Submitted to 2000 IEEE International Conference on Robotics and Automation (ICRA2000)

Backlash Compensation in Discrete Time Nonlinear Systems Using Dynamic Inversion by Neural Networks

J. Campos, F. L. Lewis, and R. Selmic.
Automation and Robotics Research Institute
The University of Texas at Arlington
7300 Jack Newell Blvd. S,
Ft. Worth, Texas 76118-7115
jcampos@arrirs04.uta.edu

Keywords: neural networks, backlash compensation, discrete time neural network learning, dynamic inversion by neural networks.

Abstract

A dynamics inversion compensation scheme is designed for control of nonlinear discrete-time systems with input backlash. The compensator uses backstepping technique with neural networks (NN) for inverting the backlash nonlinearity in the feedforward path. The technique provides a general procedure for using NN to determine the dynamics preinverse of an invertible discrete time dynamical system. A discrete-time tuning algorithm is given for the NN weights so that the backlash compensation scheme becomes adaptive, guaranteeing bounded tracking and backlash errors, and also bounded parameter estimates. A rigorous proof of stability and performance is given and a simulation example verifies performance. Unlike standard discrete-time adaptive control techniques, no certainty equivalence (CE) assumption is needed.

1 Introduction

Robotic systems often have nonlinearities in the actuator such as deadzone, backlash, saturation, etc. This includes *xy*-positioning tables, robot manipulators, overhead crane mechanism, and more. The deadzone characteristic is a non-smooth nonlinearity, which models diverse physical imperfections: biases to prevent inflow-outflow or heating-cooling overlaps, aggregate effects of dry friction, etc. The difference between toothspace and tooth width in mechanical system is known as backlash and it is necessary to allow two gears mesh without jamming. Any amount of backlash greater that the minimum amount necessary to ensure satisfactory meshing of gears can result in instability in dynamics situations and position errors in gear trains. In fact, there are many applications such as instrument differential gear trains and servomechanisms that require the complete elimination of backlash in order to function properly. Saturation and friction are among other nonlinearities frequently present in robotic actuator dynamics. Our main concern is backlash. In most applications the backlash parameters are unknown, which represent a challenge for the control design engineer. Proportional-derivative (PD) controllers have been observed to result in limit cycles if the actuators have deadzone or backlash. To overcome the PD controller limitations, several techniques have been applied to compensate for the actuator nonlinearities. These techniques include adaptive control, fuzzy logic and neural networks. Recently, in seminal work rigorously derived adaptive schemes have been given for actuator nonlinearity compensation [20]. Backlash compensation is addressed in [21]. For dynamic system in the Lagrangian form, deadzone compensation using neural networks is given in [18].

Many systems with actuator nonlinearities such as deadzone and backlash are modeled in discrete time. An example of deadzone in biomedical control is the functional neuromuscular stimulation for restoring motor function by directly activating paralyzed muscles [2]. Moreover, for implementation in digital controllers, a discrete-time actuator nonlinearity compensator is needed. For example, to address discrete-time deadzone compensation, an adaptive control approach has been proposed in the seminal work [20][22]. Also a fuzzy logic (FL) deadzone compensation discrete time scheme is proposed in [3].

The use of neural networks (NN) has accelerated in recent years in many areas, including feedback control applications. Particularly important in NN control are the *universal function approximation capabilities* of NN systems [1][5][6][8][12][15][16]. NN systems offer significant advantages over adaptive control, including no requirement for linearity in the parameters assumptions and no need to compute a regression matrix for each specific system. Dynamics inversion using NN is presented in [10][14], where NN is used for cancellation of inversion error. A continuous time dynamic inversion approach using NN for backlash compensation is presented in [17]. A compensated inverse dynamics approach using adaptive and robust control technique is presented in [19].

In this paper we show how to design a motion tracking controller for discrete-time multi-input Lagrangian mechanical systems with unknown input backlash. The general case of nonsymmetric backlash is treated. A rigorous design procedure is given that results in a PD tracking loop with an adaptive NN in the feedforward loop for dynamic inversion of the backlash nonlinearity. The NN feedforward compensator is adapted in such a way as to estimate on-line the backlash inverse. Unlike standard discrete-time adaptive control techniques, no centainty equivalence (CE) assumption is needed since the tracking error and the estimation error, are weighted in the same Lyapunov function. The approach is similar to that in [7], but additional complexities arise due to the

*Research supported by NFS Grant ECS-9521673, Texas ATP Grant 003656-027, and ARO Grant 39657-MA.

fact that the backlash compensator is in the feedforward loop.

2 Dynamics of an mn-th order MIMO system

Consider an mnth-order multi-input and multi-output discrete-time system given by

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ &\vdots \\ x_{n-1}(k+1) &= x_n(k) \\ x_n(k+1) &= f(x(k)) + t(k) + d(k), \end{aligned} \tag{1}$$

where $x(k) = [x_1(k), x_2(k)...., x_n(k)]^T$ with $x_i(k) \in \Re^n$; $i = 1,2,...,n$, $u(k) \in R^m$, and $d(k) \in R^m$ denotes a disturbance vector acting on the system at the instant k with $\|d(k)\| \le d_M$ a known constant. The actuator output $t(k)$ is related to the control input $u(k)$ through the backlash nonlinearity. $t(k) = Backlash(u(k))$ as discussed in the next section. Given a desired trajectory $x_{nd}(k)$ and its delayed values, define the tracking error as

$$e_n(k) = x_n(k) - x_{nd}(k) . \tag{2}$$

It is typical in robotics to define a so-called the filtered tracking error, as $r(k) \in R^m$, and given by

$$r(k) = e_n(k) + l_1 e_{n-1}(k) + \cdots + l_{n-1} e_1(k) , \tag{3}$$

where $e_{n-1}(k), \ldots, e_1(k)$ are the delayed values of the error $e_n(k)$, and $l_1, \ldots, l_{n-1}$ are constant matrices selected so that $\left| z^{n-1} + l_1 z^{n-2} + \cdots + l_{n-1} \right|$ is stable or Hurwitz (i.e. $e_n(k) \to 0$ exponentially as $r(k) \to 0$). Equation (3) can be further expressed as

$$r(k+1) = e_n(k+1) + l_1 e_{n-1}(k+1) + \cdots + l_{n-1} e_1(k+1). \tag{4}$$

Using Eq. (1) in Eq. (4), the dynamics of the mn-th order MIMO system can be written in terms of the tracking error as

$$\begin{aligned} r(k+1) = {} & f(x(k)) - x_{nd}(k+1) + l_1 e_{n-1}(k) + \cdots + l_{n-1} e_1(k) + \\ & + t(k) + d(k). \end{aligned} \tag{5}$$

