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Research Paper

Hyperbolic Dynamics in Two Dimensional Maps

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Abstract: *In this paper, we study the theory of hyperbolic dynamical systems, especially the dynamics of two-dimensional Hénon map. A remarkable fact is that the system with complicated orbit behavior (like the "horseshoe") can be structurally stable that hyperbolicity is a necessary condition for structural stability. We discuss the horizontal and vertical slabs in a diffeomorphism on which the induced dynamics is topologically conjugate to a full shift. Using these slabs, we provide the sufficient condition for chaos on the Hénon map*

Keywords: Slabs, Horseshoe, Structural stability, Hyperbolic.

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1. Introduction:

Hyperbolic sets [1] are used as differential models for chaotic dynamical systems. The concept of hyperbolicity was introduced first by G.A. Hedlund and E. Hopf in studying geodesic flows on surfaces with negative curvature. The systematic study of hyperbolic systems was initiated by S. Smale, A. Andronov, L. Pontryagin and D.V. Anosov. The Smale horseshoe represents a first attempt to globalize the idea of a hyperbolic fixed point: the invariant set of the horseshoe mapping has two distinguished directions, one in which the differential of the mapping is consistently contracting vectors and the other in which the differential of the mapping is consistently expanding vectors. The contracting and expanding directions are given by subspaces that are preserved by the differential, and which vary continuously with the base point.
In this paper, we present a method for detecting the hyperbolic invariant sets of Hénon map. This method relies on the use of geometrical objects, referred as "slab", whose behavior under iteration can be controlled effectively and the dynamics of Hénon map is hyperbolic.

2. Preliminaries:

The Smale Horseshoe 2.1:

The aim of this section is to describe an important application of symbolic dynamics, which played a considerable role in the development of the subject of chaos [2].We start with the unit square $S = [0, 1] \times [0, 1]$a *vertical rectangle* (respectively a *horizontal rectangle*) a sub-rectangle with sides parallel to the coordinate axes and having the vertical (respectively, horizontal) sides of length 1.

Consider the map $f : S \to \mathbb{R}^2$, performing first a uniform vertical dilation and then a uniform horizontal contraction of the square S (by factors $\mu > 2$ and respectively $\lambda \in (0, 1/2)$), followed by a clockwise folding so that the resulting shape is a horseshoe, whose intersection with S consists of two vertical rectangles V_1 and V_2 as Figure 1.

Reversing dilation and contraction and then folding (counter clockwise) we get a map g, whose geometric effect is shown in Figure 2.

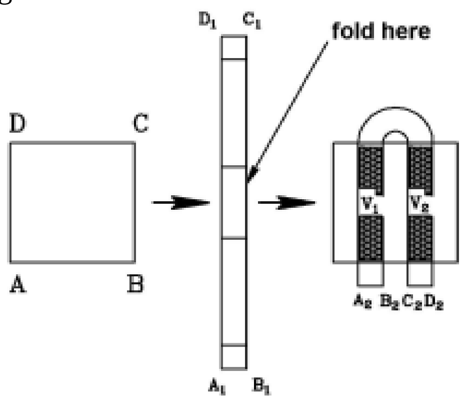

Figure 1: The action of f.

In fact,$g|f(S) \cap S = (f|f(S) \cap S)^{-1}$. Then $f^{-1}(S \cap f(S)) = S \cap f^{-1}(S)$ constitutes of two horizontal rectangles,

$$H_1 = [0, 1] \times [a, a + 1/\mu]$$
$$H_2 = [0, 1] \times [b, b + 1/\mu]$$

such that

$$f(H_1) = V_1 = [c, c + \lambda] \times [0,1]$$
$$f(H_2) = V_2 = [d, d + \lambda] \times [0,1].$$

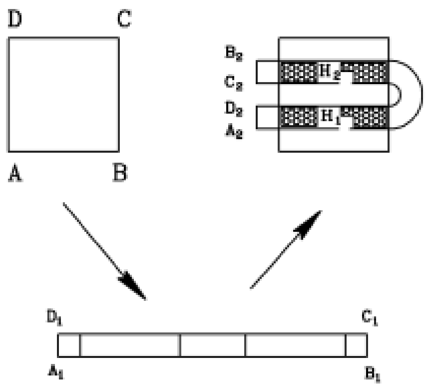

Figure 2: The action of$g$.

