categories as presented in [4]. As usual, over regular separated schemes, this will coincide with the already existing Witt groups (of locally free modules).

Thirdly, out of the long exact sequence, we shall elaborate a spectral sequence for a filtration of triangulated categories. Applying this to schemes, we shall establish a Gersten-type spectral sequence, analogous, once again, to the existing one in K-theory. This has very easy new and powerful consequences on the Gersten complex over schemes of low Krull dimension (until dimension 4) and opens the way to deal with those problems in higher dimension. Among "those problems", we do include the Gersten conjecture.

Fourthly, finally and obviously, the result : "*derived Witt group equals usual Witt group*" creates a new way for the computation of some Witt groups through results on the derived categories. Let me simply mention in this direction the possibility of computing the Witt group of the projective space over a field using well-known identifications of its derived category. This will re-prove the famous and pretty hard theorem of Arason $\big(\mathrm{W}(\mathbb{P}^n_k) \simeq \mathrm{W}(k)\big)$ and, more importantly, will allow similar computations for those projective schemes for which these determinations of the derived categories have already been performed.

To put it in a nutshell, we install a machinery that allows a serious treatment of the Witt groups within the well-known and powerful strategies of K-theory. But, of course, one has to check first that this new Witt group has something to do with the one we all love. This is the goal of the present article.

Let us overview briefly its content. After the presentations made in paragraphs 1 and 2, paragraphs 3 and 4 are devoted to the proof of the main result $\omega_{\mathcal{E}} : \mathrm{W}_{\mathrm{us}} \overset{\sim}{\to} \mathrm{W}_{\mathrm{der}}$. Surjectivity of $\omega_{\mathcal{E}}$ depends upon the possibility of reducing complexes (a non-split version of "surgery on complexes" as topologists would say) and is presented in § 3. Injectivity depends upon a general triangular result (*stably neutral forms are neutral*) established in part I of this series and upon a simple reformulation of the classical sub-lagrangian reduction (slogan : "$L^{\perp}/L$") displayed in lemma 4.1.

Paragraph 5 is dedicated to shifted or higher Witt groups, whose definitions are recalled, explained and detailed in remark 5.1. We also mention a slight amelioration of the techniques of reduction used in paragraph 3. This shows how a link could be made between odd-indexed Witt groups and groups of *formations* as introduced by Ranicki (see [12]) or Pardon (see [9]). In proposition 5.3, we give generators of the odd-indexed Witt groups but relations are not considered here. A complete treatment of this question though important is not of crucial interest in this first presentation. So we choose to focus on the usual Witt group for the obvious reasons.

Finally, we also establish the reassuring result : *Over a commutative local ring (e.g. a field) in which 2 is a unit, the only one of the four shifted Witt groups which is not trivial is the usual one* (theorem 5.6).

## Table of content

## 1. The usual Witt group.

**1.1. References.** Standard references for exact categories are [5], [8], [11] or appendix A of [13].

**1.2. Definition.** Let $\mathcal{E}$ be an exact category and let $* : \mathcal{E} \to \mathcal{E}$ be a contravariant exact functor such that there exists an isomorphism

$$\pi : \mathrm{Id} \overset{\sim}{\to} * \circ *$$

satisfying $(\pi_M)^* \circ \pi_{M^*} = \mathrm{Id}_{M^*}$ for all object $M$ of $\mathcal{E}$. The triple $(\mathcal{E}, *, \pi)$ is called an *exact category with duality.*

**1.3. Additive definitions.** Let $(\mathcal{E}, *, \pi)$ be an exact category with duality. This is in particular an additive categories with duality (see [10]). In this context, there are classical notions that we recall here briefly. A *symmetric space* is a pair $(P, \varphi)$ where $\varphi : P \overset{\sim}{\to} P^*$ is a symmetric isomorphism, meaning that $\varphi^* \circ \pi_P = \varphi$. It is common to refer to $\varphi$ as *the form* or sometimes *the symmetric form* over $P$.

Let $(P, \varphi)$ and $(Q, \psi)$ be symmetric spaces. The *orthogonal sum* of these spaces is the symmetric space $(P \oplus Q, \left(\begin{smallmatrix} \varphi & 0 \\ 0 & \psi \end{smallmatrix}\right))$ and is denoted by $(P, \varphi) \perp (Q, \psi)$. An *isometry* between $(P, \varphi)$ and $(Q, \psi)$ is an isomorphism $h : P \overset{\sim}{\to} Q$ such that $h^* \psi h = \varphi$. Isometry is of course an equivalence relation on symmetric spaces and preserves orthogonal sum.

Assume that $\mathcal{E}$ is essentially small. We define the *Witt monoid* of $\mathcal{E}$ to be the set of isometry classes of symmetric spaces, with orthogonal sum. We denote it by

$$(\mathrm{MW}(\mathcal{E}), \perp).$$

"M" stands for "Monoid".

**1.4. Definition.** Let $(\mathcal{E}, *, \pi)$ be an exact category with duality. A symmetric space $(P, \varphi)$ is called *metabolic* if it possesses a *lagrangian*, that is a pair $(L, \alpha)$ where $\alpha : L \rightarrowtail P$ is an admissible monomorphism such that the following sequence is exact:

$$L \overset{\alpha}{\rightarrowtail} P \overset{\alpha^* \varphi}{\twoheadrightarrow} L^*.$$

It is easy to check that a space isometric to a metabolic one is metabolic and that the sum of two metabolic spaces is again metabolic. In other words, we can consider the sub-monoid of $\mathrm{MW}(\mathcal{E})$ consisting of isometry classes of metabolic spaces, which we denote by

$$\mathrm{NW}(\mathcal{E}).$$

"N" stands for "Neutral".

**1.5. Remark.** Let $N \subset M$ be an inclusion of abelian monoids. Recall that the quotient monoid $M/N$ is the set of equivalence classes of elements of M under the following relation: $x \sim y$ if and only if there exists $n_1, n_2 \in N$ such that $x + n_1 = y + n_2$. The monoid $M/N$ is a group as soon as for any element $x \in M$ there exists an element $y \in M$ such that $x + y \in N$.

It is easy to see that if $(P, \varphi)$ is a symmetric space then the space $(P, \varphi) \perp (P, -\varphi)$ is metabolic.

**1.6. Definition.** Let $(\mathcal{E}, *, \pi)$ be an essentially small exact category with duality. The *Witt group* of $\mathcal{E}$ is defined to be the quotient

$$\mathrm{W}(\mathcal{E}) = \frac{\mathrm{MW}(\mathcal{E})}{\mathrm{NW}(\mathcal{E})}$$

(see remark 1.5). It is an abelian group. The class of symmetric space $(P, \varphi)$ in $\mathrm{W}(\mathcal{E})$ is denoted by $[P, \varphi]$ and we say that two symmetric spaces are *Witt-equivalent* if their classes in $\mathrm{W}(\mathcal{E})$ coincide. We call it the *usual* Witt group and denote it by $\mathrm{W}_{\mathrm{us}}(\mathcal{E})$ when we want to distinguish it from the derived one.

**1.7. Exercise.** Define the natural notion of *morphism of exact categories with duality* and check that the Witt group is a covariant functor. Repeat this exercise for all the notions introduced hereafter.

**1.8. Example.** Let $X$ be a scheme. Denote by $\mathcal{E}(X)$ the category of $\mathcal{O}_X$-modules which are locally free of finite rank. It is an exact category with the exact structure inherited from the abelian category of (quasi-coherent) $\mathcal{O}_X$-modules. Explicitly, a sequence is exact if it is locally a (necessarily split) short exact sequence of $\mathcal{O}_{X,x}$-modules for all $x \in X$. The functor

$$\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X) \,:\, \mathcal{E}(X) \to \mathcal{E}(X)$$

is a duality. By definition, the *usual Witt group* of the scheme $X$ is $\mathrm{W}_{\mathrm{us}}(X) = \mathrm{W}\big(\mathcal{E}(X)\big)$.

## 2. The derived Witt group.

**2.1. Once for all.** Let $K$ denote a triangulated category and $T$ be its *translation automorphism.* For the basic notions of triangulated categories, the reference is [15, chapter 10] or [14]. For the derived category of an exact category, the reference is [8] or [5]. For triangulated categories like we use them, see also § 0 of [2].

**2.2. Definition.** An additive contravariant functor $\# : K \to K$ is said to be *exact* if $T \circ \# \cong \# \circ T^{-1}$ and if for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

the following triangle is exact:

$$C^{\#} \xrightarrow{v^{\#}} B^{\#} \xrightarrow{u^{\#}} A^{\#} \xrightarrow{T(w^{\#})} T(C^{\#}).$$

Suppose, moreover, that there exists an isomorphism of functors

$$\varpi : \mathrm{Id} \xrightarrow{\sim} \# \circ \#$$

such that $\varpi_{T(M)} = T(\varpi_M)$ and $(\varpi_M)^{\#} \circ \varpi_{M^{\#}} = \mathrm{Id}_{M^{\#}}$ for any object $M$ of $K$. Then the triple $(K, \#, \varpi)$ is called a *triangulated category with duality.*

**2.3. Definition.** Symmetric spaces, orthogonal sums and isometries are defined in the same way as in 1.3 above. So, we get the Witt monoid $\mathrm{MW}(K)$. The question is now to define an analogue to metabolic spaces. To avoid confusion, we call them *neutral.*

A symmetric space $(P, \varphi)$ in a triangulated category with duality $(K, \#, \varpi)$ is said to be *neutral* if it possesses a *lagrangian.* In this context, a *lagrangian* is a triple $(L, \alpha, z)$ where the morphisms $\alpha : L \to P$ and $z : T^{-1}(L^{\#}) \to L$ satisfy those two conditions :

(1) the following triangle is exact :

$$T^{-1}(L^{\#}) \xrightarrow{z} L \xrightarrow{\alpha} P \xrightarrow{\alpha^{\#}\varphi} L^{\#}$$

(2) $T^{-1}(z^{\#}) = \varpi_L \circ z$, which we shall abbreviate by : $s$ is symmetric for $T^{-1} \circ \#$.

This is, from the psychological point of view, the analogue of definition 1.4, namely the existence of a symmetric exact triangle instead of a symmetric exact sequence.

As before, we define $\mathrm{NW}(K)$ to be the sub-monoid of $\mathrm{MW}(K)$ consisting in neutral symmetric spaces and we define the *Witt group* of $K$ to be

$$\mathrm{W}(K) = \frac{\mathrm{MW}(K)}{\mathrm{NW}(K)}.$$

**2.4. Remark.** Actually, we have at our disposal a collection of Witt groups, $\mathrm{W}^n(K)$ for $n \in \mathbb{Z}$, called the *shifted Witt groups* defined in [2]. The above one is only $\mathrm{W}^0(K)$. We focus on this group when $K$ is the derived category of some exact category (see 2.8 below). Nevertheless, we shall reconsider shifted Witt groups in § 5.

**2.5. Exercise.** Let $(L, \alpha, z)$ be a lagrangian of a symmetric space $(P, \varphi)$. Show that $z = 0$ if and only if $\alpha$ is a monomorphism if and only if $P \simeq L \oplus L^\#$ and $\varphi \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & \beta \end{smallmatrix}\right)$ for some morphism $\beta : L^\# \to L$ such that $\beta^\# = \beta$. Such a space is called *split neutral* (or *split metabolic*). When $\beta = 0$ it is called *hyperbolic*.