3 Backlash Nonlinearity and Backlash Inverse

The backlash nonlinearity is shown in Fig. 1, and the mathematical model for the discrete time case is given by [28, 35]

$$t(k+1) = B(t(k), u(k), u(k+1)) = \\ = \begin{cases} m \cdot u(k), & \begin{array}{l} if\ u(k+1) > 0\ and\ u(k) = m \cdot t(k) - m \cdot d_+ \\ if\ u(k+1) < 0\ and\ u(k) = m \cdot t(k) - m \cdot d_- \end{array} \\ 0, & otherwise \end{cases} . \tag{6}$$

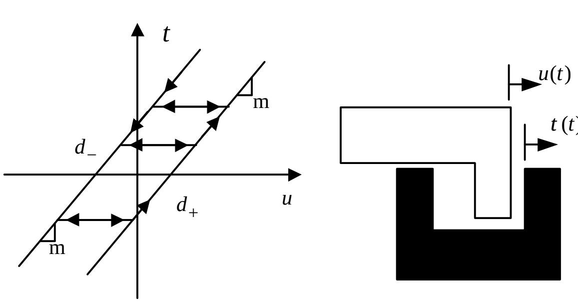


Fig. 1: (a) Backlash Model.
(b) Backlash in Mechanical connections.

It can be seen that backlash is a first-order velocity driven dynamic system, with inputs $u(k)$ and $u(k+1)$, and state $t(k)$. It contains its own dynamics, therefore its compensation requires the design of a dynamic compensator [17].

Whenever the motion $u(k)$ changes its direction, the motion $t(k)$ is delayed from motion of $u(k)$. The objective of the backlash compensator is to make this delay as small as possible, i.e. to make the throughput from $u(k)$ to $t(k)$ be the unity. The backlash precompensator needs to generate the inverse of the backlash nonlinearity. The backlash inverse function is shown in Fig. 2.

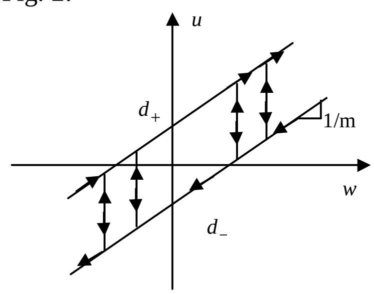


Fig. 2: Backlash Inverse

The dynamics of the NN backlash compensator is given by

$$u(k+1) = B_{inv}(u(k), w(k), w(k+1)) , \tag{7}$$

The backlash inverse characteristic shown in the Fig. 2 can be decomposed into two functions [17]: a direct feedforward term plus an additional *modified backlash inverse* term as shown in Fig. 3. This decomposition allows design of a compensator that has a better structure when a NN is used in the feedforward path.

4 Discrete Time NN Backlash Compensator

The discrete time NN backlash compensator is designed using the backstepping technique [9]. In this section we will show how to tune the NN weights on-line so that the tracking error is guaranteed small and all internal states are bounded. It is assumed that the actuator output $t(k)$ is measurable.

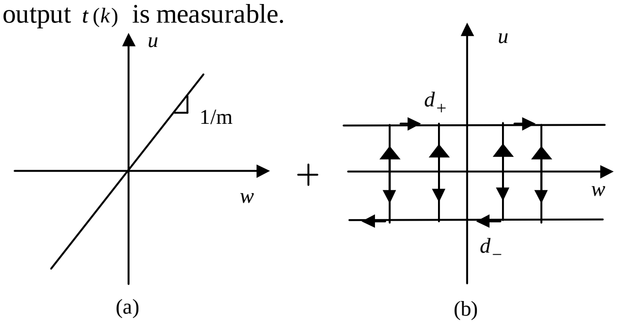


Fig. 3: Backlash inverse decomposition.

4.1 Dynamics of Nonlinear System with Backlash

Equation (1) is in the companion form and represents a large class of multi-input multi-output (MIMO) nonlinear systems. The overall system dynamics consist of (1) and the backlash dynamics (6).

The following assumptions are needed and they are true in every practical situation and are standard in the existing literature.

Assumption 1 (Bounded disturbance): The unknown disturbance satisfies $\|d(k)\| \le d_M$, with d_M a known positive constant.

Assumption 2 (Bounded estimation error): The nonlinear function is assumed to be unknown, but a fixed estimate $\hat{f}(x(k))$ is assumed known such that the functional estimation error, $\tilde{f}(x(k)) = f(x(k)) - \hat{f}(x(k))$, satisfies $\|\tilde{f}(x(k))\| \le f_M(x(k))$, for some known bounding function $f_M(x(k))$.

This assumption is not unreasonable [11][12], as in practical systems the bound $f_M(x(k))$ can be computed knowing the upper bound on payload masses, frictional effects, and so on.

Assumption 3 (Bounded desired trajectories): The desired trajectory is bounded in the sense, for instance that

$$\left\| \begin{matrix} x_{1d}(k) \\ x_{2d}(k) \\ \vdots \\ x_{nd}(k) \end{matrix} \right\| \le X_d .$$

4.2 Backstepping Controller

A robust compensation scheme for unknown terms in $f(x(k))$ is provided by selecting the tracking controller

$$\begin{aligned} t_{des}(k) = K_v \cdot r(k) - \hat{f}(x(k)) + x_{nd}(k+1) - l_1 \cdot e_{n-1}(k) + \\ - l_2 \cdot e_{n-2}(k) - \cdots l_{n-1} \cdot e_1(k), \end{aligned} \quad (8)$$

with $\hat{f}(x(k))$ an estimate for the nonlinear terms $f(x(k))$. The feedback gain matrix $K_v > 0$ is often selected diagonal. The problem of finding $\hat{f}(x(k))$ is not the main concern of this paper, so it is considered to be available. This function $f(x(k))$ can be estimated using adaptive control techniques [10] or neural network controllers [13].

