Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physics Letters A 318 (2003) 526–536
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Determination of boundaries of unsteady oscillatory zone in asymptotic solutions of non-integrable dispersive wave equations

G.A. El [a,*], V.V. Khodorovskii [b], A.V. Tyurina [a]

[a] *School of Mathematical and Information Sciences, Coventry University, Coventry, UK*
[b] *Information Science Department, St. Petersburg State University of Culture and Arts, St. Petersburg, Russia*

Received 9 September 2003; accepted 18 September 2003

Communicated by V.M. Agranovich

Abstract

We propose a simple general method for analytic determination of the boundaries of the expanding nonlinear oscillation zone occurring in the decay of a step problem for non-integrable dispersive wave equations. A remarkable feature of the method is that it essentially uses only the dispersionless limit and the linear dispersion relation of the original nonlinear dispersive wave system. A concrete example pertaining to collisionless plasma dynamics is considered and complete agreement with the results of earlier numerical simulations is demonstrated.

Keywords: Finite-amplitude waves; Dispersive shock; Modulation equations

1. Introduction

Since the discovery by Sagdeev of the oscillatory structure of collisionless shocks in plasma [1], the problem of their analytic description have been attracting a great deal of attention of both mathematicians and physicists. This interest is partially explained by the fact that the phenomenon of generation of nonlinear oscillations in the vicinity of the gradient catastrophe point is quite ubiquitous in dispersive media and its applications range from space plasma physics to bubbly fluid dynamics and fiber optics. Considered in the general context of classical theory of conservation laws, such waves in presence of small dissipation demonstrate global properties characteristic for classical shocks: the Rankine–Hugoniot transition conditions are valid for them and dissipation determines only their width. In other words, although dispersion dramatically affects the fine structure of stationary collisionless shocks their speed of propagation and the transition conditions follow directly from inviscid dispersionless conservation laws.

This is true, however, only for a steady regime when nonlinearity, dispersion and dissipation balance each other and the collisionless shock has constant width. Contrastingly, the case when dispersion prevails over dissipation cannot be treated by a simple consideration of mass, momentum and energy balance at the shock transition. The reason for that is that *the boundaries of the dispersive dissipationless shock di-*

* Corresponding author.
E-mail address: g.el@coventry.ac.uk (G.A. El).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2003.09.060

verge with time, i.e., instead of the shock speed s defined by the balance of mass one has now two different speeds $s_1 > s_2$ determining motion of the dispersive shock boundaries. Thus, such dispersive shocks are unsteady and require a separate study. The predominantly dispersive dynamics is of considerable interest by itself and also in many cases it can be considered as an unsteady intermediate asymptotics in a general setting when the small dissipation is taken into account.

In a weakly nonlinear case when the original system can be approximated by one of exactly integrable equations, the study of the dispersive shocking phenomenon has lead to discovery of a new class of mathematical problems which can be broadly described as semiclassical limits in the integrable systems. The rigorous methods developed by Lax, Levermore and Venakides [2,3] who studied the semiclassical asymptotics in the inverse scattering transform for the KdV equation allowed to reduce the KdV initial value problem to integrating the corresponding Whitham modulation equations [4] with special matching conditions (the formulation proposed earlier by Gurevich and Pitaevskii [5] on a basis of a more universal, albeit more heuristic, reasoning). The methods of Lax, Levermore and Venakides as well as direct formulation of Gurevich and Pitaevskii have been extended to many other integrable equations. The characteristic feature of both approaches is that owing to the integrable nature of the problem the determination of the boundaries of the oscillatory zone becomes an intrinsic part of constructing the whole solution. In fact, as we show in this Letter, the problem of determination of the boundaries can be solved separately using some very general assumptions about qualitative behaviour of the characteristics of the Whitham equations for the problem under study.

In non-integrable case, when exact solution is not available in principle, the possibility to determine the dispersive shock boundaries allows to put the entire problem of the dispersive shock dynamics in the classical setting when the shock is 'built in' the solution of the Euler equations of ideal hydrodynamics. In this Letter we, by adopting the asymptotic 'averaged' formulation of the problem from the integrable systems theory, propose a simple general method for analytic determination of the boundaries of the expanding nonlinear oscillation zone occurring in the decay of a step for non-integrable dispersive wave equations. A remarkable feature of the method is that it essentially uses only the dispersionless limit and the linear dispersion relation of the original system.

In the last section, as a concrete example, we obtain the boundaries of the dispersive shock in a non-integrable system describing fully nonlinear flows in a two-temperature collisionless plasma. Our analytic results are in a complete agreement with the results of earlier direct numerical simulations of the same problem in [6].