**2.6. The homotopy category.** Let $(\mathcal{E}, *, \pi)$ be an exact category with duality in the sense of definition 1.2. Let $K = \mathrm{K}^{\mathrm{b}}(\mathcal{E})$ be the homotopy category of $\mathcal{E}$, whose objects are bounded chain complexes of objects of $\mathcal{E}$ and whose morphisms are chain complexes morphisms up to homotopy. The triangulated structure of $K$ is given by the cone construction (*confer* [15, paragraph 1.5]), that is : a triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

is said to be *exact* if it is isomorphic to the triangle of the mapping cone construction over $u$, that is the following triangle over $u$, where $C(u)$ or $\mathrm{Cone}(u)$ stands for *cone of u*:

$$
\begin{array}{ccccccccc}
 & & & & \text{degree } n & & & & \\
 & & & & \vdots\downarrow & & & & \\
A = & \cdots \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n\perp 1} & \xrightarrow{d_{n-1}} & \cdots \\
u\downarrow & & \downarrow u_{n+1} & & \downarrow u_n & & \downarrow u_{n-1} & & \\
B = & \cdots \longrightarrow & B_{n+1} & \xrightarrow{d'_{n+1}} & B_n & \xrightarrow{d'_n} & B_{n\perp 1} & \xrightarrow{d'_{n-1}} & \cdots \\
v\downarrow & & \downarrow \binom{0}{1} & & \downarrow \binom{0}{1} & & \downarrow \binom{0}{1} & & \\
C(u) := & \cdots \longrightarrow & A_n \oplus B_{n+1} & \xrightarrow{\begin{pmatrix} \perp d_n & 0 \\ \perp u_n & d'_{n+1} \end{pmatrix}} & A_{n\perp 1} \oplus B_n & \xrightarrow{\begin{pmatrix} \perp d_{n-1} & 0 \\ \perp u_{n-1} & d'_n \end{pmatrix}} & A_{n\perp 2} \oplus B_{n\perp 1} & \longrightarrow & \cdots \\
w\downarrow & & \downarrow (\perp 1\ 0) & & \downarrow (\perp 1\ 0) & & \downarrow (\perp 1\ 0) & & \\
T(A) := & \cdots \longrightarrow & A_n & \xrightarrow[\perp d_n]{} & A_{n\perp 1} & \xrightarrow[\perp d_{n-1}]{} & A_{n\perp 2} & \longrightarrow & \cdots
\end{array}
$$

Moreover $K$ is naturally endowed with a duality. If we denote a complex by $(E_i, d_i)_{i\in\mathbb{Z}}$, then its dual would be the complex

$$\big((E_i, d_i)_{i\in\mathbb{Z}}\big)^\# := \big((E_{\perp i})^*, (d_{\perp i+1})^*\big)_{i\in\mathbb{Z}}.$$

On morphisms, with the same notations, $\#$ is given by

$$\big((f_i)_{i\in\mathbb{Z}}\big)^\# := \big((f_{\perp i})^*\big)_{i\in\mathbb{Z}}.$$

This functor is well defined (up to homotopy), is exact and is a duality. The identification $\varpi : \mathrm{Id}_K \overset{\sim}{\to} \#\circ\#$ is given by

$$\varpi_E = \big((\varpi_E)_i\big)_{i\in\mathbb{Z}} := \big(\pi_{E_i}\big)_{i\in\mathbb{Z}}$$

for any complex $E = (E_i)_{i\in\mathbb{Z}}$ in $K$.

The homotopy category is not satisfactory because the functor $\mathcal{E} \to \mathrm{K}^{\mathrm{b}}(\mathcal{E})$, sending everything in degree 0, does not send an exact sequence to an exact triangle (exercise). To pass over this problem, one usually inverts quasi-isomorphisms. That's our next step.

**2.7. Semi-saturated exact categories.** Recall from [8, def. 1.11] that an exact category $\mathcal{E}$ is called *semi-saturated* if every weakly split epimorphism is an admissible epimorphism. In other words, if $p : E \to F$ is a morphism in $\mathcal{E}$ for which there exists another morphism $\sigma : F \to E$ such that $p \circ \sigma = \mathrm{Id}_F$, then one requires $p$ to be an admissible epimorphism.

For example, a saturated exact category (= idempotent complete = pseudo-abelian = Karoubian = every idempotent is the projection on a direct summand) is semi-saturated. For example, if $X$ is a scheme, the category $\mathcal{E}(X)$ defined in 1.8 is saturated and hence semi-saturated. So, for us, semi-saturation is a weak condition.

A semi-saturated exact category $\mathcal{E}$ admits a fully faithful embedding into an abelian category $\mathcal{A}$ such that:

(1) $\mathcal{E} \subset \mathcal{A}$ is closed under extension;
(2) the exact functor $\mathcal{E} \to \mathcal{A}$ reflects exact sequences;
(3) every morphism in $\mathcal{E}$ which is an epimorphism in $\mathcal{A}$ is an admissible epimorphism in $\mathcal{E}$.

In our example $\mathcal{E} = \mathcal{E}(X)$, we already have $\mathcal{A} =$ the category of (quasi-coherent) $\mathcal{O}_X$-modules.

**2.8. The derived category.** Let $\mathcal{E}$ be a semi-saturated exact category. Define $\mathrm{D^b}(\mathcal{E})$ to be the localization of $\mathrm{K^b}(\mathcal{E})$ with respect to quasi-isomorphisms. Recall, to be sure, that a *quasi-isomorphism* is a morphism of complexes in $\mathrm{K^b}(\mathcal{E})$ whose mapping cone is acyclic. A bounded complex

$$\cdots \to 0 \to 0 \to P_m \xrightarrow{d_m} P_{m\perp 1} \xrightarrow{d_{m-1}} P_{m\perp 2} \xrightarrow{d_{m-2}} \cdots \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \to 0 \to 0 \cdots$$

is said to be *acyclic* if $d_m$ is an admissible monomorphism and if the shorter complex

$$\cdots \to 0 \to 0 \to 0 \to P_{m\perp 1}/P_m \xrightarrow{\bar{d}_{m-1}} P_{m\perp 2} \xrightarrow{d_{m-2}} \cdots \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \to 0 \to 0 \cdots$$

is acyclic, where $P_m \overset{d_m}{\rightarrowtail} P_{m\perp 1} \overset{p}{\twoheadrightarrow} P_{m\perp 1}/P_m$ is a short exact sequence and $\bar{d}_{m\perp 1} : P_{m\perp 1}/P_m \to P_{m\perp 2}$ is induced by $d_{m\perp 1}$, meaning that $\bar{d}_{m\perp 1}\, p = d_{m\perp 1}$. To start this inductive definition, we say that the zero complex is acyclic. Since we are on bounded complexes, this is the same as saying that one can reduce the complex from the other side and, in turn, is the same as saying (general definition) that every differential $d_i$ factors as $d_i = \alpha_{i\perp 1} \circ \beta_i$ for short exact sequence:

$$Q_i \overset{\alpha_i}{\rightarrowtail} P_i \overset{\beta_i}{\twoheadrightarrow} Q_{i\perp 1} \qquad i \in \mathbb{Z}.$$

It is easy to see that the structure of triangulated category with duality obtained on $\mathrm{K^b}(\mathcal{E})$ "localizes" to $\mathrm{D^b}(\mathcal{E})$. We still denote by $\#$ the duality on $\mathrm{D^b}(\mathcal{E})$, which could be called the *derived functor* of $* : \mathcal{E} \to \mathcal{E}$ even if it is a weak acceptation of that concept. The identification $\varpi$ localizes as well. We obtain a triangulated category with duality:

$$\big(\mathrm{D^b}(\mathcal{E}), \#, \varpi\big).$$

Applying definition 2.3 to this triangulated category with duality, we obtain what we call *the derived Witt group*:

$$\mathrm{W_{der}}(\mathcal{E}, *, \pi) := \mathrm{W}\big(\mathrm{D^b}(\mathcal{E}), \#, \varpi\big).$$

Now, consider the functor

$$c_0 : \mathcal{E} \to \mathrm{D^b}(\mathcal{E})$$

which sends an object to the complex concentrated in degree 0 and does the same with the morphisms. It is easy to check (and well known) that it sends exact sequences to exact triangles. Namely, if

$$E \underset{\alpha}{\rightarrowtail} F \underset{\beta}{\twoheadrightarrow} G$$

is an exact sequence in $\mathcal{E}$ then there exists a morphism $w : c_0(G) \to T\big(c_0(E)\big)$ in $\mathrm{D^b}(\mathcal{E})$ such that the following triangle is exact:

$$c_0(E) \xrightarrow{c_0(\alpha)} c_0(F) \xrightarrow{c_0(\beta)} c_0(G) \xrightarrow{w} T\big(c_0(E)\big).$$

It is an easy exercise to verify that if the starting exact sequence is the one characterizing a lagrangian (in the sense of definition 1.4), then the triangle obtained with this construction satisfies the properties of definition 2.3 (with $z = -T^{\perp 1}(w)$). Thus, we have the following

**2.9. Proposition.** *Let $(\mathcal{E},*,\pi)$ be a semi-saturated exact category with duality and let $\big(\mathrm{D^b}(\mathcal{E}),\#,\varpi\big)$ be the associated derived category with duality. Then the functor $c_0:\mathcal{E}\to\mathrm{D^b}(\mathcal{E})$ induces a well defined and natural homomorphism of groups :*

$$\omega_{\mathcal{E}}\;:\mathrm{W_{us}}(\mathcal{E})\to\mathrm{W_{der}}(\mathcal{E}).$$

**2.10. Exercise.** Let $(\mathcal{E},*,\pi)$ be a semi-saturated exact category with duality such that $\frac{1}{2}\in\mathcal{E}$ (see 3.1).

(1) Let $(P,\varphi)$ be a neutral form in $\mathrm{D^b}(\mathcal{E})$ and let $(L,\alpha,z)$ be a lagrangian of this form. Show that the symmetric space $(P,\varphi)$ is uniquely determined up to isometry by $L$ and $z:T^{\perp 1}(L^{\#})\to L$ such that $T^{\perp 1}(z^{\#})=z$. *Solution: theorem 1.6 of [2].*

(2) Let $(L,z)$ be such a pair. Suppose that $z$ is moreover a good old morphism of complexes (no fractions) and that $z^*_{\perp i\perp 1}=z_i$ for all $i\in\mathbb{Z}$. Prove that the cone of $z$ possesses a symmetric form $\chi$ given in each degree by

$$\mathrm{Cone}(z)_i=(L_{\perp i})^*\oplus L_i\xrightarrow{\chi_i=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}}L_i\oplus(L_{\perp i})^*=\big(\mathrm{Cone}(z)_{\perp i}\big)^*=\big(\mathrm{Cone}(z)^{\#}\big)_i$$

for $i\in\mathbb{Z}$, which satisfies $w=v^{\#}\chi$ where $v:L\to\mathrm{Cone}(z)$ and $w:\mathrm{Cone}(z)\to L^{\#}$ are the morphisms of the cone construction described in 2.6 :

$$T^{\perp 1}(L^{\#})\xrightarrow{\;z\;}L\xrightarrow{\;v\;}\mathrm{Cone}(z)\xrightarrow{\;w\;}L^{\#}.$$

Show that such a form is *not* hyperbolic, in general, even if each $\chi_i$ looks pretty hyperbolic.

(3) Suppose that $z$ is only symmetric up to homotopy (instead of $z^*_{\perp i\perp 1}=z_i$). Write the entire exact triangle over $z$ obtained from the cone construction together with the symmetric form on $\mathrm{Cone}(z)$ satisfying condition (1) of definition 2.3 The form might be more complicated when $z$ is only symmetric in $\mathrm{D^b}(\mathcal{E})$.