Using (8) as a control input, the system dynamics in (5) can be rewritten as

$$r(k+1) = K_v \cdot r(k) + \tilde{f}(x(k)) + d(k) . \quad (9)$$

The next theorem is the first step in the backstepping design; and it shows that the desired control law (8) will keep the filtered tracking error small.

Theorem 1 (Control law for outer tracking loop).

Considered the system given by equation (1). Provided that assumptions 1 and 2 hold, let the control action by provided by (8) with $0 < K_v < I$ being a design parameter.

Then the filtered tracking error $r(k)$ is UUB. Moreover, the filtered tracking error $r(k)$ can be made arbitrarily small by increasing the fixed control gains K_v.

Proof.

Let us consider the following Lyapunov function candidate

$$L_1(k) = r(k)^T r(k) . \quad (10)$$

The first difference is

$$\begin{aligned} \Delta L_1(k) &= r(k+1)^T r(k+1) - r(k)^T r(k) = \\ &= \left(K_v r(k) + \tilde{f}(x(k)) + d(k)\right)^T \left(K_v r(k) + \tilde{f}(x(k)) + d(k)\right) + \\ &- r(k)^T r(k). \end{aligned} \quad (11)$$

$\Delta L_1(k)$ is negative if the following is satisfied

$$\left\|K_v r(k) + \tilde{f}(x(k)) + d(k)\right\| \le K_{v\max}\|r(k)\| + f_M + d_M < \|r(k)\|$$

$$\Rightarrow (1 - K_{v\max})\|r(k)\| > f_M + d_M ,$$

which is true as long as

$$\|r(k)\| > \frac{f_M + d_M}{1 - K_{v\max}} . \quad (12)$$

Therefore, $\Delta L_1(k)$ is negative outside a compact set. According to standard Lyapunov theory extension [11], this demonstrates the UUB of $r(k)$.

4.3 NN Backlash Compensation using Dynamic Inversion

Theorem 1 gives the control law that guarantees stability in term of the filtered tracking error assuming that no nonlinearity besides the system nonlinear function plus some bounded external disturbances are present. In the presence of unknown backlash nonlinearity, the desired and actual value of the control signal $t(k)$ will be different. A dynamics inversion technique by neural networks is used for compensation of the inversion error [10][14][17]

The actuator output given by (8) is the desired signal. The complete error system dynamics can be found defining the error

$$\tilde{t}(k) = t_{des}(k) - t(k) . \quad (13)$$

Using the desired control input (8), under the presence of unknown backlash the system dynamics (5) can be rewritten as

$$r(k+1) = K_v \cdot r(k) + \tilde{f}(x(k)) + d(k) - \tilde{t}(k) . \quad (14)$$

Evaluating (13) at the following time interval

$$\begin{aligned} \tilde{t}(k+1) &= t_{des}(k+1) - t(k+1) = \\ &= t_{des}(k+1) - B(t(k), u(k), u(k+1)), \end{aligned} \quad (15)$$

which together with (14) represents the complete system error dynamics.

The dynamics of the backlash nonlinearity can be written as [17]

$$t(k+1) = j(k) , \quad (16)$$

$$j(k) = B(t(k), u(k), u(k+1)) , \quad (17)$$

where $j(k)$ is a pseudo-control input [10][14][17]. In the case of known backlash, the ideal backlash inverse is given by

$$u(k+1) = B^{-1}(u(k), t(k), j(k)) . \quad (18)$$

Since the backlash and therefore its inverse are not known, one can only approximate the backlash inverse as

$$\hat{u}(k+1) = \hat{B}^{-1}(\hat{u}(k), t(k), \hat{j}(k)) . \quad (19)$$

The backlash dynamics can now be written as

$$\begin{aligned} t(k+1) &= B(t(k), \hat{u}(k), \hat{u}(k+1)) \\ &= \hat{B}(t(k), \hat{u}(k), \hat{u}(k+1)) + \tilde{B}(t(k), \hat{u}(k), \hat{u}(k+1)) , \\ &= \hat{j}(k) + \tilde{B}(t(k), \hat{u}(k), \hat{u}(k+1)) \end{aligned} \quad (20)$$

where $\hat{j}(k) = \hat{B}(t(k), \hat{u}(k), \hat{u}(k+1))$ and therefore its inverse $\hat{u}(k+1) = \hat{B}^{-1}(t(k), \hat{u}(k), \hat{j}(k))$. The unknown function $\tilde{B}(t(k), \hat{u}(k), \hat{u}(k+1))$, which represents the backlash inversion error, will be approximated using a neural network.

Based on NN approximation property, the backlash inversion can be represented as

$$\tilde{B}(t(k), \hat{u}(k), \hat{u}(k+1)) = W(k)^T s\left(V^T x_{nn}(k)\right) + e(k)\,, \tag{21}$$

where the NN input vector is chosen to be $x_{nn}(k) = \left[1 \quad r(k)^T \quad x_d(k)^T \quad \tilde{t}(k)^T \quad t(k)^T\right]^T$, and $e(k)$ represents the NN approximation error. It can be seen that the first layer of weights is not time dependant since it is selected randomly at initial time to provide a basis [1] and then it is kept constant through the tuning process.

Define the weights estimation error as

$$\tilde{W}(k) = W(k) - \hat{W}(k)\,, \tag{22}$$

where $\hat{W}(k)$ is the estimate of the ideal NN weights $W(k)$.

In order to design a stable closed-loop system with backlash compensation, one selects a nominal backlash inverse $\hat{u}(k+1) = \hat{f}(k)$ and pseudo-control input as

$$\hat{f}(k) = -K_b \tilde{t}(k) + t_{des}(k+1) + \hat{W}(k)^T s\left(V^T x_{nn}(k)\right), \tag{23}$$

where $K_b > 0$ is a design parameter.

