Substructural Logics on Display

Rajeev Goré, *Automated Reasoning Project*[1] *and Department of Computer Science, Australian National University, Canberra, ACT, 0200, Australia, Email: rpg@arp.anu.edu.au*

Abstract

Substructural logics are traditionally obtained by dropping some or all of the structural rules from Gentzen's sequent calculi LK or LJ. It is well known that the usual logical connectives then split into more than one connective. Alternatively, one can start with the (intuitionistic) Lambek calculus, which contains these multiple connectives, and obtain numerous logics like: exponential-free linear logic, relevant logic, BCK logic, and intuitionistic logic, in an incremental way. Each of these logics also has a classical counterpart, and some also have a "cyclic" counterpart. These logics have been studied extensively and are quite well understood.

Generalising further, one can start with intuitionistic Bi-Lambek logic, which contains the dual of every connective from the Lambek calculus. The addition of the structural rules then gives Bi-linear, Bi-relevant, Bi-BCK and Bi-intuitionistic logic, again in an incremental way. Each of these logics also has a classical counterpart, and some even have a "cyclic" counterpart. These (bi-intuitionistic and bi-classical) extensions of Bi-Lambek logic are not so well understood. Cut-elimination for Classical Bi-Lambek logic, for example, is not completely clear since some cut rules have side conditions requiring that certain constituents be empty or non-empty.

The Display Logic of Nuel Belnap is a general Gentzen-style proof theoretical framework designed to capture many different logics in one uniform setting. The beauty of display logic is a general cut-elimination theorem, due to Belnap, which applies whenever the rules of the display calculus obey certain, easily checked, conditions. The original display logic, and its various incarnations, are not suitable for capturing bi-intuitionistic and bi-classical logics in a uniform way. We remedy this situation by giving a single (cut-free) Display calculus for the Bi-Lambek Calculus, from which all the well-known (bi-intuitionistic and bi-classical) extensions are obtained by the incremental addition of structural rules to a constant core of logical introduction rules.

We highlight the inherent duality and symmetry within this framework obtaining "four proofs for the price of one". We give algebraic semantics for the Bi-Lambek logics and prove that our calculi are sound and complete with respect to these semantics. We show how to define an alternative display calculus for bi-classical substructural logics using negations, instead of implications, as primitives.

Borrowing from other display calculi, we show how to extend our display calculus to handle bi-intuitionistic or bi-classical substructural logics containing the forward and backward modalities familiar from tense logic, the exponentials of linear logic, the converse operator familiar from relation algebra, four negations, and two unusual modalities corresponding to the non-classical analogues of Sheffer's "dagger" and "stroke", all in a modular way.

Using the Gaggle Theory of Dunn we outline relational semantics for the binary and unary intensional connectives, but make no attempt to do so for the extensional connectives, or the exponentials.

Finally, we flesh out a suggestion of Lambek to embed intuitionistic logic using two unusual "exponentials", and show that these "exponentials" are essentially tense logical modalities, quite at odds with the usual exponentials.

Using a refinement of the display property, you can pick and choose from these possibilities to construct a display calculus for your needs.

Keywords: display logic, substructural logics, linear logic, relevant logic, BCK-logic, bi-Heyting logic, intuitionistic substructural tense logics, intuitionistic modal logics, proof theory, gaggle theory.

[1] Work partially supported by a visiting fellowship from the Swiss National Science Foundation to the Institute for Applied Mathematics and Computer Science, University of Bern, Switzerland, and the Australian Research Council via a Queen Elizabeth II Fellowship.

1 Introduction

Substructural logics [22] are logics which are obtained by dropping some of Gentzen's original structural rules from his calculi LK and LJ. It is well known that the usual logical connectives then split into more than one connective. Alternatively, one can start with the (intuitionistic) Lambek calculus [44], which contains these multiple connectives, and obtain numerous logics like: exponential-free linear logic, relevant logic, BCK logic, and intuitionistic logic, in an incremental way. Each of these logics also has a classical counterpart. Whether the logic is intuitionistic or classical usually depends on whether we allow the right hand side of sequents to contain single formulae or multiple formulae.

Traditionally, these logics have been studied from different philosophical viewpoints which focus on particular structural rules, such as "logics without weakening" (relevant logics) [3], "logics without contraction" [61], or as "logics with limited weakening and contraction" (linear logic) [30]. Each of these communities has collected a vast number of results about their favourite logics.

Augmenting the Lambek calculus with the duals of the traditional logical connectives gives the Bi-Lambek calculus [45]. The addition of the structural rules then gives Bi-linear, Bi-relevant, Bi-BCK and Bi-intuitionistic logic, again in an incremental way. Each of these logics also has a classical counterpart. The logics without exchange also have cyclic counterparts. Most of the traditional communities mentioned above *have* investigated the dual connectives which seemed interesting, from their particular viewpoint, to varying degrees of success. Examples are: the study of "right-difference" and "left-difference" in relevant logic [3, page 357], the unsatisfactory attempt to add a dual intensional disjunction to BCK-logic [60], the question of cut-elimination in non-commutative substructural logics [74], the intricacies of cut-elimination for full intuitionistic linear logic [41, 12], and the (unsettled?) question of full cut-elimination in classical Bi-Lambek logic [40]. Only recently have we seen attempts to obtain a general picture of these logics using uniform calculi, specifically designed to cater for these "structural" variations [8, 18, 62, 29, 63, 13, 58, 48, 57, 73].

The Display Logic [8] of Nuel Belnap is one such general (Gentzen-style) proof theoretical framework designed to capture many different logics in one uniform setting. The beauty of Display Logic is that display calculi enjoy a general cut-elimination theorem, which applies whenever the rules of the display calculus obey certain, easily checked, conditions. The original display logic [8, 10] and its various refinements are designed for particular logics like linear logic [9], modal and tense logics [79, 43, 34], relevant logics [67], or the logic of relation algebras [35]. Although all these *logics* can be seen as substructural logics, these display calculi are not suitable for capturing bi-intuitionistic and bi-classical substructural (modal) logics in a uniform way. We remedy this situation by giving a single generalised (cut-free) display calculus for intuitionistic Bi-Lambek logic, from which all the well-known extensions are obtained by the incremental addition of structural rules to a *constant core* of logical introduction rules. The addition of four further structural rules gives the classical counterpart of each of these logics.

Specifically, we obtain cut-free display calculi for intuitionistic and classical versions of: non-associative Bi-Lambek logic, non-commutative full linear logic (= Bi-Lambek logic), cyclic Bi-Lambek logic, Bi-linear logic, Bi-relevant logic, Bi-BCK logic, and Bi-Intuitionistic (Heyting) logic. By adding only some of the "classicality" structural rules we obtain Lambek's BL1a and BL1b [45]. In Section 13 we also outline an idea which *may* allow us to display the Basic Logic of Sambin, Battilotti and Faggian [73].

By extending the algebraic semantics [59] for the Lambek calculus to the intuitionistic and classical extensions of Bi-Lambek logics, we prove that our display calculi are sound and complete with respect to these algebraic semantics.

Our general display calculus enjoys a refinement of the display property which allows us to pick and choose from (slight variants of) the various existing display calculi to construct a display calculus for specific needs, allowing extensions of our display calculus to handle intuitionistic or classical substructural logics extended with the forward and backward modalities familiar from tense logic, the exponentials of linear logic, the converse operator familiar from relation algebra, four negations, and two unusual modalities corresponding to the non-classical analogues of Sheffer's "dagger" and "stroke", all in an incremental, modular way. For example, we show how to define a display calculus for bi-classical substructural logics using negations, instead of implications, as primitives.

Using the Gaggle Theory of Dunn [24] we outline relational semantics for many of the binary and unary intensional connectives, but make no attempt to extend these semantics to the extensional connectives, or to the exponentials. Finally, we flesh out a suggestion of Lambek to embed intuitionistic logic into any of our substructural logics using two unusual "exponentials", and show that these "exponentials" are essentially tense logical modalities, quite at odds with the usual exponentials of linear logic.

As far as I am aware, there is no other unified account of all of these bi-intuitionistic and bi-classical, substructural, modal/tense, "exponentiated" logics, in a cut-free manner. However, in a companion paper [36] we show how to generalise these ideas and obtain these display calculi using the Gaggle Theory of Dunn [24].

The major innovation in this paper is the use of additional structural connectives to capture the inherent duality of every logic using dual sets of display postulates [36]. By doing so, we obtain a refined display property and complete a quest begun in [31] and [32]. Discussions with numerous people have helped (see the acknowledgements), but the most important sources of inspiration have been Lambek's [45] categorial analysis of Bi-Lambek logic, and Dunn's [26] instructions on how *not* to get confused by residuation and Galois connections.

The display calculus presented here allows you to pick and choose certain sets of logical connectives at will, according to your needs, as long as you respect the (limited) display property. But the choice is not totally free since some connectives come in non-divisible collections (partial-gaggles). Further work is required to give a system with total freedom, but see Section 13 for a tentative solution.

The paper is structured as follows: In Section 1.1 we describe the **syntax** of our logics. In Section 2 we describe **display logic**, the modifications that we need for handling substructural logics, and the actual rules of our calculus for the **binary connectives**. In Section 3 we describe the **symmetry and duality** inherent in the logics and the display calculi. In Section 4 we describe **bi-intuitionistic substructural logics**. In Section 5 we describe **bi-classical substructural logics**, and also the semi-classical logics BL1a and BL1b of Lambek [45]. In Section 6.1 we extend the **algebraic semantics** for substructural logics to substructural logics with dual connectives, and prove that our calculi are **sound and complete** with respect to these semantics. We also outline an alternative syntactic method that simply simulates the traditional intuitionistic systems given by Ono [59]. In Section 6.2: we consider **relational semantics** for the intensional binary connectives. In Section 7 we show that **explicit negations** can also be brought into the generalised display framework. In Section 8 we consider the non-classical versions of Sheffer's "**dagger**" and "**stroke**" operations. In Section 9 we show how to add **modalities and converse**. In Section 10 we flesh out Lambek's suggestion for obtaining exponentials (essentially from modalities) in order to **embed Intuitionistic logic** into any of the substructural logics displayed here. In Section 11 we show how to obtain the **exponentials** of linear logic. In Section 12 we discuss **related work** on general proof systems. In Section 13 we mention some **further work**. In Section 14 we discuss our results and conclude.

Acknowledgements: This work began in 1995 [31]. Since then, numerous people have helped to crystallise those ideas into what is presented here. In particular, I am extremely grateful to: Nuel Belnap for encouragement, J Michael Dunn for numerous email discussions on gaggle theory and for pointers to the literature; Hiroakira Ono for encouragement, pointers to the literature and for many enlightening discussions on cut-elimination; Giovanni Sambin for inviting me to Padua to speak on display logic, for encouraging me to "write down all you know about display logic" and for many lively discussions on duality, distribution and nested turnstiles; Lev Gordeev for pointers to the literature; two anonymous reviewers for suggesting many many improvements; Jeremy Dawson for machine-checking many of the proofs; Stéphane Demri for kindly proof-reading an earlier version; and James Harland for showing me how to write upside-down ampersands. Thanks also to Greg Restall (whose paper [66] also broached some of these issues) for many useful discussions.

1.1 Bi-Intuitionistic and Bi-Classical Substructural Logics

The **formulae** of our logics are built from a set of propositional variables $\{p, q, r, \cdots\}$ and four constants $\mathbf{1}, \mathbf{0}, \top, \bot$ with the help of numerous logical connectives using the following BNF grammar where A and B stand for formulae:

$$
\begin{array}{rcl}
A\ B & ::= & p \mid \top \mid \bot \mid \mathbf{1} \mid \mathbf{0} \mid \Diamond A \mid \Box A \mid \blacklozenge A \mid \blacksquare A \mid \smile A \mid\ !A \mid\ ?A \mid \\
 & & A \rightarrow B \mid A \leftarrow B \mid A \otimes B \mid A \oplus B \mid A \prec B \mid A \succ B \mid \\
 & & A \wedge B \mid A \vee B \mid A \succ\!\!\sim\!\!\prec B \mid A <\!\!\sim\!\!> B
\end{array}
$$

In the first row above, the white diamond and white box are the traditional forward looking modalities, the black diamond and black box are their backward looking tense logical counterparts, and the curved "smile" is converse. The exclamation mark and question mark are $\Box$-like and $\Diamond$-like modalities from linear logic [30]. In the second row, the two arrows without tails are substructural versions of classical implication, the two arrows without heads are their algebraic duals. Some authors have investigated these connectives under the name of "co-implication". Negations can be defined from the arrows and their duals using the constants $\mathbf{0}$ and $\mathbf{1}$ respectively. The circled times and circled plus connectives are types of conjunction and disjunction. In the third row, the wedge-like and vee-like connectives are types of conjunction and disjunction, and the two strange connectives with two tails and two heads are non-classical analogues of Sheffer's "stroke" and "dagger".

In the sequel we often speak of "intensional" and "extensional" connectives since the

n-ary intensional connectives have Kripke-style semantics in terms of, usually $n+1$-ary, "accessibility relations", and the extensional connectives often, but not always, have semantics in terms of the usual set-theoretic notions of union and intersection. The following map shows some of these properties. The "but-not" reading is suggested by Lambek [45]:

symbol	type	reading	symbol	type	reading
$\rightarrow$	intensional	if ... then ...	$\succ$	intensional	not ... but ...
$\leftarrow$	intensional	... if ...	$\prec$	intensional	... but not ...
$\otimes$	intensional	conjunction	$\oplus$	intensional	disjunction
$\wedge$	extensional	conjunction	$\vee$	extensional	disjunction
$\smile$	intensional	converse			
$\Diamond$	intensional	sometime future	$\Box$	intensional	always future
$\blacklozenge$	intensional	sometime past	$\blacksquare$	intensional	always past

Since substructural logics are motivated by proof theory, the logics are usually formulated using Gentzen systems. Standard Gentzen systems for intuitionistic substructural logics can be found in [19, 20, 59]. The systems for classical substructural logics can be deduced from these, although the cut rule is usually not so easy to deduce. I am not aware of a good source for traditional Gentzen systems for substructural logics containing dual logical connectives, but there seems to be some question as to their exact (cut-free) formulation [1, 40, 41, 12]. The display calculi presented here hopefully fill this gap.

2 Display Logic

Display Logic [8] is a very powerful Gentzen-style formulation, due to Nuel Belnap, which generalises Gentzen's notion of structures using multiple, complex, structural connectives. The name comes from a fundamental property of all display calculi which allows us to choose any particular constituent of a sequent and to make it the whole of the right or left side by moving other constituents to the other side. There are now numerous display calculi in the literature, each specialised to capture some specific family of logics like: classical tense logics [79, 43, 34], various intuitionistic logics [38, 81], relevant logics [67] and even the logic of relation algebras [35]. Needless to say, we must assume that the reader is familiar with display calculi to some extent.

In our display calculi, **structures** are built using the following **structural connectives** and BNF grammar rules where A stands for any formula and X, Y, and later V, W and Z, stand for structures:

structural connectives	
nullary	I Φ
unary	$\circ$ $\bullet$ $\sharp$ $\flat$ ©
binary	; $<$ $>$

$$X\ Y ::= A \mid \Phi \mid \mathrm{I} \mid \circ X \mid \bullet X \mid \sharp X \mid \flat X \mid ©X \mid X ; Y \mid X > Y \mid X < Y$$

Note that, unlike in standard Gentzen systems, comma is not a structural connective in the sequel, and no structural connective is automatically polyvalent.

A **sequent** is simply an expression of the form $X \vdash Y$ with X the **antecedent** and Y the **succedent**. A **sequent rule** is defined from sequents in the usual way with the **premisses** above the line and the **conclusion** below the line.

2.1 Display Postulates

The "display property" is a by-product of special, *reversible* rules, called the **display postulates** (also called display equivalences), which allow us to manipulate structures. These are shown in Figure 1. All these display postulates can be derived using the procedure outlined in [36].

In Figure 1, the notation rp(.,.), gc(.,.) and dgc(.,.) respectively stand for the algebraic notions known as "residuated pairs", "Galois connections" and "dual Galois connections", all restricted to operations that are unary [24]. The ternary notation rp(.,.,.) is just a generalisation of "residuated pairs" to "residuated triples" when the operations under consideration are binary, and the notation drp(.,.,.) is its dual [24]. In the unary case, we cannot distinguish between "residuated pairs" and "dual residuated pairs" hence there is no drp(.,.).

$$\mathrm{dgc}(\sharp,\flat)\ \frac{\sharp X \vdash Y}{\overline{\flat Y \vdash X}} \qquad \mathrm{gc}(\sharp,\flat)\ \frac{X \vdash \sharp Y}{\overline{Y \vdash \flat X}}$$

$$\mathrm{rp}(\circ,\circ)\ \frac{\circ X \vdash Y}{\overline{X \vdash \circ Y}} \qquad \mathrm{rp}(\bullet,\bullet)\ \frac{\bullet X \vdash Y}{\overline{X \vdash \bullet Y}}$$

$$\mathrm{rp}(;,>,<)\ \begin{array}{c} X \vdash Z < Y \\ \hline\hline X \,;\, Y \vdash Z \\ \hline\hline Y \vdash X > Z \end{array} \qquad \mathrm{drp}(;,>,<)\ \begin{array}{c} Z < Y \vdash X \\ \hline\hline Z \vdash X \,;\, Y \\ \hline\hline X > Z \vdash Y \end{array}$$

FIG. 1. Display Postulates

The rules rp$(\bullet,\bullet)$ and rp$(\circ,\circ)$ are from [79], the gc$(\sharp,\flat)$ and dgc$(\sharp,\flat)$ rules are from [65], and the rules rp(; , > , <) and drp(; , > , <) are from [31]. Not all of these structural connectives or display postulates will be used immediately.

These rules allow us to dis-assemble and re-assemble any structure bigger than a formula so that any particular part of the structure can be made the whole of the antecedent, or the whole of the succedent, depending on its "polarity". The exact side where a particular occurrence of a substructure finally rests depends upon whether it is an "antecedent part" or a "succedent part". We formalise this as follows.

Let $\pm_j$ stand for one of $+$ and $-$, $j \geq 1$. For every n-ary structural connective s, we assign a **tonicity vector** $\mathrm{tn}(s, \pm_1, \cdots, \pm_n)$ according to the table below:

tn(; , +, +)	tn(< , +, −)	tn(> , −, +)
tn(∘, +)	tn(•, +)	
tn(♯, −)	tn(♭, −)	
tn(©, +)		

Sometimes, we abuse notation and write $\mathrm{tn}(s, j, \pm_j)$ to indicate that "s has tonicity $\pm_j$ at the j-th place", $1 \leq j \leq n$. For example, tn(< , 2, −).

In a sequent $V \vdash W$, the structure V is an **antecedent part** and the structure W is a **succedent part**. Then, given that the structure $s(X_1, \ldots, X_n)$ with outermost structural connective s is an antecedent [respectively succedent] part, the substructure X_j is:

1. an antecedent [respectively succedent] part if $\mathrm{tn}(s, j, +)$
2. a succedent [respectively antecedent] part if $\mathrm{tn}(s, j, -)$.

Intuitively, if $s(X_1, \cdots, X_n)$ is an antecedent part, then its j-th argument is an antecedent part [succedent part] if s has tonicity $+$ [$-$] in its j-th position.

Two sequents s and s' are **structurally equivalent** if we can pass from one to the other (and back) using only the display postulates shown above. The name Display Logic comes from the following theorem (which does not hold for subformulae!).

Theorem 2.1 (Belnap) *For every sequent s and every antecedent/succedent part X of s, there is a structurally equivalent sequent s' that has X (alone) as its antecedent/succedent. X is said to be "displayed" in s' [8].*

Proof. By inspection of the display postulates. ∎

Example 2.2 *The following derivation, read upwards, shows how to display the (succedent part) B in the sequent $(A > B) > C \vdash D$ as the whole of the succedent in $A\ ;(C < D) \vdash B$ where the annotations at the right name the rule used:*

$$\cfrac{\cfrac{\cfrac{A\ ;(C < D) \vdash B}{C < D \vdash A > B}\ \mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )}{C \vdash (A > B)\ ;\ D}\ \mathrm{drp}(\ ;\ ,\ >\ ,\ <\ )}{(A > B) > C \vdash D}\ \mathrm{drp}(\ ;\ ,\ >\ ,\ <\ )$$

In the sequel, we just use (dp) as an annotation to mean "some display postulate".

2.2 Logical Rules

Once we have the display postulates and the display theorem, we can give a core set of logical and structural rules which will be common to all our display calculi.

In Figure 2 we give the logical rules that introduce the logical constants and the binary logical connectives into the antecedent and the succedent. Note that every introduced principal formula is "displayed". We extend this set to include unary modalities, exponentials, negations, and Sheffer's stroke and dagger in later sections. Thus some structural connectives are not used in Figure 2.

Also, for every intensional logical connective c, one rule is always a simple "rewrite" in which some structural connective in the *premiss* turns into the logical connective c in the *conclusion*, with everything else *remaining the same* [31]. For example, in the rule $(\otimes \vdash)$, the structural connective $;$ turns into the logical connective $\otimes$, while in the rule $(\vdash \oplus)$, the structural connective $;$ turns into the logical connective $\oplus$.