2. General setting

We consider a decay of an initial discontinuity problem for a system describing fully nonlinear flows in dissipationless dispersive medium. In a general form such a system can be conventionally represented as

$$\partial_t \mathbf{U} = \mathbf{K}_N\big(\mathbf{U}, \partial_x \mathbf{U}, \partial_{xx}^2 \mathbf{U}, \ldots\big), \tag{1}$$

where $\mathbf{U}$ and $\mathbf{K}$ are vectors and N is the order of the system with respect to the spatial variable. In this Letter we restrict ourselves with the important subclass of such systems with $N = 4$ and the real-valued linear dispersion relation $\omega = \omega_0(k)$, where ω, is the frequency and k is the wavenumber. Also we assume the system (1) to have at least four conservation laws of the form

$$\partial_t P_j + \partial_x Q_j = 0. \tag{2}$$

We define the dispersionless limit of the system (1) in the following way. We introduce new independent variables $X = \epsilon x$, $T = \epsilon t$, where $\epsilon \ll 1$ is a small parameter and then formally tend ϵ to zero. Then, to the leading order we obtain a quasilinear system which is the dispersionless limit of the system (1). Let this limit have the form of the Euler hydrodynamic equations for ideal gas

$$\begin{aligned} &\partial_T \rho + \partial_X(\rho v) = 0, \\ &\partial_T v + v\partial_X v + \frac{c_s^2(\rho)}{\rho}\partial_X \rho = 0, \end{aligned} \tag{3}$$

where ρ is the density, v is the velocity, and $c_s(\rho)$ is the 'sound speed' in the corresponding 'gas dynamics'. For convenience of explanation we will also suppose that the three first conservative densities P_j in (2) can be associated with the hydrodynamic density, velocity and momentum: $P_1 = \rho$, $P_2 = v$, $P_3 = \rho v$.

The described subclass of systems (1) is quite broad and includes some known integrable models such as defocusing nonlinear Schrödinger equation and Kaup–Boussinesq system [7]. As physically important examples of non-integrable systems that possess the described general properties one can indicate the Green–Naghdi system for fully nonlinear shallow water gravity waves [8] which also describes the waves in bubbly fluids [9], the generalized nonlinear Schrödinger equation describing propagation of nonlinear waves in photorefractive materials [10], the systems for nonlinear ion-acoustic and magnetoacoustic waves in collisionless plasma [4,11], and many others. In this Letter we will not be concerned with the integrability properties of the system under consideration. Instead, we will show that having in disposal only the dispersionless limit and the linear dispersion relation (in a somewhat extended form) it is possible to obtain some asymptotically exact results pertaining to a global dynamics of fully nonlinear waves in the system (1).

We consider the initial data for the system (1) in the form of a step for the variables ρ and v appearing in the dispersionless limit (3):

$$t = 0: \quad \begin{cases} \rho = \rho_1, & v = v_1, \quad x \geqslant 0, \\ \rho = \rho_2, & v = v_2, \quad x < 0, \end{cases} \tag{4}$$

where $\rho_{1,2}$ and $v_{1,2}$ are some constants.

Since our aim in this work is to study the boundaries of the dispersive shock it is necessary to extract for our consideration only the admissible set of discontinuities producing *a single dispersive shock* as a result of the decay. For the dispersive shocks moving to the right (in the frame moving with velocity v_2) such discontinuities are distinguished by the relationship

$$v_1 - \int_{\rho_0}^{\rho_1} \frac{c_s(\rho)}{\rho}\, d\rho = v_2 - \int_{\rho_0}^{\rho_2} \frac{c_s(\rho)}{\rho}\, d\rho, \tag{5}$$

where ρ_0 is a constant. This transition relationship has been for the first time formulated in [6] and then derived using characteristics in [12,13].

3. Modulation description of dispersive shocks and natural matching conditions

Because of lack of integrability for most systems describing finite-amplitude waves the rigorous results concerning existence and uniqueness of solutions can hardly be expected in this area. The natural approach then is to make some plausible assumptions about the general structure of the solution of our interest and then, to explore possible consequences of these assumptions. The results of such an approach can be validated by comparison with available direct numerical simulations and by consistency of the weakly nonlinear asymptotics of the obtained solution with the exact results for the corresponding integrable system.

We formulate the dispersive shock problem for a non-integrable system by adopting the resulting 'averaged' setting from the theory of integrable systems, i.e., by direct application of the Whitham method to the system (1) in the conservative form (2) and then by postulating appropriate boundary conditions.