Using the proposed controller shown in Fig. 4, the error dynamics can be written as

$$\begin{aligned} &\tilde{t}(k+1) = t_{des}(k+1) - \hat{f}(k) + \tilde{B}(t(k), \hat{u}(k), \hat{u}(k+1)) = \\ &K_b \tilde{t}(k) - \hat{W}(k)^T s\left(V^T x_{nn}(k)\right) + W(k)^T s\left(V^T x_{nn}(k)\right) + e(k). \end{aligned} \tag{24}$$

Using (22),

$$\tilde{t}(k+1) = K_b \tilde{t}(k) + \tilde{W}(k)^T s\left(V^T x_{nn}(k)\right) + e(k)\,. \tag{25}$$

The next theorem is our main result and it shows how to tune the neural network weights so the tracking error $r(k)$ and backlash estimation error $\tilde{t}(k)$ achieve small values while the NN weights estimation errors $\tilde{W}(k)$ are bounded.

Theorem 2 (Control law for backstepping loop).

Consider the system given by (1). Provided that assumptions 1, 2, and 3 hold, let the control action $\hat{f}(k)$ by provided by (23) with $K_b > 0$ being a design parameter.

Let $u(k+1) = j(k)$, and the estimated NN weights be provided by the NN tuning law

$$\begin{aligned} &\hat{W}(k+1) = \hat{W}(k) + a\ s(k) r(k+1)^T + a\ s(k) \tilde{t}(k+1)^T + \\ &- \Gamma \left\| I - a\ s(k) s(k)^T \right\| \hat{W}(k), \end{aligned} \tag{26}$$

where $a > 0$ is a constant learning rate parameter or adaptation gain, $\Gamma > 0$ is a design parameter, and for simplicity purposes $s(V^T x_{nn}(k))$ is represented as $s(k)$. Then, the filtered tracking error $r(k)$, the backlash estimation error $\tilde{t}(k)$, and the NN weight estimation error $\tilde{W}(k)$ are UUB, provided the following conditions hold:

1) $0 < a\ s(k)^T s(k) < 1/2$, (27)

2) $0 < \Gamma < 1$, (28)

3) $0 < K_v < I$ and $K_{v\max} < \dfrac{1}{\sqrt{h+2}}$, (29)

where

$$b = K_v^{-1}(2I - K_v) + \left(1 - a\ s(k)^T s(k)\right) K_v^{-T} K_v^{-1} > 0\,. \tag{30}$$

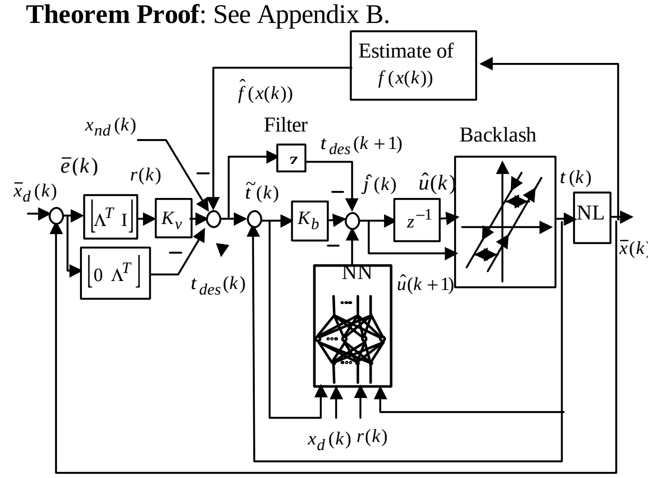
$$\begin{aligned} r &= \left(1 - a\ s(k)^T s(k)\right) I + \\ &- b^{-1} \left(a\ s(k)^T s(k) + \Gamma \left\| I - a\ s(k) s(k)^T \right\| \right)^2 > 0. \end{aligned} \tag{31}$$

$$\begin{aligned} h &= \left(1 + a\ s(k)^T s(k)\right) I + \\ &+ r^{-1} \left(a\ s(k)^T s(k) + \Gamma \left\| I - a\ s(k) s(k)^T \right\| \right)^2 > 0. \end{aligned} \tag{32}$$

Moreover, the filtered tracking error $r(k)$ and the backlash estimation error can be made arbitrarily small by increasing the fixed control gains K_v and K_b, respectively.

Note that condition (30) is true because of (27) and (29). Note also that (32) is satisfied because of conditions (27) and (31). Proof for condition (31) is given in Appendix A.

Theorem Proof: See Appendix B.



Notes: $\Lambda = [\, l_{n-1} \quad l_{n-2} \quad \cdots \quad l_1]$, $\bar{x}(k) = [x_1(k) \quad x_2(k \quad \cdots \quad x_n(k]^T$

Fig. 4: Discrete Time NN Backlash Compensator.

Remarks:

It is important to note that in this theorem there is no certainty equivalence (CE) assumption, in contrast to standard work in discrete-time adaptive control. In the latter, a parameter identifier is first selected and the parameter estimation errors are assumed small. In the tracking proof, it is assumed that the parameter estimates are exact (the CE assumption), and a Lyapunov function is selected that weights only the tracking error to demonstrate close-loop stability and tracking performance. This approach is used for instance in [22]. By contrast, in our proof, the Lyapunov function in the Appendix B is of the form

$$\begin{aligned} J(k) &= \left[r(k) + \tilde{t}(k)\right]^T \cdot \left[r(k) + \tilde{t}(k)\right] + r(k)^T r(k) + \\ &+ \frac{1}{a} tr\left\{\tilde{W}(k)^T \cdot \tilde{W}(k)\right\} > 0, \end{aligned} \tag{B.4}$$

which weights the tracking error $r(k)$, backlash estimation error $\tilde{t}(k)$ and the NN weight estimation error $\tilde{W}(k)$. This requires an exceedingly complex proof, but obviates the need for any sort of CE assumption. It also allows the parameter-tuning algorithm to be derived during the proof process, not selected a priori in an ad hoc manner. This is akin to the proof of [7], but additional complexities arise due to the fact that the backlash compensator NN system is in the feedforward loop.

The third term in (26) is a discrete-time version of Narendra's e-mod, which is required to provide robustness due to the coupling in the proof between tracking error, backlash error terms and weight estimation error terms in the Lyapunov function. This is called 'forgetting terms' in NN weight-tuning algorithms. These are required in that context to prevent parameter overtraining.