Thus the logical connectives come in pairs, with each component of a pair captured by the same structural connective, but in different (antecedent or succedent) positions

Intensional Introduction Rules

$(\mathbf{0}\vdash)\ \ \mathbf{0}\vdash\Phi$

$(\vdash\mathbf{0})\ \ \dfrac{Z\vdash\Phi}{Z\vdash\mathbf{0}}$

$(\mathbf{1}\vdash)\ \ \dfrac{\Phi\vdash Z}{\mathbf{1}\vdash Z}$

$(\vdash\mathbf{1})\ \ \Phi\vdash\mathbf{1}$

$(\otimes\vdash)\ \ \dfrac{A\ ;\ B\vdash Z}{A\otimes B\vdash Z}$

$(\vdash\otimes)\ \ \dfrac{X\vdash A\quad Y\vdash B}{X\ ;\ Y\vdash A\otimes B}$

$(\oplus\vdash)\ \ \dfrac{A\vdash X\quad B\vdash Y}{A\oplus B\vdash X\ ;\ Y}$

$(\vdash\oplus)\ \ \dfrac{Z\vdash A\ ;\ B}{Z\vdash A\oplus B}$

$(\leftarrow\vdash)\ \ \dfrac{A\vdash X\quad Y\vdash B}{A\leftarrow B\vdash X < Y}$

$(\vdash\leftarrow)\ \ \dfrac{Z\vdash A < B}{Z\vdash A\leftarrow B}$

$(\prec\vdash)\ \ \dfrac{A < B\vdash Z}{A\prec B\vdash Z}$

$(\vdash\prec)\ \ \dfrac{X\vdash A\quad B\vdash Y}{X < Y\vdash A\prec B}$

$(\rightarrow\vdash)\ \ \dfrac{X\vdash A\quad B\vdash Y}{A\rightarrow B\vdash X > Y}$

$(\vdash\rightarrow)\ \ \dfrac{Z\vdash A > B}{Z\vdash A\rightarrow B}$

$(\succ\vdash)\ \ \dfrac{A > B\vdash Z}{A\succ B\vdash Z}$

$(\vdash\succ)\ \ \dfrac{A\vdash X\quad Y\vdash B}{X > Y\vdash A\succ B}$

Extensional Introduction Rules

$(\bot\vdash)\ \ \bot\vdash\mathrm{I}$

$(\vdash\bot)\ \ \dfrac{Z\vdash\mathrm{I}}{Z\vdash\bot}$

$(\vdash\top)\ \ \dfrac{\mathrm{I}\vdash Z}{\top\vdash Z}$

$(\vdash\top)\ \ \mathrm{I}\vdash\top$

$(\vee\vdash)\ \ \dfrac{A\vdash Z\quad B\vdash Z}{A\vee B\vdash Z}$

$(\vdash\vee)\ \ \dfrac{Z\vdash A}{Z\vdash A\vee B}\qquad\dfrac{Z\vdash B}{Z\vdash A\vee B}$

$(\wedge\vdash)\ \ \dfrac{A\vdash Z}{A\wedge B\vdash Z}\qquad\dfrac{B\vdash Z}{A\wedge B\vdash Z}$

$(\vdash\wedge)\ \ \dfrac{Z\vdash A\quad Z\vdash B}{Z\vdash A\wedge B}$

Fig. 2. Logical Rules: Intensional and Extensional Connectives

as shown below. Unlike in traditional Gentzen systems, antecedent and succedent positions do not correspond exactly to left hand and right hand sides of sequents.

Structural Overloading

structural connective	$>$	$<$	;	I	Φ
antecedent position	$\succ$	$\prec$	$\otimes$	$\top$	$\mathbf{1}$
succedent position	$\rightarrow$	$\leftarrow$	$\oplus$	$\perp$	$\mathbf{0}$

There is a simple mnemonic for remembering this overloading: the shape $<$ appears in both the shapes $\leftarrow$ and $\prec$, while the shape $>$ appears in both the shapes $\rightarrow$ and $\succ$. Similarly, the shape of Φ is a superposition of the shapes of the two logical constants $\mathbf{1}$ and $\mathbf{0}$, and the shape of I is a superposition of the shapes of the two logical constants $\top$ and $\perp$ (well ... almost).

The rules for the intensional connectives $\rightarrow$, $\leftarrow$, $\succ$, $\prec$, $\otimes$, $\oplus$ all involve one of the structural connectives. The rules for the extensional connectives $\wedge$, $\vee$ involve no structural connective. Since no residuals for the extensional conjunction or disjunction are postulated, the extensional connectives do not obey any distribution principle [7]. That is, we do not have $A \wedge (B \vee C) \vdash (A \wedge B) \vee C$ or $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$ [7, page 32]. However, due to the nature of the extensional rules and the display postulates, we do have $A \otimes (B \vee C) \vdash (A \otimes B) \vee (A \otimes C)$, but do not have $A \otimes (B \vee C) \vdash (A \otimes B) \vee C$, which we leave as simple exercises for the reader (requiring approximately ten rule applications each). For a historical discussion regarding the blocking of distribution see also [7, page 34].

2.3 Limited Display Property

The ability to display any substructure has been the primary focus of extant display calculi, but this need not be so. We now formalise a notion of a limited display property based on the observation that each of our display postulates contains structural connectives in only certain (antecedent, succedent or both) positions.

Given a sequent $X \vdash Y$ we can adorn every antecedent/succedent part occurrence c of a structural connective with "a"/"s". Doing so gives us the **adorned** structural connective occurrences in $X \vdash Y$. This notion is extended to sequent rules in the obvious way. Intuitively, if c is adorned with "a" then it will be displayed as the main connective on the antecedent side.

An adorned connective occurrence c^a/c^s **respects** a display postulate if the structural connective c appears as the outermost structural connective in an antecedent/ succedent position in that display postulate. That is, this display postulate can be applied to display the substructures of this occurrence of c. For example, $;^a$ respects rp(; , $>$, $<$), but both $\bullet^a$ and $\bullet^s$ respect rp($\bullet, \bullet$).

Then, a given sequent $V \vdash W$ **respects** a collection of display postulates if every adorned structural connective occurrence of $V \vdash W$ respects some display postulate in the collection.

Proposition 2.3 (limited display) *For every collection $\mathcal{C}$ of display postulates, for every sequent s that respects $\mathcal{C}$, and for every antecedent/succedent part X of s, there is a structurally equivalent sequent s' that has X (alone) as its antecedent/succedent.*

Proof. By definition. ∎

It may seem that the above definitions are just tricks since they restrict sequents to those that will obey the limited display property. But there is a point to this limited display property as shown below.

Consider the display postulate $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$ *only* from Figure 1, the introduction rules for $\otimes$, $\rightarrow$, $\leftarrow$ from Figure 2, and a sequent $X \vdash Y$ which respects $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$.

Suppose that we now attempt to obtain a proof for this sequent by using these introduction rules and the display postulate $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$ in a *backward* manner. Since all occurrences of the structural connectives $\ ;\ $, $>$ and $<$ in $X \vdash Y$ respect $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$, we will be able to unravel substructures until we display some formula A. If all formulae in $X \vdash Y$ are built from primitive propositions and logical constants using only $\otimes$, $\rightarrow$, $\leftarrow$, then the main connective of the displayed A will either have an applicable "rewrite" introduction rule which (read upwards) converts the main connective into a structural connective respecting $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$, or will have a non "rewrite" rule that may or may not be applicable. The upshot is that if a derivation fails, then it is not because of the *missing* display postulates from Figure 1.

Thus $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$, together with the introduction rules for $\otimes$, $\rightarrow$, $\leftarrow$ is a display calculus on its own right for sequents $X \vdash Y$ that respect $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$ and which contain formulae from the language restricted to $\otimes$, $\rightarrow$ and $\leftarrow$ only. In particular, it is a display calculus for all end-sequents of the form $A \vdash B$ where A and B are built from primitive propositions and constants using only $\otimes$, $\rightarrow$, and $\leftarrow$.

Similarly, the display postulate $\mathrm{drp}(\ ;\ ,\ >\ ,\ <\ )$ and the introduction rules for $\oplus$, $\succ$, $\prec$, form a display calculus for sequents $X \vdash Y$ that respect $\mathrm{drp}(\ ;\ ,\ >\ ,\ <\ )$ and that contain formulae from the language restricted to $\oplus$, $\succ$, $\prec$.

The ability to separate the larger calculus into smaller calculi and still retain the (limited) display property is not possible in previous display calculi since they build in the inherent duality using $*$.

If, for example, you do not need $\oplus$, $\succ$ and $\prec$, then just leave out $\mathrm{drp}(\ ;\ ,\ >\ ,\ <\ )$, and leave out the introduction rules for $\oplus$, $\succ$ and $\prec$. But beware! You now only have a limited display property, so you must restrict yourself to end-sequents that respect the collection $\mathcal{C}$ of display postulates that are present. Needless to say, in this example, your formulae must not contain any occurrences of $\oplus$, $\succ$ and $\prec$.

2.4 Structural Rules

The two rules shown below are common to all Display Logics. Note that we can cut on formulae only.

$$(\mathrm{id})\ \ p \vdash p \qquad\qquad (\mathrm{cut})\ \ \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}$$

The structural rules shown in Figure 3 imbue the structural connectives with the properties that are shared by their corresponding logical entities. For example, associativity of $\otimes$ and $\oplus$ come from the rules ($\vdash$ Ass) and (Ass $\vdash$) while their commutativity comes from the rules ($\vdash$ Com) and (Com $\vdash$). The associativity of $\ ;\ $ in an antecedent position does not imply the associativity of $\ ;\ $ in a succedent position since the display postulates $\mathrm{rp}(\ ;\ ,\ >\ ,\ <\ )$ and $\mathrm{drp}(\ ;\ ,\ >\ ,\ <\ )$ *cannot* interact. In the display calculi of Belnap [8, 9] and Wansing [79], such properties are intimately tied together due to an extra structural connective $*$, see [10, 32, 43].

Basic Structural Rules

$$(\text{Ver-I})\ \frac{\mathrm{I} \vdash X}{Y \vdash X} \qquad (\text{Efq-I})\ \frac{X \vdash \mathrm{I}}{X \vdash Y}$$

$$(\Phi^{+}_{-} \vdash)\ \begin{array}{c} X \ ; \Phi \vdash Y \\ \hline\hline X \vdash Y \\ \hline\hline \Phi \ ; X \vdash Y \end{array} \qquad (\vdash \Phi^{+}_{-})\ \begin{array}{c} X \vdash \Phi \ ; Y \\ \hline\hline X \vdash Y \\ \hline\hline X \vdash Y \ ; \Phi \end{array}$$

Further Intensional Structural Rules

$$(\text{Wk} \vdash)\ \frac{X \vdash Z}{X \ ; Y \vdash Z} \quad \frac{Y \vdash Z}{X \ ; Y \vdash Z} \qquad (\vdash \text{Wk})\ \frac{Z \vdash X}{Z \vdash X \ ; Y} \quad \frac{Z \vdash Y}{Z \vdash X \ ; Y}$$

$$(\text{Ctr} \vdash)\ \frac{X \ ; X \vdash Z}{X \vdash Z} \qquad (\vdash \text{Ctr})\ \frac{Z \vdash X \ ; X}{Z \vdash X}$$

$$(\text{Ass} \vdash)\ \begin{array}{c} X \ ; (Y \ ; Z) \vdash W \\ \hline\hline (X \ ; Y) \ ; Z \vdash W \end{array} \qquad (\vdash \text{Ass})\ \begin{array}{c} W \vdash (X \ ; Y) \ ; Z \\ \hline\hline W \vdash X \ ; (Y \ ; Z) \end{array}$$

$$(\text{Com} \vdash)\ \frac{Y \ ; X \vdash Z}{X \ ; Y \vdash Z} \qquad (\vdash \text{Com})\ \frac{Z \vdash Y \ ; X}{Z \vdash X \ ; Y}$$

FIG. 3. Basic and Intensional Structural Rules

The basic structural rules of Figure 3 are standard since they force **1** and **0** to behave as identities for $\otimes$ and $\oplus$ respectively, and make $\top$ and $\bot$ the familiar "verum" and "falsum" constants respectively.

Associativity is included via the rules ($\vdash$ Ass) and (Ass $\vdash$) in most of the logics we mention, so we rarely mention this in their names. Omitting these structural rules for associativity immediately gives the non-associative versions of all of our logics.

2.5 Grishin's Rules, Yetter's Rules and Derived Rules

The rules shown in Figure 4 are originally due to Yetter [84] and Grishin [39].[2] Yetter's rules capture "cyclic" but non-commutative versions of linear logic while Grishin's rules turn all our intuitionistic substructural logics into classical ones.

We can add various parts of Grishin's rules, if we wish. But if all of them are present, we can add the *invertible* versions shown at the bottom of Figure 4 instead.

2.6 Various Display Calculi

Let δBiL be the display calculus consisting of (in reverse order of appearance):

[2] Notice that all these are forms of "associativity", a fact also noticed by Allwein and Dunn [2, page 518].

Yetter's Rules in Structural Form

$$(\text{Yet} \vdash)\ \frac{Y\ ;\ X \vdash \Phi}{X\ ;\ Y \vdash \Phi} \qquad (\vdash \text{Yet})\ \frac{\Phi \vdash Y\ ;\ X}{\Phi \vdash X\ ;\ Y}$$

Grishin's Rules in Structural Form

$$(\text{Grn}a(i) \vdash)\ \frac{X > (Y\ ;\ Z) \vdash W}{(X > Y)\ ;\ Z \vdash W} \qquad (\vdash \text{Grn}a(i))\ \frac{W \vdash (X\ ;\ Y) < Z}{W \vdash X\ ;\ (Y < Z)}$$

$$(\text{Grn}a(ii) \vdash)\ \frac{(X\ ;\ Y) < Z \vdash W}{X\ ;\ (Y < Z) \vdash W} \qquad (\vdash \text{Grn}a(ii))\ \frac{W \vdash X > (Y\ ;\ Z)}{W \vdash (X > Y)\ ;\ Z}$$

$$(\text{Grn}b(i) \vdash)\ \frac{(X > Y)\ ;\ Z \vdash W}{X > (Y\ ;\ Z) \vdash W} \qquad (\vdash \text{Grn}b(i))\ \frac{W \vdash X\ ;\ (Y < Z)}{W \vdash (X\ ;\ Y) < Z}$$

$$(\text{Grn}b(ii) \vdash)\ \frac{X\ ;\ (Y < Z) \vdash W}{(X\ ;\ Y) < Z \vdash W} \qquad (\vdash \text{Grn}b(ii))\ \frac{W \vdash (X > Y)\ ;\ Z}{W \vdash X > (Y\ ;\ Z)}$$

Compact Form of Grishin's Rules

$$(\text{Grn}(i) \vdash)\ \frac{X > (Y\ ;\ Z) \vdash W}{\overline{(X > Y)\ ;\ Z \vdash W}} \qquad (\vdash \text{Grn}(i))\ \frac{W \vdash (X\ ;\ Y) < Z}{\overline{W \vdash X\ ;\ (Y < Z)}}$$

$$(\text{Grn}(ii) \vdash)\ \frac{(X\ ;\ Y) < Z \vdash W}{\overline{X\ ;\ (Y < Z) \vdash W}} \qquad (\vdash \text{Grn}(ii))\ \frac{W \vdash X > (Y\ ;\ Z)}{\overline{W \vdash (X > Y)\ ;\ Z}}$$

Fig. 4. Further Structural Rules

- the basic structural rules from Figure 3,
- the rules (id) and (cut) from Section 2.4
- the intensional and extensional logical rules from Figure 2
- the display postulates rp(; , > , <), drp(; , > , <) from Figure 1.

We can now form a display calculus for a particular substructural logic (extending Lambek's original Sentential Calculus) in a modular, incremental way by choosing further structural rules from Figure 3 and Figure 4, according to Table 1, where: a rule name like Com means the pair of rules ($\vdash$ Com) and (Com $\vdash$), the rule ($\vdash$ Com) stands for just the latter of this pair, and where "drvd" indicates that these rules are derived (from the rules Com). Similarly, Grn(a) means all the rules Grn$a(i)$ and Grn$a(ii)$, and Grn(b) means all the rules Grn$b(i)$ and Grn$b(ii)$.

(Associative) Bi-Lambek logic and Non-commutative Bi-Linear logic are different names for the same logic [47]. Lambek's BL2 is Classical Bi-Lambek logic, while

Version of Logic	Intuitionistic					Classical
Choice of structural rules	Ass	Com	Ctr	Wk	Yet	Grn
Non-associative Bi-Lambek						✓
BL1 = Bi-Lambek = Non-commutative Bi-Linear	✓					✓ BL2
Lambek's BL1a	✓					Grn(a)
Lambek's BL1b	✓					Grn(b)
Cyclic Non-commutative Bi-Linear	✓				✓	✓
Bi-Linear logic	✓	✓			drvd	✓
Non-commutative Bi-Relevant	✓		✓			✓
(Semi-Commutative) Bi-Relevant	✓	($\vdash$ Com)	✓			✓
Bi-Affine (or Bi-BCK)	✓			✓		✓
Bi-Heyting	✓	✓	✓	✓	drvd	✓

TABLE 1. Various Displayed Substructural Logics

Lambek's BL3 (not shown) is Cyclic Classical Bi-Lambek logic and is obtained by adding Yet to BL2 [45].

Any display calculus extending δBiL by the addition of any collection of structural rules from Figures 3 and 4 enjoys the following theorems.

Lemma 2.4 *For any formula A, the sequent $A \vdash A$ is provable.*

PROOF. By induction on the formation of A. ∎

Theorem 2.5 *Every "rewrite" rule is invertible: if the conclusion of the rule is provable then so are each of the premisses [34, 31].*

PROOF. We give only the case for $(\otimes \vdash)$ since the others are all analogous:

$$(\otimes\vdash)\ \ \frac{A\ ;\ B \vdash Z}{A \otimes B \vdash Z} \qquad\qquad \dfrac{\dfrac{A \vdash A \quad B \vdash B}{A\ ;\ B \vdash A \otimes B}\,(\vdash \otimes) \qquad \dfrac{\text{hypothesis}}{A \otimes B \vdash Z}}{A\ ;\ B \vdash Z}\,(\text{cut})$$

Invertibility of $(\otimes \vdash)$

∎

The beauty of Display Logic is the following generic cut-elimination theorem.

Theorem 2.6 (cut-elimination) *If there is a proof of the sequent $X \vdash Y$, then there is a cut-free proof of $X \vdash Y$ (in the same display calculus) [8].*

Proof. The rules obey Belnap's conditions C1-C8 [8]. ∎

Belnap's proof eliminates *any* application of cut, not just the highest one, as is traditional in most cut-elimination proofs. Such a view of cut-elimination was independently initiated by Mints [56]. An instance of Belnap's proof to eliminate highest cuts can be found in [35, Appendix]. Wansing [80] has proved strong cut-elimination for Display Logic using ideas from the strong-normalisation proofs for lambda calculi. That is, Wansing first shows how to "reduce", or eliminate completely, a (not necessarily highest) cut instance, as long as in one of the subproofs of this cut, the sequence of inferences between the introduction of the cut-formula and this cut is cut-free. There is always one such cut instance since the highest cut is guaranteed to meet this restriction. Wansing then shows that *any* sequence of such reductions must terminate, and that the resulting proof is cut-free.

3 Symmetry and Duality Rules O.K. ?

As stated previously, the intensional logical connectives come in pairs, with each component of a pair captured by the same structural connective, but when the structural connective is in different (antecedent or succedent) positions.

Below we define three maps δ, $\vartriangle$ and σ that capture three transformations on formulae, structures, rules and proofs. The Greek letter σ is for "symmetry" and the two deltas are for "duality" [43]. All missing entries are assumed to be the identity map.

$$\delta : \quad (X \vdash Y)^\delta := (Y^\delta \vdash X^\delta)$$

	$(X\ c\ Y)^\delta := (Y^\delta\ c^\delta\ X^\delta)$					$(A)^\delta$	
$c \updownarrow c^\delta$	$\rightarrow$	$\leftarrow$	$\wedge$	$\otimes$	$>$	$\top$	$\mathbf{1}$
	$-\!\!<$	$>\!\!-$	$\vee$	$\oplus$	$<$	$\bot$	$\mathbf{0}$

Summary of δ: exchange antecedent and succedent, swap the order of the arguments of *every* binary connective, replace c by c^δ, and replace the constant A by the constant A^δ, using the correspondences shown in the vertical columns of the table above right. All other connectives, constants and primitive propositions are mapped to themselves by δ. This transformation is also present in Basic Logic [73], where it is called "symmetry". For an explanation of why the two terms are different, see Section 12.5.7.

$$\vartriangle : \quad (X \vdash Y)^\vartriangle := (Y^\vartriangle \vdash X^\vartriangle)$$

	$(X\ c\ Y)^\vartriangle := (X^\vartriangle\ c^\vartriangle\ Y^\vartriangle)$					$(A)^\vartriangle$	
$c \updownarrow c^\vartriangle$	$\rightarrow$	$\leftarrow$	$\wedge$	$\otimes$		$\top$	$\mathbf{1}$
	$>\!\!-$	$-\!\!<$	$\vee$	$\oplus$		$\bot$	$\mathbf{0}$

Summary of $\vartriangle$: exchange antecedent and succedent, replace c by $c^\vartriangle$, and replace the constant A by the constant $A^\vartriangle$, using the correspondences shown in the vertical columns of the table above right. All other connectives, constants and primitive propositions are mapped to themselves by $\vartriangle$. In particular, do *not* swap the order of arguments of *any* binary connective. This transformation is explicitly mentioned in [45, page 214].

$$\sigma: \quad (X \vdash Y)^\sigma := (X^\sigma \vdash Y^\sigma)$$

	$(X \; c \; Y)^\sigma := (Y^\sigma \; c^\sigma \; X^\sigma)$					$(A)^\sigma$	
$c \updownarrow c^\sigma$	$\rightarrow$ $\leftarrow$	$-\!\!<$ $>\!\!-$			$>$ $<$		

Summary of σ: swap the order of the arguments of *every* binary connective, and replace c by c^σ, using the correspondences shown in the vertical columns of the table above right. All other connectives, constants and primitive propositions are mapped to themselves by σ. In particular, do not exchange antecedent and succedent of a sequent. This distinction is not present in Basic Logic, see Section 12.5.7.

For $\alpha \in \{\delta, \Delta, \sigma\}$, we extend these maps in the obvious way to sequent rules:

$$\left(\frac{X_1 \vdash Y_1 \cdots X_n \vdash Y_n}{X \vdash Y} \right)^\alpha \quad := \quad \left(\frac{(X_1 \vdash Y_1)^\alpha \cdots (X_n \vdash Y_n)^\alpha}{(X \vdash Y)^\alpha} \right)$$

thereby extending the "transforms" (maps) to proofs.