Notice that the actions of $f|H_1$ and$f|H_2$ are affine, more precisely

$$f|H_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ -\mu a \end{pmatrix}$$
$$f|H_2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d+\lambda \\ \mu b + 1 \end{pmatrix}.$$

The above discussion leads us to the following result:

Lemma 2.2:(i) If V is a vertical rectangle, then $f(V) \cap S$ is a union of two vertical rectangles, one included in V_1 and the other in V_2, each of width λ (width of V).

(ii) If H is a horizontal rectangle, then $g(H) \cap S$ is a union of two horizontal rectangles, one included in H_1 and the other in H_2, each of height $\frac{1}{\mu}$ (height of H).

When the map f is iterated, some points leave S. We shall show that the set

$$\Lambda = \bigcap_{k \in \mathbb{Z}} f^k(S)$$

of all points whose orbits remain in S, is a Cantor set, on which f shows a Devaney chaotic behavior. For this, we need an alternative description of Λ, via an inductive process. The starting point is the equality

$$H_1 \cup H_2 = f^{-1}\big(S \cap f(S)\big) = V_1 \cup V_2.$$

By Lemma 2.2, the set $f^{-2}(S \cap f(S) \cap f^2(S))$ consists of 2^2 horizontal strips, (each of height μ^{-2}), mapped by f^2 into 2^2 vertical strips (each of height λ^2). See Figures 3 and 4.

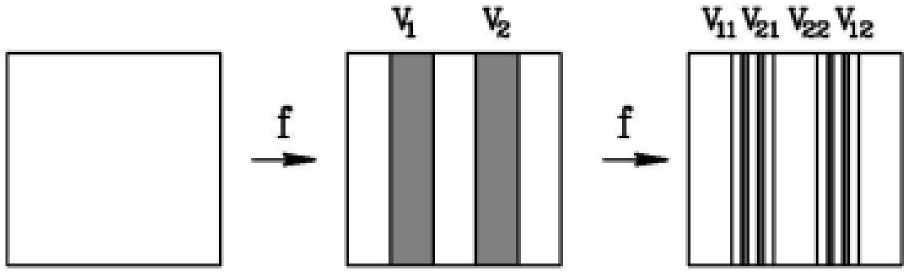

Figure 3: Iterating f.

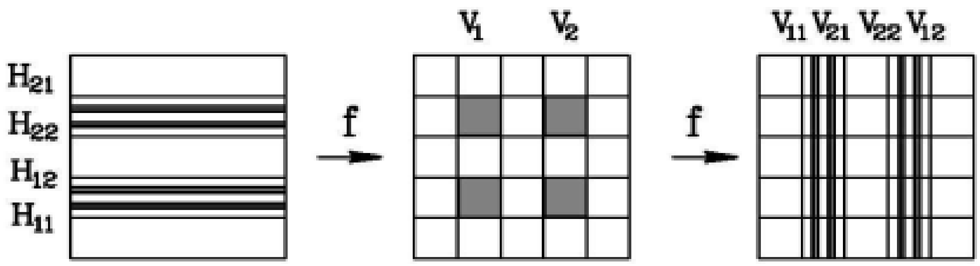

Figure 4: Iterating$f$ on $f^{-2}(S \cap f(S) \cap f^2(S))$.

Proceeding inductively, we infer that $f^{-n}(S \cap f(S) \cap \cdots \cap f^n(S))$ consists of 2^n horizontal rectangles (each of height μ^{-n}), which are mapped bijectively by f^n into 2^n vertical rectangles (each of width λ^n).

Therefore, the set Λ can be represented as

$$\Lambda = \Lambda_- \cup \Lambda_+$$

where

$$\Lambda_- = \bigcap_{n=1}^{\infty} f^{-n}\,(S \cap f(S) \cap \cdots \cap f^n(S))$$

represents the intersection of all horizontal rectangles which appear in the above iterative process and

$$\Lambda_+ = \bigcap_{n=0}^{\infty} (S \cap f(S) \cap \cdots \cap f^n(S))$$

represents the intersection of all vertical rectangles. As a consequence, Λ is the intersection of a decreasing sequence of compact sets,

$$K_n = f^{-n}(S) \cap \cdots \cap f^{-1}(S) \cap S \cap f(S) \cdots \cap f^n(S),$$

which consists of 2^{2n} rectangles of length λ^n and height μ^{-n}.