The main premise in our construction is that the dispersive shock is locally described by the one-phase periodic travelling solution of system (1):

$$f = f(\theta), \qquad \theta = kx - \omega t,$$
$$f(\theta + 2\pi) = f(\theta), \tag{6}$$

where k is the wave number and ω is the frequency. The variable f is one of the components of the vector $\mathbf{U}$, its choice is obvious in each particular case. All the remaining components of the vector $\mathbf{U}$ are expressed in terms of f by algebraic expressions. The travelling wave solution is parametrized by a number of independent integrals of motion (four in our case), which, on a large scale, depend on $X = \epsilon x$, $T = \epsilon t$, where $\epsilon \ll 1$. Their variations are governed by the averaged equations

$$\partial_T \bar{P}_j + \partial_X \bar{Q}_j = 0, \quad j = 1, \dots, 4. \tag{7}$$

Averaging is done over the solution (6)

$$\bar{F}(X, T) = \frac{1}{2\pi} \int_0^{2\pi} F(\theta)\, d\theta, \tag{8}$$

where $F(\theta) \equiv F(f(\theta))$. The averaged equations, thus, describe slow modulations in the travelling wave solution. One more modulation equation, which, of course, is consistent with the closed system (7) is provided by the wave number conservation law

$$\partial_T k + \partial_X \omega = 0, \tag{9}$$

which is a compatibility condition in the Whitham theory and can be used instead of any of the modulation equations (7) [14]. Its consistency with the modulation system (7) can be often verified directly [4].

Now we have to choose an appropriate system of dependent variables. This choice is crucial in the integrable systems theory where a unique system of variables, Riemann invariants, exists, which makes it possible to effectively integrate the modulation system (see [15] and references therein). In the case when the Riemann invariants are not available, the advantages of any distinguished system of variables are not so decisive although the 'right' choice can seriously facilitate calculations.

The 'hydrodynamic' nature of the system (1) having Euler equations (3) as a dispersionless limit suggests natural choice of the basis modulation variables: $\bar\rho$, $\bar v$, $A^2 = \overline{\rho v} - \bar\rho\bar v$, k. The variable A^2 can be viewed as a measure of intensity of the oscillations. In the absence of oscillations, apparently, $A^2 = 0$, $\bar\rho = \rho$, $\bar v = v$, and the two first integrals of the Whitham system should become consistent with the dispersionless limit of the original system (3).

This degeneration can occur in two ways:

(a) through the linear vanishing amplitude wave limit, when $A^2 \to 0$, $k = O(1)$;
(b) through the solitary wave limit, when $A^2 \to 0$, $k \to 0$, $A^2/k = O(1)$.

The first type of transition is realized at the trailing edge of the dispersive shock, and the second one—at the leading edge (to be definite we imply here the negative dispersion in the system, in the positive dispersion case the structure of the dispersive shock is inverted (see, for instance, [16])).

Remark. The travelling wave frequency ω is expressed in terms of the basis modulation variables by nonlinear dispersion relation

$$\omega = \omega\big(k, \bar\rho, \bar v, A^2\big). \tag{10}$$

In the linear limit this relation assumes the form

$$\omega = \omega_0(k, \bar\rho, \bar v), \tag{11}$$

and can be obtained directly from the original system (1) by linearization against the slowly varying mean background:

$$U_j \approx \bar U_j + a_j e^{i(kx-\omega t)}, \quad a_j \ll 1. \tag{12}$$

Now we apply the following setting, which we adopt from the integrable systems theory (see, for instance, [15]). There is, of course, an underlying basic assumption that the solution of the problem of our interest exists in some broad sense which will be clear from what follows. For convenience we explicitly itemize our major assumptions and some their important implications.

- We assume the space–time of the asymptotic as $\epsilon \to 0$ solution to the initial value problem (1), (4) to be split into three domains: $(-\infty, X_2(T))$, $[X_2(T), X_1(T)]$, $(X_1(T), +\infty)$, in which the solution is governed by different equations;
- Outside the dispersive shock region $[X_2(t); X_1(t)]$ the solution is governed by the dispersionless limit of the modulation system, i.e., by Euler equations of ideal gas dynamics (3).
- Inside the dispersive shock domain $[X_2(t); X_1(t)]$ we replace the original equations by the system of the averaged conservation laws (7).
- The solutions of the inner (Whitham (7)) and outer (Euler (3)) systems are then subject to matching conditions at the (unknown) boundaries $X_{1,2}(T)$. We require the natural continuity matching conditions to be satisfied [12]

$$X = X_{1,2}(T):\quad A^2 = 0,$$
$$\bar\rho = \rho_E(X,T), \quad \bar v = v_E(X,T), \tag{13}$$

subject to additional restrictions reflecting the different way of transition to the smooth flow at the trailing and the leading edges:

$$X \to X_2(T):\quad k = O(1), \tag{14}$$
$$X \to X_1(T):\quad A^2/k = O(1). \tag{15}$$

Here $(\rho_E(X,T), v_E(X,T))$ stands for the solution of the Euler system (3) with the initial or boundary conditions for the original system (1).