**2.11. Example.** Let $X$ be a scheme. The exact category with duality $\mathcal{E}(X)$ introduced in 1.8 has a derived category $\mathrm{D^b}\big(\mathcal{E}(X)\big)$ which Witt group shall be called the *derived Witt group of the scheme $X$* :

$$\mathrm{W_{der}}(X):=\mathrm{W}\Big(\mathrm{D^b}\big(\mathcal{E}(X)\big)\Big).$$

**2.12. Remark.** This notion is already introduced in [1, § 2] in which the notations are slightly different but do not affect $\mathrm{W}^0$ (see remark 5.1 below). In [*loc. cit.*, thm. 4.29], we establish that $\omega$ is an isomorphism when $\mathcal{E}=\mathcal{P}(R)$ is the category of finitely generated projective $R$-modules over a commutative (noetherian) ring $R$ containing $\frac{1}{2}$. This is an additive version (this exact category $\mathcal{P}(R)$ is split) of the more general theorem stated here. The result of [*loc. cit.*] does not apply to non-affine schemes.

## 3. Reducing complexes.

**3.1. Definition.** Let $\mathcal{E}$ be an additive category. We say that $\frac{1}{2}\in\mathcal{E}$ if any morphism $f$ in $\mathcal{E}$ can be written in an unique way as $f=g+g$. In this case, we abbreviate $g=\frac{1}{2}f$.

**3.2. Theorem.** *Let $\mathcal{E}$ be a semi-saturated exact category with duality such that $\frac{1}{2}\in\mathcal{E}$. Then the homomorphism $\omega_{\mathcal{E}}:\mathrm{W_{us}}(\mathcal{E})\to\mathrm{W_{der}}(\mathcal{E})$ is surjective.*

**3.3. Proof.** First, it is easy to see that any symmetric space in $\mathrm{D^b}(\mathcal{E})$ is represented by a symmetric quasi-isomorphism (see 3.6). Then, we prove that it is Witt-equivalent to a form over a shorter complex (see 3.7 and main lemma 3.9). Then an easy induction gives the conclusion. The proof of theorem 3.2 will occupy the end of paragraph 3.

**3.4. Definition.** Let $P = (P_i)_{i\in\mathbb{Z}}$ be a bounded complex. We say that $P$ is *supported in* $[m,n]$ where $m,n\in\mathbb{Z}$ if $P_i = 0$ when $i > m$ or $n > i$:

$$P = \qquad \cdots \to 0 \to 0 \to P_m \xrightarrow{d_m} P_{m\perp 1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} P_n \to 0 \to 0 \cdots$$

**3.5. Definition.** A morphism of complexes $s : P \to P^{\#}$ is said to be *strongly symmetric* if $s^{*}_{\perp i} = s_i$ for all $i\in\mathbb{Z}$.

**3.6. First step.** Let $x \in \mathrm{W}\big(\mathrm{D}^{\mathrm{b}}(\mathcal{E})\big)$. Then, using only isometries and $\frac{1}{2}$, it is easy to see that $x$ is equal to a class $[P,s]$ in $\mathrm{W}\big(\mathrm{D}^{\mathrm{b}}(\mathcal{E})\big)$ where $P$ and $s$ are as follows:

(1) $P$ is a (bounded) complex and $s : P \to P^{\#}$ is a morphism of complexes (no fractions)
(2) $s$ is a quasi-isomorphism;
(3) the form $s$ is strongly symmetric.

The proof will consist in an induction on the length of the support of $P$. The following lemma is the easy part.

**3.7. Lemma.** *Let $(P,s)$ be as in 3.6. Suppose that $P$ is supported in $[m,-n]$, with $m > n \geq 0$. Then $(P,s)$ is isomorphic in $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$ to a symmetric space $(Q,t)$ such that*

(1) *$(Q,t)$ is as in 3.6;*
(2) *$Q$ is supported in $[n,-n]$.*

**3.8. Proof.** The proof for $m = n+1$ gives the general case. Assume that $n \geq 1$ and consider $(P,s)$:

$$\begin{array}{ccccccccccccc}
P = \cdots 0 & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & \cdots & \xrightarrow{d_{-n+1}} & P_{\perp n} & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
\downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle s_n} & & & & \downarrow{\scriptstyle s_n{}^{*}} & & & & \\
P^{\#} = \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_{\perp n}{}^{*} & \xrightarrow[d_{-n+1}{}^{*}]{} & \cdots & \xrightarrow[d_n{}^{*}]{} & P_n{}^{*} & \xrightarrow[d_{n+1}{}^{*}]{} & P_{n+1}{}^{*} & \longrightarrow & 0 \cdots
\end{array}$$

The assumption that $s$ is a quasi-isomorphism means that the mapping cone $C(s)$ is acyclic. A direct computation and the definition of acyclic complexes in an exact category give that $d_{n+1}$ is an admissible monomorphism. Choose an exact sequence

$$P_{n+1} \underset{d_{n+1}}{\rightarrowtail} P_n \underset{q_n}{\twoheadrightarrow} Q_n.$$

Since $d_n\, d_{n+1} = 0$ and $s_n\, d_{n+1} = 0$, there exist morphisms $e_n : Q_n \to P_{n\perp 1}$ and $t_n : Q_n \to P_{\perp n}{}^{*}$ such that the following diagram commutes:

$$\begin{array}{ccc}
P_n & \xrightarrow{d_n} & P_{n\perp 1} \\
 & \searrow^{q_n} \quad Q_n \quad \nearrow_{\exists e_n} & \\
\downarrow{\scriptstyle s_n} & \swarrow_{\exists t_n} & \downarrow{\scriptstyle s_{n\perp 1}} \\
P_{\perp n}{}^{*} & \xrightarrow[d_{\perp n+1}{}^{*}]{} & P_{\perp n+1}{}^{*}.
\end{array}$$

The relation $s_{n\perp 1}\, e_n = d_{\perp n+1}{}^{*}\, t_n$ follows from the commutativity of the rest of the diagram and the fact that $q$ is an epimorphism. Now, it is easy to show that

$$\begin{array}{ccccccccccccc}
Q := \cdots 0 & \longrightarrow & Q_n & \xrightarrow{e_n} & P_{n\perp 1} & \xrightarrow{d_{n-1}} & \cdots & \xrightarrow{d_{-n+2}} & P_{\perp n+1} & \xrightarrow{d_{-n+1}} & P_{\perp n} & \longrightarrow & 0 \cdots \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t_n} & & \downarrow{\scriptstyle s_{n-1}} & & & & \downarrow{\scriptstyle s_{n-1}{}^{*}} & & \downarrow{\scriptstyle t_n{}^{*}} & & \\
Q^{\#} = \cdots 0 & \longrightarrow & P_{\perp n}{}^{*} & \xrightarrow[d_{-n+1}{}^{*}]{} & P_{\perp n+1}{}^{*} & \xrightarrow[d_{-n+2}{}^{*}]{} & \cdots & \xrightarrow[d_{n-1}{}^{*}]{} & P_{n\perp 1}{}^{*} & \xrightarrow[e_n{}^{*}]{} & Q_n{}^{*} & \longrightarrow & 0 \cdots
\end{array}$$

is a symmetric morphism. Actually, the complexes $P$ and $Q$ are quasi-isomorphic through:

$$\begin{array}{ccccccccccccc}
P = \cdots 0 & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n\perp 1} & \xrightarrow{d_{n-1}} & \cdots & \xrightarrow{d_{-n+1}} & P_{\perp n} & \longrightarrow & 0 \cdots \\
{\scriptstyle q}\downarrow & & \downarrow & & {\scriptstyle q_n}\downarrow & & \| & & & & \| & & \\
Q = \cdots 0 & \longrightarrow & 0 & \longrightarrow & Q_n & \xrightarrow[e_n]{} & P_{n\perp 1} & \xrightarrow[d_{n-1}]{} & \cdots & \xrightarrow[d_{-n+1}]{} & P_{\perp n} & \longrightarrow & 0 \cdots
\end{array}$$

We leave to the reader to check that this is a quasi-isomorphism directly from the definition.

An immediate computation gives $q^{\#}\, t\, q = s$. So, first, $t$ is a quasi-isomorphism (and $(Q,t)$ is as in 3.6) and secondly the spaces $(P,s)$ and $(Q,t)$ are isometric. This ends the proof of the lemma when $n \geq 1$.

Suppose $n = 0$. Then the form $(P,s)$ is simply

$$\begin{array}{ccccccccc}
\cdots 0 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle s=s^*} & & \downarrow & & \\
\cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_0}^* & \xrightarrow[d^*]{} & {P_1}^* & \longrightarrow & 0 \cdots
\end{array}$$

and the proof contains a little change. As before, since $s$ is a quasi-isomorphism, its cone is acyclic. The $d$ is an admissible monomorphism that we can complete in an exact sequence as before:

$$P_1 \underset{d}{\rightarrowtail} P_0 \underset{p}{\twoheadrightarrow} Q.$$

Since $s\, d = 0$ there exists a unique morphism $\tilde{t} : Q \to {P_0}^*$ such that $\tilde{t}\, p = s$. But now $d^*\, \tilde{t}\, p = d^*\, s = 0$ and then (since $p$ is an epimorphism) $d^*\, \tilde{t} = 0$. On the other side, $*$ being exact, the following sequence is exact:

$$Q^* \underset{p^*}{\rightarrowtail} {P_0}^* \underset{d^*}{\twoheadrightarrow} {P_1}^*.$$

The relation $d^*\, \tilde{t} = 0$ induces the existence of a unique morphism $t : Q \to Q^*$ such that $\tilde{t} = p^*\, t$. Observe that $p^*\, t\, p = \tilde{t}\, p = s$. Dualizing this last equation, we get $p^*\, t^*\, p = s^* = s$ by hypothesis. This implies that $t^* = t$ by uniqueness of $\tilde{t}$ and $t$.

Now it is clear that $(P,s)$ is quasi-isomorphic to $c_0(Q,t)$ through the following quasi-isomorphism $q : P \to c_0(Q)$:

$$\begin{array}{ccccccccc}
\cdots 0 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle p} & & \downarrow & & \\
\cdots 0 & \longrightarrow & 0 & \longrightarrow & Q & \longrightarrow & 0 & \longrightarrow & 0 \cdots
\end{array}$$

Since $q^{\#}\, c_0(t)\, q = s$, $c_0(t)$ is a quasi-isomorphism (and then $t : Q \to Q^*$ is an isomorphism). ♯

**3.9. Main lemma.** Let $(P,s)$ be a symmetric space as in 3.6 and suppose that $P$ is supported in $[n,-n]$ with $n \geq 1$. Then there exists a symmetric space $(Q,t)$ such that

(1) $(Q,t)$ is as in 3.6;
(2) $Q$ is supported in $[n,-(n-1)]$;
(3) $[P,s] = -[Q,t]$ in $\mathrm{W}\big(\mathrm{D^b}(\mathcal{E})\big)$.

**3.10. Proof.** Suppose $n \geq 2$.

Let $(P,s)$ be

$$\begin{array}{ccccccccccc}
P = \cdots 0 & \longrightarrow & P_n & \xrightarrow{d_n} & P_{n\perp 1} & \xrightarrow{d_{n-1}} & \cdots & \xrightarrow{d_{-n+1}} & P_{\perp n} & \longrightarrow & 0 \cdots \\
{\scriptstyle s}\downarrow & & {\scriptstyle s_n}\downarrow & & {\scriptstyle s_{n-1}}\downarrow & & & & \downarrow{\scriptstyle s_{-n}} & & \\
P^{\#} = \cdots 0 & \longrightarrow & {P_{\perp n}}^* & \xrightarrow[{d_{-n+1}}^*]{} & {P_{\perp n+1}}^* & \xrightarrow[{d_{-n+2}}^*]{} & \cdots & \xrightarrow[{d_n}^*]{} & {P_n}^* & \longrightarrow & 0 \cdots
\end{array}$$

For readability purpose, we omit the subscripts for the differentials $d_i$ as well as for the morphisms $s_i$. They are forced by the subscripts of the objects they are linking. Those who want to restore these subscripts are recalled that, by assumption, ${s_{-i}}^* = s_i$ for all $i \in \mathbb{Z}$ and that we make an extensive use of this fact.