Consequently, Λ is a totally disconnected, compact and perfect set (and thus homeomorphic to the Cantor Ternary Set).The map induced by f on Λ (also denoted f) is known in literature as *Smale horseshoe* (or simply, the *horseshoe*).

The trajectories of different points of Λ can be codified by indicating in which of the rectangles H_1 and H_2 the iterates of different order land. That is done by using the function

$$\Phi : \Lambda \to \Sigma(2), \quad \Phi(x) = (t_n)_n,$$

where $t_n = 1$ if $f^n(x) \in H_1$ and $t_n = 2$ if $f^n(x) \in H_2$.
The picture of chaotic behavior of the "horseshoe" persists under small perturbations (i.e., it is structurally stable).

3. Detection of Hyperbolic Sets:

Let $f : K = \bar{B}^u(0,1) \times \bar{B}^s(0,1) \to \mathbb{R}^n$ be a diffeomorphism [3] onto its image, where u, s are two positive integers with $u + s = n$; here $\bar{B}^u(0,1)$ and $\bar{B}^s(0,1)$ are the closed unit balls in $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively.

Definition3.1: A *horizontal slice* in K is the graph of a function $h : B^u(0,1) \to K$ satisfying a Lipschitz condition

$\|h(x_1) - h(x_2)\| \le \mu_h \|x_1 - x_2\|$ for all $x_1,\ x_2 \in \bar{B}^u(0,1)$,where $0 \le \mu_h < \infty$.

Similarly, a *vertical slice*in K is the graph of a function $v : \bar{B}^s(0,1) \to K$satisfying Lipschitz condition

$\|v(y_1) - v(y_2)\| \le \mu_v \|y_1 - y_2\|$ for all $y_1,\ y_2 \in \bar{B}^s(0,1)$,where $0 \le \mu_v < \infty$.

We now want to thicken these horizontal and vertical slices to n-dimensional horizontal and vertical "slabs".

Definition 3.2: Fix some $0 \le \mu_h < \infty$ and let h be a horizontal slice of Lipschitz contstant μ_h. A *horizontal slab*[4] about h is a set $H \subset$K together with a homeomorphism $c_H \bar{B}^u(0,1) \times D^s \to H$, where D^s is a s-dimensional topological disk contained in $\bar{B}^s(0,1)$, satisfying the following three properties:

(i) There is a point $y_0 \in \mathrm{int} D^s$ such that $c_H(x, y_0) = (x, h(y))$;

(ii) For every point $y \in \partial D^s$ there exists a horizontal slice h_y from $\bar{B}^u(0,1)$ into K, with Lipschitz constant μ_h, such that $c_H(x, y) = (x, h_y(x))$;

(iii) For every point $x \in \bar{B}^u(0,1), c_H(\{x\} \times D^s)$ is contained in the plane $\{x\} \times \mathbb{R}^s$.

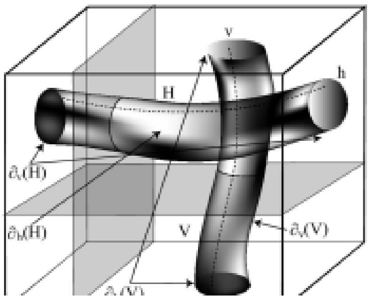

Figure 5: Horizontal and vertical slabs.

By definition, the horizontal boundary $\partial_h H$ of the horizontal slab H is the image of $\partial \bar{B}^u(0,1) \times D^s$ through c_H. The vertical boundary $\partial_v H$ of the horizontal slab H is the image of $\bar{B}^u(0,1) \times D^s$ through c_H. We note that the horizontal boundary of a horizontal slab is made of horizontal slices. We also note that the vertical boundary of a horizontal slab is parallel to the plane $\{0\} \times \mathbb{R}^s$ as shown in Figure 5.

We define vertical slabs similarly.