The conditions (13)–(15) represent a natural extension of the Gurevich–Pitaevskii conditions formulated for the KdV equation in [5], and then for the defocusing NLS equation and the Kaup–Boussinesq system in [16,17].

- We assume hyperbolicity of the modulation system (7) for the solutions of our interest. In the context of the dispersive shock problem the hyperbolicity implies the modulational stability of the dispersive shock. Another implication of hyperbolicity is the possibility of using the classical characteristics method.
- The boundaries $X_{1,2}$ are defined for the solution of the matching problem by the kinematic conditions:

$$\frac{dX_2}{dT} = \frac{\partial \omega_0}{\partial k}\bigg|_{A^2=0}, \tag{16}$$

$$\frac{dX_1}{dT} = \lim_{k\to 0} \frac{\omega}{k}\bigg|_{A^2/k=O(1)}, \tag{17}$$

that is the trailing edge moves with the group velocity of the trailing linear wave packet while the leading edge is identified with the position of the leading solitary wave in the dispersive shock.

Since the order of the Whitham system (7) is equal to four while the Euler system (3) is of the second order, the boundaries of the dispersive shock $X = X_{1,2}$ are the multiple (double) characteristics of the Whitham system. This is why one cannot specify the values of k and A^2/k at the trailing and the leading edges correspondingly (see (14), (15)) and only two functions $\rho_E(X,T)$ and $v_E(X,T)$ can be prescribed at each boundary.

The governing equations (7), (3) and the initial conditions (4) are invariant with respect to the linear transformation $X \to cX$, $T \to cT$. Therefore, the problem under consideration is self-similar, i.e., we have only one independent variable $s = X/T$. Hence the boundaries of the dispersive shock are the straight lines

$$X_{1,2} = s_{1,2}T \tag{18}$$

and the matching conditions (13) assume the form

$$s = s_2: \quad A^2 = 0, \quad \bar\rho = \rho_2, \quad \bar v = v_2, \tag{19}$$

$$s = s_1: \quad A^2 = 0, \quad \bar\rho = \rho_1, \quad \bar v = v_1. \tag{20}$$

Also from (14), (15) we have

$$s \to s_2: \quad k = O(1), \tag{21}$$

$$s \to s_1: \quad A^2/k = O(1). \tag{22}$$

We note in conclusion that the parameter ϵ formally introduced in the definition of the slow variables X and T appears naturally in the solution of the original system (1) as a ratio of the characteristic scale of oscillations to the oscillation zone width and, therefore, is proportional to t^{-1}. Thus our definition of the edges is asymptotically accurate as $t \to \infty$.

4. Determination of the edges

Our task now is to determine the constants s_1 and s_2, which are the self-similar coordinates (speeds) of the dispersive shock edges, in terms of the initial discontinuity parameters $\rho_{1,2}$, $v_{1,2}$. Using definitions of the edges (16)–(18) we obtain

$$s_2 = \frac{\partial \omega_0}{\partial k}(k_2, \rho_2, v_2), \tag{23}$$

where k_2 is the value of the wavenumber at the trailing edge, and

$$s_1 = \lim_{k\to 0} \frac{\omega}{k}(\zeta_1, \rho_1, v_1), \tag{24}$$

where ζ_1 is the value of the variable $\zeta = A^2/k$ at the leading edge.

As we have already mentioned, owing to the fact that the edges of the dispersive shock are the double characteristics of the Whitham system, the values k_2, ρ_2, v_2 as well as ζ_1, ρ_1, v_1 are not independent. Thus our task of obtaining $s_{1,2}$ is reduced to finding dependencies

$$k_2 = k_2(\rho_2, v_2) \quad \text{for any given } \rho_1, v_1$$

and

$$\zeta_1 = \zeta_1(\rho_1, v_1) \quad \text{for any given } \rho_2, v_2$$

compatible with the modulation system considered in the limits $A^2 \to 0$, $k = O(1)$ and $A^2 \to 0$, $\zeta = O(1)$, correspondingly.