We can formalise these dualities and symmetry as follows.

Proposition 3.1 *For every structure X, sequent s, and rule ρ, we have $(X^\alpha)^\alpha = X$, $(s^\alpha)^\alpha = s$, and $(\rho^\alpha)^\alpha = \rho$ for $\alpha \in \{\delta, \Delta, \sigma\}$.*

Proposition 3.2 *For every logical connective c,*

1. *for every $\alpha \in \{\delta, \Delta\}$, the rule ρ is the rule for $(\vdash c)/(c \vdash)$ iff the rule ρ^α is the rule for $(c^\alpha \vdash)/(\vdash c^\alpha)$ respectively (note switch of sides w.r.t. $\vdash$)*
2. *the rule ρ is the rule for $(\vdash c)/(c \vdash)$ iff the rule ρ^σ is the rule for $(\vdash c^\sigma)/(c^\sigma \vdash)$ respectively (note no switch of sides w.r.t. $\vdash$).*

Proof. Pure case analysis, all checked by the implementation of Jeremy Dawson. ∎

Proposition 3.3 *For every structural rule with name* Str

1. *for every $\alpha \in \{\delta, \Delta\}$, the rule ρ is the rule for $(\vdash \mathrm{Str})/(\mathrm{Str} \vdash)$ iff the rule ρ^α is the rule for $(\mathrm{Str} \vdash)/(\vdash \mathrm{Str})$ respectively (note switch of sides w.r.t. $\vdash$)*
2. *the rule ρ is the rule for $(\vdash \mathrm{Str})/(\mathrm{Str} \vdash)$ iff the rule ρ^σ is the rule for $(\vdash \mathrm{Str})/(\mathrm{Str} \vdash)$ respectively (note no switch of sides w.r.t. $\vdash$).*

Proof. Pure case analysis, all checked by the implementation of Jeremy Dawson. In some structural rules with two parts like the rule Wk, one part of the rule maps onto another part of the same rule under some transforms. ∎

Theorem 3.4 *If* Π *is a proof of* $X \vdash Y$ *then* $(\Pi)^\alpha$ *is a proof of* $(X \vdash Y)^\alpha$*, for* $\alpha \in \{\delta, \Delta, \sigma\}$.

PROOF. By induction on the length of the given proof Π. The base cases are the introduction rules for the constants $\top$, $\bot$, $\mathbf{1}$ and $\mathbf{0}$, which fall under Proposition 3.2, and the rule (id), which is invariant under each transform. The cut rule is also invariant under all transformations. Now apply Proposition 3.2 and Proposition 3.3∎

For the transformation δ, an analogous theorem is also proved for classical displayed tense logic in [43] and for Basic Logic in [73]. The following theorems, found in conjunction with Jeremy Dawson, appear to be new.

Theorem 3.5 *Let "." be composition of the transformations* δ, Δ, σ *when they are applied to sequents of* δBiL*, and let* ι *be the identity transformation on sequents. Thus, if* s *is a sequent of* δBiL*, then* $\delta.\ \Delta$ *represents the sequent* $(s^\delta)^\Delta$.

Then,

(a) for all $x, y, z \in \{\delta, \Delta, \sigma\}$: $(x.y).z = x.(y.z)$

(b) for all $x \in \{\delta, \Delta, \sigma\}$: $x.x = \iota$

(c) for all $x, y, z \in \{\delta, \Delta, \sigma\}$: $x.y = y.x$

(d) for all distinct (choices of) $x, y, z \in \{\delta, \Delta, \sigma\}$: $x.y = z$

(e) for all distinct (choices of) $x, y, z \in \{\delta, \Delta, \sigma\}$: $(x.y).z = \iota$

PROOF.

(a) follows immediately from the definition of composition

(b) follows by the definitions of the transformations

(c) is proved by induction on formation of sequents

(d) is proved by induction on formation of sequents

(e) follows from (d) and (b).

∎

Since the application of any transformation to rules is just its application to the premisses and conclusion, we obtain:

Corollary 3.6 *In words:*

(a) composition of proof transformations is associative from the very notion of composition itself

(b) the proof transformations are self inverses (w.r.t. ι*)*

(c) composition of proof transformations is commutative

(d) composing any two distinct *proof transformations from* δ, Δ *and* σ *gives the third transformation*

(e) composing, in any order, any three distinct *proof transformations from* δ, Δ *and* σ *does nothing.*

Corollary 3.7 *The proof transformations* $\{\delta,\ \Delta,\ \sigma,\ \iota\ \}$*, together with the operation "." form an (Abelian) group of order four.*

Thus, we can obtain "four proofs for the price of one", as also observed by Lambek [45]. We make use of this phenomenon in later sections.

Currently, I can offer no further insights, although category theorists will no doubt be able to do so since this is obviously a commuting diagram of morphisms on proofs. The irony here is that we will later show strong connections between our display calculi and Dunn's Gaggle Theory, so named because it is a name that "is like a group, but which suggests a certain amount of complexity and disorder" [24, page 2].

4 Intuitionistic Substructural Logics

We now briefly summarise the properties of various intuitionistic substructural logics and map out some relationships between the various connectives. *These relationships are well-known, particularly to the researchers working in the various substructural communities.* But it may be of interest to see them spelled out in one setting.

There are four natural negations within the generalised display framework [45]. The easiest way to gain access to them is via the following definitions [45]:

$$\begin{array}{lcllcl} A^{\mathbf{0}} & := & (A \rightarrow \mathbf{0}) & \qquad {}^{\mathbf{0}}A & := & (\mathbf{0} \leftarrow A) \\ {}^{\mathbf{1}}A & := & (\mathbf{1} \prec A) & \qquad A^{\mathbf{1}} & := & (A \succ \mathbf{1}) \end{array}$$

We include them in the discussion below since some of the structural rules affect them, but do not affect the binary connectives from which they are defined.

4.1 (Associative) Bi-Lambek or Non-Commutative Bi-Linear Logic

The Lambek Calculus can be obtained from Gentzen's LK by dropping all structural rules, and adding two extra constants **1** and **0** [59, 45], although this was not how it was originally formulated [44]. By adding the dual of every logical connective we obtain (Associative) Bi-Lambek logic [45, 39]. This intuitionistic Bi-Lambek logic contains the following connectives where the vertical pairs are Δ-duals:

$\rightarrow$	$\leftarrow$	$\otimes$	$\wedge$	$\mathbf{1}$	$\top$	$.^{\mathbf{0}}$	${}^{\mathbf{0}}.$
$\succ$	$\prec$	$\oplus$	$\vee$	$\mathbf{0}$	$\bot$	$.^{\mathbf{1}}$	${}^{\mathbf{1}}.$

We also have gc($.^{\mathbf{0}}$,${}^{\mathbf{0}}.$) and dgc($.^{\mathbf{1}}$,${}^{\mathbf{1}}.$) giving

$A \vdash \mathbf{0} \leftarrow (A \rightarrow \mathbf{0})$ and $A \vdash (\mathbf{0} \leftarrow A) \rightarrow \mathbf{0}$
$\mathbf{1} \prec (A \succ \mathbf{1}) \vdash A$ and $(\mathbf{1} \prec A) \succ \mathbf{1} \vdash A$.

¿From here on, the addition of further structural rules causes some of the connectives to collapse (become indistinguishable proof theoretically). In our tables, we show this by omitting the vertical line between indistinguishable connectives.

4.2 Cyclic Non-Commutative Bi-Linear Logic

The addition of Yetter's rules [84] gives us commutativity for ; in certain contexts, namely when the whole of the other side is Φ, see Figure 4. Consequently we get a horizontal collapse of the negations, indicated by the absence of a vertical line between the indistinguishable connectives in the table below, but not the full collapse of the residuals of $\otimes$ and $\oplus$ from which these negations are defined:

$\rightarrow$	$\leftarrow$	$\otimes$	$\wedge$	$\mathbf{1}$	$\top$	$.^{\mathbf{0}}$ $^{\mathbf{0}}.$
$\succ$	$\prec$	$\oplus$	$\vee$	$\mathbf{0}$	$\perp$	$.^{\mathbf{1}}$ $^{\mathbf{1}}.$

4.3 Bi-Linear Logic

The addition of commutativity Com for ; on both sides (Figure 4) destroys the distinction between the two residuals of $\otimes$ and $\oplus$ giving the following distinct connectives:

$\rightarrow$ $\leftarrow$	$\otimes$	$\wedge$	$\mathbf{1}$	$\top$	$.^{\mathbf{0}}$ $^{\mathbf{0}}.$
$\succ$ $\prec$	$\oplus$	$\vee$	$\mathbf{0}$	$\perp$	$.^{\mathbf{1}}$ $^{\mathbf{1}}.$

There is thus only one "implication", but there is also a dual from the but-not family.

Basic Logic also contains "$\rightarrow$" (our $\rightarrow$ *and* $\leftarrow$) and "$\leftarrow$" (our $\succ$ *and* $\prec$). Bi-Linear logic has also been proposed and studied by Hyland and de Paiva [41] and Braüner and de Paiva [12], who call it Full Intuitionistic Linear Logic (FILL). The cut-elimination proof in [12] takes some thirteen pages.

4.4 (Non-Commutative and Semi-Commutative) Bi-Relevant Logic

Since we have Com, we get the collapse described in Bi-Linear Logic. The addition of contraction Ctr for ; on both sides gives us $A \vdash A \otimes A$ as well as $A \oplus A \vdash A$ [45, 26].

Many relevant logics have a non-commutative $\otimes$ but a commutative $\oplus$, which is why Belnap's original display calculus [8], and Restall's modification of it [67] both have ($\vdash$ Com) built into the display postulates. Our display calculus is more liberal since we can obtain the same effect using only structural rules. That is, postulating ($\vdash$ Com), but not (Com $\vdash$), makes ; commutative in succedent positions (on the right hand side), but does not automatically force commutativity for ; in antecedent positions (on the left hand side), giving us:

$\rightarrow$	$\leftarrow$	$\otimes$	$\wedge$	$\mathbf{1}$	$\top$	$.^{\mathbf{0}}$	$^{\mathbf{0}}.$
$\succ$ $\prec$		$\oplus$	$\vee$	$\mathbf{0}$	$\perp$	$.^{\mathbf{1}}$ $^{\mathbf{1}}.$	

Mike Dunn has pointed out to me that the duals of the arrows have been studied by the relevant logic community under the names "right-difference" and "left-difference" [3, page 357], [54]. As one of the referees pointed out, most relevant logics assume distribution of $\wedge$ over $\vee$, so we do not actually get relevant logics unless we regain distribution. For ways to do so, see [7], [35] or [67].

4.5 Bi-BCK or Bi-Affine Logic

The addition of Com gives us the picture of Bi-Linear Logic. The omission of Ctr but the addition of Wk gives us $A \otimes A \vdash A$ and $A \vdash A \oplus A$, and the collapse of the constants:

$\rightarrow$ $\leftarrow$	$\otimes$	$\wedge$	$\mathbf{1}$ $\top$	$.^{\mathbf{0}}$ $^{\mathbf{0}}.$
$\succ$ $\prec$	$\oplus$	$\vee$	$\mathbf{0}$ $\perp$	$.^{\mathbf{1}}$ $^{\mathbf{1}}.$

Hiroakira Ono has informed me that he and Y Komori tried to introduce $\oplus$ into BCK but were not able to do so to their satisfaction. Thus our formulation may be of interest to this community.

4.6 Bi-Intuitionistic Logic

The addition of Com, Ctr and Wk gives us an almost complete collapse of the intensional and extensional connectives:

$\to \quad \leftarrow$	$\otimes \quad \wedge$	$\mathbf{1} \quad \top$	${}_{\cdot}\mathbf{0} \quad \mathbf{0}_{\cdot}$
$\succ \quad \prec$	$\oplus \quad \vee$	$\mathbf{0} \quad \bot$	${}_{\cdot}\mathbf{1} \quad \mathbf{1}_{\cdot}$

This logic has been studied extensively, predominently by Polish logicians. See [75], [68], and [45, page 216] for references.

5 Classical Substructural Logics

Having sketched the various bi-intuitionistic substructural logics we can display, we now examine the effects of Grishin's rules. We work mostly in classical δBiL, so most of our comments apply to all bi-classical extensions.

Generalising a notion from Ono [59], a substructural logic is **classical** if both:

1. $(\mathbf{0} \leftarrow A) \to \mathbf{0} \dashv\vdash A \dashv\vdash \mathbf{0} \leftarrow (A \to \mathbf{0})$
2. $\mathbf{1} \prec (A \succ \mathbf{1}) \dashv\vdash A \dashv\vdash (\mathbf{1} \prec A) \succ \mathbf{1}$

The directions

$$(\mathbf{0} \leftarrow A) \to \mathbf{0} \dashv A \vdash \mathbf{0} \leftarrow (A \to \mathbf{0})$$
$$\mathbf{1} \prec (A \succ \mathbf{1}) \vdash A \dashv (\mathbf{1} \prec A) \succ \mathbf{1}$$

follow by simple (dp) moves since our built-in negations from the implicational family are Galois connected, and our built-in negations from the but-not family are dually Galois connected, as mentioned in Section 4.1. The other directions require Grishin's rules as shown below.

¿From Grn(a) and Grn(b), by multiplying on the outermost or innermost side of $\vdash$ by Φ we can derive the following[3]:

$$\dfrac{\dfrac{\dfrac{\dfrac{(X > \Phi) > \Phi \vdash Y}{\Phi \vdash (X > \Phi) \; ; \; Y}\,(\text{dp})}{\Phi \vdash X > (\Phi \; ; \; Y)}\,(\vdash \text{Grn}b(ii))}{X \; ; \; \Phi \vdash \Phi \; ; \; Y}\,(\text{dp})}{X \vdash Y}\,(\vdash \Phi^{+}_{-}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{X \vdash Y}{X \; ; \; \Phi \vdash \Phi \; ; \; Y}\,(\vdash \Phi^{+}_{-})}{(X \; ; \; \Phi) < Y \vdash \Phi}\,(\text{dp})}{X \; ; \; (\Phi < Y) \vdash \Phi}\,(\text{Grn}a(ii) \vdash)}{X \vdash \Phi < (\Phi < Y)}\,(\text{dp})$$

[3] Multiplying asymetrically does not seem to be useful.

There are two analogues which are not shown in detail. All four derivations are invertible, and the directions shown require $\mathrm{Grn}(a)$, so call them $(\mathrm{Grn}a\Phi^{+}_{-})$, while the reverse directions require $\mathrm{Grn}(b)$, so call them $(\mathrm{Grn}b\Phi^{+}_{-})$ as shown below:

$$(\mathrm{Grn}a\Phi^{+}_{-}) \quad \text{and} \quad (\mathrm{Grn}b\Phi^{+}_{-})$$

$$\frac{X \vdash Y}{(X > \Phi) > \Phi \vdash Y} \qquad \frac{X \vdash Y}{\Phi < (\Phi < X) \vdash Y} \qquad \frac{X \vdash Y}{X \vdash \Phi < (\Phi < Y)} \qquad \frac{X \vdash Y}{X \vdash (Y > \Phi) > \Phi}$$

Now the four missing directions of points 1. and 2. for classicality are easy to prove, as shown next. The proof for $(\mathbf{0} \leftarrow A) \rightarrow \mathbf{0} \vdash A$ is shown below, on the left. This proof can be converted into a proof of $(\mathbf{0} \leftarrow A^{\alpha}) \rightarrow \mathbf{0} \vdash A^{\alpha}$ where A^{α} is one of A^{δ}, A^{Δ} or A^{σ} simply by uniformly replacing A by A^{α}. Now, the required directions for classicality come for free via duality, symmetry and Theorem 3.4 since

$$(A \vdash \mathbf{1} \prec (A \succ \mathbf{1})) = ((\mathbf{0} \leftarrow A^{\delta}) \rightarrow \mathbf{0} \vdash A^{\delta})^{\delta}$$
$$(A \vdash (\mathbf{1} \prec A) \succ \mathbf{1}) = ((\mathbf{0} \leftarrow A^{\Delta}) \rightarrow \mathbf{0} \vdash A^{\Delta})^{\Delta}$$
$$(\mathbf{0} \leftarrow (A \rightarrow \mathbf{0}) \vdash A) = ((\mathbf{0} \leftarrow A^{\sigma}) \rightarrow \mathbf{0} \vdash A^{\sigma})^{\sigma}$$

For example, the proof shown below right is the δ-dual of the proof below left:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash (A > \Phi) > \Phi}\,(\mathrm{Grn}a\Phi^{+}_{-})}{(A > \Phi)\ ;\ A \vdash \Phi}\,(\mathrm{dp})}{(A > \Phi)\ ;\ A \vdash \mathbf{0}}\,(\vdash \mathbf{0})}{A > \Phi \vdash \mathbf{0} < A}\,(\mathrm{dp})}{A > \Phi \vdash \mathbf{0} \leftarrow A}\,(\vdash\leftarrow) \quad \mathbf{0} \vdash \Phi}{(\mathbf{0} \leftarrow A) \rightarrow \mathbf{0} \vdash (A > \Phi) > \Phi}\,(\rightarrow\vdash)}{(\mathbf{0} \leftarrow A) \rightarrow \mathbf{0} \vdash A}\,(\mathrm{Grn}b\Phi^{+}_{-})$$

$$\dfrac{\dfrac{\Phi \vdash \mathbf{1} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^{\delta} \vdash A^{\delta}}{\Phi < (\Phi < A^{\delta}) \vdash A^{\delta}}\,(\mathrm{Grn}a\Phi^{+}_{-})}{\Phi \vdash A^{\delta}\ ;\ (\Phi < A^{\delta})}\,(\mathrm{dp})}{\mathbf{1} \vdash A^{\delta}\ ;\ (\Phi < A^{\delta})}\,(\mathbf{1} \vdash)}{A^{\delta} > \mathbf{1} \vdash \Phi < A^{\delta}}\,(\mathrm{dp})}{A^{\delta} \succ \mathbf{1} \vdash \Phi < A^{\delta}}\,(\succ\, \vdash)}{\Phi < (\Phi < A^{\delta}) \vdash \mathbf{1} \prec (A^{\delta} \succ \mathbf{1})}\,(\vdash\, \prec)}{A^{\delta} \vdash \mathbf{1} \prec (A^{\delta} \succ \mathbf{1})}\,(\mathrm{Grn}b\Phi^{+}_{-})$$

Note that both proofs make essential use of both $\mathrm{Grn}(a)$ and $\mathrm{Grn}(b)$, hence they need (full) Grn. All other rules used in the proofs are rules of the basic calculus δBiL, hence these proofs also suffice in all extensions that include Grn. But, since we do not use Ass in any of them, they hold even in non-associative classical Bi-Lambek logic. That is, (our formulation of) Grishin's rules convert every intuitionistic substructural logic into its classical counterpart.

Lambek [47] points out some lattice models for Classical Bi-Linear logic and notes that such logics have also been investigated in [39, 1, 59, 11].

5.1 Effect of Grishin's Rules

Using Grishin's rules we can prove each of the following in all our calculi [45, page 222]:

$$A \rightarrow B \quad \dashv\vdash \quad (A \rightarrow \mathbf{0}) \oplus B \qquad\qquad B \leftarrow A \quad \dashv\vdash \quad B \oplus (\mathbf{0} \leftarrow A)$$
$$B \prec A \quad \dashv\vdash \quad B \otimes (\mathbf{1} \prec A) \qquad\qquad A \succ B \quad \dashv\vdash \quad (A \succ \mathbf{1}) \otimes B$$

In particular, in Classical (Bi-Intuitionistic) logic, the two negations of Bi-Intuitionistic logic collapse into one "not", with $A \rightarrow B$ and $B \leftarrow A$ collapsing to "not-A or B", and $B \prec A$ and $A \succ B$ collapsing to "not-A and B", as expected.

5.2 Further Derivations Using Grishin's Rules

¿From $(\mathrm{Grn}b\Phi^{+}_{-})$ and $(\mathrm{Grn}a\Phi^{+}_{-})$ we can derive:

$$
\cfrac{\cfrac{\cfrac{\Phi < Y \vdash X > \Phi}{\Phi \vdash (X > \Phi)\ ;\ Y}\ (\mathrm{dp})}{(X > \Phi) > \Phi \vdash Y}\ (\mathrm{dp})}{X \vdash Y}\ (\mathrm{Grn}b\Phi^{+}_{-})
\qquad\qquad
\cfrac{\cfrac{\cfrac{Y > \Phi \vdash \Phi < X}{(Y > \Phi)\ ;\ X \vdash \Phi}\ (\mathrm{dp})}{X \vdash (Y > \Phi) > \Phi}\ (\mathrm{dp})}{X \vdash Y}\ (\mathrm{Grn}b\Phi^{+}_{-})
$$

using only (dp) moves. We get only two derivations since the others give the same derived rules. Furthermore they are invertible (in the presence of all Grn rules) since we use only (dp) moves giving:

$$
(\mathrm{Grn} <\ \ >)\ \ \frac{\Phi < Y \vdash X > \Phi}{\overline{X \vdash Y}}
\qquad\qquad
(\mathrm{Grn} >\ \ <)\ \ \frac{Y > \Phi \vdash \Phi < X}{\overline{X \vdash Y}}
$$

It is also possible to derive these directly by multiplying on the outermost or innermost side of $\vdash$ by Φ. Multiplying on asymmetric sides does not seem to be useful.

These are essentially laws of contraposition since we can instantiate $X \vdash Y$ as $A \vdash B$ and derive:

$$
\frac{{}^{\mathbf{1}}B \vdash A^{\mathbf{0}}}{\overline{A \vdash B}}
\qquad\qquad
\frac{B^{\mathbf{1}} \vdash {}^{\mathbf{0}}A}{\overline{A \vdash B}}
$$

5.3 Lambek's BL1a and BL1b

Lambek [45, 216-219] considers two semi-classical non-commutative substructural logics under the names BL1a and BL1b, which he axiomatises using two equations (a) and (b) due to Grishin. Our version of Grishin's rules capture these equations exactly, and so we can display these logics as well, as shown in Table 1.