Definition 3.3: Fix $0 \leq \mu_v < \infty$ and let v be a vertical slice of Lipschitz contstant μ_v. A *vertical slab*[4] is a set V⊂K together with a homeomorphism $c_V: D^u \times \bar{B}^s(0,1) \to W$, where D^u is a topological disk contained in $\bar{B}^u(0,1)$, satisfying the following three properties:

(i) There is a point $x_0 \in \text{int} D^u$ such that $c_V(x_0, y) = (v(y), y)$;

(ii) For every point $x \in \partial D^u$ there exists a vertical slice v_x from $\bar{B}^s(0,1)$ into K, with Lipschitz constant μ_v, such that $c_V(x,y) = (v_x(y), y)$;

(iii) For every point $y \in \bar{B}^s(0,1)$, $c_V(D^u \times \{y\})$ is contained in the plane $\mathbb{R}^u \times \{y\}$.

The horizontal boundary $\partial_h V$ of the horizontal slab W is the image of $D^u \times \partial \bar{B}^s(0,1)$ through c_V. The vertical boundary $\partial_v V$ of the horizontal slab V is the image of $\partial D^u \times \bar{B}^s(0,1)$ through c_V (Figure 5).

We shall be interested in the relative position of horizontal slabs and vertical slabs.

Definition 3.4 [5]**:** Let H_1 and H_2 be horizontal slabs. We say that H_2*intersects* H_1*fully* if $H_2 \subset H_1$ and $\partial_v H_2 \subset \partial_v H_1$. Similarly, V_1 and V_2 are vertical slabs. We say that V_2*intersects* V_1 *fully* if $V_2 \subset V_1$ and $\partial_h V_2 \subset \partial_h V_1$.

Now we provide sufficient conditions for chaos using horizontal and vertical slabs. Let H_i, where $i \in \{1, 2, \ldots, N\}$, be a collection of mutually disjoint horizontal slabs in K, of Lipschitz constant μ_h, and $V_i, i \in \{1, 2, \ldots, N\}$ be a collection of mutually disjoint vertical slabs in K, of Lipschitz constant μ_v.
We make the following assumptions on the diffeomorphism f and on the first order iterates of horizontal and vertical slabs:

(A) For each $i \in \{1, 2, \ldots, N\}, f(H_i) = V_i$, and $0 < \mu_h \mu_v < 1$;

(B) If μ_h-horizontal slab H intersects the μ_h-horizontal slab H_i fully, then $f^{-1}(H \cap H_j)$ is a horizontal slab intersecting H_j fully, for all j, and

$$d\left(f^{-1}(H) \cap H_j\right) \leq v_h d(H),$$

for some $0 < v_h < 1$; if a μ_v-vertical slab V intersects the μ_v-vertical slab V_i fully, then $f\left(V \cap V_j\right)$ is a vertical slab intersecting V_j fully, for all j, and

$$d\left(f(V) \cap V_j\right) \leq v_v d(V),$$

for some $0 < \nu_v < 1$ (Figure 6).

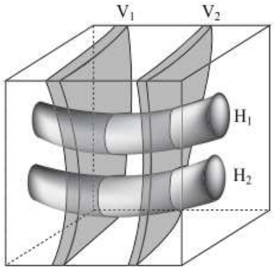

Figure 6: Horizontal slabs that are mapped into vertical slabs.

Theorem3.5: Suppose that f satisfies the conditions (A) and (B).Then there exists an invariant Cantor set Λ for f in $\bigcup_{i=1}^{N} H_i \cap \bigcup_{i=1}^{N} V_i$ on which the induced dynamics is topologically conjugate to a full shift on N symbols.

The proof of this theorem can be found in [6].