5 Simulation Results

In this section, the discrete-time NN backlash compensator is simulated on a digital computer. It is found to be very efficient at canceling the deleterious effects of actuator backlash.

5.1 Simulation

We simulate the response for the known plant with input backlash, both with and without the NN compensator. Consider the following nonlinear plant

$$x_1(k+1) = x_2(k) \ ,$$

$$x_2(k+1) = -\frac{3}{16}\left[\frac{x_1(k)}{1+x_2^2(k)}\right] + x_2(k) + u(k) \ .$$

The deadband widths for the backlash nonlinearity were selected as $d_+ = d_- = 0.2$ and the slope as $m = 0.5$.

5.1.1 Trajectory Tracking

In this subsection we simulate the trajectory tracking performance of the system for sinusoidal reference signals. The reference signal used was selected to be

$$x_d(k) = \sin(w \cdot t_k + f) \ , \qquad w = 0.5, f = p/2 \ .$$

The sampling period was selected as $T = 0.001 s$.

Figure 5 shows the system response without backlash using a standard PD controller. The PD controller does a good job on the tracking which is achieved at about 2 seconds. Figure 6 shows the system response with input backlash. The system backlash destroys the tracking and the PD controller by itself is not capable of compensating for that. Figure 7 shows the same situation but using the proposed discrete-time NN backlash compensator. The backlash compensator takes care of the system backlash and the tracking is achieved in less than 0.5 seconds.

Remarks:

It is important to mention that one of the drawbacks of this technique is that the future value of the desired torque $t_{des}(k)$ is needed in (24) for the calculations. This implies that the future value of the unknown nonlinear function estimate $\hat{f}(x(k))$ is needed. Since the main concern of this approach is to compensate for backlash, the nonlinear function estimate $\hat{f}(x(k))$ is assumed to be available by any other methodology as shown in the controller diagram given by Fig. 4. The estimate of future values for

Fig. 5: PD controller without backlash .

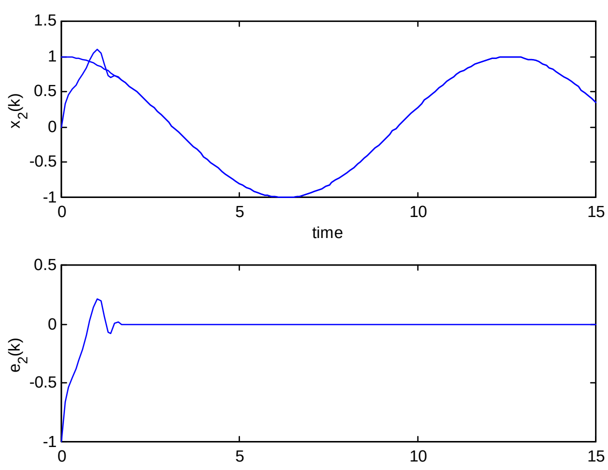


Fig. 6: PD controller with backlash .

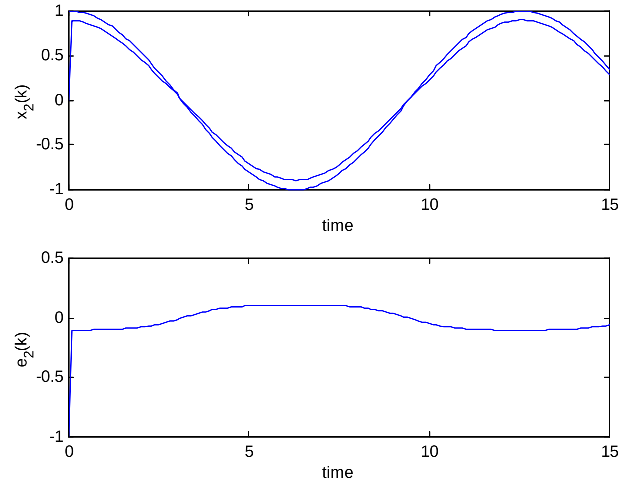
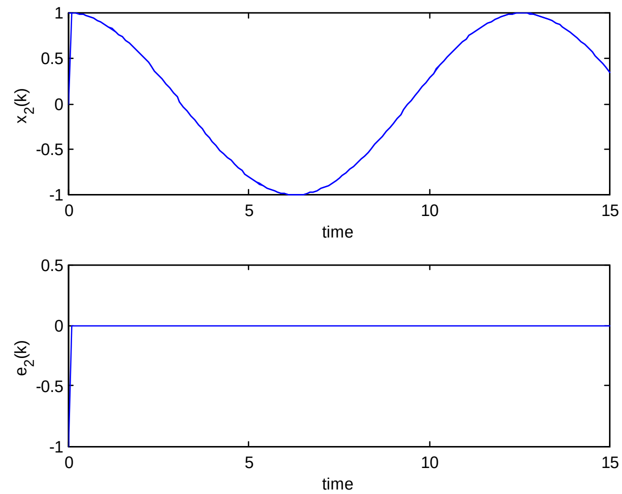


Fig. 7: PD controller with backlash compensation .



6 Conclusion

A discrete-time dynamic inversion compensation has been proposed for backlash compensation in nonlinear systems. The compensator uses the backstepping technique with neural networks (NN) for inverting the backlash nonlinearity in the feedforward path. It was shown how to tune the NN weights in discrete-time so that the unknown backlash parameters are learned on-line, resulting in a discrete-time adaptive backlash compensator. Using discrete-time nonlinear stability techniques, the tuning algorithm was rigorously shown to guarantee small tracking errors as well as bounded parameter estimates. Since the tracking error, backlash error and the parameter estimation error are weighted in the same Lyapunov function, no certainty equivalence assumption is needed.

Appendix A:

Note: For simplicity purposes, from now on we will omit the k sub-index. So, every variable is suppose to have a k sub-index unless specified otherwise. This statement is valid only for the proofs shown in the appendices.

Proof of Condition (31).