In BL1a, we have

$$
{}^{\mathbf{1}}C \vdash C^{\mathbf{0}} \qquad\qquad C^{\mathbf{1}} \vdash {}^{\mathbf{0}}C
$$

and in BL1b, we have

$$
C^{\mathbf{0}} \vdash {}^{\mathbf{1}}C \qquad\qquad {}^{\mathbf{0}}C \vdash C^{\mathbf{1}}
$$

although Lambek [45] only considers the first of each row. To prove the sequents in the first row, use the laws of contraposition mentioned above by putting $A = B = C$, but remember that these are invertible only when all Grn rules are present. To prove the sequents in the second row use $(\mathrm{Grn}b\Phi^{+}_{-})$ to build up the succedent [antecedent] in the first [second] sequent, then simplify using introduction rules and (dp) moves.

We thus have Lambek's result [45, page 222] that in Classical Bi-Lambek logic, there are only two negations, since ${}^{\mathbf{1}}C \dashv\vdash C^{\mathbf{0}}$ and ${}^{\mathbf{0}}C \dashv\vdash C^{\mathbf{1}}$.

6 Semantics

The search for semantics for substructural logics is now a major field of research [21]. There are four major traditions in this work: algebraic semantics, relational semantics, (pre)topological semantics and categorial semantics.

We begin with algebraic semantics from Ono [59] since Ono explicitly connects these algebraic semantics with many of the other semantics. We then use the Gaggle Theory [24] of Dunn to work out a ternary semantics for each collection (partial-gaggle) of operations we have seen. Allwein and Dunn [2, page 543] have also considered this aspect, but have also given semantics for the extensional connectives, and this aspect is the most difficult (when distribution is missing).

6.1 Algebraic Semantics

Algebraic semantics for substructural logics have been investigated by Došen [19, 20], and Ono and colleagues [59]. We generalise definitions from [59] to cater for the duality inherent in our display calculi. We use the same connectives for the algebraic and logical parts to minimise clutter. Ono uses the term "Full Lambek algebra" and shortens it to FL-algebra. But "full" is also used by the linear logic community to mean the presence of dual connectives, so we avoid FL, and use BiL instead.

A structure

$$\mathbf{A} = \langle A, \leq, \rightarrow, \leftarrow, \wedge, \otimes, \mathbf{1}, \top, \succ, \prec, \vee, \oplus, \mathbf{0}, \bot \rangle$$

is a BiL-algebra (short for Bi-Lambek algebra) if:

1. $\langle A, \leq, \vee, \wedge, \top, \bot \rangle$ is a lattice with least element $\bot = \top \succ \top = \top \prec \top$ and greatest element $\top = \bot \rightarrow \bot = \bot \leftarrow \bot$
2. (a) $\langle A, \otimes, \mathbf{1} \rangle$ is a monoid with identity $\mathbf{1} \in A$
   (b) $\langle A, \oplus, \mathbf{0} \rangle$ is a co-monoid with co-identity $\mathbf{0} \in A$
3. (a) $z \otimes (x \vee y) \otimes w = (z \otimes x \otimes w) \vee (z \otimes y \otimes w)$ for every $x, y, z, w \in A$
   (b) $z \oplus (x \wedge y) \oplus w = (z \oplus x \oplus w) \wedge (z \oplus y \oplus w)$ for every $x, y, z, w \in A$
4. (a) $x \otimes y \leq z$ iff $x \leq z \leftarrow y$ iff $y \leq x \rightarrow z$, for every $x, y, z \in A$
   (b) $z \leq x \oplus y$ iff $x \succ z \leq y$ iff $z \prec y \leq x$, for every $x, y, z \in A$.

Let $\mathbf{A}$ be a BiL algebra. Then [59]:

1. $\mathbf{A}$ is a BiL$_e$ algebra if $\langle A, \otimes, \mathbf{1} \rangle$ and $\langle A, \oplus, \mathbf{0} \rangle$ are commutative
2. $\mathbf{A}$ is a BiL$_w$ algebra if
   (a) $\mathbf{0} = \bot$ and $\mathbf{1} = \top$ and
   (b) $x \otimes y \leq x$ and $y \otimes x \leq x$ and $x \leq y \oplus x$ and $x \leq x \oplus y$ for every $x, y \in A$
3. $\mathbf{A}$ is a BiL$_c$ algebra if $x \otimes x \leq x$ and $x \leq x \oplus x$ for every $x \in A$.
4. $\mathbf{A}$ is **classical** if $((\mathbf{0} \leftarrow x) \rightarrow \mathbf{0}) = x = (\mathbf{0} \leftarrow (x \rightarrow \mathbf{0}))$ and $((\mathbf{1} \prec x) \succ \mathbf{1}) = x = (\mathbf{1} \prec (x \succ \mathbf{1}))$, for every $x \in A$.

To prove soundness of δ**BiL** with respect to BiL-algebras we translate sequents to equations using the function τ shown below:

$$\tau(X \vdash Y) := \tau_1(X) \leq \tau_2(Y)$$

$$\begin{array}{lcllcl}
\tau_1(X \,;\, Y) & := & \tau_1(X) \otimes \tau_1(Y) & \tau_2(X \,;\, Y) & := & \tau_2(X) \oplus \tau_2(Y) \\
\tau_1(X < Y) & := & \tau_1(X) \prec \tau_2(Y) & \tau_2(X < Y) & := & \tau_2(X) \leftarrow \tau_1(Y) \\
\tau_1(X > Y) & := & \tau_2(X) \succ \tau_1(Y) & \tau_2(X > Y) & := & \tau_1(X) \rightarrow \tau_2(Y) \\
\tau_1(\mathrm{I}) & := & \top & \tau_2(\mathrm{I}) & := & \bot \\
\tau_1(\Phi) & := & \mathbf{1} & \tau_2(\Phi) & := & \mathbf{0} \\
\tau_1(A) & := & A & \tau_2(A) & := & A
\end{array}$$

Theorem 6.1 (soundness) *If the sequent $X \vdash Y$ is provable in δBiL then the equation $\tau(X \vdash Y)$ is BiL-valid (true in all BiL-algebras).*

Proof. The proof is by induction on the length of the given proof of the sequent $X \vdash Y$. The base cases are the sequents $\Phi \vdash \mathbf{1}$, $\mathrm{I} \vdash \top$, $\mathbf{0} \vdash \Phi$ and $\bot \vdash \mathrm{I}$, which respectively become the BiL-valid equations $\mathbf{1} \leq \mathbf{1}$, $\top \leq \top$, $\mathbf{0} \leq \mathbf{0}$ and $\bot \leq \bot$. For the induction step we show for every sequent rule that: if the τ-translations of the premisses are BiL-valid then so is the τ-translation of the conclusion.

We give only the case for one of the display postulates since the other rules are fairly obvious. Thus the triple rule below left τ-translates into the equation-triple on the right, and this is part of the definition of BiL-algebras.

$$\mathrm{rp}(\,;\,,\ >\,,\ <\,)\ \begin{array}{c} X \vdash Z < Y \\ \hline\hline X \,;\, Y \vdash Z \\ \hline\hline Y \vdash X > Z \end{array} \qquad \begin{array}{c} \tau_1(X) \leq \tau_2(Z) \leftarrow \tau_1(Y) \text{ iff} \\ \tau_1(X) \otimes \tau_1(Y) \leq \tau_2(Z) \text{ iff} \\ \tau_1(Y) \leq \tau_1(X) \rightarrow \tau_2(Z) \end{array}$$

∎

Theorem 6.2 (completeness) *If the equation $\tau(X \vdash Y)$ is BiL-valid (true in all BiL-algebras) then the sequent $X \vdash Y$ is provable in δBiL.*

Proof. We prove that the Lindenbaum algebra formed from the provably equivalent **formulae** of δBiL is a BiL-algebra [35, 59]. ∎

A more insightful alternate proof is to show that δBiL can simulate Ono's [59, pages 261-262] sequent system. The proof proceeds by induction on the length of the given proof in Ono's system, modulo some complications necessary to handle different notations, namely the replacement of Ono's "$\rightarrow$", "," and "$\supset$" by our "$\vdash$", " ; " and "$\leftarrow$" respectively (see [37] for details). For example, below we show how Ono's rule for ($\supset\vdash$), our ($\leftarrow\vdash$), can be simulated in δBiL:

$$(\supset\vdash)\ \dfrac{\Gamma \vdash \alpha \qquad \Delta, \beta, \Sigma \vdash \theta}{\Delta, \alpha \supset \beta, \Gamma, \Sigma \vdash \theta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{Induction Hypothesis}\\ \Gamma \vdash \alpha\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\text{Induction Hypothesis}\\ (\Delta \ ; \beta) \ ; \Sigma \vdash \theta\end{array}}{\Delta \ ; \beta \vdash \theta < \Sigma}\,(\mathrm{dp})}{\beta \vdash \Delta > (\theta < \Sigma)}\,(\mathrm{dp})}{\beta \leftarrow \alpha \vdash (\Delta > (\theta < \Sigma)) < \Gamma}\,(\leftarrow\vdash)}{(\beta \leftarrow \alpha) \ ; \Gamma \vdash \Delta > (\theta < \Sigma)}\,(\mathrm{dp})}{\Delta \ ; (\beta \leftarrow \alpha \ ; \Gamma) \vdash \theta < \Sigma}\,(\mathrm{dp})}{(\Delta \ ; \beta \leftarrow \alpha \ ; \Gamma) \ ; \Sigma \vdash \theta}\,(\mathrm{dp})$$

Note that the end-sequent above right requires a particular way of associating the elements of the antecedents, but we know that we can obtain this particular way of associating by repeated application of associativity Ass.

Adding further structural rules like Com, Wk and Ctr will give the obvious extension of this completeness result with respect to the algebras BiL_e, BiL_w, and BiL_c, respectively. Došen [19, page 371] gives many examples of further structural rules and their algebraic counterparts, but does not consider the classical substructural logics in any detail [19, page 365].

6.2 Relational Semantics

As noted by Došen [21], substructural logics have been motivated by proof-theoretical considerations rather than semantical considerations. But relational semantics for substructural logics have also been considered in the literature. Most trace their origins to various semantics for Relevant Logic [19, 20, 70, 28, 76, 52] or to the work of Jónsson and Tarski on Boolean algebras with operators [42]. In this section we use Dunn's Gaggle Theory [24] to outline a ternary Kripke-style semantics for our display calculi. In a sequel [36] we show how to display a "gaggle", thereby completing a proof-theoretic, model-theoretic and algebraic picture for substructural logics.

Dunn's Gaggle Theory [24] is a general method for obtaining Kripke-style semantics for algebraic logics, where an n-ary logical connective is associated with an $n+1$-ary relation over a set U of points or possible worlds [42].

The following definitions are a slight modification of those from Dunn [24]. Dunn's definitions do not quite work out since he sometimes forgets to multiply the signs of a "trace" by the output sign in order to obtain the "tonicity" (explained shortly).

Let $(A, \leq, \bot, \top)$ be a lattice with least and greatest elements $\top$ and $\bot$, respectively. Let f be an n-ary operation on this lattice. The operation f is **isotonic in the j-th position** if for all $a, b \in A$,

$$(a \leq b) \Rightarrow f(a_1, \cdots, a_{j-1}, a, a_{j+1}, \cdots, a_n) \leq f(a_1, \cdots, a_{j-1}, b, a_{j+1}, \cdots, a_n)$$

and is **antitonic in the j-th position** if for all $a, b \in A$,

$$(a \leq b) \Rightarrow f(a_1, \cdots, a_{j-1}, b, a_{j+1}, \cdots, a_n) \leq f(a_1, \cdots, a_{j-1}, a, a_{j+1}, \cdots, a_n).$$

Let $\mathrm{tn}(f, j, +)$ and $\mathrm{tn}(f, j, -)$ stand for the fact that the operation f is isotonic or antitonic in the j-th position respectively, and let $\pm$ stand for one of $+$ or $-$.

Then, for any n-ary operation f, the notation

$$(\pm_1, \pm_2, \ldots, \pm_j, \ldots, \pm_n) \mapsto \pm_{n+1}$$

is a **trace** t_f for f iff, for all $1 \leq j \leq n$:

$$(\pm_j, \pm_{n+1}) = \begin{cases} (+,+) \text{ if } \mathrm{tn}(f,j,+) \text{ and } (a_j = \top \Rightarrow f(a_1, \ldots, a_j, \ldots, a_n) = \top) \\ (-,-) \text{ if } \mathrm{tn}(f,j,+) \text{ and } (a_j = \bot \Rightarrow f(a_1, \ldots, a_j, \ldots, a_n) = \bot) \\ (-,+) \text{ if } \mathrm{tn}(f,j,-) \text{ and } (a_j = \bot \Rightarrow f(a_1, \ldots, a_j, \ldots, a_n) = \top) \\ (+,-) \text{ if } \mathrm{tn}(f,j,-) \text{ and } (a_j = \top \Rightarrow f(a_1, \ldots, a_j, \ldots, a_n) = \bot). \end{cases}$$

Depending on the tonicity of f at the j-th place, the second part of each condition in the definition above just states that f must behave in a particular way when its j-th argument takes one of the values $\top$ or $\bot$. Some intuitions, suggested by one of the reviewers, may be useful. Intuitively, the first two entries above say that, if the operation f is isotonic in its j-th position, then it can behave in two ways. One way is that if we make its j-th argument $\top$, then the value of the whole operation becomes $\top$. The second is that if we make its j-th argument $\bot$, then the value of the whole operation becomes $\bot$. An operation that is antitonic in its j-th position behaves in exactly the opposite way. Since these values are the greatest and least elements of the lattice, the operation f is said to "respect the bounds" [24].

Note that for each $1 \leq j \leq n$, the operation f has tonicity $\mathrm{tn}(f, j, \pm_j \pm_{n+1})$ where the product of two signs is obtained using $+- = -+ = -$, and $++ = -- = +$. That is, f has tonicity $(\pm_1\pm_{n+1}, \pm_2\pm_{n+1}, \ldots, \pm_j\pm_{n+1}, \ldots, \pm_n\pm_{n+1})$.

Since we have already explored algebraic semantics for substructural logics we abuse notation by sometimes speaking of our symbols as operations, sometimes as functions, and sometimes as logical connectives. For example, the "functions" that interest us have the following traces and tonicities:

Binary Functions

function	tonicity	trace
$\otimes$	$(+,+)$	$(-,-) \mapsto -$
$\rightarrow$	$(-,+)$	$(-,+) \mapsto +$
$\leftarrow$	$(+,-)$	$(+,-) \mapsto +$
$\oplus$	$(+,+)$	$(+,+) \mapsto +$
$\succ$	$(-,+)$	$(+,-) \mapsto -$
$\prec$	$(+,-)$	$(-,+) \mapsto -$

Following Dunn [24]:

1. A structure $\mathcal{T} := (A, \leq, OP)$ is a **tonoid** where
   (a) $(A, \leq)$ is a non-empty partially ordered set, and
   (b) OP is a collection of n-ary operations, each having a trace (so that each function respects the bounds in at least one way).

2. The n-ary function g is a **contrapositive** (with respect to the j-th position) of another n-ary function f when their traces t_f and t_g satisfy:

$$\begin{array}{lrcl} \text{if} & t_f & = & (\pm_1, \pm_2, \ldots, \pm_j, \ldots, \pm_n) \mapsto \pm_{n+1} \\ \text{then} & t_g & = & (\pm_1, \pm_2, \ldots, -\pm_{n+1}, \ldots, \pm_n) \mapsto -\pm_j \,. \end{array}$$

3. If $\pm_{n+1} = -$, we write $S(f, a_1, a_2, \ldots, a_n, b)$ for $f(a_1, \ldots, a_n) \leq b$.
   If $\pm_{n+1} = +$, we write $S(f, a_1, a_2, \ldots, a_n, b)$ for $b \leq f(a_1, \ldots, a_n)$.
4. Two functions f and g satisfy the **Abstract Law Of Residuation** (in their j-th place) when:
   (a) f and g are contrapositives (with respect to their j-th place), and
   (b) $S(f, a_1, a_2, \ldots, a_j, \ldots, a_n, b)$ iff $S(g, a_1, a_2, \ldots, b, \ldots, a_n, a_j)$.
5. Two functions $f, g \in OP$ are **relatives** when they satisfy the Abstract Law Of Residuation in some position.
6. The family of operations OP is **founded** when there is a distinguished operation $f \in OP$, the **head**, such that any other operation $g \in OP$ is a relative of f.
7. And finally, a **partial gaggle** is a tonoid $\mathcal{T} := (A, \leq, OP)$ in which OP is a founded family.

Given a tonoid $\mathcal{T} := (A, \leq, OP)$, a **frame** for $\mathcal{T}$ [26] is a structure $(U, \sqsubseteq, \langle R_i \rangle_{i \in I})$ where U is a non-empty set (of points, worlds or setups), $\sqsubseteq$ is a partial order over U, and each R_i is a relation such that there is a 1-1 correspondence between OP and I[4], and for each $f \in OP$, if the degree of f is n, then the corresponding relation $R_i \subseteq U^{n+1}$. Thus each n-ary operation is associated with an $n+1$-ary relation [42].

To describe tonoids in terms of frames, Dunn uses the following conventions [24]:

1. If R is an $n+1$-ary relation, then for $1 \leq j \leq n$:

$$\begin{array}{lcl} R^{-j}(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n, \beta) & := & R(\alpha_1, \ldots, \beta, \ldots, \alpha_n, \alpha_j) \\[1ex] \overline{R}(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n, \beta) & := & U^{n+1} \setminus R(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n, \beta). \end{array}$$

2. Two traces t and t' of the same arity **agree** if they have the same output sign, and they **clash** if they have different output signs.
3. Given an n-ary operation f with trace

$$t = (\pm_1, \pm_2, \ldots, \pm_n) \mapsto \pm_{n+1}$$

   (a) If $\pm_{n+1} = +$ then $C_t(\alpha_1 \in X_1, \ldots, \alpha_n \in X_n, R(\alpha_1, \ldots, \alpha_n, \beta))$ is

$$\forall \alpha_1, \ldots, \alpha_n : \sigma(\alpha_1 \in X_1) \text{ or } \ldots \text{ or } \sigma(\alpha_n \in X_n) \text{ or } R(\alpha_1, \ldots, \alpha_n, \beta)$$

$$\text{where each component } \sigma(\alpha_j \in X_j) \text{ is } \begin{cases} \alpha_j \in X_j \text{ if } \pm_j = + \\[1ex] \alpha_j \in \overline{X_j} \text{ if } \pm_j = - \end{cases}$$

[4] This I has nothing to do with the structural connective I from our display calculi.

(b) If $\pm_{n+1} = -$ then $C_t(\alpha_1 \in X_1, \ldots, \alpha_n \in X_n, R(\alpha_1, \ldots, \alpha_n, \beta))$ is

$$\exists \alpha_1, \ldots, \alpha_n : \sigma(\alpha_1 \in X_1) \text{ and } \ldots \text{ and } \sigma(\alpha_n \in X_n) \text{ and } R(\alpha_1, \ldots, \alpha_n, \beta)$$

$$\text{where each component } \sigma(\alpha_j \in X_j) \text{ is } \begin{cases} \alpha_j \in X_j \text{ if } \pm_j = - \\ \\ \alpha_j \in \overline{X_j} \text{ if } \pm_j = + \end{cases}$$

Theorem 6.3 (Dunn [26]) *Let* $\mathbf{T}$ *be a family of* n *traces, all of the same arity* n*, where there is some trace* t_0 *in* $\mathbf{T}$ *such that every trace* t' *in* $\mathbf{T}$ *can be obtained from* t_0 *by contraposition. Let* $(U, \sqsubseteq, R_0)$ *be a frame, where* R_0 *is a relation on* U *of degree* $n+1$*. Define the* n*-ary operation* F_0 *on subsets of* U *as follows:*

$$F_0(X_1, \ldots, X_n) := \{\beta \in U \mid C_{t_0}(\alpha_1 \in X_1, \ldots, \alpha_n \in X_n, R_0(\alpha, \ldots, \alpha_n, \beta))\}$$

For a given trace t *that is a contraposition of* t_0 *in the* j*-th place, define the* n*-ary operation* G_j *on subsets of* U *as:*

$$G_j(X_1, \ldots, X_n) := \{\beta \in U \mid C_t(\alpha_1 \in X_1, \ldots, \alpha_n \in X_n, R_0'(\alpha, \ldots, \alpha_n, \beta))\}$$

where

R_0' *is* R_0^{-j} *if the trace outputs of* t_0 *and* t *agree, and*
R_0' *is* $(\overline{R_0})^{-j}$ *if the trace outputs of* t_0 *and* t *clash.*

Then the operation F_0 *has trace* t_0*,* G_j *has trace* t*, and* F_0 *and* G_j *satisfy the Abstract Law of Residuation with respect to their* j*-th place.*

Once we do this for every j*,* $1 \leq j \leq n$*, we have a partial-gaggle of operations with traces assigned appropriately.*

This theorem gives us a way to construct a set of operations $F_0, G_1, \cdots, G_{n-1}$, on subsets of U, that match the given family of traces $\mathbf{T}$. Since the operations from a partial-gaggle form such a family of traces, we have

Corollary 6.4 (Dunn [26]) *Every partial-gaggle* $\mathcal{T} := (A, \leq, OP)$ *can be represented using a frame* $(U, \sqsubseteq, \langle R_i \rangle_{i \in I})$ *for* $\mathcal{T}$*.*

The relations $\langle R_i \rangle_{i \in I}$ are constructed using the recipe of Theorem 6.3 where F_0 is the operation on U which comes from the trace t_0 of the head f_0 of the given OP.