Now we provide sufficient conditions for hyperbolicity on the chaotic set given by Theorem 3.5. For each point $(x_0, y_0) \in \bigcup_{i=1}^{N} H_i$ we define a *stable cone*,

$$C^s_{(x_0,y_0)} = \{(\xi,\eta) \in \mathbb{R}^u \times \mathbb{R}^s;\ \|\eta\| \leq \mu_h \|\xi\|\}.$$

Similarly, for each $(x_0, y_0) \in \bigcup_{i=1}^{N} V_i$ we define an *unstable cone*,

$$C^u_{(x_0,y_0)} = \{(\xi,\eta) \in \mathbb{R}^u \times \mathbb{R}^s;\ \|\xi\| \leq \mu_v \|\eta\|\}.$$

We make the following assumption on the diffeomorphism f and on its first order iterates:

$(B')Df$ maps each unstable cone $C^u_{(x_0,y_0)}$ within the unstable cone $C^u_{f(x_0,y_0)}$ and Df^{-1}maps each stable cone $C^s_{(x_0,y_0)}$ within the stable cone $C^s_{f^{-1}(x_0,y_0)}$. Moreover, there exist $0 < \lambda < 1 - \mu_h\mu_v$ such that if

$$Df(x_0,y_0)(\xi_0,\eta_0) = (\xi_1,\eta_1) \text{ and } Df^{-1}(x_0,y_0)(\xi_0,\eta_0) = (\xi_{-1},\eta_{-1}),$$

Then $\|\eta_1\| \geq (1/\lambda)\|\eta_0\|$ and$\|\xi_{-1}\| \geq (1/\lambda)\|\xi_0\|$.

This condition(B') implies condition (B). From the above discussion we obtain:

Theorem3.6 [6]**:** Suppose that satisfies conditions (A) and (B′) from above. Then there exists an invariant Cantor set Λ for $\bigcup_{i=1}^{N} H_i \cap \bigcup_{i=1}^{N} V_i$ as in Theorem 3.5, such that f restricted to Λ is hyperbolic, and it is conjugate to a full shift on N symbols.

4. Hyperbolic Set for the Hénon Map:

In 1976, the French astronomer M. Hénon [7] has initiated the study of the asymptotic dynamics of a family of smooth maps

$$H_{(a,b)}: \mathbb{R}^2 \to \mathbb{R}^2, \qquad H_{(a,b)}(x,y) = (1 + y - ax^2, bx)$$

where $a > 0$ and b real parameters.

Such a map $H_{(a,b)}$ is invertible for $b \neq 0$, with its inverse given by

$$H^{-1}_{(a,b)}(x',y') = \left(\frac{y'}{b}, -1 + x' + a\left(\frac{y'}{b}\right)^2\right).$$

The Jacobian of $H_{(a,b)}$ is $-b$, so $H_{(a,b)}$ changes area by a factor of $|b|$. In particular, it is dissipative for $0 < |b| < 1$. Also $H_{(a,b)}$ reverses orientation for $b > 0$ and preserves orientation for $b < 0$. For $b = 0$ and $a < 2$, $H_{(a,b)}(x, y) = (1 - y - ax^2, 0)$ maps all of $\mathbb{R}^2$ onto the x-axis. We should notice that $H_{(a,b)}$ is topologically conjugate with $H^{-1}_{(a/b^2,\,1/b)}$. Indeed, the transformation of the plane $h(x, y) = (-y, -x)$ is a topological conjugacy between the two maps, since

$$h \circ H_{(a,b)}(x, y) = (-bx, -1 - y + ax^2)$$
$$\text{and } H^{-1}_{(a/b^2,\,1/b)} \circ h(x, y) = \left(\frac{-x}{1/b}, -1 - y + \frac{a}{b^2}\left(\frac{-x}{1/b}\right)^2\right).$$

This shows that the study of the dynamics for $|b| > 1$ can be reduced, via conjugacy, to the study of the dynamics for $|b| < 1$.

Hénon investigated numerically the dynamics of the map for the choice of parameters $a = 1.4$and $b = 0.3$. For this choice, a visualization of an attractive set (called *Hénon attractor*) as shown in Figure 7, on which the induced dynamics appears to be chaotic [8].

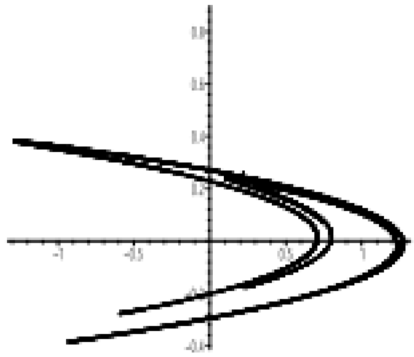

Figure 7: The Hénon attractor obtained by iterating one generic point.