Because of condition (27), we have that $\left(1-a\ s^T s\right)I > \frac{1}{2}I$. Also using (27), (28) we have that

$$\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)^2 < \frac{1}{4}I$$

Using (29) we have that $b > I$ (i.e. $b^{-1} < I$). Then we can conclude that $b^{-1}\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)^2 < \frac{1}{4}I$. Finally, using this last result we can show that

$$r = \left(1-a\ s^T s\right)I - b^{-1}\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)^2 > \frac{1}{4}I > 0.$$

Appendix B:

Proof of Theorem 2.

For simplicity purposes let us rewrite the system dynamics as

$$r_{k+1} = K_v \cdot r + D - \tilde{t}. \tag{B.1}$$

where $D = \tilde{f} + d$. And let us rewrite the backlash dynamics as

$$\tilde{t}_{k+1} = K_b \tilde{t} + \tilde{W}^T s\left(V^T x_{nn}\right) + e. \tag{B.2}$$

Using the following Lyapunov function candidate

$$L = \begin{bmatrix} r^T & \tilde{t}^T \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ \tilde{t} \end{bmatrix} + \frac{1}{a} tr\left(\tilde{W}^T \tilde{W}\right) > 0. \tag{B.3}$$

This can be rewritten as

$$L = 2r^T r + 2r^T \tilde{t} + \tilde{t}^T \tilde{t} + \frac{1}{a} tr\left(\tilde{W}^T \tilde{W}\right) = L_1 + L_2 + L_3 + L_4. \tag{B.4}$$

Taking the first difference

$$\Delta L_1 = 2r_{k+1}{}^T r_{k+1} - 2r^T r =$$
$$= -2r^T[I - K_v^T K_v]r + 4r^T K_v^T D - 4r^T K_v^T \tilde{t} + 2D^T D - 4D^T \tilde{t} + 2\tilde{t}^T \tilde{t}.$$
$$\Delta L_2 = 2r_{k+1}{}^T \tilde{t}_{k+1} - 2r^T \tilde{t} = 2r^T K_v^T K_b \tilde{t} + 2r^T K_v^T \tilde{W}^T s + 2r^T K_v^T e +$$
$$+ 2D^T K_b \tilde{t} + 2D^T \tilde{W}^T s + 2D^T e - 2\tilde{t}^T K_b^T K_b \tilde{t} - 2\tilde{t}^T \tilde{W}^T s - 2\tilde{t}^T e +$$
$$- 2r^T \tilde{t}.$$
$$\Delta L_3 = \tilde{t}_{k+1}{}^T \tilde{t}_{k+1} - \tilde{t}^T \tilde{t} = \tilde{t}^T K_b^T K_b \tilde{t} + 2\tilde{t}^T K_b^T \tilde{W}^T s + 2\tilde{t}^T K_b^T e +$$
$$+ s^T \tilde{W} \tilde{W}^T s + 2e^T \tilde{W}^T s + e^T e - \tilde{t}^T \tilde{t}.$$
$$\Delta L_4 = \frac{1}{a} tr\left(\tilde{W}_{k+1}^T \tilde{W}_{k+1} - \tilde{W}^T \tilde{W}\right) =$$
$$= \frac{1}{a} tr\left(\hat{W}_{k+1}{}^T \hat{W}_{k+1} + W^T W - 2W^T \hat{W}_{k+1} - \tilde{W}^T \tilde{W}\right).$$

Pick $K_b = (I + K_v^{-1})^T = I + K_a$ and define the term $b = 2K_a + (1 - a\ s^T s)K_a^T K_a - I > 0$ (condition (30)) which is true as long as $K_v^{-1} < I$ (condition (29)). It can be seen that $b > I$ and b is a diagonal matrix since K_a is diagonal.

Combining all terms and select the tuning law given by (B.5) and using equations (B.1) and (B.2)

$$\hat{W}_{k+1} = \hat{W} + a\ s \cdot r_{k+1}^T + a\ s \cdot \tilde{t}_{k+1}^T - \Gamma\left\|I - a\ s\ s^T\right\|\hat{W}. \tag{B.5}$$

$$\Delta L = -r^T[2I - (2 + a\ s^T s)K_v^T K_v]r + 2(2 + a\ s^T s)r^T K_v^T D +$$
$$+ (2 + a\ s^T s)D^T D + 2(1 + a\ s^T s)r^T K_v^T e + (1 + a\ s^T s)e^T e +$$
$$+ 2(1 + a\ s^T s)e^T D - (I + b)\tilde{t}^T t - 2r^T K_v^T \tilde{t} + 2a\ s^T s r^T \tilde{t} +$$
$$+ 2(1 + a\ s^T s)D^T K_a \tilde{t} - 2D^T \tilde{t} + 2a \tilde{t}^T K_a^T s^T s \tilde{W}^T s +$$
$$+ 2(1 + a\ s^T s)\tilde{t}^T K_a - 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|\tilde{t}^T K_a^T W^T s +$$
$$+ 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|\tilde{t}^T K_a^T \tilde{W}^T s + 2a\ s^T s r^T K_v^T \tilde{W}^T s +$$
$$- (1 - a\ s^T s)s^T \tilde{W} \tilde{W}^T s + 2a\ s^T s\ e^T \tilde{W}^T s + 2a\ s^T s r^T K_v^T \tilde{W}^T s +$$
$$- (1 - a\ s^T s)s^T \tilde{W} \tilde{W}^T s + 2a\ s^T s\ e^T \tilde{W}^T s + 2a\ s^T \tilde{W} s^T s D +$$
$$+ 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|D^T \tilde{W}^T s + 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|r^T K_v^T \tilde{W}^T s +$$
$$- 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|r^T K_v^T W^T s - 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|D^T W^T s +$$
$$+ 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|e^T \tilde{W}^T s - 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|e^T W^T s +$$
$$- 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|s^T \tilde{W} \hat{W}^T s + \frac{1}{a} tr\Big\{- 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|\hat{W}^T \hat{W} +$$
$$+ 2W^T \Gamma\left\|I - a \cdot s \cdot s^T\right\|\hat{W} + \Gamma^2\left\|I - a \cdot s \cdot s^T\right\|^2 \hat{W}^T \hat{W}\Big\}$$