To see how all this is connected to our display calculi observe that

1. If a function/logical-connective c has a trace output $\pm_{n+1} = +$, then that connective's right introduction rule $(\vdash c)$ is a "rewrite" rule. Conversely, if a function/connective c has a trace output $\pm_{n+1} = -$, then that connective's left introduction rule $(c \vdash)$ is a "rewrite" rule.
2. Since every "rewrite" introduction rule is invertible, the structural connective is an exact proxy for the logical connective introduced from it [34].
3. The logical connectives we deal with form families (partial-gaggles) of operations with some operation as the head. The necessary contraposition (read residuation) conditions which these connectives must satisfy are captured *indirectly* by their

structural proxies via the display postulates. But since the structural connective is an exact proxy for the *head* connective, the logical connectives all inherit the residuation properties via the "rewrite" introduction rules. Below we show these partial-gaggles with the head as the first operation, and also associate a relation with that operation:

partial-gaggle (head first)	associated ternary relation
$(\otimes, \rightarrow, \leftarrow)$	$R_{\otimes}(x, y, z)$
$(\oplus, \succ, \prec)$	$S_{\oplus}(x, y, z)$

We can thus give a relational semantics for all these intensional connectives as below where X and Y are subsets of U and α, β, and χ are elements of U:

$$
\begin{aligned}
X \otimes Y &= \{\chi \in U \mid \exists \alpha \in X \text{ and } \exists \beta \in Y \text{ and } R_{\otimes}(\alpha, \beta, \chi)\} \\
X \rightarrow Y &= \{\chi \in U \mid \forall \alpha, \beta, \text{ if } R_{\otimes}(\alpha, \chi, \beta) \text{ and } \alpha \in X, \text{ then } \beta \in Y\} \\
X \leftarrow Y &= \{\chi \in U \mid \forall \alpha, \beta, \text{ if } R_{\otimes}(\chi, \beta, \alpha) \text{ and } \beta \in Y, \text{ then } \alpha \in X\}
\end{aligned}
$$

$$
\begin{aligned}
X \oplus Y &= \{\chi \in U \mid \forall \alpha, \beta, \text{ if } \overline{S}_{\oplus}(\alpha, \beta, \chi) \text{ then, } \alpha \in X \text{ or } \beta \in Y\} \\
X \succ Y &= \{\chi \in U \mid \exists \alpha \in \overline{X} \text{ and } \exists \beta \in Y \text{ and } \overline{S}_{\oplus}(\alpha, \chi, \beta)\} \\
X \prec Y &= \{\chi \in U \mid \exists \beta \in \overline{Y} \text{ and } \exists \alpha \in X \text{ and } \overline{S}_{\oplus}(\chi, \beta, \alpha)\}
\end{aligned}
$$

Since gaggle-theory grew out of relevant logics, it is not surprising that the first three of these are the traditional clauses from the Routley-Meyer [70] semantics for relevant logics. The last three are for their duals, but their definitions are couched in terms of $\overline{S}_{\oplus}$. Aesthetically it would be nice to have the dual of $\otimes$ "look forwards" with a relation since $\otimes$ itself "looks backwards" against one. Defining

$$S'_{\oplus} := ((\overline{S}_{\oplus})^{-1})^{-2}$$

gives us

$$
\begin{aligned}
X \oplus Y &= \{\chi \in U \mid \forall \alpha, \beta, \text{ if } S'_{\oplus}(\chi, \alpha, \beta) \text{ then, } \alpha \in X \text{ or } \beta \in Y\} \\
X \succ Y &= \{\chi \in U \mid \exists \alpha \in \overline{X} \text{ and } \exists \beta \in Y \text{ and } S'_{\oplus}(\beta, \alpha, \chi)\} \\
X \prec Y &= \{\chi \in U \mid \exists \beta \in \overline{Y} \text{ and } \exists \alpha \in X \text{ and } S'_{\oplus}(\alpha, \chi, \beta)\}
\end{aligned}
$$

Other authors have also given ternary relation semantics for many of these logical operators. Some of them treat $\otimes$ as a "forward" looking binary modality while we have treated it as a "backward" looking binary modality. There is nothing deep in this difference, this is just the ternary analogue of the fact that $\diamond$ can be given an interpretation as a forward looking modality, or as a backward looking modality. As Dunn [24] (Section 5) says, "one person's relation is another person's converse". The crux is to keep the same perspective uniformly.

Dunn [26, page 85] shows that each of our structural rules corresponds to a certain property of the ternary relation. For example, commutativity is reflected by a condition on frames which says: $R(\alpha, \beta, \gamma)$ implies $R(\beta, \alpha, \gamma)$. The existence of $\mathbf{1}$ and $\mathbf{0}$ as identities for $\otimes$ and $\oplus$ respectively can also be accommodated.

For the logics with distribution, we can give simple semantics for $\wedge$ and $\vee$ as set intersection and union respectively. For the logics without distribution, there are various alternatives [2, 77][5] but the details must await another occasion. A simple alternative is to use the fact that many of these logics can be characterised by semi-lattice-ordered monoids [61, 19, 59] as is done by MacCaull [48]. MacCaull treats $\wedge$ in a similar manner to $\otimes$ using another ternary relation $I(x, y, z)$ with the meaning $x \wedge y \leq z$ just as we can read $R_{\otimes}(x, y, z)$ as meaning $x \otimes y \leq z$. For our setting we would need another ternary relation $J(x, y, z)$, say, to interpret $\vee$ as $x \leq y \vee z$.

6.3 Pretopologies

Giovanni Sambin has given semantics for many substructural logics using pretopologies [72, 71]. Ono [59, page 259] concludes that "We have finally reached the conclusion that quantales, (intuitionistic) phase structures, pretopologies and complete semi-lattice-ordered-monoids are essentially the same thing.". Since, in Section 6.1, we have made fairly detailed connections between our work and the work of Ono [59], we omit a detailed comparison with pretopologies.

6.4 Categorial Semantics

It is clear (see [45]) that all of the preceding sections can be described in categorial terms since we started with residuation and Galois-connections which are instances of the categorial notion of adjunctions. It is also well-known that substructural logics have categorial semantics [46, 45, 68, 51]. But currently I can offer no deeper insights in this respect, except the following naive intuitions.

The intensional parts of our display calculi consist of two dual halves, each initially independent of each other. One half is defined in terms of ; in an antecedent position, and the other in terms of ; in a succedent position. But they cannot interact.

Once we add the rules (Φ^+_-), and the rules (Ver-I) and (Efq-I), we begin to make connections between these two dual halves. And the more structural rules we add, the more they interact. In the end, they become a Bi-Heyting algebra. And if we add Grishin's rules then even this collapses to a Boolean algebra.

Lambek [46] already speaks of variables and co-variables. I have no idea if this is related to the notions of positive and negative knowledge of Allwein and Dunn [2], but the analogy is tempting. That is, suppose that the variables p_i of our language told us about the things that are (known) true. Then the co-variables q_i would tell us about the things that are (known) false. A state of our knowledge would be composed of a set of pairs (p_i, q_i), one pair for each index i.

7 Explicit Negations

It is well-known that we can define Classical Lambek logic using implications, or negations, as primitives [1, 40]. But there is some controversy as to the exact formulation for cut-elimination. To clarify these issues, and to show how to introduce other negations into our display calculi, we now consider explicit negations.

There are four natural negations within Intuitionistic Bi-Lambek logic. The easiest way to gain access to them is via the following definitions [45]:

$$\begin{array}{rcl@{\qquad\qquad}rcl} A^{\mathbf{0}} & := & (A \rightarrow \mathbf{0}) & {}^{\mathbf{0}}A & := & (\mathbf{0} \leftarrow A) \\ {}^{\mathbf{1}}A & := & (\mathbf{1} \prec A) & A^{\mathbf{1}} & := & (A \succ \mathbf{1}) \end{array}$$

[5] Thanks to Mike Dunn for pointing this out to me.

Introduction Rules for Negation

$$({}^{\mathbf{0}}. \vdash)\ \frac{Z \vdash A}{{}^{\mathbf{0}}A \vdash \flat Z} \qquad\qquad (\vdash {}^{\mathbf{0}}.)\ \frac{Z \vdash \flat A}{Z \vdash {}^{\mathbf{0}}A}$$

$$(.^{\mathbf{0}} \vdash)\ \frac{Z \vdash A}{A^{\mathbf{0}} \vdash \sharp Z} \qquad\qquad (\vdash .^{\mathbf{0}})\ \frac{Z \vdash \sharp A}{Z \vdash A^{\mathbf{0}}}$$

$$({}^{\mathbf{1}}. \vdash)\ \frac{\flat A \vdash Z}{{}^{\mathbf{1}}A \vdash Z} \qquad\qquad (\vdash {}^{\mathbf{1}}.)\ \frac{A \vdash Z}{\flat Z \vdash {}^{\mathbf{1}}A}$$

$$(.^{\mathbf{1}} \vdash)\ \frac{\sharp A \vdash Z}{A^{\mathbf{1}} \vdash Z} \qquad\qquad (\vdash .^{\mathbf{1}})\ \frac{A \vdash Z}{\sharp Z \vdash A^{\mathbf{1}}}$$

Structural Rules for Negation

$$(\sharp \vdash)\ \begin{array}{c} X > \Phi \vdash Y \\ \hline\hline \sharp X \vdash Y \end{array} \qquad\qquad (\vdash \sharp)\ \begin{array}{c} Y \vdash X > \Phi \\ \hline\hline Y \vdash \sharp X \end{array}$$

$$(\flat \vdash)\ \begin{array}{c} \Phi < X \vdash Y \\ \hline\hline \flat X \vdash Y \end{array} \qquad\qquad (\vdash \flat)\ \begin{array}{c} Y \vdash \Phi < X \\ \hline\hline Y \vdash \flat X \end{array}$$

FIG. 5. Logical and Structural Rules for Negations

An alternative approach is to use explicit structural connectives for these negations [65] with appropriate introduction rules as shown in the top part of Figure 5. These negations can be turned into the "natural" negations mentioned above by adding structural rules which enforce the above definitions for these negations, as shown in the bottom half of Figure 5. Alternatively, these rules can be seen as derivations of the display postulates gc($\sharp, \flat$) and dgc($\sharp, \flat$) rules from our more general set up.

The structural overloading, tonicities and traces are shown below:

Structural Overloading

structural connective	$\sharp$	$\flat$
antecedent position	$.^{\mathbf{1}}$	${}^{\mathbf{1}}.$
succedent position	$.^{\mathbf{0}}$	${}^{\mathbf{0}}.$

Unary Negations		
function	tonicity	trace
$.^{\mathbf{0}}$	$(-)$	$(-) \mapsto +$
${}^{\mathbf{0}}.$	$(-)$	$(-) \mapsto +$
$.^{\mathbf{1}}$	$(-)$	$(+) \mapsto -$
${}^{\mathbf{1}}.$	$(-)$	$(+) \mapsto -$

The relational semantics obtained from Gaggle Theory gives the clauses:

$$\begin{aligned} X^{\mathbf{0}} &= \{\chi \in U \mid \forall \alpha,\ \text{if } \alpha \in X,\ \text{then } R(\alpha, \chi)\} \\ {}^{\mathbf{0}}Y &= \{\chi \in U \mid \forall \alpha,\ \text{if } \alpha \in Y,\ \text{then } R(\chi, \alpha)\} \end{aligned}$$

which are explained by Dunn [27] in terms of a relation "perp". That is, R connects "incompatible" points. Their duals are:

$$\begin{array}{rcl} X^{\mathbf{1}} & = & \{\chi \in U \mid \exists \alpha \in \overline{X} \text{ and } \overline{S}(\alpha, \chi)\} \\ {}^{\mathbf{1}}Y & = & \{\chi \in U \mid \exists \alpha \in \overline{Y} \text{ and } \overline{S}(\chi, \alpha)\} \end{array}$$

7.1 Yetter's Rules Using Explicit Negations

Inspecting the structural rules for $\sharp$ and $\flat$ in Figure 5, observe that the following are all derivable and *invertible*:

$$\begin{array}{c} \sharp X \vdash Y \\ \hline\hline X > \Phi \vdash Y \\ \hline\hline \Phi \vdash X \,;\, Y \end{array} \qquad \begin{array}{c} Y \vdash \sharp X \\ \hline\hline Y \vdash X > \Phi \\ \hline\hline X \,;\, Y \vdash \Phi \end{array}$$

$$\begin{array}{c} \Phi \vdash Y \,;\, X \\ \hline\hline \Phi < X \vdash Y \\ \hline\hline \flat X \vdash Y \end{array} \qquad \begin{array}{c} Y \,;\, X \vdash \Phi \\ \hline\hline Y \vdash \Phi < X \\ \hline\hline Y \vdash \flat X \end{array}$$

But then, in the presence of Yetter's rules (Figure 4), we can pass from the bottom-most sequents of the top pair, to the top-most sequents of the bottom pair (respectively), or vice-versa. For example, from $\Phi \vdash X \,;\, Y$ to $\Phi \vdash Y \,;\, X$ in the left column. The derived rules are:

$$(\text{Yet-}\sharp\flat \vdash)\ \begin{array}{c} \sharp X \vdash Y \\ \hline\hline \flat X \vdash Y \end{array} \qquad (\vdash \text{Yet-}\flat\sharp)\ \begin{array}{c} Y \vdash \sharp X \\ \hline\hline Y \vdash \flat X \end{array}$$

as an *alternative* formulation of Yetter's rules in the explicit negation framework. Why alternative ? Because Yet can be derived via the inverted derivations using $(\text{Yet-}\sharp\flat \vdash)$ and $(\vdash \text{Yet-}\flat\sharp)$.

Referring to Section 4.2, the reader is invited to now prove ${}^{\mathbf{1}}A \dashv\vdash A^{\mathbf{1}}$ and $A^{\mathbf{0}} \dashv\vdash {}^{\mathbf{0}}A$, using $(\text{Yet-}\sharp\flat)$, but without using (Yet), thereby illustrating the collapse of two of the negations as mentioned in the table of Section 4.2.

7.2 Grishin's Rules Using Explicit Negations

Using the derived rules, $(\text{Grn} < \ >)$ and $(\text{Grn} > \ <)$, from Section 5.2, together with the structural rules for $(\vdash \sharp)$, $(\sharp \vdash)$ and $(\vdash \flat)$, $(\flat \vdash)$, from Figure 5, we can obtain:

$$\begin{array}{c} X \vdash Y \\ \hline\hline \flat Y \vdash \sharp X \end{array} \qquad \begin{array}{c} X \vdash Y \\ \hline\hline \sharp Y \vdash \flat X \end{array}$$

These two immediately give ${}^{\mathbf{1}}A \dashv\vdash A^{\mathbf{0}}$ and $A^{\mathbf{1}} \dashv\vdash {}^{\mathbf{0}}A$ so we again have only two negations, but due to a different collapse of connectives relative to that given by Yetter's rules. See Section 4.2.

We can now immediately use the display postulates for $\sharp$ and $\flat$ to derive:

GrnDNeg

$$\frac{X \vdash Y}{\overline{\sharp\sharp X \vdash Y}} \qquad \frac{X \vdash Y}{\overline{X \vdash \flat\flat Y}} \qquad \frac{X \vdash Y}{\overline{\flat\flat X \vdash Y}} \qquad \frac{X \vdash Y}{\overline{X \vdash \sharp\sharp Y}}$$

as an *alternative* formulation of Grishin's rules in the explicit negation framework.

The name GrnDNeg indicates the visual nature of "double negation", which is the traditional way of obtaining classical logic from intuitionistic logic. We have shown that the principle is quite general when viewed from the perspective of explicit negations. This does not imply an involution because the structural connectives are overloaded, so for example, the two occurrence of $\sharp$ mean different things in $\sharp\sharp X$.

7.3 Classicality Using Explicit Negations

Finally, using Grn, and multiplying on the outermost and innermost sides of $\vdash$ by Φ, we can derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{X \vdash Z \; ; \; (\flat Y)}{X \vdash Z \; ; \; (\Phi < Y)}}{X \vdash (Z \; ; \; \Phi) < Y}\,(\mathrm{Grn})}{X \; ; \; Y \vdash Z \; ; \; \Phi}}{X \; ; \; Y \vdash Z} \qquad \dfrac{\dfrac{\dfrac{\dfrac{Z \vdash X \; ; \; Y}{\Phi \; ; \; Z \vdash X \; ; \; Y}}{X > (\Phi \; ; \; Z) \vdash Y}}{(X > \Phi) \; ; \; Z) \vdash Y}\,(\mathrm{Grn})}{(\sharp X) \; ; \; Z \vdash Y}$$

Analogously we can derive rules to display the other components as well, deriving a new set of display postulates with classicality built into them:

$$(\mathrm{Grn\ DP} \vdash)\ \dfrac{\dfrac{X \vdash Z \; ; \; \flat Y}{\overline{X \; ; \; Y \vdash Z}}}{\overline{Y \vdash \sharp X \; ; \; Z}} \qquad (\vdash \mathrm{Grn\ DP})\ \dfrac{\dfrac{Z \; ; \; \flat Y \vdash X}{\overline{Z \vdash X \; ; \; Y}}}{\overline{\sharp X \; ; \; Z \vdash Y}}$$

At first glance it may seem that we do not have the display property since we cannot immediately display the Y' in $X' \; ; \; \sharp Y' \vdash Z'$ directly using the above only. We can display $\sharp Y'$ as shown below left

$$\dfrac{\sharp Y' \vdash \sharp X' \; ; \; Z'}{X' \; ; \; \sharp Y' \vdash Z'} \qquad \dfrac{\dfrac{\flat(\sharp X' \; ; \; Z') \vdash Y'}{\sharp Y' \vdash \sharp X' \; ; \; Z'}\,(\mathrm{dp})}{X' \; ; \; \sharp Y' \vdash Z'}\,(\mathrm{dp})$$

Then appealing to the display postulates for $\flat$ and $\sharp$ from Figure 1 gives us the display property as shown above right.

What have we achieved ? There is some controversy regarding the exact form of the cut rules, and their elimination, in the traditional Gentzen systems for Classical Bi-Lambek logic. The system of Lambek [45], uses implications as primitives, and does not seem to enjoy cut-elimination [40, page 256,footnote 3]. The system of Hudelmaier and Schroeder-Heister [40], uses negations instead of implications as primitives, but enjoys only a limited cut-elimination theorem.

We have presented two solutions.

The first is our formulation of Classical Bi-Lambek logic from Table 1 since it enjoys cut-elimination. It uses implications as primitives.

The second is outlined next. Take the display calculus for Classical Bi-Lambek logic from Table 1 and:

1. replace rp(; , > , <) and drp(; , > , <) by (Grn DP $\vdash$) and ($\vdash$ Grn DP)
2. replace the original Grishin's rules (Grn) by (GrnDNeg) from Section 7.2
3. replace the introduction rules for $\rightarrow$, $\leftarrow$, $\succ$ and $\prec$ by the negation introduction rules from Figure 5
4. define $C \rightarrow A := C^{\mathbf{0}} \oplus A$, $A \leftarrow C := A \oplus C^{\mathbf{1}}$, $C \succ A := C^{\mathbf{0}} \otimes A$ and $A \prec C := A \otimes C^{\mathbf{1}}$ as indicated by Lambek [45, page 222].

The resulting display calculus not only uses negations as primitives, but also "builds-in" classicality via the new display postulates (Grn DP). Since it also obeys Belnap's conditions, it also enjoys cut-elimination. These rules generalise the displayed classical substructural logics of Wansing [79, Section 5] in which there is only one negation, obtained using $*$, since uniformly putting $\sharp := *$ and $\flat := *$ gives exactly Wansing's display postulates.

8 Sheffer's "stroke" and "dagger"

There are two unusual connectives $\succ\!\!\prec$ and $<\!\!\sim\!\!>$ with the traces:

$$\succ\!\!\prec\ : (+,+) \mapsto - \qquad\qquad <\!\!\sim\!\!>\ : (-,-) \mapsto +$$

in the binary display set up which do not appear to have been investigated in depth, at least in terms of sequent systems[6]. These symbols are supposed to represent the "negative" nature of these connectives via the common $\sim$ part. The other parts, arrow heads or arrow tails, come from the implication family or the but-not family. If we use the structural connective $--$ to stand for these connectives in antecedent and succedent positions, then the associated display postulates are [32, page 30]:

$$\mathrm{dgc}(--,--,--)\ \begin{array}{c} Z--Y \vdash X \\ \hline\hline X--Y \vdash Z \\ \hline\hline X--Z \vdash Y \end{array} \qquad\qquad \mathrm{gc}(--,--,--)\ \begin{array}{c} X \vdash Z--Y \\ \hline\hline Z \vdash X--Y \\ \hline\hline Y \vdash X--Z \end{array}$$

Thus, $--$ is its own (dual) Galois-connection, in all places.

These connectives can be given the following introduction rules:

$$(\succ\!\!\prec\ \vdash)\ \frac{A--B \vdash Z}{A \succ\!\!\prec B \vdash Z} \qquad\qquad (\vdash\ \succ\!\!\prec)\ \frac{A \vdash X \quad B \vdash Y}{X--Y \vdash A \succ\!\!\prec B}$$

$$(<\!\!\sim\!\!>\ \vdash)\ \frac{X \vdash A \quad Y \vdash B}{A <\!\!\sim\!\!> B \vdash X--Y} \qquad\qquad (\vdash\ <\!\!\sim\!\!>)\ \frac{Z \vdash A--B}{Z \vdash A <\!\!\sim\!\!> B}$$

The functions have the following tonicities and traces:

---

[6] Michael Dunn has informed me that Allwein and Dunn [2, page 543] mention them explicitly as substructural analogues of the classical logic "truth functions" NAND and NOR. They also give the residuation conditions (effectively our display postulates) mentioned here. He has also informed me that Curry, and Gonzalo Reyes have studied the intuitionistic version (there is only one) of these connectives.

Binary Functions		
function	tonicity	trace
$>\!\!\sim\!\!<$	$(-,-)$	$(+,+) \mapsto -$
$<\!\!\sim\!\!>$	$(-,-)$	$(-,-) \mapsto +$

Then, the display postulates shown here and the introduction rules for these connectives form a display calculus in their own right, which enjoys cut-elimination à la Belnap. Adding them to the previous set up preserves cut-elimination and the full and limited display properties.