In this section, we will show that for (a, b) near $(5, 0.3)$, the Hénon map has a hyperbolic invariant set. For computation reasons, we shall work with the map $F_{(a,b)}$ obtained from $H_{(a,b)}$ after a coordinate change $x = x/a$ and $y = by/a$,

$$F_{(a,b)} = (a + by - x^2, x).$$

Clearly, $F_{(a,b)}$ is topologically conjugate with $H_{(a,b)}$, because $\phi(x, y) = \left(\frac{x}{a}, \frac{by}{a}\right)$, we have

$$H_{(a,b)} \circ \emptyset = H_{(a,b)}\left(\frac{x}{a}, \frac{by}{a}\right) = \left(1 + \frac{b}{a}y - \frac{x^2}{a}, \frac{b}{a}x\right)$$
$$\text{and } \emptyset \circ F_{(a,b)} = \emptyset(a + by - x^2, x) = \left(1 + \frac{b}{a}y - \frac{x^2}{a}, \frac{b}{a}x\right),$$

that is, $H_{(a,b)} \circ \emptyset = \emptyset \circ F_{(a,b)}$.

We fix $(a, b) = (5, 0.3)$and, for simplification, we denote $F_{(5,0.3)}$ by$F$. Its inverse is

$$F^{-1}(x, y) = (y, (-a + x + y^2)/b).$$

Define the square

$$S = \{(x, y);\ |x| \leq 3, |y| \leq 3\}.$$

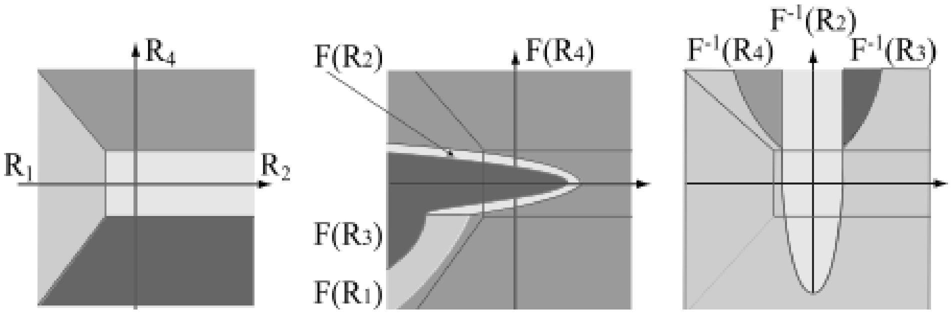

Figure 8: Decompositions of the square S into regions.

Theorem 4.1: The following hold true:

(i) The nonwandering set $\Omega(F)$ is nonempty and contained in S;

(ii) The maximal invariant set Λ of S,

$$\Lambda = \bigcap_{n\in\mathbb{Z}} F^{-n}(S),$$

is a Cantor set, on which the induced dynamics is topologically conjugate to the full shift on two symbols;

(iii) F is hyperbolic onΛ.

Proof:(i) It suffices to prove that all points outside S are wandering points and there exist some nonwandering points inside S. For reason, we divide the plane into four regions as specified below (Figure 8 (left)):

$$R_1 = \{(x,y); x \leq \min(-|y|, 3)\},$$
$$R_2 = \{(x,y); x \geq -3, |y| \leq 3\},$$
$$R_3 = \{(x,y); x \geq -|y|, y \leq -3\},$$
$$R_4 = \{(x,y); x \geq -|y|, y \geq 3\}.$$

We want to look at the images of these four regions under $F$and F^{-1}. The image $F(R_1)$ of R_1 under F is bounded by two arcs of parabola $x = a \pm by - y^2$, $-\infty < y \leq -3$, and a line segment $y = -3$, $-4.9 \leq x \leq -3.1$. The image $F(R_2)$ of R_2 under F is bounded by two arcs of parabola $x = a \pm 3b - y^2, -3 < y < \infty$ and a line segment$y = -3, -4.9 \leq x \leq -3.1$.