Define $(Q,t)$ to be

$$\begin{array}{ccccccccccccccccc}
 & & & & \text{(degree } n) & & & & & & & & & & \text{(degree } -n) & & \\
Q := \cdots 0 & \longrightarrow & P_n & \xrightarrow{\binom{s}{d}} & {P_{-n}}^* \oplus P_{n-1} & \xrightarrow{(0\ d)} & P_{n-2} & \overset{d}{\cdots\to} & P_{-n+2} & \xrightarrow{d} & P_{-n+1} & \longrightarrow & 0 & \longrightarrow & 0\cdots \\
{\scriptstyle t}\downarrow & & \downarrow & & \downarrow{\scriptstyle (d^*\ -s)} & & \downarrow{\scriptstyle -s} & \cdots & \downarrow{\scriptstyle -s} & & \downarrow{\scriptstyle \binom{d}{-s}} & & \downarrow & & \\
Q^\# = \cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_{-n+1}}^* & \xrightarrow{d^*} & {P_{-n+2}}^* & \overset{d^*}{\cdots\to} & {P_{n-2}}^* & \xrightarrow{\binom{0}{d^*}} & P_{-n} \oplus {P_{n-1}}^* & \xrightarrow{(s\ d^*)} & {P_n}^* & \longrightarrow & 0\cdots
\end{array}$$

We are going to prove that $(P,s)-(Q,t)$ is neutral in $\mathrm{D^b}(\mathcal{E})$, which is enough since $(Q,t)$ obviously satisfies conditions (1) and (2) of the lemma. Actually, to see that $t$ is a quasi-isomorphism, simply compute the mapping cone of $t$ and observe that it is isomorphic (not only quasi-isomorphic or homotopically equivalent but even isomorphic as a complex) to the mapping cone of $s$. This easy part is left to the reader. Up to small sign tricks, the isomorphisms are the obvious ones.

The rest of the proof is a little bit technical but involves no particular difficulties. The reader can very well check every step, the only knowledge required (and already used) being the construction of the mapping cone.

First of all, we introduce a new complex $M$ and a morphism $z : T^{-1}M^\# \to M$. Namely, let

$$\begin{array}{cccccccccccccc}
 & & \text{(degree } n) & & & & & & & & \text{(degree } -n) & & \\
T^{-1}M^\# = \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{-d} & P_{n-1} & \longrightarrow \cdots \xrightarrow{-d} & P_{-n+2} & \xrightarrow{-d} & P_{-n+1} & \longrightarrow & 0\cdots \\
{\scriptstyle z}\downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle s_{-n}\, d_{-n+1}} & & \\
M := \cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_{-n+1}}^* & \xrightarrow{d^*} & {P_{-n+2}}^* & \longrightarrow \cdots \xrightarrow{d^*} & {P_{n-1}}^* & \xrightarrow{d^*} & {P_n}^* & \longrightarrow & 0\cdots
\end{array}$$

Observe that $T^{-1}(z^\#)$ is the same map except that the first zero map is replaced by ${d_{-n+1}}^* \circ {s_{-n}}^*$ and that the last map (from $P_{-n+1}$ to ${P_n}^*$) is zero. Actually the two chain morphisms $z$ and $T^{-1}z^\#$ are homotopic through the homotopy (we use the strong symmetry of $s$):

$$\begin{array}{cccccccccccc}
T^{-1}M^\# = \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{-d} & P_{n-1} & \longrightarrow \cdots \xrightarrow{-d} & P_{-n+2} & \xrightarrow{-d} & P_{-n+1} & \longrightarrow 0\cdots \\
 & & & \swarrow & & \swarrow{\scriptstyle s} & & \cdots\qquad\cdots & & \swarrow{\scriptstyle s} & & \\
M = \cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_{-n+1}}^* & \xrightarrow{d^*} & {P_{-n+2}}^* & \longrightarrow \cdots \xrightarrow{d^*} & {P_{n-1}}^* & \xrightarrow{d^*} & {P_n}^* & \longrightarrow 0\cdots
\end{array}$$

This means that $z = T^{-1}(z^\#)$ in $\mathrm{K^b}(\mathcal{E})$ and *a fortiori* in $\mathrm{D^b}(\mathcal{E})$. Now, consider the exact triangle obtained by the cone construction:

$$T^{-1}M^\# \xrightarrow{z} M \xrightarrow{z_1} Z \xrightarrow{z_2} M^\#.$$

Explicitly, the complex $Z$ and the morphisms $z_1$, $z_2$ are:

$$
\begin{array}{lcccccccccccc}
T^{-1}M^{\#} = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_n & \xrightarrow{-d} & P_{n-1} & \overset{\cdots -d \cdots}{\dashrightarrow} & P_{-n+2} & \xrightarrow{-d} & P_{-n+1} & \longrightarrow 0\cdots \\
\downarrow z & & & \downarrow & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow sd & \\
M = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_{-n+1}}^* & \xrightarrow{d^*} & {P_{-n+2}}^* & \overset{\cdots d^* \cdots}{\dashrightarrow} & {P_{n-1}}^* & \xrightarrow{d^*} & {P_n}^* & \longrightarrow 0\cdots \\
\downarrow z_1 & & & \downarrow & & \downarrow \binom{0}{1} & & \downarrow \binom{0}{1} & & \downarrow \binom{0}{1} & & \downarrow 1 & \\
Z := C(z) = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{\binom{d}{0}} & P_{n-1} \oplus {P_{-n+1}}^* & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d^* \end{smallmatrix}\right)} & P_{n-2} \oplus {P_{-n+2}}^* & \overset{\cdots \left(\begin{smallmatrix} d & 0 \\ 0 & d^* \end{smallmatrix}\right) \cdots}{\dashrightarrow} & P_{-n+1} \oplus {P_{n-1}}^* & \xrightarrow{(-sd\ d^*)} & {P_n}^* & \longrightarrow 0\cdots \\
\downarrow z_2 & & & \downarrow -1 & & \downarrow (-1\ 0) & & \downarrow (-1\ 0) & & \downarrow (-1\ 0) & & \downarrow & \\
M^{\#} = & \cdots 0 & \longrightarrow & P_n & \xrightarrow[d]{} & P_{n-1} & \xrightarrow[d]{} & P_{n-2} & \underset{\cdots d \cdots}{\dashrightarrow} & P_{-n+1} & \longrightarrow & 0 & \longrightarrow 0\cdots
\end{array}
$$

There is a symmetric form $\chi$ on $Z$ defined by

$$
\begin{array}{lcccccccccc}
Z = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{\binom{d}{0}} & P_{n-1} \oplus {P_{-n+1}}^* & \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d^* \end{smallmatrix}\right)} \cdots \xrightarrow{\left(\begin{smallmatrix} d & 0 \\ 0 & d^* \end{smallmatrix}\right)} & P_{-n+1} \oplus {P_{n-1}}^* & \xrightarrow{(-sd\ d^*)} & {P_n}^* & \longrightarrow 0\cdots \\
\downarrow \chi & & & \downarrow -1 & & \downarrow \left(\begin{smallmatrix} s & -1 \\ -1 & 0 \end{smallmatrix}\right) & & \downarrow \left(\begin{smallmatrix} s & -1 \\ -1 & 0 \end{smallmatrix}\right) & & \downarrow -1 & \\
Z^{\#} = & \cdots 0 & \longrightarrow & P_n & \xrightarrow[\left(\begin{smallmatrix} -d^* s \\ d \end{smallmatrix}\right)]{} & {P_{-n+1}}^* \oplus P_{n-1} & \xrightarrow[\left(\begin{smallmatrix} d^* & 0 \\ 0 & d \end{smallmatrix}\right)]{} \cdots \xrightarrow[\left(\begin{smallmatrix} d^* & 0 \\ 0 & d \end{smallmatrix}\right)]{} & {P_{n-1}}^* \oplus P_{-n+1} & \xrightarrow[(d^*\ 0)]{} & {P_n}^* & \longrightarrow 0\cdots
\end{array}
$$

One can verify that $\chi \circ z_1 = z_2^{\#}$, which means that the triangle

$$T^{-1}M^{\#} \xrightarrow{z} M \xrightarrow{z_1} Z \xrightarrow{{z_1}^{\#}\chi} M^{\#}.$$

is exact. By definition 2.3, the space $(Z, \chi)$ is neutral (actually, this one is already neutral in $\mathrm{K}^{\mathrm{b}}(\mathcal{E})$). By the way, this gives a solution to exercise 2.10, part (3).

It is now sufficient to prove that, in $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$, the two following spaces are isometric:

$$(Z, \chi) \simeq (P, s) - (Q, t).$$

Note that $Z \not\simeq P \oplus Q$ in $\mathrm{K}^{\mathrm{b}}(\mathcal{E})$, but only in $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$, as we are going to see. The strategy is the following. We are going to establish a certain number of equalities true in $\mathrm{K}^{\mathrm{b}}(\mathcal{E})$, labelled from (1) to (5). Then, we are going to use the fact that $s$ and $t$ are quasi-isomorphisms to deduce new equalities in $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$. The final statement being $(Z, \chi) \simeq (P, s) - (Q, t)$ as announced.

*Ex nihilo*, let's consider the following morphism $a : T^{-1}(Q^{\#}) \to P$, that we immediately present in the triangle of its mapping cone:

$$T^{-1}(Q^{\#}) \xrightarrow{a} P \xrightarrow{a_1} A \xrightarrow{a_2} Q^{\#}$$

$$
\begin{array}{lccccccccccccc}
 & & & \text{degree } n & & & & & & & & \text{degree } -n & & \\
 & & & \vdots & & & & & & & & \vdots & & \\
T^{-1}(Q^{\#}) = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & {P_{-n+1}}^* & \xrightarrow{-d^*} \cdots \xrightarrow{-d^*} & {P_{n-2}}^* & \xrightarrow{\left(\begin{smallmatrix} 0 \\ -d^* \end{smallmatrix}\right)} & P_{-n} \oplus {P_{n-1}}^* & \xrightarrow{(-s\ -d^*)} {P_n}^* & \longrightarrow 0\cdots \\
\downarrow a & & & \downarrow & & \downarrow & & \downarrow 0 & & \downarrow 0 & & \downarrow (1\ 0) & \downarrow & \\
P = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{d} & P_{n-1} & \xrightarrow{d} & P_{n-2} & \xrightarrow{d} \cdots \xrightarrow{d} & P_{-n+1} & \xrightarrow{d} & P_{-n} & \longrightarrow 0 & \longrightarrow 0\cdots \\
\downarrow a_1 & & & \downarrow 1 & & \downarrow \binom{0}{1} & & \downarrow \binom{0}{1} & & \downarrow \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) & & \downarrow \binom{0}{1} & \downarrow & \\
A := C(a) = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{\binom{0}{d}} & P^*_{-n+1} \oplus P_{n-1} & \longrightarrow & P^*_{-n+2} \oplus P_{n-2} & \xrightarrow[\left(\begin{smallmatrix} d^* & 0 \\ 0 & d \end{smallmatrix}\right) \cdots]{} \cdots \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ d^* & 0 \\ 0 & d \end{smallmatrix}\right)} & P_{-n} \oplus P^*_{n-1} \oplus P_{-n+1} & \xrightarrow{\left(\begin{smallmatrix} s & d^* & 0 \\ -1 & 0 & d \end{smallmatrix}\right)} & P^*_n \oplus P_{-n} & \longrightarrow 0 & \longrightarrow 0\cdots \\
 & & & & & & {\scriptstyle\left(\begin{smallmatrix} d^* & 0 \\ 0 & d \end{smallmatrix}\right)} & & & & & & & \\
\downarrow a_2 & & & \downarrow & & \downarrow (-1\ 0) & & \downarrow (-1\ 0) & & \downarrow \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right) & & \downarrow (-1\ 0) & \downarrow & \\
Q^{\#} = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_{-n+1}}^* & \xrightarrow[d^*]{} & {P_{-n+2}}^* & \xrightarrow[d^* \cdots]{} \cdots \xrightarrow[\left(\begin{smallmatrix} 0 \\ d^* \end{smallmatrix}\right)]{} & P_{-n} \oplus {P_{n-1}}^* & \xrightarrow[(s\ d^*)]{} & {P_n}^* & \longrightarrow 0 & \longrightarrow 0\cdots
\end{array}
$$