Completing squares for $K_a \tilde{t}$, $\tilde{t}$ and $\tilde{W}^T s$. Define $r = \left(1 - a\ s^T s\right)I - b^{-1}\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)^2 > 0$ (condition 31) and then simplifying

$$\Delta L = -r^T[2I - (3 + a\ s^T s)K_v^T K_v - \left(a\ s^T s\right)^2 + 2a\ s^T s K_v]r +$$
$$+ 2(1 + a\ s^T s)r^T K_v^T (D + e) + 4r^T K_v^T D + 2D^T D +$$
$$- \Big\{K_a \tilde{t} - b^{-1}\Big[(1 + a\ s^T s)(D + e) +$$
$$- \left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)\tilde{W}^T s + \Gamma\left\|I - a\ s\ s^T\right\|W^T s\Big]\Big\}^T \cdot b \cdot$$
$$\Big\{K_a \tilde{t} - b^{-1}\Big[(1 + a\ s^T s)(D + e) +$$
$$- \left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)\tilde{W}^T s + \Gamma\left\|I - a\ s\ s^T\right\|W^T s\Big]\Big\}^T +$$
$$- \left[\tilde{t} - a\ s^T s r + D + K_v r\right]^T \left[\tilde{t} - a\ s^T s r + D + K_v r\right] - 2a\ s^T s r^T D +$$
$$+ (1 + a\ s^T s)\left(1 + \frac{1 + a\ s^T s}{b}\right)(D + e)^T (D + e) +$$
$$- \Big\{\tilde{W}^T s - r^{-1}\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right) \cdot$$
$$\Big[K_v r + \left(1 + b^{-1}\left(1 + a\ s^T s\right)\right)(D + e) - b^{-1}\Gamma\left\|I - a\ s\ s^T\right\|W^T s\Big]\Big\}^T r$$
$$\Big\{\tilde{W}^T s - r^{-1}\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right) \cdot$$
$$\Big[K_v r + \left(1 + b^{-1}\left(1 + a\ s^T s\right)\right)(D + e) - b^{-1}\Gamma\left\|I - a\ s\ s^T\right\|W^T s\Big]\Big\} +$$
$$+ r^{-1}\left(a\ s^T s + \Gamma\left\|I - a\ s\ s^T\right\|\right)^2 \cdot$$
$$\Big[K_v r + \left(1 + b^{-1}\left(1 + a\ s^T s\right)\right)(D + e) - b^{-1}\Gamma\left\|I - a\ s\ s^T\right\|W^T s\Big]^T \cdot$$
$$\Big[K_v r + \left(1 + b^{-1}\left(1 + a\ s^T s\right)\right)(D + e) - b^{-1}\Gamma\left\|I - a\ s\ s^T\right\|W^T s\Big] +$$
$$+ b^{-1}\Gamma^2\left\|I - a\ s\ s^T\right\|^2 s^T W W^T s +$$
$$- 2b^{-1}(1 + a\ s^T s)\Gamma\left\|I - a\ s\ s^T\right\|(D + e)^T W^T s +$$
$$- 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|r^T K_v^T W^T s - 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|(D + e)^T W^T s +$$
$$- 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|s^T \tilde{W} \hat{W}^T s + \frac{1}{a} tr\Big\{- 2\Gamma\left\|I - a \cdot s \cdot s^T\right\|\hat{W}^T \hat{W} +$$
$$+ 2W^T \Gamma\left\|I - a \cdot s \cdot s^T\right\|\hat{W} + \Gamma^2\left\|I - a \cdot s \cdot s^T\right\|^2 \hat{W}^T \hat{W}\Big\}.$$

Putting the term $-2\Gamma\left\|I - a \cdot s^T s\right\|s^T \tilde{W} \hat{W}^T s$ back on the trace term and bounding the trace term.

$$\frac{1}{a} tr\Big\{- 2a\Gamma\left\|I - a\ s\ s^T\right\|\tilde{W}^T s\ s^T \hat{W} + 2\Gamma\left\|I - a\ s\ s^T\right\|\tilde{W}^T \hat{W} +$$
$$+ \left\|I - a\ s\ s^T\right\|^2 \hat{W}^T \Gamma^T \Gamma \hat{W}\Big\} < -\frac{\left\|I - a\ s\ s^T\right\|^2}{a}\Big\{\Gamma(2 - \Gamma)\left\|\tilde{W}\right\|^2 - \Gamma^2 W_M^2\Big\}$$

Define

$$h = \left(1+a\ s^T s\right)I + r^{-1}\left(a\ s^T s + \Gamma\left\| I - a\ s\ s^T \right\|\right)^2 > 0 \quad \text{(cond. (32))}$$

and

$$g = h + b^{-1} r^{-1}\left(1+a\ s^T s\right)\left(a\ s^T s + \Gamma\left\| I - a\ s\ s^T \right\|\right)^2 > 0$$

$$r_1 = 2 - \left(a\ s^T s\right)^2 + 2a\ s^T s K_{V\min} - (h+2)K_{v\max}^2 > 0$$

which are true because of conditions (27),(29),(30),(31) and (32)

$$r_2 = \Gamma\left\| I - a \cdot s \cdot s^T \right\| \left[\left\| b^{-1} r^{-1}\right\| \left(a\ s^T s + \Gamma\left\| I - a\ s\ s^T \right\|\right)^2 + 1\right] \cdot$$

$$s_M W_M K_{v\max} + gK_{v\max} e_M + \left[(g+2)K_{v\max} + a\ s_M^2\right] D_M$$

$$r_3 = g\left[1 + \left\| b^{-1}\right\| \left(1+a\ s^T s\right)\right](D_M + e_M)^2 + 2D_M^2 +$$

$$+ 2\Gamma\left\| I - a \cdot s \cdot s^T \right\| \left[1 + \left\| b^{-1}\right\| \left(1+a\ s^T s\right)\right] \cdot$$

$$\left[1 + \left\| r^{-1} b^{-1}\right\| \left(a\ s^T s + \Gamma\left\| I - a\ s\ s^T \right\|\right)^2\right] \cdot (D_M + e_M) W_M s_M +$$

$$+ \frac{\left\| I - a \cdot s \cdot s^T \right\|^2}{a} \Gamma^2 W_M^2 +$$

$$\left\| b^{-1}\right\| \Gamma^2 \left\| I - a\ s\ s^T \right\|^2 \left[1 + \left\| r^{-1} b^{-1}\right\| \left(a\ s^T s + \Gamma\left\| I - a\ s\ s^T \right\|\right)^2\right] s_M^2 W_M^2$$