Additional restrictions on possible values of the modulation parameters at the edges of the self-similar dispersive shock are imposed by the transition relationship (5), which implies at the trailing edge: $v_2 =$

$v_2(\rho_2)$ for given ρ_1, v_1; and at the leading edge: $v_1 = v_1(\rho_1)$ for given ρ_2, v_2.

4.1. Trailing edge

We consider the reduction of the modulation system in the limit when intensity of oscillations $A^2 = \overline{\rho v} - \bar\rho\bar v$ vanishes, while the wave number k remains finite (see (19), (21)). Then, according to general properties described in Section 3 the modulation system becomes consistent with the dispersionless limit (3) and assumes the degenerate form

$$\partial_T\bar\rho + \partial_X(\bar\rho\bar v) = 0, \tag{25}$$

$$\partial_T\bar v + \bar v\partial_X\bar v + \frac{c_s^2(\bar\rho)}{\bar\rho}\partial_X\bar\rho = 0, \tag{26}$$

$$\partial_T k + \partial_X\omega_0(k, \bar\rho, \bar v) = 0. \tag{27}$$

The system (25)–(27) has three different families of characteristics. Two of them are the usual Euler hydrodynamics characteristics $dX/dT = V_\pm$ defined by the characteristic velocities [14]

$$V_\pm = \bar v \pm c_s(\bar\rho), \tag{28}$$

while the third one is the linear wave characteristic $dX/dT = \partial\omega_0/\partial k$, which is the double characteristic of the full Whitham system in the linear limit. It is important that this characteristic depends not only on the wave number k but also on the mean flow parameters $\bar\rho, \bar v$ owing to the general form of the linear dispersion relation (11).

Now we are looking for the integral $k = k(\bar\rho, \bar v)$ of the system (25)–(27) subject to the additional restriction

$$\bar v - \int_{\rho_0}^{\bar\rho}\frac{c_s(\rho)}{\rho}\,d\rho = v_1 - \int_{\rho_0}^{\rho_1}\frac{c_s(\rho)}{\rho}\,d\rho \equiv C, \tag{29}$$

which in view of the boundary conditions (19), (21) provides consistency with the dispersive shock condition (5).

First we note that the restriction (29) coincides with the relationship $\bar v(\bar\rho)$ between the density and the velocity in the simple compression wave solution ($r_-(\bar\rho, \bar v) = C$) of the Euler equations (25), (26) [14]. So, substituting (29) into (25), (27) we obtain

$$\partial_T\bar\rho + V(\bar\rho)\partial_X\bar\rho = 0, \tag{30}$$

$$\partial_T k + \partial_X\Omega_0(k, \bar\rho) = 0, \tag{31}$$

where

$$V(\bar\rho) = V_+\big(\bar\rho, \bar v(\bar\rho)\big) = c_s(\bar\rho) + \int_{\rho_0}^{\bar\rho}\frac{c_s(\rho)}{\bar\rho}\,d\rho + C,$$

$$\Omega_0(k, \bar\rho) = \omega_0\big(k, \bar\rho, \bar v(\bar\rho)\big). \tag{32}$$

Thus, the characteristic integral we are looking for has the form $k = k(\bar\rho, \bar v(\bar\rho)) \equiv k_0(\bar\rho)$. Substituting this into the system (30), (31), we arrive at the ordinary differential equation

$$\frac{dk_0}{d\bar\rho} = \frac{\partial\Omega_0/\partial\bar\rho}{V(\bar\rho) - \partial\Omega_0/\partial k}, \qquad k_0(\rho_1) = 0. \tag{33}$$

The initial condition $k_0(\rho_1) = 0$ follows from the boundary conditions (19), (20) and implies that no dispersive shock is generated if the boundary values at both edges are equal (if $\rho_2 = \rho_1$ then automatically $v_2 = v_1$ by (5)). Integrating (33) we find $k_0(\bar\rho)$. Then the self-similar coordinate of the trailing edge according to (23) is found as