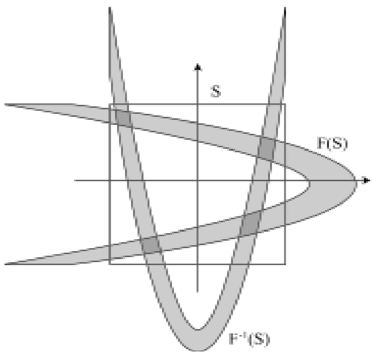

Figure 9 Horseshoe for the Hénon map

The image $F(R_3)$ of R_3 under F is bounded by two arcs of parabola $x = a + by - y^2$, $-\infty < y \leq -3$, and $x = a + 3b - y^2$, $-3 \leq y < \infty$. The image $F(R_4)$ of R_4 under F is bounded by two arcs of parabola $x = a + 3b - y^2, -3 < y < \infty$, and$x = a - by - y^2, -\infty < y \leq -3$ (Figure 8(centre)) .

The image $F^{-1}(R_1)$ of R_1 under F^{-1} is bounded by three arcs of parabola$y = (-a + x + x^2)/b$, $-\infty < x < -3$, $y = (-a - 3 + x^2)/b$, $-3 \leq x \leq 3$and $y = (-a - x + x^2)/b$, $3 \leq x < \infty$. The image $F^{-1}(R_2)$ of R_2 under F^{-1} is bounded by an arc of parabola $y = (-a - 3 + x^2)/b, -3 \leq x \leq$

3, and two semi-lines$x = -3, \frac{1}{3} \leq y < \infty$. The image $F^{-1}(R_3)$ of R_3 under F^{-1} is bounded by an arc of parabola $y = (-a - x + x^2)/b, -3 \leq x < \infty$ and a line segment $x = 3, 1/3 \leq y < \infty$. The image $F^{-1}(R_4)$ of R_4 under F^{-1} is bounded by an arc of parabola $y = (-a + x + x^2)/b, -\infty < x < -3$,and a line segment$x = -3, 1/3 \leq y < \infty$ (Figure 8 (right)).

The outcome of this construction is that $F(R_1) \subset R_1$, $F(R_2 \cup R_3) \subset R_1 \cup R_2$,$F^{-1}(R_3 \cup R_4) \subset R_4$ and$F^{-1}(R_2) \subset R_2 \cup R_3 \cup R_4$ and$F^{-1}(R_2) \cap R_2 \subset S$ (Figure 9).

Since $a + by - x^2 < x$ for all (x, y) with $x \leq -|y|$, we have that x is strictly decreasing along F-orbits in R_1. Similarly, y is strictly increasing along F^{-1} orbits in R_4. These two facts imply that all points in R_1 and R_4 are wandering, since they all escape to infinity at least one direction. From this and from the above properties, it follows that $\Omega(F) \subset R_2 \cap F^{-1}(R_2) \subset S$. This proves half ofstatement(i) of the theorem.

We prove the other half together with (ii) and (iii). For this purpose, we apply Theorem 3.6.One should notice the horseshoe shape determined by the first iterate of f on S (see Figure 9). We need to prove that the maximal invariant set within the square S is a Cantor set with hyperbolic structure.

Note that $F^{-1}(S) \cap S$ consists of two vertical slabs, bounded by arcs of parabola $y = (\pm 3 - a + x^2)/b$ or $x = \pm\sqrt{by + a \mp 3}$, and by four horizontal line segments $\sqrt{1.1} \leq |x| \leq 7.1$, $y = -3$, and $\sqrt{2.9} \leq x \leq \sqrt{8.9}$, $y = 3$. Also note that $S \cap F(S)$ consists of two horizontal slabs, bounded by arcs of parabola $x = a \pm 3b - y^2$or $y = \pm\sqrt{a \pm 3b - x}$, and four vertical line segments $x = -3$, $\sqrt{2.9} \leq |y| \leq \sqrt{8.9}$, and $x = 3$, $\sqrt{1.1} \leq |y| \leq 7.1$. We shall then keep in mind that for all points in $F^{-1}(S) \cap S$, we have that $|x| > 1$ and for all points in $S \cap f(S)$, we have that $|y| > 1$.