We claim that there exists an homotopy equivalence $\rho : A \overset{\sim}{\to} Z$ and we give explicitly $\rho$ and $\rho^{\perp 1}$:

$$
\begin{array}{ccccccccccc}
A = \cdots 0 & \longrightarrow & P_n & \xrightarrow{\left(\begin{smallmatrix}0\\ d\end{smallmatrix}\right)} & {P_{\perp n+1}}^* \oplus P_{n\perp 1} & \xrightarrow{\left(\begin{smallmatrix}d^* & 0\\ 0 & d\end{smallmatrix}\right)} \cdots \xrightarrow{\left(\begin{smallmatrix}0 & 0\\ d^* & 0\\ 0 & d\end{smallmatrix}\right)} & P_{\perp n} \oplus {P_{n\perp 1}}^* \oplus P_{\perp n+1} & \xrightarrow{\left(\begin{smallmatrix}s & d^* & 0\\ \perp 1 & 0 & d\end{smallmatrix}\right)} & {P_n}^* \oplus P_{\perp n} & \longrightarrow & 0\cdots \\
\rho \big\downarrow & & \big\downarrow \perp 1 & & \big\downarrow \left(\begin{smallmatrix}0 & \perp 1\\ 1 & 0\end{smallmatrix}\right) \cdots & & \big\downarrow \left(\begin{smallmatrix}0 & 0 & \perp 1\\ 0 & 1 & 0\end{smallmatrix}\right) & & \big\downarrow (1\ s) & & \\
Z = \cdots 0 & \longrightarrow & P_n & \xrightarrow{\left(\begin{smallmatrix}d\\ 0\end{smallmatrix}\right)} & P_{n\perp 1} \oplus {P_{\perp n+1}}^* & \longrightarrow \cdots \xrightarrow{\left(\begin{smallmatrix}d & 0\\ 0 & d^*\end{smallmatrix}\right)} & P_{\perp n+1} \oplus {P_{n\perp 1}}^* & \xrightarrow{(\perp sd\ d^*)} & {P_n}^* & \longrightarrow & 0\cdots \\
\rho^{\perp 1} \big\downarrow & & \big\downarrow \perp 1 & & \big\downarrow \left(\begin{smallmatrix}0 & 1\\ \perp 1 & 0\end{smallmatrix}\right) \cdots & & \big\downarrow \left(\begin{smallmatrix}\perp d & 0\\ 0 & 1\\ \perp 1 & 0\end{smallmatrix}\right) & & \big\downarrow \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right) & & \\
A = \cdots 0 & \longrightarrow & P_n & \xrightarrow[\left(\begin{smallmatrix}0\\ d\end{smallmatrix}\right)]{} & {P_{\perp n+1}}^* \oplus P_{n\perp 1} & \xrightarrow[\left(\begin{smallmatrix}d^* & 0\\ 0 & d\end{smallmatrix}\right)]{} \cdots \xrightarrow[\left(\begin{smallmatrix}0 & 0\\ d^* & 0\\ 0 & d\end{smallmatrix}\right)]{} & P_{\perp n} \oplus {P_{n\perp 1}}^* \oplus P_{\perp n+1} & \xrightarrow[\left(\begin{smallmatrix}s & d^* & 0\\ \perp 1 & 0 & d\end{smallmatrix}\right)]{} & {P_n}^* \oplus P_{\perp n} & \longrightarrow & 0\cdots
\end{array}
$$

One has $\rho\,\rho^{\perp 1} = \mathrm{Id}_Z$ and it is easy to find the homotopy that insures $\rho^{\perp 1}\rho \sim \mathrm{Id}$. Using this isomorphism and the (cone) exact triangle of $a$, we get an exact triangle:

$$
(\Delta) \qquad T^{\perp 1}(Q^{\#}) \xrightarrow{a} P \xrightarrow{i} Z \xrightarrow{\pi} Q^{\#}.
$$

where we have baptized $i := \rho\, a_1$ and $\pi := a_2\, \rho^{\perp 1}$. Explicitly, they are

$$
\begin{array}{ccccccccccccc}
P = \cdots 0 & \longrightarrow & P_n & \xrightarrow{d} & P_{n\perp 1} & \xrightarrow{d} \cdots \xrightarrow{d} & P_{\perp n+2} & \xrightarrow{d} & P_{\perp n+1} & \xrightarrow{d} & P_{\perp n} & \longrightarrow & 0\cdots \\
i \big\downarrow & & \big\downarrow \perp 1 & & \big\downarrow \left(\begin{smallmatrix}\perp 1\\ 0\end{smallmatrix}\right) & & \big\downarrow \left(\begin{smallmatrix}\perp 1\\ 0\end{smallmatrix}\right) & & \big\downarrow \left(\begin{smallmatrix}\perp 1\\ 0\end{smallmatrix}\right) & & \big\downarrow s & & \\
Z = \cdots 0 & \longrightarrow & P_n & \xrightarrow{\left(\begin{smallmatrix}d\\ 0\end{smallmatrix}\right)} & P_{n\perp 1} \oplus {P_{\perp n+1}}^* & \xrightarrow{\left(\begin{smallmatrix}d & 0\\ 0 & d^*\end{smallmatrix}\right)} \cdots \longrightarrow & P_{\perp n+2} \oplus {P_{n\perp 2}}^* & \xrightarrow{\left(\begin{smallmatrix}d & 0\\ 0 & d^*\end{smallmatrix}\right)} & P_{\perp n+1} \oplus {P_{n\perp 1}}^* & \xrightarrow{(\perp sd\ d^*)} & {P_n}^* & \longrightarrow & 0\cdots \\
\pi \big\downarrow & & \big\downarrow & & \big\downarrow (0\ \perp 1) & & \big\downarrow (0\ \perp 1) & & \big\downarrow \left(\begin{smallmatrix}d & 0\\ 0 & \perp 1\end{smallmatrix}\right) & & \big\downarrow \perp 1 & & \\
Q^{\#} = \cdots 0 & \longrightarrow & 0 & \longrightarrow & {P_{\perp n+1}}^* & \xrightarrow[d^*]{} \cdots \longrightarrow & {P_{n\perp 2}}^* & \xrightarrow[\left(\begin{smallmatrix}0\\ d^*\end{smallmatrix}\right)]{} & P_{\perp n} \oplus {P_{n\perp 1}}^* & \xrightarrow[(s\ d^*)]{} & {P_n}^* & \longrightarrow & 0\cdots
\end{array}
$$

We consider now a very last morphism; let $j := \chi^{\perp 1}\,\pi^{\#} : Q \to Z$ be the following:

$$
\begin{array}{ccccccccccccc}
Q = \cdots 0 & \longrightarrow & P_n & \xrightarrow{\left(\begin{smallmatrix}s\\ d\end{smallmatrix}\right)} & {P_{\perp n}}^* \oplus P_{n\perp 1} & \xrightarrow{(0\ d)} & P_{n\perp 2} & \xrightarrow{d} \cdots \longrightarrow & P_{\perp n+1} & \longrightarrow & 0 & \longrightarrow & 0\cdots \\
j \big\downarrow & & \big\downarrow 1 & & \big\downarrow \left(\begin{smallmatrix}0 & 1\\ \perp d^* & s\end{smallmatrix}\right) & & \big\downarrow \left(\begin{smallmatrix}1\\ s\end{smallmatrix}\right) & & \big\downarrow \left(\begin{smallmatrix}1\\ s\end{smallmatrix}\right) & & \big\downarrow & & \\
Z = \cdots 0 & \longrightarrow & P_n & \xrightarrow[\left(\begin{smallmatrix}d\\ 0\end{smallmatrix}\right)]{} & P_{n\perp 1} \oplus {P_{\perp n+1}}^* & \xrightarrow[\left(\begin{smallmatrix}d & 0\\ 0 & d^*\end{smallmatrix}\right)]{} & P_{n\perp 2} \oplus {P_{\perp n+2}}^* & \xrightarrow[\left(\begin{smallmatrix}d & 0\\ 0 & d^*\end{smallmatrix}\right)]{} \cdots \longrightarrow & P_{\perp n+1} \oplus {P_{n\perp 1}}^* & \xrightarrow[(\perp sd\ d^*)]{} & {P_n}^* & \longrightarrow & 0\cdots
\end{array}
$$

We give now three equalities of morphisms of complexes and leave their verification to the reader:

$$
\begin{array}{lr}
(1) & i^{\#} \circ \chi \circ i = s, \\
(2) & j^{\#} \circ \chi \circ j = t, \\
(3) & \pi \circ j = t.
\end{array}
$$

Now, only up to homotopy, we have:

$$
(4) \qquad i^{\#} \circ \chi \circ j = 0.
$$

This is easy and we are pretty sure that our reader would have found the homotopy : $Q \overset{[+1]}{\dashrightarrow} P^{\#}$ , which is zero except in degree $n-1$ where it is $(\,1\;0\,)$. The second (and last) equality that we have to establish in $\mathrm{K}^{\mathrm{b}}(\mathcal{E})$ is

$$s \circ a = 0. \tag{5}$$

Set the homotopy $\epsilon : T^{-1}(Q^{\#}) \overset{[+1]}{\dashrightarrow} P^{\#}$ to be :

$$
\begin{array}{l}
T^{-1}(Q^{\#}) = \cdots 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow P_{-n+1}{}^{*} \xrightarrow{-d^{*}} \cdots \longrightarrow P_{n-2}{}^{*} \xrightarrow{\left(\begin{smallmatrix} 0 \\ -d^{*}\end{smallmatrix}\right)} P_{-n} \oplus P_{n-1}{}^{*} \xrightarrow{(-s\;\; -d^{*})} P_{n}{}^{*} \longrightarrow 0 \cdots \\
\qquad \Big\downarrow \epsilon\,[+1] \qquad\qquad \text{(diagonal maps: } 1 : P_{-n+1}{}^{*} \to P_{-n+1}{}^{*},\; 1 : P_{n-2}{}^{*} \to P_{n-2}{}^{*},\; (0\;1) : P_{-n} \oplus P_{n-1}{}^{*} \to P_{n-1}{}^{*},\; 1 : P_{n}{}^{*} \to P_{n}{}^{*}) \\
P^{\#} = \cdots 0 \longrightarrow P_{-n}{}^{*} \xrightarrow[d_{-n+1}{}^{*}]{} P_{-n+1}{}^{*} \xrightarrow[d_{-n+2}{}^{*}]{} \cdots \xrightarrow[d_{n-2}{}^{*}]{} P_{n-2}{}^{*} \xrightarrow[d_{n-1}{}^{*}]{} P_{n-1}{}^{*} \xrightarrow[d_{n}{}^{*}]{} P_{n}{}^{*} \longrightarrow 0 \cdots
\end{array}
$$

Then, a direct verification gives

$$s\,a + d^{*}\epsilon + \epsilon\bar{d} = 0$$

where $d^{*}$ (respectively $\bar{d}$) denotes the differential in $P^{\#}$ (respectively in $T^{-1}(Q^{\#})$).