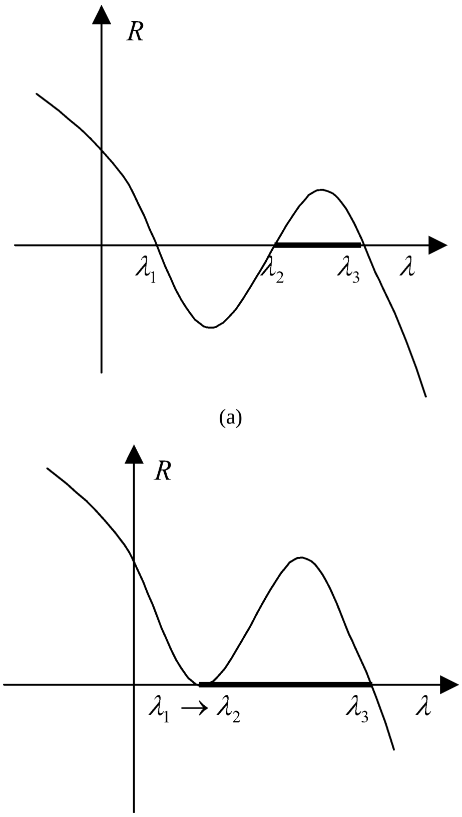
$$s_2 = \frac{\partial\Omega_0}{\partial k}(k_2, \rho_2), \tag{34}$$

where $k_2 = k_0(\rho_2)$. We note that the values ρ_1, v_1 enter the expression (34) via the constant C in (32) and the initial condition for $k_0(\bar\rho)$ in (33).

4.2. Leading edge

The position of the leading edge (24) can be, in principle, determined in a similar way by considering the limit of the modulation system at $A^2 \to 0$ $\zeta = O(1)$, where $\zeta = A^2/k$. The chosen system of modulation variables $\bar\rho$, $\bar v$, ζ, however, is not as convenient for this purpose, since obtaining the equation for ζ from the modulation system (7) in the solitary wave limit is a technically cumbersome problem (see [12]), which, in addition, does not shed much light on the limiting structure of the Whitham equations.

This complexity can be bypassed by introducing a new system of the basis modulation variables in which, as we will see, the theory of the leading edge will become in essence equivalent to that for the trailing edge. Although it might look as a technical task we will see that the method we are proposing is quite general and leads to an elegant and clear description of the modulated finite-amplitude *solitary wave trains* in terms of the *linear dispersion relation*.



(b)

Fig. 1. Potential curve $R(\lambda)$: (a) general configuration, (b) solitary wave configuration.

First we recall that the modulation variables locally represent a set of independent parameters specifying the travelling wave solution (6) of the original system (1). This solution is usually specified by the ordinary differential equation of the form

$$(kf_\theta)^2 = R(f), \qquad \theta = kx - \omega t,$$
$$f(\theta + 2\pi) = f(\theta), \tag{35}$$

where f is a component of the vector $\mathbf{U}$ and all the remaining components are expressed in terms of f by algebraic expressions. To be definite, we assume the potential curve $R(\lambda)$ to have three real roots $\lambda_1 < \lambda_2 < \lambda_3$ (see Fig. 1(a)):

$$R(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)G^2(\lambda),$$
$$G(\lambda) \neq 0, \tag{36}$$

where $G(\lambda)$ is a 'good' function. Then the 2π-periodic solution of (35) oscillate between the roots λ_2 and λ_3. The wavenumber and the mean values in this travelling wave are given by the integrals

$$k = \pi \left(\int_{\lambda_2}^{\lambda_3} \frac{d\lambda}{\sqrt{R(\lambda)}} \right)^{-1},$$
$$\bar{F} = \frac{k}{\pi} \int_{\lambda_2}^{\lambda_3} \frac{F(\lambda)\, d\lambda}{\sqrt{R(\lambda)}}. \tag{37}$$

Here $F(f) \equiv F(\theta(f))$ (cf. (8)), where F is any function of the vector $\mathbf{U}$ considered for the solution (35). All modulation variables in the system such as $\bar{\rho}$, $\bar{v}$, A^2, ζ, etc., are expressed in terms of the integrals (37). The following general asymptotics are valid in the solitary wave configuration:

$$\lambda_2 \to \lambda_1: \quad k \sim \left| \frac{1}{\ln(\lambda_2 - \lambda_1)} \right| \to 0, \quad \bar{F} \to F(\lambda_2). \tag{38}$$

Now we introduce an alternative (conjugated) system of modulation variables which is more convenient when studying the solitary wave limit $\lambda_2 \to \lambda_1$ in the modulation system. We define the conjugated wavenumber and the conjugated mean value as

$$\tilde{k} = \pi \left(\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\sqrt{-R(\lambda)}} \right)^{-1},$$
$$\langle F \rangle = \frac{\tilde{k}}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{F(\lambda)\, d\lambda}{\sqrt{-R(\lambda)}}. \tag{39}$$

As a matter of fact, any of these quantities can be taken as a modulation variable instead of any one from the set (37). The corresponding asymptotics for the conjugated variables are (cf. (38))

$$\lambda_2 \to \lambda_1: \quad \tilde{k} \to \tilde{k}_s = O(1), \quad \langle F \rangle \to F(\lambda_2). \tag{40}$$