Proposition 3.1, Proposition 3.2 and Theorem 3.4 can be extended to these connectives using the "duality" maps:

$$>\!\!\sim\!\!< \;\overset{\sigma}{\longleftrightarrow}\; >\!\!\sim\!\!< \;\overset{\delta\ \Delta}{\longleftrightarrow}\; <\!\!\sim\!\!> \;\overset{\sigma}{\longleftrightarrow}\; <\!\!\sim\!\!>$$

These connectives form a partial-gaggle on their own, as shown below, and hence have a ternary semantics in terms of the two associated ternary relations $R_{<\sim>}$ for $<\!\!\sim\!\!>$ and $S_{>\sim<}$ for $>\!\!\sim\!\!<$:

partial-gaggle (head first)	associated ternary relation
($<\!\!\sim\!\!>$)	$R_{<\sim>}(x, y, z)$
($>\!\!\sim\!\!<$)	$S_{>\sim<}(x, y, z)$

$$\begin{array}{rcl}
X <\!\!\sim\!\!> Y & = & \{\chi \in U \mid \forall\alpha, \beta,\ \text{if } \overline{R_{<\sim>}}(\alpha, \beta, \chi) \text{ then, } \alpha \in \overline{X} \text{ or } \beta \in \overline{Y}\} \\
 & = & \{\chi \in U \mid \forall\alpha, \beta,\ \text{if } \alpha \in X \text{ and } \beta \in Y \text{ then, } R_{<\sim>}(\alpha, \beta, \chi)\} \\
X >\!\!\sim\!\!< Y & = & \{\chi \in U \mid \exists\alpha \in \overline{X} \text{ and } \exists\beta \in \overline{Y} \text{ and } S_{>\sim<}(\alpha, \beta, \chi)\}
\end{array}$$

Jeremy Dawson has pointed out the following simple derivation showing that these connectives are inherently commutative:

$$\cfrac{\cfrac{\cfrac{X - -Y \vdash Z}{Z - -Y \vdash X}\,(\text{dp})}{Z - -X \vdash Y}\,(\text{dp})}{Y - -X \vdash Z}\,(\text{dp})$$

9 Modalities

Wansing [79] has given a display calculus for classical tense logic (see also [43, 34]). Here we show how to generalise Wansing's framework so that modalities can be added to any substructural logic.

Consider the structural and logical introduction rules given in Figure 6. These are different from the traditional display rules for *classical* modal logics [79, 43] because there is no direct connection between $\Diamond$ and $\Box$, and between $\blacklozenge$ and $\blacksquare$, via $*$, since there is no $*$. There is, however, a direct connection between $\blacklozenge$ and $\Box$, and between $\Diamond$ and $\blacksquare$.

These operations have the following tonicities and traces:

$$(\blacklozenge \vdash)\ \frac{\bullet A \vdash X}{\blacklozenge A \vdash X} \qquad \mathrm{rp}(\bullet,\bullet)\ \frac{\bullet X \vdash Y}{\overline{\overline{X \vdash \bullet Y}}} \qquad (\vdash \blacklozenge)\ \frac{X \vdash A}{\bullet X \vdash \blacklozenge A}$$

$$(\Box \vdash)\ \frac{A \vdash X}{\Box A \vdash \bullet X} \qquad (\vdash \Box)\ \frac{X \vdash \bullet A}{X \vdash \Box A}$$

$$(\blacksquare \vdash)\ \frac{A \vdash X}{\blacksquare A \vdash \circ X} \qquad \mathrm{rp}(\circ,\circ)\ \frac{\circ X \vdash Y}{\overline{\overline{X \vdash \circ Y}}} \qquad (\vdash \blacksquare)\ \frac{X \vdash \circ A}{X \vdash \blacksquare A}$$

$$(\Diamond \vdash)\ \frac{\circ A \vdash X}{\Diamond A \vdash X} \qquad (\vdash \Diamond)\ \frac{X \vdash A}{\circ X \vdash \Diamond A}$$

FIG. 6. Display Rules for Modalities

Unary Modalities		
function	tonicity	trace
$\blacklozenge$	$(+)$	$(-) \mapsto -$
$\Box$	$(+)$	$(+) \mapsto +$
$\Diamond$	$(+)$	$(-) \mapsto -$
$\blacksquare$	$(+)$	$(+) \mapsto +$

Once again, they form a display calculus in their own right, and enjoy cut elimination and both the display and the limited display property. Furthermore, their addition to any other display calculi from this paper preserves all these properties.

Since there is no way to distinguish between rp(., .) and drp(., .) when the operations are unary, the duality maps are quite liberal. For each duality map there are two possibilities that preserve Proposition 3.1, Proposition 3.2 and Theorem 3.4:

$$\delta\ \ \Delta \begin{cases} \text{either} & \Diamond \longleftrightarrow \blacksquare \quad \blacklozenge \longleftrightarrow \Box \quad \bullet \longleftrightarrow \bullet \quad \circ \longleftrightarrow \circ \\ \text{or} & \Diamond \longleftrightarrow \Box \quad \blacklozenge \longleftrightarrow \blacksquare \quad \bullet \longleftrightarrow \circ \end{cases}$$

$$\sigma \begin{cases} \text{either} & \Diamond \longleftrightarrow \blacklozenge \quad \Box \longleftrightarrow \blacksquare \quad \bullet \longleftrightarrow \circ \\ \text{or the identity map} & \end{cases}$$

These pairs of connectives also form partial-gaggles of their own:

partial-gaggle (head first)	associated ternary relation
$(\blacklozenge, \Box)$	$R_{\blacklozenge}(x, y)$
$(\Diamond, \blacksquare)$	$S_{\Diamond}(x, y)$

¿From the gaggle-theory we obtain:

$$\begin{aligned} \blacklozenge Y &= \{\chi \in U \mid \exists \beta \in Y \text{ and } R(\beta, \chi)\} \\ \Box Y &= \{\chi \in U \mid \forall \beta \text{ if } R(\chi, \beta) \text{ then } \beta \in Y\} \end{aligned}$$

Display Postulates for Converse

$$\mathrm{rp}(\copyright, \copyright)\ \frac{\copyright X \vdash Y}{\overline{\overline{X \vdash \copyright Y}}}$$

Structural Rules for Converse

$$(\copyright\copyright \vdash)\ \frac{X \vdash Y}{\overline{\overline{\copyright\copyright X \vdash Y}}} \qquad (\vdash \copyright\copyright)\ \frac{X \vdash Y}{\overline{\overline{X \vdash \copyright\copyright Y}}}$$

Introduction Rules for Converse

$$(\smile\vdash)\ \frac{\copyright A \vdash Z}{\smile A \vdash Z} \qquad (\vdash \smile)\ \frac{Z \vdash \copyright A}{Z \vdash \smile A}$$

Additional Structural Rules for Converse

$$(\copyright\Phi \vdash)\ \frac{\Phi \vdash X}{\overline{\overline{\copyright\Phi \vdash X}}} \qquad (\vdash \copyright\Phi)\ \frac{X \vdash \Phi}{\overline{\overline{X \vdash \copyright\Phi}}}$$

$$(\copyright\mathrm{I} \vdash)\ \frac{\mathrm{I} \vdash X}{\overline{\overline{\copyright\mathrm{I} \vdash X}}} \qquad (\vdash \copyright\mathrm{I})\ \frac{X \vdash \mathrm{I}}{\overline{\overline{X \vdash \copyright\mathrm{I}}}}$$

$$(\ \mathrm{dist}\ \vdash)\ \frac{\copyright(X\ ;\ Y) \vdash Z}{\overline{\overline{(\copyright Y)\ ;\ (\copyright X) \vdash Z}}} \qquad (\vdash\ \mathrm{dist}\ )\ \frac{Z \vdash \copyright(X\ ;\ Y)}{\overline{\overline{Z \vdash (\copyright Y)\ ;\ (\copyright X)}}}$$

FIG. 7. Logical and Structural Rules for Converse

which are the standard clauses we expect from the binary Kripke semantics for tense logics. The clauses

$$\begin{array}{rcl} \blacksquare Y & = & \{\chi \in U \mid \forall\beta,\ \text{if } \overline{S}(\beta,\chi) \text{ then, } \beta \in Y\} \\ \Diamond Y & = & \{\chi \in U \mid \exists\beta \in Y \text{ and } \overline{S}(\chi,\beta)\} \end{array}$$

seem non-standard but defining $S' := \overline{S}$ gives

$$\begin{array}{rcl} \blacksquare Y & = & \{\chi \in U \mid \forall\beta,\ \text{if } S'(\beta,\chi) \text{ then, } \beta \in Y\} \\ \Diamond Y & = & \{\chi \in U \mid \exists\beta \in Y \text{ and } S'(\chi,\beta)\} \end{array}$$

as we expect from the binary Kripke semantics for tense logics. See also Dunn's [25] for further details.

9.1 Converse

Many substructural logics admit a natural analogue of the converse operation $\smile$ familiar from relation algebras [50]. The display rules shown in Figure 7 for this operation essentially come from [35], although you will need to make some adjustments. Thus converse is an isotonic, self-dual operation of period two. That is, it has the tonicity and traces shown below:

Unary Converse

function	tonicity	trace
$\smile$	$(+)$	$(+) \mapsto +$
$\smile$	$(+)$	$(-) \mapsto -$

and also satisfies $\smile\smile A = A$.

Further structural rules [35] can be added so that converse satisfies:

$$\begin{array}{rclcrcl} \smile \mathbf{1} & = & \mathbf{1} & \quad & \smile \mathbf{0} & = & \mathbf{0} \\ \smile \top & = & \top & & \smile \bot & = & \bot \\ \smile (A \otimes B) & = & (\smile B \otimes \smile A) & & \smile (A \oplus B) & = & (\smile B \oplus \smile A) \end{array}$$

Note that most of these additional structural rules can be unidirectional since their inverses can be derived with the help of the (dp) rules because converse is self-dual. Since $\smile$ is a self-dual and is isotonic, all duality maps must use the identity map to preserve Proposition 3.1, Proposition 3.2 and Theorem 3.4.

10 Embedding Intuitionistic Logic Via Modalities

Lambek [45, page 233] and Belnap [7, page 38] have observed that the main purpose of the "modalities" ! and ? of Girard's linear logic [30] is to embed intuitionistic logic into linear logic. Belnap [9] has already given a display calculus for such modalities, and also observed that the embedding is really a trick since a true residual implication for extensional conjunction immediately gives distribution [7, page 38]. Here we follow Lambek, who suggests that Girard's "!" should be left adjoint to "?" [45, page 233] when viewed from the perspective of category theory.

We have already seen two sets of unary connectives which form "adjoint functors", namely $(\blacklozenge, \Box)$ and $(\Diamond, \blacksquare)$. The upshot is that under Lambek's suggestion, the modality ! is really a diamond-operator (either $\blacklozenge$ or $\Diamond$) and the modality ? is really a box-operator (respectively either $\Box$ or $\blacksquare$): the colour switch here is crucial since neither $(\blacklozenge, \blacksquare)$ nor $(\Diamond, \Box)$ are adjoint functors.

Thus Lambek's ! and ? are not true duals, but residuals. These modalities are at odds with Girard's ones where ! can be seen as a $\Box$ and ? as a $\Diamond$. Moreover, given our introduction rules for $\wedge$ and $\vee$ from Figure 2, Lambek's modalities obey $!(A \vee B) \dashv\vdash (!A \vee !B)$, which as one referee pointed out, should make "linear logic people squirm". We therefore avoid Girard's notation and use $(\blacklozenge, \Box)$ instead of "!" and "?". We could equally well have used $(\Diamond, \blacksquare)$ instead.

In Figure 8 we give the rules for the modalities. The display postulates make $\bullet$ into a unary, isotonic structural connective [79]. The structural rules Dens and Pers are sufficient to enforce $\blacklozenge A \to \blacklozenge\blacklozenge A$ and $\blacklozenge A \to A$ as well as their "resi-dual contra-positives" $\Box\Box A \to \Box A$ and $A \to \Box A$ [43]. These structural rules for $\bullet$ force the underlying Kripke reachability relation to be "dense" [79, 43, 34] and "persistent" [38], hence the names.[7]

Now, following Lambek [45], we define a new metalevel turnstile $\vdash_i$ to simulate

[7] I suspect that using "persistence" may be using the Gödel translation as is done in [38].

Display Postulates for Lambek's Modalities

$$\text{rp}(\bullet,\bullet)\ \frac{\bullet X \vdash Y}{\overline{\overline{X \vdash \bullet Y}}}$$

Structural Rules for Lambek's Modalities

$$(\text{ Dens} \bullet \vdash)\ \frac{\bullet \bullet X \vdash Y}{\bullet X \vdash Y} \qquad (\vdash\ \text{Dens} \bullet)\ \frac{X \vdash \bullet \bullet Y}{X \vdash \bullet Y}$$

$$(\text{ Pers} \bullet \vdash)\ \frac{X \vdash Y}{\bullet X \vdash Y} \qquad (\vdash\ \text{Pers} \bullet)\ \frac{X \vdash Y}{X \vdash \bullet Y}$$

Introduction Rules for Lambek's Modalities

$$(\blacklozenge \vdash)\ \frac{\bullet A \vdash Z}{\blacklozenge A \vdash Z} \qquad (\vdash \blacklozenge)\ \frac{Z \vdash A}{\bullet Z \vdash \blacklozenge A}$$

$$(\Box \vdash)\ \frac{A \vdash Z}{\Box A \vdash \bullet Z} \qquad (\vdash \Box)\ \frac{Z \vdash \bullet A}{Z \vdash \Box A}$$

Additional Structural Rules for Lambek's Modalities

$$(\bullet\Phi^{+}_{-} \vdash)\ \frac{\Phi \vdash X}{\overline{\overline{\bullet\Phi \vdash X}}} \qquad (\vdash \bullet\Phi^{+}_{-})\ \frac{X \vdash \Phi}{\overline{\overline{X \vdash \bullet\Phi}}}$$

$$(\bullet\text{Com} \vdash)\ \frac{\bullet Y \ ;\ X \vdash Z}{\overline{\overline{X\ ;\ \bullet Y \vdash Z}}} \qquad (\vdash \bullet\text{Com})\ \frac{Z \vdash \bullet Y\ ;\ X}{\overline{\overline{Z \vdash X\ ;\ \bullet Y}}}$$

$$(\bullet\text{Wk } 1 \vdash)\ \frac{\bullet X \vdash Z}{\bullet X\ ;\ \bullet Y \vdash Z} \qquad (\vdash \bullet\text{Wk } 1)\ \frac{Z \vdash \bullet X}{Z \vdash \bullet X\ ;\ \bullet Y}$$

$$(\bullet\text{Wk } 2 \vdash)\ \frac{\bullet Y \vdash Z}{(\bullet X)\ ;\ (\bullet Y) \vdash Z} \qquad (\vdash \bullet\text{Wk } 2)\ \frac{Z \vdash \bullet Y}{Z \vdash (\bullet X)\ ;\ (\bullet Y)}$$

$$(\bullet\text{Ctr} \vdash)\ \frac{(\bullet X)\ ;\ (\bullet X) \vdash Z}{\bullet X \vdash Z} \qquad (\vdash \bullet\text{Ctr})\ \frac{Z \vdash (\bullet X)\ ;\ (\bullet X)}{Z \vdash \bullet X}$$

Fig. 8. Logical and Structural Rules for Lambek's Modalities

intuitionistic turnstile, and also define intuitionistic implication $\Rightarrow$:

$$(A_1 \vdash_i A_2) \quad := \quad (\blacklozenge A_1 \vdash A_2) \tag{10.1}$$

$$(A_1 \Rightarrow A_2) \quad := \quad (\blacklozenge A_1 \to \Box A_2) \tag{10.2}$$

As Belnap [7, page 38] and Lambek [45, page 233] observe, we have embedded intuitionistic logic if we can create a "residual" for extensional conjunction. That is, if the invertible rule ($\vdash_i\!\to$), shown below left, is admissible. Via the above translation, this requirement is really as shown below right:

$$(\vdash_i\!\to)\ \frac{A \wedge B \vdash_i C}{\overline{B \vdash_i A \Rightarrow C}} \qquad\qquad \frac{\blacklozenge(A \wedge B) \vdash C}{\overline{\blacklozenge B \vdash (\blacklozenge A) \to (\Box C)}}$$

The proofs of the right hand side derivations are easy:

$$
\begin{prooftree}
\AxiomC{$\blacklozenge B \vdash \blacklozenge A \to \Box C$}
\AxiomC{$\blacklozenge A \vdash \blacklozenge A$}
\AxiomC{$C \vdash C$}
\RightLabel{$(\Box \vdash)$}
\UnaryInfC{$\Box C \vdash \bullet C$}
\RightLabel{$(\to\vdash)$}
\BinaryInfC{$\blacklozenge A \to \Box C \vdash \blacklozenge A > (\bullet C)$}
\RightLabel{(cut)}
\BinaryInfC{$\blacklozenge B \vdash \blacklozenge A < (\bullet C)$}
\RightLabel{(dp)}
\UnaryInfC{$\blacklozenge A \ ; \ \blacklozenge B \vdash \bullet C$}
\RightLabel{$(\otimes \vdash)$}
\UnaryInfC{$\blacklozenge A \otimes \blacklozenge B \vdash \bullet C$}
\RightLabel{(dp)}
\UnaryInfC{$\bullet(\blacklozenge A \otimes \blacklozenge B) \vdash C$}
\RightLabel{$(\blacklozenge \vdash)$}
\UnaryInfC{$\blacklozenge(\blacklozenge A \otimes \blacklozenge B) \vdash C$}
\AxiomC{shown as (a)}
\UnaryInfC{$\blacklozenge(A \wedge B) \vdash \blacklozenge(\blacklozenge A \otimes \blacklozenge B)$}
\RightLabel{(cut)}
\BinaryInfC{$\blacklozenge(A \wedge B) \vdash C$}
\end{prooftree}
$$

Proof of $\blacklozenge(A \wedge B) \vdash C$ assuming $\blacklozenge B \vdash \blacklozenge A \to \Box C$

$$
\begin{prooftree}
\AxiomC{shown as (b)}
\UnaryInfC{$\blacklozenge(\blacklozenge A \otimes \blacklozenge B) \vdash \blacklozenge(A \wedge B)$}
\AxiomC{trivial}
\UnaryInfC{$\bullet(\blacklozenge A \ ; \ \blacklozenge B) \vdash \blacklozenge(\blacklozenge A \otimes \blacklozenge B)$}
\RightLabel{(cut)}
\BinaryInfC{$\bullet(\blacklozenge A \ ; \ \blacklozenge B) \vdash \blacklozenge(A \wedge B)$}
\AxiomC{$\blacklozenge(A \wedge B) \vdash C$}
\RightLabel{(cut)}
\BinaryInfC{$\bullet(\blacklozenge A \ ; \ \blacklozenge B) \vdash C$}
\RightLabel{(dp)}
\UnaryInfC{$\blacklozenge A \ ; \ \blacklozenge B \vdash \bullet C$}
\RightLabel{$(\vdash \Box)$}
\UnaryInfC{$\blacklozenge A \ ; \ \blacklozenge B \vdash \Box C$}
\RightLabel{(dp)}
\UnaryInfC{$\blacklozenge B \vdash \blacklozenge A < \Box C$}
\RightLabel{$(\vdash\to)$}
\UnaryInfC{$\blacklozenge B \vdash \blacklozenge A \to \Box C$}
\end{prooftree}
$$

Proof of $\blacklozenge B \vdash \blacklozenge A \to \Box C$ assuming $\blacklozenge(A \wedge B) \vdash C$

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{(Pers $\vdash$)}
\UnaryInfC{$\bullet A \vdash A$}
\RightLabel{(Wk 1 $\vdash$)}
\UnaryInfC{$\bullet A \ ; \ \bullet B \vdash A$}
\RightLabel{($\blacklozenge \otimes \blacklozenge \vdash$)}
\UnaryInfC{$\blacklozenge A \otimes \blacklozenge B \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{(Pers $\vdash$)}
\UnaryInfC{$\bullet B \vdash B$}
\RightLabel{(Wk 2 $\vdash$)}
\UnaryInfC{$\bullet A \ ; \ \bullet B \vdash B$}
\RightLabel{($\blacklozenge \otimes \blacklozenge \vdash$)}
\UnaryInfC{$\blacklozenge A \otimes \blacklozenge B \vdash B$}
\RightLabel{($\vdash \wedge$)}
\BinaryInfC{$\blacklozenge A \otimes \blacklozenge B \vdash A \wedge B$}
\RightLabel{($\vdash \blacklozenge$)}
\UnaryInfC{$\bullet(\blacklozenge A \otimes \blacklozenge B) \vdash \blacklozenge(A \wedge B)$}
\RightLabel{($\blacklozenge \vdash$)}
\UnaryInfC{$\blacklozenge(\blacklozenge A \otimes \blacklozenge B) \vdash \blacklozenge(A \wedge B)$}
\end{prooftree}
$$

(b)

$$
\begin{prooftree}
\AxiomC{$\bullet A \ ; \ \bullet B \vdash X$}
\RightLabel{(dp)}
\UnaryInfC{$\bullet B \vdash (\bullet A) > X$}
\RightLabel{($\blacklozenge \vdash$)}
\UnaryInfC{$\blacklozenge B \vdash (\bullet A) > X$}
\RightLabel{(dp)}
\UnaryInfC{$\bullet A \ ; \blacklozenge B \vdash X$}
\RightLabel{(dp)}
\UnaryInfC{$\bullet A \vdash X < (\blacklozenge B)$}
\RightLabel{($\blacklozenge \vdash$)}
\UnaryInfC{$\blacklozenge A \vdash X < (\blacklozenge B)$}
\RightLabel{(dp)}
\UnaryInfC{$\blacklozenge A \ ; \blacklozenge B \vdash X$}
\RightLabel{($\otimes \vdash$)}
\UnaryInfC{$\blacklozenge A \otimes \blacklozenge B \vdash X$}
\end{prooftree}
$$

derived rule ($\blacklozenge \otimes \blacklozenge \vdash$)

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{($\wedge \vdash$)}
\UnaryInfC{$A \wedge B \vdash A$}
\RightLabel{($\vdash \blacklozenge$)}
\UnaryInfC{$\bullet(A \wedge B) \vdash \blacklozenge A$}
\AxiomC{$B \vdash B$}
\RightLabel{($\wedge \vdash$)}
\UnaryInfC{$A \wedge B \vdash B$}
\RightLabel{($\vdash \blacklozenge$)}
\UnaryInfC{$\bullet(A \wedge B) \vdash \blacklozenge B$}
\RightLabel{($\vdash \otimes$)}
\BinaryInfC{$\bullet(A \wedge B) \ ; \ \bullet(A \wedge B) \vdash \blacklozenge A \otimes \blacklozenge B$}
\RightLabel{(Ctr$\blacklozenge \vdash$)}
\UnaryInfC{$\bullet(A \wedge B) \vdash \blacklozenge A \otimes \blacklozenge B$}
\RightLabel{($\vdash \blacklozenge$)}
\UnaryInfC{$\bullet \bullet (A \wedge B) \vdash \blacklozenge(\blacklozenge A \otimes \blacklozenge B)$}
\RightLabel{(Dens $\bullet \vdash$)}
\UnaryInfC{$\bullet(A \wedge B) \vdash \blacklozenge(\blacklozenge A \otimes \blacklozenge B)$}
\RightLabel{($\blacklozenge \vdash$)}
\UnaryInfC{$\blacklozenge(A \wedge B) \vdash \blacklozenge(\blacklozenge A \otimes \blacklozenge B)$}
\end{prooftree}
$$

(a)

Note that each of Pers , Dens , Wk 1, Wk 2 and Ctr$\bullet$ are absolutely essential. The commutativity rules $\bullet$Com are needed to prove $A \vdash_i ((A \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0})$. The reader is invited to check that $((A \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0}) \vdash_i A$ fails. Since all other moves in these proofs use rules of δBiL, this embedding works in all our displayed substructural logics.