Now, jump in the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$ and invert happily $s$ and $t$ which are known to be quasi-isomorphisms. We obtain

$$a = 0$$

directly from (5). Therefore the triangle $(\Delta)$ is split, meaning that

$$P \overset{i}{\rightarrowtail} Z \overset{\pi}{\twoheadrightarrow} Q^{\#}$$

is a split short exact sequence. Using (3), we have the following commutative diagram :

$$
\begin{array}{ccccc}
P & \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & P \oplus Q & \xrightarrow{(\,0\;1\,)} & Q \\
\Big\| & & \Big\downarrow {\scriptstyle (\,i\;j\,)} & & \simeq \Big\downarrow t \\
P & \xrightarrow[i]{} & Z & \xrightarrow[\pi]{} & Q^{\#}.
\end{array}
$$

Since both lines are split exact, the morphism $h := (\,i\;j\,) : P \oplus Q \to Z$ is an isomorphism. Since the beginning, we know that $Z$ is equipped with a neutral form $\chi$. Using $h$, we have that

$$\big(P \oplus Q, h^{\#}\,\chi\,h\big)$$

is neutral. To conclude, compute $h^{\#}\,\chi\,h$ using (1), (2) and (4) :

$$h^{\#}\,\chi\,h = \begin{pmatrix} i^{\#} \\ j^{\#} \end{pmatrix} \circ \chi \circ (\,i \quad j\,) = \begin{pmatrix} i^{\#}\,\chi\,i & i^{\#}\,\chi\,j \\ j^{\#}\,\chi\,i & j^{\#}\,\chi\,j \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}.$$

This gives the result when $n \geq 2$. For $n = 1$, some cone constructions may have 3 factors in degree 0 (when $n$ was $\geq 2$, $n-1$ and $-n+1$ were distinct integers. This is no more the case when $n = 1$). For instance, $(Q, t)$ should be defined as :

$$
\begin{array}{ccccccccc}
 & & & & \text{(degree 0)} & & & & \\
 & & & & \vdots\downarrow & & & & \\
Q := \cdots 0 & \longrightarrow & P_{1} & \xrightarrow{\left(\begin{smallmatrix} s \\ d \end{smallmatrix}\right)} & P_{-1}{}^{*} \oplus P_{0} & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
t \Big\downarrow & & \Big\downarrow & & \Big\downarrow \left(\begin{smallmatrix} 0 & d \\ d^{*} & -s \end{smallmatrix}\right) & & \Big\downarrow & & \\
Q^{\#} = \cdots 0 & \longrightarrow & 0 & \longrightarrow & P_{-1} \oplus P_{0}{}^{*} & \xrightarrow[(\,s\;\;d^{*}\,)]{} & P_{1}{}^{*} & \longrightarrow & 0 \cdots
\end{array}
$$

Nevertheless, the argument is exactly the same and the reader may find the exact triangles (over $z$ and over $a$) as well as relations (1) to (5) directly. Then, the end of the proof is the same (see 3.12 if needed). Our main lemma is now proven. $\sharp$

**3.11. Last step.** The end of the proof of theorem 3.2 is now obvious. Using alternatively lemma 3.11 and lemma 3.9 (for $m = n+1$), we reduce any element of $\mathrm{W}\big(\mathrm{D}^{\mathrm{b}}(\mathcal{E})\big)$ to the class of a form (as in 3.6) over a complex concentrated in degree zero. Such a quasi-isomorphism is necessarily a symmetric isomorphism in $\mathcal{E}$. ♯

**3.12. Back entrance to proof 3.10.** The idea of that proof is very simple. Actually, the complexes $Q$, $M$, and other morphisms $z$, $a$, that we take out of our pocket in proof 3.10 are produced by a general argument. Consider a complex like the $P$ that we have at the beginning of 3.10. Then the following map :

$$\begin{array}{ccccccccccc} L := & \cdots 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & P_{\perp n} & \longrightarrow & 0 \cdots \\ {\scriptstyle \nu_1}\big\downarrow & & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle 1} & & \\ P = & \cdots 0 & \longrightarrow & P_n & \xrightarrow{d_n} \cdots \xrightarrow{d_{-n+2}} & P_{\perp n+1} & \xrightarrow{d_{-n+1}} & P_{\perp n} & \longrightarrow & 0 \cdots \end{array}$$

is a sub-lagrangian of the symmetric space $(P, s)$ (in the sense of [2, paragraph 3]). This makes use, of course, of the fact that $n \geq 1$ and that $s$ is a real morphism of complexes.

Then, we can follow the sub-lagrangian construction described in [*loc. cit.*] and especially construct the diagram $\Omega$ of 3.1 [*loc. cit.*]. With those notations, $P$ is $P$, their $\varphi$ is our $s$ and the morphism $\eta_0$ is easily found. Our $z$ (used in 3.10) is the $w = \eta_0\,\nu_0$ of [*loc. cit.*]. Our morphism $a$ is $\nu_1 \circ T^{\perp 1}\eta_2$ in the homotopy category, which we prove to be zero in the derived category. The morphism $t$ is guessed to match in the diagram $\bar{\Omega}$ that appears in [*loc. cit.*, 3.5].

However, we cannot use formally the results of [*loc. cit.*] for two reasons. First, they depend upon the assumption that the considered triangulated category is "noetherian" and this is not necessary here. Second reason, the constructions in [*loc. cit.*] are not explicit and have to be written here into technical details to be sure that $Q$ is really a shorter complex. So, this conceptual complement is only added to explain and strengthen the proof.

As we shall see in § 5, it is useful to understand this reduction of complexes because it applies to shifted dualities too.

## 4. The isomorphism between usual and derived.

**4.1. Lemma.** *Let $(\mathcal{E}, *)$ be a semi-saturated exact category with duality such that $\frac{1}{2} \in \mathcal{E}$. Let $\mathcal{A}$ be an abelian category as in 2.7. Consider a complex $P$ in $\mathcal{E}$ of the form :*

$$P = \qquad \cdots 0 \longrightarrow P_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_1^* \circ \psi} {P_1}^* \xrightarrow{\alpha_2^*} \cdots \xrightarrow{\alpha_n^*} {P_n}^* \longrightarrow 0 \cdots$$

*where $(P_0, \psi)$ is a symmetric space in $\mathcal{E}$. Suppose that the complex $P$, seen in $\mathcal{A}$, is acyclic except in degree zero. Denote by $\mathrm{H}_0(P)$ the homology of $P$ in degree zero. Denote by $\Psi : P \xrightarrow{\sim} P^{\#}$ the obvious form, that is identity in each degree but $\psi$ in degree zero. Then, all the following hold :*

1. $\mathrm{H}_0(P)$ *is in* $\mathcal{E}$*;*
2. $P^{\#}$ *has also homology concentrated in degree zero and* $\mathrm{H}_0(P^{\#}) \cong \big(\mathrm{H}_0(P)\big)^*$*;*
3. $\mathrm{H}_0(\Psi)$ *induces a symmetric form on* $\mathrm{H}_0(P)$*;*
4. *most important :* $[\mathrm{H}_0(P), \mathrm{H}_0(\Psi)] = -[P_0, \psi]$ *in* $\mathrm{W}_{\mathrm{us}}(\mathcal{E})$*;*
5. *If $P'$ is of the same form as $P$ and is also acyclic except in degree 0 and if $h : P \to P'$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$, then $\mathrm{H}_0(h)$ is an isometry between the symmetric spaces $\big(\mathrm{H}_0(P), \mathrm{H}_0(\Psi)\big)$ and $\big(\mathrm{H}_0(P'), \mathrm{H}_0(\Psi')\big)$.*

**4.2. Proof.** Suppose $n \geq 2$. In the abelian category $\mathcal{A}$, the morphism $\alpha_n^*$ is an epimorphism because $\mathrm{H}_{\perp n}(P) = 0$. By 2.7, part (3), it is an admissible epimorphism in $\mathcal{E}$. Therefore $\alpha_n$ is an admissible monomorphism and one can easily get rid of $P_n$ and shorten the complex, keeping the same homology in degree zero and the same structure. So we are reduced to the case $n = 1$:

$$\begin{array}{ccccccccc}
P = \cdots 0 & \longrightarrow & P_1 & \xrightarrow{\alpha} & P_0 & \xrightarrow{\alpha^*\,\psi} & P_1^* & \longrightarrow & 0\cdots \\
\Psi\downarrow & & \downarrow 1 & & \downarrow \psi & & \downarrow 1 & & \\
P^{\#} = \cdots 0 & \longrightarrow & P_1 & \xrightarrow[\psi\,\alpha]{} & P_0^* & \xrightarrow[\alpha^*]{} & P_1^* & \longrightarrow & 0\cdots
\end{array}$$

But, as before, $\alpha^*$ is an admissible epimorphism and $\alpha$ is an admissible monomorphism. The pair $(P_1, \alpha)$ is called a sub-lagrangian of the symmetric space $(P_0, \psi)$ in the **usual sense**. Therefore, one can apply the usual sub-lagrangian construction which is nothing but taking the homology in degree zero of the above complex. This is classical and is not included here (see [6, § 2 prop. 4 p. 128 and § 4 thm. 3 p.140 and its proof]). For instance, the orthogonal $P_1^{\perp}$ of $P_1$ is by definition $\ker(\alpha^*\,\psi)$ which is in $\mathcal{E}$. The homology is then $P_1^{\perp}/P_1$ and is in $\mathcal{E}$ because the map $P_1 \longrightarrow P_1^{\perp}$ is an admissible monomorphism (still using the semi-saturation). The form induced by $\psi$ on $P_1^{\perp}/P_1$ is clearly $\mathrm{H}_0(\Psi)$ and one has property (4) as usual.

The rest of the proof is left as an easy exercise. ♯

**4.3. Theorem.** *Let $\mathcal{E}$ be a semi-saturated exact category with duality such that $\frac{1}{2} \in \mathcal{E}$. Endow $\mathrm{D^b}(\mathcal{E})$ with the induced duality as in paragraph 2. Then the natural homomorphism:*

$$\omega_{\mathcal{E}} : \mathrm{W_{us}}(\mathcal{E}) \longrightarrow \mathrm{W}\big(\mathrm{D^b}(\mathcal{E})\big)$$

*sending everything in degree zero is an isomorphism.*

**4.4. Proof.** In view of theorem 3.2, we only have to prove injectivity of $\omega_{\mathcal{E}}$. Let $x \in \ker(\omega_{\mathcal{E}})$. Then $x = [Q, \varphi]$ for some symmetric space $(Q, \varphi)$ in $\mathcal{E}$. By hypothesis, $\omega_{\mathcal{E}}(x) = [c_0(Q), c_0(\varphi)] = 0$ in $\mathrm{W}\big(\mathrm{D^b}(\mathcal{E})\big)$ using the notations of 2.8. This means that $\big(c_0(Q), c_0(\varphi)\big)$ is stably neutral in $\big(\mathrm{D^b}(\mathcal{E}), \#\big)$.