We want to prove that conditions (A) and $(B')$of Theorem 3.6 are satisfied. Since the horizontal curves bounding the horizontal slabs and the vertical curves bounding the vertical slabs are C^1-functions, the corresponding Lipschitz constants can be chosen anything larger than or equal to the maximal value of the slope of the graph (in absolute value), that is,

$$\mu_h = \sup_{|x| \leq 3} 1/\left(2\sqrt{a \pm 3b - x}\right) = 1/(2\sqrt{1.1}) \approx 0.476,$$

and

$$\mu_v = 1/1.7 > \sup_{|y| \leq 3} b/\left(2\sqrt{by + a \mp 3}\right) = 0.15/\sqrt{1.1} \approx 0.143.$$

Choose $0 < \lambda = 1/1.7 \approx 0.588 < 1 - \mu_h \mu_v \approx 0.72$. By construction, the mapping F takes each vertical slab onto a horizontal slab, which shows (A). Now let

$$(x_0, y_0) \in F^{-1}(S) \cap S, \ (\xi_0, \eta_0) \in C^u_{(x_0, y_0)} \text{ and} (\xi_1, \eta_1) = DF(x_0, y_0)(\xi_0, \eta_0)$$

We have $|\xi_0| \geq (1/\mu_v)|\eta_0| \geq |\eta_0|$, and

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} -2x_0 & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} -2x_0 \xi_0 - b\eta_0 \\ \xi_0 \end{pmatrix}.$$

Since $(x_0, y_0) \in F^{-1}(S) \cap S$, we have $|x_0| > 1$, so

$$|\xi_1| = |-2x_0 \xi_0 - b\eta_0| \geq |2x_0||\xi_0| - b|\eta_0| \geq (|2x_0| - b)|\xi_0| \geq (2 - 0.3)|\xi_0| = 1.7|\xi_0|.$$

We also have $|\eta_1| = |\xi_0|$. Thus $|\xi_1| \geq 1.7|\eta_1|$, which means that DF maps the unstable cone at (x_0, y_0) within the unstable cone at $F(x_0, y_0)$, and $|\eta_1| \geq (1/\lambda)|\eta_0|$. So the half of condition (B') concerning unstable cones is verified.Now let

$$(x_0, y_0) \in S \cap F(S), (\xi_0, \eta_0) \in C^s_{(x_0, y_0)} \text{ and} (\xi_{-1}, \eta_{-1}) = DF(x_0, y_0)^{-1}(\xi_0, \eta_0).$$

We have $|\eta_0| \geq (1/\mu_h)|\xi_0| \geq |\xi_0|$ and

$$\begin{pmatrix}\xi_{-1}\\ \eta_{-1}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1/b & -2y_0/b\end{pmatrix}\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix} = \begin{pmatrix}\eta_0\\ -\xi_0/b - 2y_0\eta_0/b\end{pmatrix}.$$

Since $(x_0, y_0) \in S \cap F(S)$, we have $|y_0| > 1$, so

$$|\eta_{-1}| = (1/b)(2|y_0| - 1)|\eta_0| \geq (1/b)(2-1)|\eta_0| = (1/0.3)|\eta_0|.$$

We also have $|\xi_{-1}| = |\eta_0|$. Thus $|\eta_{-1}| \geq (1/0.3)|\xi_{-1}| \geq (1/\mu_h)|\xi_{-1}|$, which means that DF^{-1} maps the stable cone at (x_0, y_0) within the stable cone at $DF^{-1}(x_0, y_0)$, and $|\xi_{-1}| \geq (1/\lambda)|\xi_0|$. So the other half of condition (B')concerning stable cones is verified.
We conclude the existence of an invariant set$\Lambda \subset F^{-1}(S) \cap S \cap F(S)$, which is a hyperbolic Cantor set, topologically conjugate to a full shift on two symbols. Due to the conjugacy, Λ contains fixed points, which are nonwandering points. This proves that the nonwandering set $\Omega(F)$ found in (i) is nonempty.

5. Conclusion:

We detect the hyperbolicity of some geometrical objects, referred as “slab” in this method. The consistent behavior of these slabs under iteration resembles the idea of covering relations from the one-dimensional case. It allows one to detect points with prescribed trajectories, periodic points, and invariant sets with symbolic dynamics. There are additional technical conditions to ensure that the detected invariant set is hyperbolic.

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