Substituting and bounding the term

$$\Delta J < -r_1 \|r\|^2 + 2r_2\|r\| + r_3 - \frac{\left\| I - a \cdot s \cdot s^T \right\|^2}{a}\Gamma(2-\Gamma)\left\|\tilde{W}\right\|^2$$

Completing squares for $\|r\|$

$$\Delta J < -r_1\left[\|r\| - \frac{r_2}{r_1}\right]^2 + \frac{r_2^2}{r_1} + r_3 - \frac{\left\| I - a \cdot s \cdot s^T \right\|^2}{a}\Gamma(2-\Gamma)\left\|\tilde{W}\right\|^2$$

which is negative as long as

$$\left\|\tilde{W}\right\| > \frac{1}{\left\| I - a \cdot s \cdot s^T \right\|}\sqrt{\frac{a\left(r_2^2 + r_1 r_3\right)}{r_1\Gamma(2-\Gamma)}} \quad \text{or} \quad \|r\| > \frac{r_2 + \sqrt{r_2^2 + r_1 r_3}}{r_1}$$

From the above results, ΔL is negative outside a compact set. According to a standard Lyapunov theorem extension [11], it can be concluded that the tracking error $r(k)$, the actuator error $\tilde{t}(k)$ and the NN weights estimates $\tilde{W}(k)$ are GUUB.

References

[1] A. R. Barron, "Universal approximation bounds for superposition of a sigmoidal function," *IEEE Trans. Inform. Theory*, vol. 39, no. 3, pp. 930-945, 1993.

[2] L.A. Bernotas, P. E. Crago, and H. J. Chizeck, "Adaptive control of electrically stimulated muscle," *IEEE Trans. on Biomedical Engineering* no.34, pp. 140-147, 1987.

[3] J. Campos and F. L. Lewis, "Deadzone compensation in discrete time using adaptive fuzzy logic," *Proc. IEEE Conference on Decision and Control*, pp. 2920-2926, Tampa, FL, 1998.

[4] V. Gullapalli, J. A. Franklin, and H. Benbrahuim, "Acquiring robot skills via reinforcement learning," *IEEE Cont. Syst.*, pp. 13-24, February 1994.

[5] K. Hornik, M. Stinchombe, and H. White. "Multilayer feedforward networks are universal approximators," *Neural Networks* vol. 2, pp. 359-366, 1989.

[6] B. Igelnik and Y-H. Pao, "Stochastic choice of basis functions in adaptive function approximation and the functional-link net," *IEEE Trans. on Neural Networks*, vol. 6, no. 6, pp. 1320-1329, November 1995.

[7] S. Jagannathan, and F.L. Lewis, "Discrete-Time Control of a Class of Nonlinear Dynamical Systems," *Int. Journal of Intelligent Control and Systems*, vol. 1, no. 3, pp. 297-326, 1996.

[8] B. Kosko, *Neural Networks and Fuzzy Systems*, Prentice Hall, New Jersey, 1992.

[9] M. Krstic, I. Kanellahopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*, John Wiley & Sons, New York, NY, 1995.

[10] J. Leitner, A. Calise and J. V. R. Prasad, "Analysis of adaptive neural networks for helicopter flight control," *Journal of Guidance, Control, and Dynamics*, vol. 20, no. 5, pp. 972-979, Sep.-Oct. 1997.

[11] F. L. Lewis, C. T. Abdallah, and D. M. Dawson, *Control of Robot Manipulators*, MacMillan, New York, 1993.

[12] F. L. Lewis, S. Jagannathan, and A. Yesildirek, *Neural Network Control of Robot Manipulators and Nonlinear Systems*, Taylor & Francis, Philadelphia, PA 1999.

[13] F. L. Lewis, A. Yesildirek, and K. Liu, "Multilayer Neural-Net Robot Controller with Guaranteed Tracking Performance," *IEEE Trans. Neural Networks*, vol. 7, no. 2, pp. 388-399, 1996.

[14] M. B. McFarland and A. J. Calise, "Multilayer neural network and adaptive nonlinear control of agile anti-air missiles," *Preprint*, 1999.

[15] K. S. Narendra and K. Parthasarathy, "Identification and control of dynamical systems using neural networks," *IEEE Trans. Neural Networks*, vol. 4, no. 6, pp. 982-988, 1993.

[16] N. Sadegh, "A perceptron network for functional identification and control of nonlinear systems," *IEEE Trans. Neural Networks*, vol. 4, pp.1823-1836, 1992.

[17] R. Selmic and F. L. Lewis, "Backlash Compensation in Nonlinear Systems using Dynamic Inversion by Neural Networks," *To appear at the Conference in Control and Automation*, Kona, Hawaii, August 1999.

[18] R. Selmic and F. L. Lewis, "Deadzone compensation in nonlinear systems using neural networks," *Proc. IEEE Conference Decision and Control*, Tampa, FL, 1998.

[19] Y. D. Song, T. L. Mitchell, and H. Y. Lai, "Control of a class of nonlinear uncertain systems via compensated inverse dynamics approach," *IEEE Trans. Automatic Control*, vol. 39, no. 9, pp. 1866-1871, September 1994.

[20] G. Tao and P. V. Kokotovic, *Adaptive Control of Systems with Actuator and Sensor Nonlinearities*, John Wiley & Sons, Inc., New York, 1996.

[21] G. Tao and P. V. Kokotovic, "Continuous-time adaptive control of system with unknown backlash," *IEEE Trans. Automatic Control*, vol. 40, no. 6, pp. 1083-1087, June 1995.

[22] G. Tao and P.V. Kokotovic, "Discrete-time adaptive control of nonlinear systems with unknown deadzones," *Int. J. Control*, vol. 61, no. 1, pp. 1-17, 1995.