The key part of the proof is theorem 2.5 of [2] which says that a stably neutral space is neutral in any triangulated category with duality containing $\frac{1}{2}$. That is $\big(c_0(Q), c_0(\varphi)\big)$ is isomorphic to some cone form. In other words, applying definition 2.3, we obtain easily that there exists a complex $L$ in $\mathcal{E}$, a morphism $z : T^{\perp 1}(L^{\#}) \longrightarrow L$

$$\begin{array}{ccccccccccccccc}
 & & & & \text{degree } n & & & & & & \text{degree } -n & & & & \\
T^{\perp 1}L^{\#} = \cdots 0 & \longrightarrow & 0 & \longrightarrow & {L_{\perp n \perp 1}}^* & \xrightarrow{\perp d^*} & {L_{\perp n}}^* & \longrightarrow & \cdots & \xrightarrow{\perp d^*} & {L_{n\perp 1}}^* & \xrightarrow{\perp d^*} & {L_n}^* & \longrightarrow & 0\cdots \\
z\downarrow & & \downarrow & & \downarrow z_n & & \downarrow z_{n-1} & & & & \downarrow z_{-n} & & \downarrow z_{-n-1} & & \\
L = \cdots 0 & \longrightarrow & 0 & \longrightarrow & L_n & \xrightarrow[d]{} & L_{n\perp 1} & \longrightarrow & \cdots & \xrightarrow[d]{} & L_{\perp n} & \xrightarrow[d]{} & L_{\perp n \perp 1} & \longrightarrow & 0\cdots
\end{array}$$

such that:

1. $z$ is strongly symmetric, *i.e.* $(z_{\perp i \perp 1})^* = z_i$ for all $i \in \mathbb{Z}$.
2. Let $Z = \mathrm{Cone}(z)$. There is an isometry in $\mathrm{D^b}(\mathcal{E})$ between $(Z, \chi)$ and $\big(c_0(Q), c_0(\varphi)\big)$ where $\chi = \chi^{\#} : Z \xrightarrow{\sim} Z^{\#}$ is the neutral form given by:

$$\begin{array}{ccccccccccc}
 & & & & \text{degree } n & & & & \text{degree } -n & & \\
Z := C(z)\cdots 0 & \longrightarrow & {L_{\perp n\perp 1}}^* & \xrightarrow{\begin{pmatrix} d^* \\ \perp z \end{pmatrix}} & {L_{\perp n}}^* \oplus L_n & \xrightarrow{\begin{pmatrix} d^* & 0 \\ \perp z & d \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} d^* & 0 \\ \perp z & d \end{pmatrix}} & {L_n}^* \oplus L_{\perp n} & \xrightarrow{(\perp z \;\; d)} & L_{\perp n \perp 1} & \longrightarrow & 0\cdots \\
\chi\downarrow & & \downarrow \perp 1 & & \downarrow \begin{pmatrix} 0 & \perp 1 \\ \perp 1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & \perp 1 \\ \perp 1 & 0 \end{pmatrix} & & \downarrow \perp 1 & & \\
Z^{\#} = \cdots 0 & \longrightarrow & {L_{\perp n \perp 1}}^* & \xrightarrow[\begin{pmatrix} \perp z^* \\ d^* \end{pmatrix}]{} & L_n \oplus {L_{\perp n}}^* & \xrightarrow[\begin{pmatrix} d & \perp z^* \\ 0 & d^* \end{pmatrix}]{} \cdots \xrightarrow[\begin{pmatrix} d & \perp z^* \\ 0 & d^* \end{pmatrix}]{} & L_{\perp n} \oplus {L_n}^* & \xrightarrow[(d^* \;\; \perp z^*)]{} & L_{\perp n \perp 1} & \longrightarrow & 0\cdots
\end{array}$$

Remarks: the integer $n$ does not mean anything and can be taken large enough; the strong symmetry of $z$ comes from exercise 2.10, which is recommended to the confused reader (see also [2, thm. 1.6, part (2)] if necessary).

Note that in degree zero, $\chi$ is minus the hyperbolic space over $L_0^* \oplus L_0$. Using the isomorphism $\chi$ to modify $Z$ in strictly negative degrees only, one obtains the following complex $P$:

$$P := \cdots \xrightarrow{\alpha_2 := \begin{pmatrix} d^* & 0 \\ \perp z & d \end{pmatrix}} {L_{\perp 1}}^* \oplus L_1 \xrightarrow{\alpha_1 := \begin{pmatrix} d^* & 0 \\ \perp z & d \end{pmatrix}} \underset{\text{degree } 0}{{L_0}^* \oplus L_0} \xrightarrow{\alpha_1^* \psi} {L_1}^* \oplus L_{\perp 1} \xrightarrow{\alpha_2^*} {L_2}^* \oplus L_{\perp 2} \xrightarrow{\alpha_3^*} \cdots$$

where $\psi = \begin{pmatrix} 0 & \perp 1 \\ \perp 1 & 0 \end{pmatrix}$ is a hyperbolic form over $P_0 := {L_0}^* \oplus L_0$. We check now that we are in the situation of the above lemma. The form $\Psi : P \xrightarrow{\sim} P^\#$ induced on $P$ from $\chi$ on $Z$ is identity in all degree but degree zero where it is $\psi = \begin{pmatrix} 0 & \perp 1 \\ \perp 1 & 0 \end{pmatrix}$. Moreover, this new complex $P$ is still isomorphic in $\mathrm{D^b}(\mathcal{E})$ to $c_0(Q, \varphi)$. Therefore $P$ has homology only in degree zero and this homology is isomorphic to $Q$. The lemma 4.1 says that the form $(Q, \varphi) \simeq (\mathrm{H}_0(P), \mathrm{H}_0(\Psi))$ is Witt-equivalent to $\left(L_0 \oplus {L_0}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, which is Witt-trivial. ♯

**4.5. Corollary.** *Let $\mathcal{E}$ be a semi-saturated exact category with duality such that $\frac{1}{2} \in \mathcal{E}$. Then the inverse of the natural isomorphism $\omega_{\mathcal{E}}$ is given algorithmically by the explicit constructions of 3.6, 3.8 and 3.10.*

**4.6. Proof and remark.** Since $\omega_{\mathcal{E}}$ is an isomorphism, the inverse consists simply in finding any pre-image. ♯

The procedure in 3.6, 3.8 and 3.10 was the following: choose a representation of the form $(P, s)$ where $s$ is a real quasi-isomorphism and is strongly symmetric; then reduce $(P, s)$ on the right and on the left alternatively using the explicit descriptions of $(Q, t)$ that can be found in those proofs.

In [1, § 4], it was quite a pain to construct explicitly an inverse to $\omega$ when the considered derived category was simply the homotopy category (in the split case, there was no need to invert quasi-isomorphisms). Even if the result of [*loc. cit.*] is a closed formula instead of our present algorithmic reduction, it is quite impressive how the proof of injectivity is shorter in the more general context. The result "*stably neutral implies neutral*" [2, thm 2.5], though very abstract, appears very useful here.

**4.7. Theorem.** *Let $X$ be a scheme such that $\frac{1}{2} \in \mathcal{O}_X$ (obvious sense). There is a natural isomorphism:*

$$\omega_X : \mathrm{W_{us}}(X) \longrightarrow \mathrm{W_{der}}(X).$$

*The same holds for skew-symmetric forms as well.*

**4.8. Proof.** Let $(\mathcal{E}(X), *, \mathrm{can})$ be the exact category of 1.8 where $(-)^* = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ and $\mathrm{can} : \mathrm{Id} \xrightarrow{\sim} *^2$ is the canonical (usual) identification. It suffices then to apply theorem 4.3 to the following exact categories: $(\mathcal{E}(X), *, \mathrm{can})$ and $(\mathcal{E}(X), *, -\mathrm{can})$. ♯

**4.9. Remark.** We give in [2, thm 5.1] a general localization exact sequence for triangulated categories with duality, under the hypotheses of "noetherianity" and of presence of $\frac{1}{2}$. This applies of course to localization of schemes. Consider $U \subset X$ to be an open subscheme of a noetherian scheme $X$. Denote by $\mathrm{D^b}(X)$ the derived bounded category of the exact $\mathcal{E}(X)$ considered several times up to here (see 1.8, 2.11). We can apply our localization theorem as soon as $\mathrm{D^b}(U)$ is a localization of $\mathrm{D^b}(X)$. This is the case when $X$ is regular and separated (more general conditions are presented in [13]). In that case, we have a long exact sequence:

$$\cdots \mathrm{W}^{n \perp 1}(U) \longrightarrow \mathrm{W}^n(J) \longrightarrow \mathrm{W}^n(X) \longrightarrow \mathrm{W}^n(U) \longrightarrow \mathrm{W}^{n+1}(J) \longrightarrow \cdots$$

where $J$ is the full subcategory of $\mathrm{D^b}(X)$ on those complexes which are acyclic over $U$.

We recall from [2, proposition 1.14] that the shifted (or higher) Witt groups are 4-periodic. Theorem 4.7 gives a description of half of them, namely the even-indexed Witt groups: $\mathrm{W}^0(X)$ is the same as the usual Witt group and $\mathrm{W}^2(X)$ (which is the same as $\mathrm{W}^0$ skew-symmetric) is the usual Witt group of skew-symmetric forms. We discuss odd-indexed Witt groups in next paragraph.

For more general $X$, we shall have the ideal formulation by using the derived category of all coherent $\mathcal{O}_X$-modules with Grothendieck's duality (see [4] for instance). This will appear later.

## 5. Some remarks on the shifted or higher Witt groups.

**5.1. The four Witt groups : Explaning definitions and notations.** In [2] (respectively in [1]), we associated to a given triangulated category with duality $(K, \#, \varpi)$ a collection of Witt groups $\mathrm{W}^n$, $n \in \mathbb{Z}$, (respectively $\mathrm{W}^\epsilon_n$, $n \in \mathbb{Z}$, $\epsilon = \pm 1$). They should be remembered as being simply the Witt groups for the shifted dualities $T^n \circ \#$. To be less simple, here is a series of considerations that we have to make on those *shifted* Witt groups.

(1) The functor $T^n \circ \#$ does satisfy $(T^n \circ \#)^2 \simeq \mathrm{Id}$ because $T \circ \# \cong \# \circ T^{\perp 1}$ and $\#^2 \simeq \mathrm{Id}$. But it is not always *exact* in the sense of Definition 2.2 because the functor $T : K \to K$ is only skew-exact (*i.e.* $T$ sends an exact triangle to a skew-exact triangle, that is a triangle which is exact when changing the sign of all the morphisms). Therefore, we had to deal carefully with the notion of exactness and introduced $\delta$-exact functors for $\delta = \pm 1$ (for $T^n \circ \#$, $\delta = (-1)^n$). This affects also the notion of neutral forms. This is easy to understand : in the definition of neutral symmetric spaces, we required the existence of a symmetric exact triangle, but exactness involves some sign. We do not want to re-write all this here and we refer to [2, § 1].

(2) Those Witt groups are called sometimes *shifted* Witt groups because they use shifted dualities. Their interest is that they fit into a long exact sequence of localization (see thm. 5.1 in [2]) already mentioned in remark 4.9. This explains why we also call them *higher (and lower)* Witt groups. This cohomological use justifies also the notation $\mathrm{W}^n$. To have this nice property, we make the following definition of the translation of a triangulated category with $\delta$-duality : $T(K, \#, \varpi) = (K, T \circ \#, -\delta \cdot \varpi)$. In other words, we introduce a sign when we translate a duality but we introduce no sign when we translate a skew-duality. If we start with a triangulated category with (+1)-duality $(K, \#, \varpi)$, we have then :
$$T^n(K, \#, \varpi) \overset{\text{(def)}}{=} \big( K\,,\, T^n \circ \#\,,\, (-1)^{\frac{n(n+1)}{2}} \cdot \varpi \big)$$
for all $n \in \mathbb{Z}$. We define then the *shifted Witt groups* very simply by :
$$\mathrm{W}^n(K, \#, \varpi) := \mathrm{W}\big(T^n(K, \#, \varpi)\big).$$

(3) In [1], the groups $\mathrm{W}^n$ were written $\mathrm{W}^\epsilon_n$. This notation had the handicap of being inadequate for the harmony of the localization sequence (not yet established at that time). Nevertheless, it was more expressive because the "$n$" referred to the shift $T^n \circ \#$ and the $\epsilon = \pm 1$ referred to $\epsilon$-symmetric forms, keeping the identification $\varpi : \mathrm{Id} \overset{\sim}{\to} \#^2$ unaltered. The dictionary is the following one : $\mathrm{W}^n = \mathrm{W}^\epsilon_n$ where $\epsilon = (-1)^{\frac{n(n+1)}{2}}$ (see above). We suggest to stick to the present notation $\mathrm{W}^n(K)$.