Ono [59] gives a sequent system for Girard's exponentials together with associated algebraic semantics. MacCaull [48] has given a semantics for this system using binary Kripke relations. MacCaull uses two binary relations with ! interpreted as a diamond "looking backward" along F as in tense logic, but with ? "looking forward" along G in a complex way.

11 Adding Exponentials

It is clear that Lambek's suggestion for adding exponentials is not completely satisfactory since the distribution property $!(A \vee B) \dashv\vdash (!A \vee !B)$ is a by-product of the syntactical form of the rules and the display property. It should be possible to mimic MacCaull's [48] relational calculus by using two binary relations F and G to display the exponentials via two independent structural connectives $\circ_F$ and $\circ_G$ say, each with display postulates of the form $\mathrm{rp}(\circ_F, \circ_F)$ and $\mathrm{rp}(\circ_G, \circ_G)$. But there is an existing solution due to Belnap [9], although Belnap uses a form of display logic which depends upon an involutive operation $*$. We follow his suggestion to "add the exponentially restricted rules we liked" [9, page 21] to show that his solution works in our more general framework.

We now use $!$ and $?$ since these modalities are like the exponentials of Girard [30].

We require an explicit notion of "polarity". In a structure Z, a substructure Y is a *positive [negative] part* if

1. Y is an antecedent [succedent] part when Z is an antecedent part and
2. Y is a succedent [antecedent] part when Z is a succedent part.

Following Belnap [9, page 18], a structure Y is **[dual] exponentially restricted** if whenever Y contains a formula B, if B is a positive part of Y, then B has the form $[?A]$ $!A$, and if B is a negative part of Y, then B has the form $[!A]$ $?A$. Furthermore, Y must not contain Φ as a [positive] negative part.

$$(\mathrm{Com}^{\mathrm{e}} \vdash)\ \frac{Y^{\mathrm{e}} \,;\, X \vdash Z}{X \,;\, Y^{\mathrm{e}} \vdash Z} \qquad\qquad (\vdash \mathrm{Com}^{\mathrm{d}})\ \frac{Z \vdash Y^{\mathrm{d}} \,;\, X}{Z \vdash X \,;\, Y^{\mathrm{d}}}$$

$$(\mathrm{Ctr}^{\mathrm{e}} \vdash)\ \frac{Y^{\mathrm{e}} \vdash Z}{Y^{\mathrm{e}} \,;\, Y^{\mathrm{e}} \vdash Z} \qquad\qquad (\vdash \mathrm{Ctr}^{\mathrm{d}})\ \frac{Z \vdash Y^{\mathrm{d}}}{Z \vdash Y^{\mathrm{d}} \,;\, Y^{\mathrm{d}}}$$

$$(\mathrm{Wk}^{\mathrm{e}}2 \vdash)\ \frac{X \vdash Z}{X \,;\, Y^{\mathrm{e}} \vdash Z} \qquad\qquad (\vdash \mathrm{Wk}^{\mathrm{d}}2)\ \frac{Z \vdash X}{Z \vdash X \,;\, Y^{\mathrm{d}}}$$

$$(\mathrm{Wk}^{\mathrm{e}}1 \vdash)\ \frac{X \vdash Z}{Y^{\mathrm{e}} \,;\, X \vdash Z} \qquad\qquad (\vdash \mathrm{Wk}^{\mathrm{d}}1)\ \frac{Z \vdash X}{Z \vdash Y^{\mathrm{d}} \,;\, X}$$

$$(?\vdash)\ \frac{A \vdash Y^{\mathrm{d}}}{?A \vdash Y^{\mathrm{d}}} \qquad\qquad (\vdash ?)\ \frac{X \vdash A}{X \vdash ?A}$$

$$(!\vdash)\ \frac{A \vdash X}{!A \vdash X} \qquad\qquad (\vdash !)\ \frac{Y^{\mathrm{e}} \vdash A}{Y^{\mathrm{e}} \vdash !A}$$

Y^{e} is exponentially restricted.

Y^{d} is dual exponentially restricted.

FIG. 9. Structural and Introduction Rules for Exponentials

The introduction rules in Figure 9 give the exponentials as studied by Ono [59, page 279], who also shows that they generalise Girard's exponentials in a substructural setting. Ono also gives many results connecting these calculi to quantales and phase structures. But notice that our rules are equally viable without commutativity or associativity for $;$, and furthermore, are not affected at all by whether the logic is classical or intuitionistic. The reader is invited to verify the formula versions of each of the algebraic properties 1-10 on page 280 of [59]. They are all provable in the display logic framework outlined above. By working through the details given by Ono, it should be possible to obtain a completeness theorem for our modal substructural logics with respect to "modal unital quantales" and "modal phase structures" [59].

Theorem 11.1 *The addition of exponentials does not perturb cut-elimination.*

PROOF. The calculi obtained by adding the exponential rules to the basic display calculi of previous sections obey Belnap's modified conditions of "regularity" [9]. ∎

Notice that now we speak of whole calculi since the "regularity" conditions must be checked with respect to each calculus, rather than with respect to each individual rule. But see [9] for details.

12 Background and Related Work

As one referee has pointed out, the literature in this field is vast. Here I attempt to review other work which is related to this work. My background in category theory is limited so I am sure that I have omitted a vast amount of work from this community. The following quote from Lambek [45, page 210] is worth remembering:

> *In fact, the passage from syntax to semantics may be viewed as introducing the three structural rules.*

12.1 Logical Variation Via Structural Rules

As far as I am aware, the initial proposal to separate structural rules from logical rules, and obtain many different logics by varying the structural rules, seems to be due (independently) to Došen [17, 18, 19] and Belnap [8]. Došen seeks to "characterize logical constants[8] syntactically" [18, abstract] whereas Belnap's motivation is more practical: "Display Logic is essentially a proof-theoretical tool" [8, page 379] so that "you can make up hybrid logics by mixing families as you wish" [8, page 395]. For Došen, the separation is a way to "provide an illustration of how alternative logics differ only in their *structural* rules, whereas their rules for *logical constants* are identical." [18, abstract]. For Belnap, the fact that "the same set of formula connective postulates is used for every family" is "Display Logic's way of making sense out of everyone's sense of family resemblance." [8, page 382].

12.2 The Display Postulates rp(; , > , <) and drp(; , > , <)

Belnap [8] credits the idea of using multiple structural connectives to the independent work of Mints [55] and Dunn [23]. But, as we have seen, the original display logic rules do not quite suffice, primarily because of the presence of Boolean negation in the form of the $*$ connective. For a discussion of how much can be done with $*$ see [32]. Our rules for extensional conjunction and disjunction come directly from Belnap since he explicitly mentions "the possibility of 'structure free' formula-connectives" [8, page 410], where he also notes that "distribution cannot be obtained for these formula-connectives without appeal to structural elements". For a more detailed discussion of distribution and its failure, see also [7, page 34]. Our structural connectives

[8] our logical connectives

$<$ and $>$ trace their origins to [31] where they appear as r_1 and r_2, and were motivated by residuation and dual-residuation for binary connectives, and a desire to "display" Dunn's Gaggle Theory [26]. However, a re-reading of Belnap shows that he had anticipated the possibility to "replace $*$ in each family by a pair of binary structural connectives $X - Y$ and $X - -Y$, thinking of X as positive and Y as negative substructures" [8, page 409]. Although Belnap builds commutativity into the associated display postulates [8, page 409], he immediately suggests that "one might look at the case when one refuses to postulate commutativity for $\circ$[9] on the right of turnstile" [8, page 409]. The idea of using such display postulates, together with their duals, in one uniform, yet modular, framework [31] seems to be new.

12.3 Symmetry, Duality and Dual Logical Connectives

A first attempt at displaying a substructural logic containing dual logical (and structural) connectives can be found in [33]. The fact that display calculi exhibit symmetry and duality can be found in Kracht's [43]. The results from Section 3 were found while extending Kracht's notions of duality and symmetry to handle the extra connectives of Bi-Lambek logic [45]. Although I did not realise this at the time, the σ transformation of Section 3 can be seen as a manifestation of Lambek's desire that "*the order of two letters was never to be wantonly interchanged, yet all rules were to be preserved under left-right symmetry, the guiding slogan being 'symmetry without commutativity'.*" [45, page 207]. Similarly, the $\vartriangle$ transformation can be seen as a manifestation of Lambek's "*invariance under arrow reversal, replacing ... by ... and vice-versa*" [45, page 214]. The results of Section 3 are thus manifestations of the categorial notions underlying Bi-Lambek logic.

12.4 Traditional Gentzen Systems for Classical Substructural Logics

Intuitionistic and classical substructural logics without dual connectives have been studied extensively by Došen [19, 20]. Intuitionistic substructural logics containing dual logical connectives have received some attention [41, 68], but I know of no *uniform* study of classical substructural logics containing all the dual connectives. Abrusci [1] and Hudelmaier and Schroeder-Heister [40] restrict themselves to classical exponential-free Bi-Lambek (= pure non-commutative classical linear) logic. The ease with which we obtain these classical versions from the intuitionistic versions, using an already known rule (Grishin's rule) indicates the close fit between substructural logics and our generalised display calculus. A further advantage is the simplicity of our rules compared with the rules in [40, page 254]. Incidentally, Lambek [45, page 209] points out that Grishin [39] seems to have anticipated the axioms for Bi-Linear logic in 1983 (in Russian).

12.5 General Proof Systems

There are numerous generalised "proof-theoretic" frameworks for substructural logics. Here, we briefly discuss their relationships to our work. A good discussion of various Gentzen formulations for modal logics is contained in Wansing's [79].

12.5.1 Belnap's Display Logic

The original display logic of Belnap [8] was designed to display relevant logics where the connective we call $\oplus$ is usually commutative, although "fusion" (our $\otimes$) is not.

[9] our " ; "

This explains why Belnap's original calculus builds-in commutativity for $\circ$ (our " ; ") in succedent positions. In a subsequent paper, Belnap [10] attempts to dissect the intricacies of using $*$, but some of his subsystems actually fail to have the display property, see [32]. Restall [67] remedies many of these anomalies, but also builds in commutativity for $\oplus$ in succedent positions. Wansing [79] simplifies display logic to handle unary (modal) connectives, but also uses the $*$ connective since it is a natural way to capture *classical* modal logics.

12.5.2 Došen's Systems

As stated already, Došen [18] has also considered sequent systems where logical variations are obtained from changes only in the structural rules. The main tool there is the use of nested turnstiles up to arbitrary depths, and the use of (double line) introduction rules which are explicitly stated as being invertible. Došen concludes that "the double-line rules for logical constants[10] could show that logical constants serve, so to speak, as punctuation marks for some structural features of deductions ... Implication is up to a point a substitute for the turnstile at level 0: it can reduce a deduction of level 1 to a formula of level 0. Conjunction and disjunction serve to economize: they reduce to one deduction two deduction which differ only at one place." [18, page 166]. Masini [53] uses a similar idea for intuitionistic modal logics.

In [19, 20] Došen gives sequent systems for many substructural logics, albeit without dual connectives, using "G-terms" built from formulae using the binary structural connective "," and the nullary structural connective $\varnothing$. Došen's rules build in a notion of substitution since they are applicable to an arbitrarily nested occurrence of a structure within another structure. In this way, the need for the display property does not arise. Since the only binary structural connective "," is isotonic in both arguments, there is also no need to worry about whether the occurrence is an antecedent or succedent part. It is an antecedent [succedent] part if it appears in the left [right] hand side of a sequent. Structural rules, also couched in terms of occurrences at an arbitrary nesting depth, then give different logics.

Došen's cut rule explicitly builds in the notion of substitution, thereby avoiding the need for display postulates. Cut-elimination is proved on a case by case basis, and there is no analogue of Belnap's conditions. See [74] for a correction to the cut-elimination proof in some cases.

12.5.3 Hypersequents

Hypersequents, invented independently by Pottinger [64] and Avron [4] have been used to obtain cut-free formalisations of many non-classical logics [5, 6]. Wansing [82] shows that, at least for some logics, display calculi have some advantages over hypersequents. Wansing gives a thorough description of hypersequents, so we omit details here. But I am not aware of a general cut-elimination theorem for hypersequents.

12.5.4 Labelled Deductive Systems

Gabbay's Labelled Deductive Systems [29] have been applied to obtain labelled tableaux systems for intuitionistic and classical substructural logics [13] and also to the implicational fragments of substructural modal logics [14]. I am not aware of any extensions to handle dual connectives in the same framework.

---

[10] our logical connectives

The biggest advantage of the labelled deductive systems of D'Agostino and Gabbay [13] is that they make no attempt to eliminate cut. Instead, they use it in a demand driven and analytic manner which *cannot* be simulated polynomially using the traditional rules [15]. Hence, Labelled Deductive Systems lend themselves to decision procedures and automation much more easily than display calculi, although the current work of Jeremy Dawson [16] shows that display calculi can also be mechanised using existing automated deduction tools. Note however, that the labelled deductive systems of D'Agostino and Gabbay [13] essentially compile the logical deduction problem into the algebra of labels, and then seek a solution to an algebraic equation (constraints) among the labels. Of course, the algebraic equational problem will have the same degree of difficulty as the logical problem.

12.5.5 Categorial Type Logics

The plethora of work on Categorial Type Logics[11] [57, 58] conducted by Kurtonina, Moortgat, Morrill, Oehrle, van Benthem, and many many others [57] covers Gentzen systems, natural deduction systems, lambda calculi, term rewriting systems, and even labelled deductive systems for many of the logics we have "displayed". Some are even refined versions of display logic [57, Section 4], which, following Došen [19], avoid the display postulates altogether by "compiling" them into the rules, but which still enjoy cut-elimination. There are also *modular* "residuation calculi" to handle substructural multi-modal multi-dimensional logics, with "interaction postulates" for connecting them together, and numerous completeness results with respect to ternary relational semantics [57]. However, classicality, dual connectives, the various negations, and the Sheffer "stroke" and "dagger" connectives of Section 8 do not seem to have been investigated by this community, possibly because such notions may not have a clear linguistic application.[12] They also do not seem to have pursued a general cut-elimination theorem. Their ideas on "compiling" residuation into the rules, and on building hybrid logics using residuation calculi should benefit our studies immensely.

12.5.6 Relational Proof Systems

Orlowska's Relational Proof Systems [62] have been extended to (at least) modal logics [63] and to the traditional substructural logics [48] *with exponentials*. Relational Proof Systems encode the semantics for these logics directly, using object level formulae of the form $R(x, y, z)$ where the variables x, y and z range over the points of the underlying Kripke models. The resulting proof systems are traditionally formulated as dual-tableaux systems. In a sense, Relational Proof Systems can also be seen as Labelled Deductive Systems, since the variables x, y and z name the points in the underlying semantics. I am not aware of any Relational Proof Systems for substructural logics with dual connectives, although it is obvious that adding decomposition rules for another relation S, in addition to those for R, does the job.

Relational Proof Systems also split the rules into logical and structural ones, with the logical rules remaining constant and all variations coming through the "specific rules" for the logic in question. These rules are particularly easy to invent since they are directly determined by the special properties of the underlying relational semantics we wish to capture.

[11] Pointed out to me by one of the reviewers.

[12] I am not a linguist so this is just a guess.

Some relational proof systems require no cut rules at all, but the systems presented by MacCaull [48] require an *analytic* cut rule for completeness. That is, it may be necessary to apply cut on certain formulae of the form $R(x, y, z)$ or $I(x, y, z)$, although these applications are restricted to (the finite number of) variables which *already occur* on the tableau branch. However, the question of *general* cut-elimination has not been fully resolved for Relational Proof Systems for substructural logics containing both the intensional and extensional connectives [49].

12.5.7 Basic Logic

The Basic Logic of Sambin, Battilotti and Faggian [73] is another powerful Gentzen framework for substructural logics with dual logical connectives. Below, I use quotes to introduce technical terms of Basic Logic since space forbids detailed definitions.

Basic Logic is motivated by a philosophical "principle of reflection" showing how to obtain the introduction rules for a logical connective by solving a "definitional equation" between a meta-level concept and that logical connective. Basic logic also has a general notion of "symmetry", which is identical to our δ-transform. Cut is viewed as a form substitution, and implication $\rightarrow$ is seen as an object level counterpart of the turnstile.

In Basic Logic, for every connective, one introduction rule ρ can be used to derive the other using "reflection", which, put simplistically, demands that ρ be invertible. Symmetry then gives the dual connective rules for free. The idea that one introduction rule can be derived from the other using invertibility considerations also appears independently in [31], as does the idea that "duality" gives the other half for free. The first idea is also touched upon in [8, page 415, note 14] and appears to be folklore.

Basic Logic contains rules for introducing (a formula containing) each connective into the right and left of turnstile, using comma as the only additional structural connective. The introduced formula is always "visible" as the whole of the antecedent or the succedent. Extensions of Basic Logic are obtained by "liberalising" this condition to allow the rules to introduce formulae into antecedents and succedents (contexts) which are non-empty. Liberalising contexts on the left of turnstile gives intuitionistic logics, liberalising contexts on the right of turnstile gives dual intuitionistic logics, while liberalising contexts on both sides, gives classical logics. The traditional structural rules of exchange, contraction and weakening then give further variations within these types of substructural logics.

Basic Logic enjoys a cut-elimination theorem for highest cuts which is also *general* in that the cut-elimination proof is easily extended to the extensions obtained by liberalising contexts or by adding further structural (in the sense of Basic Logic) rules. Moreover, since it is not dependent upon any display postulates, it immediately allows access to the usual decidability arguments. Such arguments are also possible in display logic [67], but are not as easy. In some ways, Basic Logic can be seen as a "compiled" version of display logic since Belnap's conditions are essentially built into the rules of Basic Logic. The cut-elimination procedure of Basic Logic involves tracking the "history" of a formula occurrence until the formula becomes principal, which is exactly Belnap's notion of tracking the "congruence class" of a formula occurrence [8, 35]. Whether Basic Logic enjoys strong cut-elimination, or enjoys cut-elimination for *any* cut as discussed at the end of Section 2, appears to be an open question.

Basic Logic does not contain all the binary logical connectives we have presented, although they can be added. In particular, Basic Logic does not make a distinction between our "→" and "←", nor between their duals. Consequently, in Basic Logic, $\delta = \Delta$ and $\sigma = \iota$, see Section 3. Any comparison of Basic Logic and (our incarnation of) Display Logic is fraught with danger because of the differences in the symbols. Here is a map to help:

Basic Logic	$\&$	$\oplus$	$\otimes$	$\invamp$	$\mathbf{1}$	$\perp$	$\top$	$\mathbf{0}$	$\leftarrow$	$\rightarrow$
Display Logic	$\wedge$	$\vee$	$\otimes$	$\oplus$	$\mathbf{1}$	$\mathbf{0}$	$\top$	$\perp$	$\prec\ \succ$	$\rightarrow\ \leftarrow$

It is clear that Basic Logic and Display Logic have many common aspects. Perhaps the best view is to follow Belnap's advice that it is generally a "good idea to let quite a few flowers bloom" [7, page 35].

13 Further Work

We now outline some avenues for further work.

13.1 Breaking Residuation Using Residuation

The major disadvantage of our display calculus is that it enforces $\mathrm{rp}(\otimes, \rightarrow, \leftarrow)$ and $\mathrm{drp}(\oplus, \succ, \prec)$. We have seen that we can have one, without the other. But what if we want to break even the connections inherent in $\mathrm{rp}(\otimes, \rightarrow, \leftarrow)$? This is possible by using multiple connectives, with indices, say $\ ;_i$, $>_i$ and $<_i$ (see [57] for similar independent ideas) and the following display postulates and rules, for $1 \leq i \leq 3$:

$$\mathrm{rp}(\ ;_i,\ >_i,\ <_i)$$

$$(\otimes\vdash)\ \frac{A\ ;_1\ B \vdash Z}{A \otimes B \vdash Z} \qquad (\vdash\rightarrow)\ \frac{Z \vdash A >_2 B}{Z \vdash A \rightarrow B} \qquad (\vdash\leftarrow)\ \frac{Z \vdash A <_3 B}{Z \vdash A \leftarrow B}$$

We no longer have $\mathrm{rp}(\otimes, \rightarrow, \leftarrow)$. But if you want full residuation again, just add the display postulate $\mathrm{rp}(\ ;_1,\ >_2,\ <_3)$.