One can see then that in the limit considered

$$\lambda_2 \to \lambda_1: \quad \langle F \rangle \to \bar{F}. \tag{41}$$

The new set of independent modulation parameters we are going to use in the solitary wave limit is: $\tilde{k}_s$, $\bar{\rho}$, $\bar{v}$. Considered in the context of the leading edge of the dispersive shock (i.e., for a specific solution) these parameters, similarly to the trailing edge case, become subject to two restrictions:

(i) a simple wave relationship (cf. (29))

$$\bar{v} - \int_{\rho_0}^{\bar{\rho}} \frac{c_s(\rho)}{\rho}\, d\rho = v_2 - \int_{\rho_0}^{\rho_1} \frac{c_s(\rho)}{\rho}\, d\rho \equiv C, \quad (42)$$

imposed by the dispersive shock transition condition (5), and

(ii) the characteristic relationship $\tilde{k}_s = \tilde{k}_s(\bar{\rho}, \bar{v})$ which should take place since the leading edge is the double characteristics of the modulation system. As a result, the relationship to be found should have the form

$$\tilde{k}_s = \tilde{k}_s(\bar{\rho}). \quad (43)$$

Now we consider the modulation system (7) in the solitary wave limit $\lambda_2 \to \lambda_1$. We recall that in this limit the intensity of oscillations A^2 vanishes and two first integrals of the modulation system become those of the dispersionless limit (3) exactly as it happens in the zero-amplitude configuration $\lambda_3 \to \lambda_2$ considered in the trailing front theory.

Using the restriction (42) we again get the simple wave equation (30) for $\bar{\rho}$. The wave number conservation law (9) requires a bit more detailed analysis. We represent it in the form

$$\partial_T(\tilde{k}\Lambda) + \partial_X(\tilde{\omega}\Lambda) = 0, \quad (44)$$

where $\Lambda = k/\tilde{k}$ and the conjugated frequency is defined as $\tilde{\omega} = \omega\tilde{k}/k$ and has the limit $\tilde{\omega} \to \tilde{\omega}_s = O(1)$ as $\lambda_2 \to \lambda_1$. Since there are only three independent variables left when $\lambda_2 = \lambda_1$, then $\tilde{\omega}_s = \tilde{\omega}_s(\tilde{k}_s, \bar{\rho}, \bar{v})$ (cf. (11)). This relationship, which is yet to be found, can be called a *solitary wave dispersion relation*.

It is convenient to rewrite equation (44) in the form

$$\Lambda(\partial_T\tilde{k} + \partial_X\tilde{\omega}) + \tilde{k}\left(\partial_T\Lambda + \frac{\tilde{\omega}}{\tilde{k}}\partial_X\Lambda\right) = 0. \quad (45)$$

In the solitary wave configuration, $\lambda_2 \to \lambda_1$, we have the asymptotics $\Lambda \sim |1/\ln(\lambda_2 - \lambda_1)| \to 0$. Assuming then $\partial_T\Lambda \sim \partial_X\Lambda \gg \Lambda$ we get to the leading order:

$$\lambda_2 = \lambda_1 \quad \text{at } \frac{dX}{dT} = \frac{\tilde{\omega}_s}{\tilde{k}_s}, \quad (46)$$

which defines the leading edge in terms of conjugated variables. The constant s_1 (see (18)) is then evaluated as

$$s_1 = \frac{\tilde{\omega}_s}{\tilde{k}_s}\bigg|_{\bar{\rho}=\rho_1, \bar{v}=v_1}. \quad (47)$$

Note that owing to the definition of $\tilde{\omega}$ formula (47) is equivalent to our original expression (24) for s_1 in terms of the frequency ω and the wavenumber k. Expression (47), however, is much more simple for actual calculations since it does not contain any singular limiting transitions.

To the first order, taking into account (43) and (30), Eq. (45) yields

$$\frac{d\tilde{k}_s}{d\bar{\rho}} = \frac{\partial\tilde{\Omega}_s/\partial\bar{\rho}}{V(\bar{\rho}) - \partial\tilde{\Omega}_s/\partial\tilde{k}_s}, \qquad \tilde{k}_s(\rho_2) = 0, \quad (48)$$

where

$$\tilde{\Omega}_s(\tilde{k}_s, \bar{\rho}) = \tilde{\omega}_s\big(\tilde{k}_s, \bar{\rho}, \bar{v}(\bar{\rho})\big). \quad (49)$$

The initial condition in (48) follows from the boundary conditions (19), (20) and implies that there is no dispersive shock generated if the boundary values at both edges are equal.