(4) There is an isomorphism of triangulated categories with $(-1)^n$-duality
$$(K, T^n \circ \#, \varpi) \overset{\sim}{\to} (K, T^{n+2} \circ \#, \varpi)$$
induced by $T : K \to K$ which gave in the old notations the isomorphism $\mathrm{W}^\epsilon_n(K) \overset{\sim}{\to} \mathrm{W}^\epsilon_{n+2}(K)$ and allowed us to reduce ourselves to four Witt groups : $\mathrm{W}^{\pm}_{0\,\text{or}\,1}(K)$. In the new notations, note that $T$ does **not** give an isomorphism between $T^n(K, \#, \varpi)$ and $T^{n+2}(K, \#, \varpi)$ because of this sign trick. We only have $T^n(K, \#, \varpi) \cong T^{n+2}(K, \#, -\varpi)$ and therefore the 4-periodicity as :
$$\mathrm{W}^n(K) \simeq \mathrm{W}^{n+4}(K)$$
(see [2, prop. 1.14]). We also keep only four Witt groups : $\mathrm{W}^0$, $\mathrm{W}^1$, $\mathrm{W}^2$ and $\mathrm{W}^3$, for instance. The group $\mathrm{W}^2$ is the old $\mathrm{W}^{\perp}_0$, that is the group of skew-symmetric spaces for the unshifted duality.

(5) As a consequence of (4) above, in the case of the derived category of an exact category with duality $(\mathcal{E}, *, \pi)$, theorem 4.3 actually identifies $\mathrm{W}\big(\mathrm{D^b}(\mathcal{E})\big)$ as being $\mathrm{W_{us}}(\mathcal{E})$ but also $\mathrm{W}^2\big(\mathrm{D^b}(\mathcal{E})\big)$ as being $\mathrm{W}^{\perp}_{\mathrm{us}}(\mathcal{E}) = \mathrm{W_{us}}(\mathcal{E}, *, -\pi)$ the usual Witt group of skew-symmetric forms. A direct group isomorphism :
$$\mathrm{W}^{\perp}_{\mathrm{us}}(\mathcal{E}) \longrightarrow \mathrm{W}^2\big(\mathrm{D^b}(\mathcal{E}), \#, \varpi\big)$$
is given by the functor $c_1 : \mathcal{E} \to \mathrm{D^b}(\mathcal{E})$ sending everything in degree 1.

**5.2. Proposition.** *Let $(\mathcal{E}, *, \pi)$ be a semi-saturated exact category with duality such that $\frac{1}{2} \in \mathcal{E}$. Endow $\mathrm{D^b}(\mathcal{E})$ with the induced duality as in paragraph 2. Then every symmetric space for the skew-duality $T \circ \#$ is Witt-equivalent to a space of the form $(P, \varphi)$ where $P$ is a complex supported in $[1,0]$:*

$$P = \cdots 0 \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow 0 \cdots$$

*and where $\varphi$ is a morphism of complexes (no fractions) from $P$ to $T(P^\#)$ which is a quasi-isomorphism such that $\varphi_0 = \varphi_1^* \circ \pi_{P_0}$.*

**5.3. Proof.** We shall omit this proof because we are not going to use this result hereafter. It is an exercise to adapt the proofs of § 2 to this shifted context. Remark 3.12 definitely helps. ♯

**5.4. Remark.** Consider a symmetric form as described above:

$$\begin{array}{ccccccccc} P = & \cdots \longrightarrow & 0 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 & \longrightarrow 0 \longrightarrow & \cdots \\ \varphi\big\downarrow & & & & \big\downarrow \varphi_1 & & \big\downarrow \varphi_1^* & & \\ T(P^\#) = & \cdots \longrightarrow & 0 & \longrightarrow & P_0^* & \xrightarrow[-d^*]{} & P_1^* & \longrightarrow 0 \longrightarrow & \cdots \end{array}$$

Saying that $\varphi$ is a quasi-isomorphism is nothing but asking for the exactness of the following complex (the mapping cone with all signs changed):

$$\cdots 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{\begin{pmatrix} d \\ \varphi_1 \end{pmatrix}} P_0 \oplus P_0^* \xrightarrow{(\varphi_1^* \; d^*)} P_1^* \longrightarrow 0 \longrightarrow 0 \cdots$$

Now, we have a symmetric space $\big(P_0 \oplus P_0^*, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\big)$ with two lagrangians $P_0$ and $P_1$. This is the link with the notion of *formation* used by Ranicki [12], Pardon [9] and others. Over a ring (or in the split exact case), see in particular [12, prop. 2.3].

As explained in the introduction, we choose to focus our attention on the un-shifted Witt group $\mathrm{W}^0$ and to identify it with the usual one because this is the classical invariant largely used and studied. This actually identifies half of the derived Witt groups, the even-indexed ones as explained in remark 5.1, point (5). It is nevertheless of big interest (mainly for computation of the 12-term localization sequence) to have a simpler description of the odd-indexed Witt groups too.

* * *

**5.5. Notation.** If $R$ is a commutative ring and a notation is defined over schemes, say $N(X)$, then $N(R)$ will denote $N\big(\mathrm{Spec}(R)\big)$.

**5.6. Theorem.** *Let $R$ be a commutative local ring in which 2 is invertible. Then, among the derived Witt groups of $R$, we have $\mathrm{W}^1(R) = 0$, $\mathrm{W}^2(R) = 0$ and $\mathrm{W}^3(R) = 0$. That is there is only one non-trivial Witt group, namely $\mathrm{W}^0(R) \cong \mathrm{W_{us}}(R)$. This holds in particular for fields of characteristic not 2.*

**5.7. Proof.** We denote by $\mathrm{D^b}(R) \overset{\text{(def)}}{=} \mathrm{D^b}\big(\mathcal{E}(R)\big) = \mathrm{K^b}\big(\mathcal{E}(R)\big)$ the derived category of finitely generated projective (free) $R$-modules, which reduces to the homotopy category because a quasi-isomorphism of bounded complexes of projective modules is a homotopy equivalence. We also denote by $\#$ the duality induced by $\mathrm{Hom}_R(-, R)$ and by $\varpi$ the isomorphism of functors $\mathrm{Id} \overset{\sim}{\to} \#^2$, defined to be the canonical identification in each degree.

The computation of $\mathrm{W}^0(R)$ (respectively $\mathrm{W}^2(R) \simeq \mathrm{W}^0\big(\mathrm{D^b}(R), \#, -\varpi\big)$) to be the usual Witt group of symmetric (respectively skew-symmetric) forms is now clear from theorem 4.3 and remark 5.1, point (5). For the classical proof that $\mathrm{W}^{\perp}(R) = 0$ see [7, cor. I.3.5, p. 7].

We give the proof that $W\big(\mathrm{D^b}(R), T \circ \#, \varpi\big) = 0$. The same for $-\varpi$ is left to the reader.

Let $\mathfrak{m}$ be the maximal ideal of $R$.

Recall that a complex of free $R$-modules of finite rank is homotopically equivalent to a so called *minimal* complex, that is a complex such that all the entries of the matrix of the differentials are in $\mathfrak{m}$. To see this (exercise) it suffices to use elementary operations and to have in mind that $R \setminus \mathfrak{m}$ is the set of units of $R$. Now, if a complex is minimal, its dual is also minimal. Moreover, a homotopy equivalence between two minimal complexes is necessarily an isomorphism. This is easy to check. In fact, it suffices to note that a $n \times n$-matrix of the form $1_n + A$ where $A$ has all entries in $\mathfrak{m}$ is invertible (because its determinant belongs to $1 + \mathfrak{m}$ and is therefore invertible). In other words, an element of $W(\mathrm{D^b}(R), T \circ \#, \varpi)$ can be represented by an isomorphism :

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & \text{degree } 0 & & & & & & \\
 & & & & & & & & \vdots & & & & & & \\
P := \cdots & \longrightarrow & P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & P_{\perp 1} & \xrightarrow{d_{-1}} & P_{\perp 2} & \longrightarrow & \cdots \\
\varphi \downarrow & & \varphi_3 \downarrow \simeq & & \varphi_2 \downarrow \simeq & & \varphi_1 \downarrow \simeq & & \varphi_1^* \downarrow \simeq & & \varphi_2^* \downarrow \simeq & & \varphi_3^* \downarrow \simeq & & \\
T(P^{\#}) = \cdots & \longrightarrow & P^*_{\perp 2} & \xrightarrow[\perp d^*_{-1}]{} & P^*_{\perp 1} & \xrightarrow[\perp d^*_0]{} & P^*_0 & \xrightarrow[\perp d^*_1]{} & P^*_1 & \xrightarrow[\perp d^*_2]{} & P^*_2 & \xrightarrow[\perp d^*_3]{} & P^*_3 & \longrightarrow & \cdots
\end{array}
$$

with $P$ a bounded minimal complex of finitely generated free $R$-modules and with $\varphi_i$ being an isomorphism for all $i \in \mathbb{Z}$. As usual, we managed $\varphi$ to be strongly symmetric since the beginning, which means here $\varphi_i = (\varphi_{\perp i+1})^*$ for all $i \in \mathbb{Z}$. Using then $\varphi_i$ to replace $P_i$ by $P^*_{\perp i+1}$ for all $i > 0$, the symmetric space $(P, \varphi)$ is isomorphic to the following :

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & & \text{degree } 0 & & & & & & \\
 & & & & & & & & \vdots & & & & & & \\
Q := \cdots & \longrightarrow & P^*_{\perp 2} & \xrightarrow{\perp d^*_{-1}} & P^*_{\perp 1} & \xrightarrow{\perp d^*_0} & P^*_0 & \xrightarrow{z_0} & P_0 & \xrightarrow{d_0} & P_{\perp 1} & \xrightarrow{d_{-1}} & P_{\perp 2} & \xrightarrow{d_{-2}} & \cdots \\
\chi \downarrow & & 1 \downarrow & & 1 \downarrow & & 1 \downarrow & \text{Watch!} & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \\
T(Q^{\#}) = \cdots & \longrightarrow & P^*_{\perp 2} & \xrightarrow[\perp d^*_{-1}]{} & P^*_{\perp 1} & \xrightarrow[\perp d^*_0]{} & P^*_0 & \xrightarrow[\perp z^*_0]{} & P_0 & \xrightarrow[d_0]{} & P_{\perp 1} & \xrightarrow[d_{-1}]{} & P_{\perp 2} & \xrightarrow[d_{-2}]{} & \cdots
\end{array}
$$

where $z_0 := d_1 \circ (\varphi_1)^{\perp 1} : P_0^* \to P_0$. Note that $z_0^* = -z_0$ from the first diagram. The form $\chi$ is the one obtained from $\varphi$ through this isometry. But this last space $(Q, \chi)$ is neutral by considering the cone construction on the following skew-symmetric morphism for $T^{\perp 1}(T\,\#) = \#$ :

$$
\begin{array}{ccccccccccccc}
 & & & & & & \text{degree } 0 & & & & & & \\
 & & & & & & \vdots & & & & & & \\
L^{\#} = \cdots & \longrightarrow & P^*_{\perp 2} & \xrightarrow{d^*_{-1}} & P^*_{\perp 1} & \xrightarrow{d^*_0} & P^*_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
z \downarrow & & \downarrow & & \downarrow & & \perp z_0 \downarrow & & \downarrow & & \downarrow & & \\
L := \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P_0 & \xrightarrow[d_0]{} & P_{\perp 1} & \xrightarrow[d_{-1}]{} & P_{\perp 2} & \xrightarrow[d_{-2}]{} & \cdots
\end{array}
$$

A direct computation gives that $(Q, \chi) = \mathrm{Cone}(L^{\#}, z)$ in the notations of [2, definition 1.10]. $\sharp$

**5.8. Remark.** This result should be compared with [3, lemma 4.2, p. 215].

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**Acknowledgements**

I would like to thank Marco Schlichting for having been the first one to go through the main proof of reduction of complexes *and not from the back door!* I am also indebted to Charles Walter and Andrew Ranicki for reading the first drafts of this article, for making valuable comments and for tempering my style when it happened to be led astray by excessive enthusiasm.

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