Similar methods work for breaking and regaining $\mathrm{drp}(\oplus, \succ, \prec)$.

In a sense, this is the ultimate substructural logic since all connectives are completely independent. Such additions should enable us to obtain hybrids of substructural logics as is done in Categorial Type Logics [57].

13.2 Can Display Logic Simulate Basic Logic ?

Suppose that we abandoned the display postulates of Figure 1 but kept all the logical introduction rules of Figure 2, and the structural rules for contraction Ctr, weakening Wk and exchange Com, as desired from Figure 3. Then, in these rules, suppose we put the following restrictions:

(i) for every rule of the form $(c \vdash)$, all substructures in the structures forming the succedent of the conclusion must be succedent parts, and

(ii) for every rule of the form $(\vdash c)$, all substructures in the structures forming the antecedent of the conclusion must be antecedent parts.

The intuition is that (i) restricts our introduction rule for $(c \vdash)$ so that the succedent of the conclusion cannot contain any substructure that "really belongs" on the antecedent side. Condition (ii) restricts our introduction rule for $(\vdash c)$ so that the antecedent of the conclusion cannot contain any substructure that "really belongs" on the succedent side. It is important that the notion of substructure be disallowed from looking inside formulae! Thus a subformula is not considered to be a substructure. The introduced formula is not only displayed, but it is also visible in the sense of Basic Logic. The Basic Logic rule for ($\vdash\rightarrow$) can be simulated by demanding that the antecedent be just Φ.

Our "side conditions" (i) and (ii) are similar to the "exponentially restricted" condition on structures used by Belnap in [9], and in Section 11. So it would seem that Belnap's modified conditions and cut-elimination proof [9] may be sufficient. But, currently, this remains a suggestion since I have not checked the details.

13.3 Can Display logic Simulate Higher Order Sequents ?

The motivation for our display postulates is based upon the desire to capture residuation and co-residuation, and their current form is clearly stated in [31]. In the light of Došen's higher-order sequents, we can also view rp(; , > , <) and drp(; , > , <) as "nested turnstiles" viz:

$$\begin{array}{c} X \vdash (Z \dashv Y) \\ \hline\hline X \; ; Y \vdash Z \\ \hline\hline Y \vdash (X \vdash Z) \end{array} \qquad\qquad \begin{array}{c} (Z \dashv Y) \vdash X \\ \hline\hline Z \vdash X \; ; Y \\ \hline\hline (X \vdash Z) \vdash Y \end{array}$$

Can we now simulate Došen's higher-order sequents in this display calculus ? In particular, does this reading of our formulation of Grishin's rules give us a way to turn Došen's higher-order sequents for intuitionistic logic into those for classical logic ?

13.4 Can Basic Logic Simulate Display Logic ?

Here is an idea of how we may be able to simulate Display Logic using Basic Logic. The essential difference between Basic Logic and Display Logic are the display postulates of Figure 1. Basic Logic does not contain extra "structural connectives" at the object level, but keeps them as "meta-linguistic signs" at the meta level. So to simulate Display Logic in Basic Logic, we have to find a way to simulate rp(; , > , <), but at the meta level rather than the object level. But if we take the "nested turnstile" idea seriously then it is clear that the display postulates will take the form already mentioned in Section 13.3. Basic Logic does not contain a reverse "nested turnstile" since it is seen as just a manifestation of the "symmetry" of Basic Logic. But since $\vdash$ can be seen as the relation "yields", why not view $\dashv$ as "excludes" ? Extending Basic Logic with this notion should enable Basic Logic to simulate Display Logic.

13.5 Display Directly Using Explicit Negations

Rather than "introducing" the negations via the rules ($\sharp\vdash$), ($\vdash\sharp$), ($\flat\vdash$) and ($\vdash\flat$), is it possible to build them into the general display framework from the start ? That is, can we capture bi-intuitionistic substructural logics using negations rather than implications as primitives. I could be wrong but the following seems a tentative argument that this cannot be done. See [32] for similar ideas.

The current display set-up uses $;^{x}$, $<^{x'}$ and $>^{x'}$ with the first on one side (x is "a" for antecedent or "s" for succedent) and the others on the opposite side (x' is "s" for succedent or "a" for antecedent). There is no direct connection between the two resulting display postulates rp(; , > , <) and drp(; , > , <).

The obvious idea is to express $<^s$ and $>^s$ by some combination of $;^s$ and the negations $\sharp^s$ and $\flat^s$. And analogously for $<^a$ and $>^a$ by some combination of $;^a$ and the negations $\sharp^a$ and $\flat^a$. But by doing so, we immediately bring in a link between the two sides, which was previously absent.

A more subtle way [32] is to express $<^s$ and $>^s$ by some combination of $;^a$ and the negations $\sharp^a$ and $\flat^a$ by embedding the $;$ under one of $\sharp^a$ and $\flat^a$. For example:

$$\frac{X \vdash \sharp(\flat Z \,;\, Y)}{\overline{X \,;\, Y \vdash Z}}$$

gives us the correct polarities for Y and Z, and the $;$ on the right is actually a $;^a$ since it is embedded by $\sharp$. But the $\sharp$ itself is $\sharp^s$ whereas we really want $\sharp^a$.

Thus it seems as if any attempt to define $<$ and $>$ in terms of the new negations will bring in some unwanted connections.

13.6 Miscellaneous

As pointed out to me by Giovanni Sambin, the absence of distribution is one of the key features of Quantum Logics. Thus it should be possible to obtain a display calculus for Quantum Logics using our display calculus.

Wansing [78] has given sequent systems for extensions of many intuitionistic substructural logics with various forms of constructive negation. It should be possible to capture these negations using Restall's display rules from Figure 5.

Dual intuitionistic logics can be seen as "refutation systems" to enumerate the "counter-theorems" of a logic [33]. Further work needs to be done to relate these to existing "refutation systems" for non-classical logics.

The title of this paper is stolen from Wansing's [83] where he gives a display calculus for many first-order classical logics using the fact that the quantifies $\forall x$ and $\exists x$ form a residuated pair. It should be fairly straightforward to extend both these works to obtain "First-Order Substructural Logics on Display".

Kracht [43] has shown that display calculi can be extended by encoding a vast number of additional axioms, meeting certain criteria, as structural rules, while still preserving general cut-elimination. Kracht's techniques should be extended to our display framework.

Lambek [45, page 214] mentions that some models for intuitionistic Bi-Lambek logic are "bi-quantales". Thus it should be possible to obtain a display calculus for the logic of bi-quantales.

The three transforms δ, Δ and σ clearly form a commuting diagram in a categorial setting. Further work is needed to connect our work to the vast amount of work on categorial models for substructural logics. There also seem to be strong connections between these transformations and the combinators from Combinatory Logic.

Jeremy Dawson and I are currently trying to find translations from Relational Proof Systems to and from Display Logic. This would give a way to check general cut-elimination for Relational Proof Systems.

It is clear that display calculi are quasi-equational systems in some sense. Are all logics displayable using our calculi quasi-equationally definable ?

14 Discussion and Conclusion

As Belnap [9, page 21] points out "The known formulations of linear logic seem to *depend on* the associativity and commutativity of $\otimes$, which is in certain respects a defect. ... We can drop these postulates and still have a coherent concept of all the connectives, including exponentials". We have shown how to extend these arguments to a huge class of substructural logics, with or without dual connectives, modalities, converse, various negations, exponentials, Sheffer's stroke and dagger, both intuitionistic and classical, in a single uniform framework.

Categorial insights are no doubt useful, but using them to obtain traditional cut-free Gentzen systems is not straightforward as witnessed by a gap in the cut-elimination claim of [45], see [40, page 256,footnote 3]. Display calculi seem to sit somewhere in between category theory and traditional Gentzen systems, capturing the symmetries and dualities inherent in the underlying categorial notions, but in a single, uniform, Gentzen-style format. Is the generalised display calculus of any use ? Hopefully we have demonstrated that it is.

Having extolled the virtues of display calculi, we must mention that display calculi are *not* a panacea. Kracht [43] has shown that current display calculi can only handle a certain subset of Sahlqvist logics. Whether display calculi for non-Sahlqvist logics can be invented is an open question.

The dependence on display postulates is often touted as a weakness of display logic. But, on the contrary, the Display Property is actually a very desirable property in many circumstances. For example, when staring at a complex sequent, it is sometimes very useful to focus on some particular substructure and ask "What does this mean ?". By displaying this substructure we can separate it from the context in which it is embedded. Often this gives some insights into what is going on. A practical, independent, demonstration of this phenomenon is the Window Inference calculus of Robinson and Staples [69].

There are four ways in which we can evaluate our work:

1. As a contribution to the study of display calculi. In this regard, our display calculus has many advantages over previous display calculi since it captures the duality inherent in substructural logics, in a modular way. The resulting limited display property allows you to build (cut-free) display calculi to suit your needs.
2. As a contribution to a general picture of bi-intuitionistic and bi-classical substructural (tense) logics with or without exponentials. In this regard we have shown how various rules achieve their effects in one uniform framework, and have also shown (or exploited) deep dualities and symmetries which give us "four proofs for the price of one". I know of no general picture of such a diverse range of logics in one uniform setting.
3. As a contribution to specific logics like relevant logic or linear logic. Here our contribution is limited since the respective communities are usually more knowledgeable. We have, however, shown how to add $\oplus$ to BCK-logic, and given double sided cut-free formulations for Classical Bi-Lambek logic, using implications or negations as primitives, albeit in a display framework.
4. As a contribution to generalised proof systems. In this regard, we believe we have made a significant contribution. It should be possible to extend the other general formalisms using these insights.

References

[1] V M Abrusci. Phase semantics and sequent calculus for pure noncommutative linear propositional logic. *Journal of Symbolic Logic*, 56:1403–1451, 1991.

[2] G Allwein and J M Dunn. Kripke models for linear logic. *Journal of Symbolic Logic*, 58(2):514–545, June 1993.

[3] A R Anderson and N D Belnap. *Entailment: The Logic of Relevance and Necessity*, volume 1. Princeton University Press, Princeton, USA, 1975.

[4] A Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52:939–951, 1987.

[5] A Avron. Using hypersequents in proof systems for non-classical logics. *Annals of Mathematics and Artificial Intelligence*, 4:225–248, 1991.

[6] A Avron. The method of hypersequents in proof theory of propositional non-classical logics. Technical Report 294-94, Institute of Computer Science, Tel Aviv University, Israel, 1994.

[7] N Belnap. Life in the undistributed middle. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, Studies in Logic and Computation, pages 31–42. Oxford University Press, 1993.

[8] N D Belnap. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.

[9] N D Belnap. Linear logic displayed. *Notre Dame Journal of Formal Logic*, 31:15–25, 1990.

[10] N D Belnap. The display problem. In Heinrich Wansing, editor, *Proof Theory of Modal Logics*, pages 79–92. Kluwer, 1996.

[11] R Blute and R A G Seely. Natural deduction and coherence for weakly distributive categories. Draft, 1991.

[12] T Braüner and V de Paiva. Cut-elimination for full intuitionistic linear logic. Draft manuscript, 199?

[13] M D'Agostino and D Gabbay. A generalization of analytic deduction via labelled deductive systems. part i: Basic substructural logics. *Journal of Automated Reasoning*, 13:243–281, 1994.

[14] M Dagostino, D Gabbay, and A Russo. Grafting modalities onto substructural implication systems. *Studia Logica*, to appear.

[15] M D'Agostino and M Mondadori. The taming of the cut. classical refutations with analytic cut. *Journal of Logic and Computation*, 4:285–319, 1994.

[16] J Dawson and R Goré. A mechanised proof system for relation algebra using display logic. (submitted), Automated Reasoning Project, 1997.

[17] K Došen. *Logical constants: An essay in proof theory.* PhD thesis, D. Phil. Dissertation, Oxford University, 1980.

[18] K Došen. Sequent-systems for modal logic. *Journal of Symbolic Logic*, 50(1):149–169, 1985.

[19] K Došen. Sequent systems and groupoid models, I. *Studia Logica*, 47:353–389, 1988.

[20] K Došen. Sequent systems and groupoid models, II. *Studia Logica*, 48:41–65, 1989.

[21] K Došen. A historical introduction to substructural logics. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, Studies in Logic and Computation, pages 1–30. Oxford University Press, 1993.

[22] K Došen and P Schroeder-Heister, editors. *Substructural Logics.* Studies in Logic and Computation. Oxford University Press, 1993.

[23] J M Dunn. A "Gentzen system" for positive relevant implication. *Journal of Symbolic Logic*, 38:356–357, 1973.

[24] J M Dunn. Gaggle theory: An abstraction of Galois connections and residuation with applications to negation and various logical operations. In *JELIA 1990: Proceedings of the European Workshop on Logics in Artificial Intelligence*, volume LNCS 478. Springer, 1991.

[25] J M Dunn. Gaggle theory applied to modal, intuitionistic, and relevance logics. In I Max and W Stelzner, editors, *Logik und Mathematik: Frege-Kolloquium Jena*, pages 335–368. de Gruyter, 1993.

[26] J M Dunn. Partial gaggles applied to logics with restricted structural rules. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, Studies in Logic and Computation, pages 63–108. Oxford University Press, 1993.

[27] J M Dunn. Perp and star: Two treatments of negation. In J Tomberlin, editor, *Philosophy of Language and Logic*, volume 7 of *Philosophical Perspectives*, pages 331–357. Ridgeview Publishing Company, Atascadero, California, USA, 1993.

[28] K Fine. Models for entailment. *Journal of Philosophical Logic*, 3:347–372, 1974.

[29] D Gabbay. *Labelled Deductive Systems*. Oxford University Press, 1996.

[30] J-Y Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[31] R Goré. Gaggles, Gentzen and Galois: A proof theory for non-classical logics. Technical Report TR-SRS-1-95, Automated Reasoning Project, Australian National University, Australia, 1995. http://arp.anu.edu.au/∼rpg.

[32] R Goré. Solving the display problem via residuation. Technical Report TR-ARP-12-95, Automated Reasoning Project, Australian National University, Australia, 1995.

[33] R Goré. A uniform display system for intuitionistic and dual intuitionistic logic. Technical Report TR-ARP-6-95, Automated Reasoning Project, Austalian National University, 1995.

[34] R Goré. On the completeness of classical modal display logic. In H Wansing, editor, *Proof Theory of Modal Logic*, volume 2 of *Applied Logic*, pages 137–140. Kluwer, 1996.

[35] R Goré. Cut-free display calculi for relation algebras. In D van Dalen and M Bezem, editors, *CSL96: Selected Papers of the Annual Conference of the European Association for Computer Science Logic*, volume LNCS 1258, pages 198–210. Springer, 1997.

[36] R Goré. Gaggles, Gentzen and Galois: Cut-free display calculi and relational semantics for algebraizable logics. Technical Report TR-ARP-07-97, Automated Reasoning Project, Australian National University, Australia, 1997.

[37] R Goré. Substructural logics on display. Technical Report TR-ARP-08-97, Automated Reasoning Project, Australian National University, Canberra, 0200, Australia, August 1997.

[38] R Goré. Intuitionistic logic redisplayed. Technical Report TR-ARP-1-95, Automated Reasoning Project, Australian National University, Australia, January, 1995.

[39] V N Grishin. On a generalization of the Ajdukiewicz-Lambek system. In *Studies in Nonclassical Logics and Formal Systems*, pages 315–343. Nauka, Moscow, 1983.

[40] J Hudelmaier and P Schroeder-Heister. Classical lambek logic. In P Baumgartner, R Hähnle, and J Posegga, editors, *TABLEAUX'95: Proceedings of the 4th International Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, number 918 in LNCS, pages 247–262. Springer, 1995.

[41] M Hyland and V de Paiva. Full intuitionistic linear logic (extended abstract). *Annals of Pure and Applied Logic*, 64:273–291, 1993.

[42] B Jónsson and A Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 73-74(891-939):127–162, 1951-52.

[43] M Kracht. Power and weakness of the modal display calculus. In H Wansing, editor, *Proof Theory of Modal Logics*, pages 92–121. Kluwer, 1996.

[44] J Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65:154–170, 1958.

[45] J Lambek. From categorial grammar to bilinear logic. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, Studies in Logic and Computation, pages 207–237. Oxford University Press, 1993.

[46] J Lambek. Logics without structural rules: Another look at cut elimination. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, volume 2 of *Studies in Logic and Computation*, pages 179–206. Oxford University Press, 1993.

[47] J Lambek. Some lattice models for bilinear logic. *Algebra universalis*, 34:541–550, 1995.

[48] W MacCaull. Relational proof systems for linear and other substructural logics. *Logic Journal of the IGPL*, 5(5):673–697, 1997.

[49] W MacCaull. Personal communication, February 1998.

[50] R D Maddux. The origin of relation algebras in the development and axiomatization of the calculus of relations. *Studia Logica*, 50(3/4):421–455, 1991.

[51] M Makkai and G E Reyes. Completeness results for intuitionistic and modal logic in a categorical setting. *Annals of Pure and Applied Logic*, 72:25–101, 1995.

[52] L Maksimova. A semantics for the calculus E of entailment. *Bulletin of Section of Logic*, 2:18–21, 1973.

[53] A Masini. 2-sequent calculus: A proof theory of modalities. *Annals of Pure and Applied Logic*, 58:229–246, 1992.

[54] R Meyer and R Routley. Algebraic analysis of entailment (I). *Logique et Analyse*, 15:407–428, 1972.

[55] G Mints. Cut-elimination theorem in relevant logics. *Journal of Soviet Mathematics*, 6:422–428, 1976. Original in Russian 1972.

[56] G Mints. Finite investigations of transfinite derivations. *Journal of Soviet Mathematics*, 10(4):548–596, 1978. translation from Zap. Nauchn. Semin LOMI 49 (1975).

[57] M Moortgat. Categorial type logics. In van Benthem and A ter Meulen, editors, *Handbook of Logic and Language*, chapter 2. Elsevier, To Appear.

[58] M Moortgat and R Oehrle. Logical parameters and linguistic variation: Lecture notes on categorial grammar, 1995. http://wwwots.let.ruu.nl/staff/moortgat/ICG/UCLA/ucla.html.

[59] H Ono. Semantics for substructural logics. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, Studies in Logic and Computation, pages 259–291. Oxford University Press, 1993.

[60] H Ono. Personal communication, February 1998.

[61] H Ono and Y Komori. Logics without the contraction rule. *Journal of Symbolic Logic*, 50:169–201, 1985.

[62] E Orlowska. Relational proof systems for relevant logics. *Journal of Symbolic Logic*, 57:1425–1440, 1992.

[63] E Orlowska. Relational proof systems for modal logics. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, volume 2 of *Studies in Logic and Computation*, pages 55–78. Oxford University Press, 1993.

[64] G Pottinger. Uniform, cut-free formulations of T, S4 and S5. *Abstract in JSL*, 48:900–901, 1983.

[65] G Restall. Proof theories for almost anything. Technical Report TR-SRS-1-95, Automated Reasoning Project, Australian National University, Australia, 1995. http://arp.anu.edu.au.

[66] G Restall. Display logic and gaggle theory. *Reports on Mathematical Logic*, 29:133–146, 1995, published in 1996.

[67] G Restall. Displaying and deciding substructural logics I: logics with contraposition. *Journal of Philosophical Logic*, 1998 (to appear).

[68] G E Reyes and H Zolfaghari. Bi-heyting algebras, toposes and modalities. *Journal of Philosophical Logic*, 25(01), 1996.

[69] P Robinson and J Staples. Formalizing a hierarchical structure of practical mathematical reasoning. *Journal of Logic and Computation*, 3:47–61, 1993.

[70] R Routley and R K Meyer. The semantics of entailment I. In H Leblanc, editor, *Truth, Syntax and Modality*, pages 199–243. North-Holland, Amsterdam, 1972-73.

[71] G Sambin. Intuitionistic formal spaces and their neighbourhoods. In R Ferro, C Bonotto, S Valentini, and A Zanardo, editors, *Logic Colloquim '88*, pages 261–285. North-Holland, 1989.

[72] G Sambin. The semantics of pretopologies. In K Došen and P Schroeder-Heister, editors, *Substructural Logics*, Studies in Logic and Computation, pages 292–307. Oxford University Press, 1993.

[73] G Sambin, G Battilotti, and C Faggian. Basic logic: reflection, symmetry, visibility. draft manuscript January 1997, (to appear in the Journal of Symbolic Logic).

[74] B Surarso and H Ono. Cut elimination in noncommutative substructural logics. *Reports on Mathematical Logic*, 30:13–29, 1996.

[75] I Urbas. Dual intuitionistic logic. Technical Report 40, Zentrum Philosophie Und Wissenschaftstheorie, Universität Konstanz, August, 1994.

[76] A Urquhart. Semantics for relevant logics. *Journal of Symbolic Logic*, 37:159–169, 1972.

[77] A Urquhart. A topological representation theorem for lattices. *Algebra Universalis*, 8:45–58, 1979.

[78] H Wansing. Informational interpretation of substructural propositional logics. *Journal of Logic, Language and Information*, 2:285–308, 1993.

[79] H Wansing. Sequent calculi for normal modal propositional logics. *Journal of Logic and Computation*, 4:125–142, 1994.

[80] H Wansing. Strong cut-elimination in display logic. *Reports on Mathematical Logic*, 29:117–131, 1995 (published 1996).

[81] H Wansing. Displaying as temporalising: Sequent systems for subintuitionistic logics. In S Akama, editor, *Logic, Language and Computation*, pages 159–178. Kluwer, Dordrecht, 1997.

[82] H Wansing. Translation of hypersequents into display sequents. *Logic Journal of the IGPL*, this volume, 1998.

[83] H Wansing. Predicate logics on display. *Studia Logica*, 1998 (to appear).

[84] D N Yetter. Quantales and (non-commutative) linear logic. *Journal of Symbolic Logic*, 55:41–64, 1990.

Received August 30, 1997