One cannot help noticing that Eqs. (33) and (48) describing relationships between the variables in the linear and the solitary wave trains are identical in terms of the dispersion relations $\Omega_0(k, \bar{\rho})$ and $\tilde{\Omega}_s(\tilde{k}_s, \bar{\rho})$. The latter, however, is yet to be found.

To obtain the solitary wave dispersion relation $\tilde{\omega}_s(\tilde{k}_s, \bar{\rho}, \bar{v})$ we observe that expressions (39) can be viewed as analogs of (37) for the conjugated travelling wave given by the equation

$$(\tilde{k}\tilde{f}_{\tilde{\theta}})^2 = -R(\tilde{f}), \qquad \tilde{\theta} = \tilde{k}\tilde{x} - \tilde{\omega}\tilde{t},$$
$$\tilde{f}(\tilde{\theta} + 2\pi) = \tilde{f}(\tilde{\theta}), \quad (50)$$

where $\tilde{x}$, $\tilde{t}$ are new independent variables. This travelling wave is associated with the same (but inverted) potential curve $R(\lambda)$ (36) so that the oscillations now occur between the roots λ_2 and λ_1. For problems associated with polynomial potential curves, the functions $f(\theta)$ and $i\tilde{f}(i\tilde{\theta})$ represent the same analytic function in the complex θ-plane, which is an elliptic function with the periods 2π and $2\pi i$ along the real and the imaginary axes.

The next observation is that the solitary wave limit $\lambda_2 \to \lambda_1$ in the original travelling wave (35) corresponds to the vanishing amplitude limit in the conjugated travelling wave equation (50) (see Fig. 1(b)) and, therefore, $\tilde{\omega}_s$ and $\tilde{k}_s$ must satisfy the *linear dispersion relation* for the dispersive hydrodynamics system conjugated to (1). This conjugated system is obtained from the original system (1) by the change of variables $\tilde{x} = ix$, $\tilde{t} = it$,

$$i\partial_{\tilde{t}}\tilde{\mathbf{U}} = \mathbf{K}_N\big(\tilde{\mathbf{U}}, i\partial_{\tilde{x}}\tilde{\mathbf{U}}, -\partial^2_{\tilde{x}\tilde{x}}\tilde{\mathbf{U}}, \ldots\big), \tag{51}$$

which is equivalent to a mere change of the dispersion sign in the original system (1). The conjugated linear dispersion relation of our interest is obtained by linearizing the system (51) in a way similar to (12), i.e., about the mean background

$$\tilde{U}_j \approx \langle\tilde{U}_j\rangle + \tilde{a}_j e^{i(\tilde{k}_s\tilde{x} - \tilde{\omega}_s\tilde{t})}, \quad \tilde{a}_j \ll 1, \tag{52}$$

and has the form

$$\tilde{\omega}_s = \tilde{\omega}_s\big(\tilde{k}_s, \langle\tilde{\rho}\rangle, \langle\tilde{v}\rangle\big). \tag{53}$$

Since the components of the vector $\mathbf{U}$ are expressed in terms of the variable f (as well as the components of $\tilde{\mathbf{U}}$ in terms of $\tilde{f}$) by relationships not containing θ explicitly and the operator $\mathbf{K}_N$ is the same in (1) and (51), the functions $\rho(f)$, $v(f)$ and $\tilde{\rho}(\tilde{f})$, $\tilde{v}(\tilde{f})$ should be identical, i.e., $\rho(z) = \tilde{\rho}(z)$, $v(z) = \tilde{v}(z)$. Then it follows from (41) that in the limit

$$\lambda_2 \to \lambda_1: \quad \langle\tilde{\rho}\rangle \to \bar{\rho}, \quad \langle\tilde{v}\rangle \to \bar{v}. \tag{54}$$

Therefore, the solitary wave dispersion relation has the form $\tilde{\omega}_s = \tilde{\omega}_s(\tilde{k}_s, \bar{\rho}, \bar{v})$ and can be obtained from the original linear dispersion relation (11) by the formal change

$$k \to i\tilde{k}_s, \quad \omega_0 \to i\tilde{\omega}_s. \tag{55}$$

Remark. We emphasize that all the obtained relationships between original and conjugated averaged variables essentially represent algebraic identities between integrals of the form (37) and (39) associated with given potential curve $R(\lambda)$ and do not imply any connection between their spatio-temporal dynamics in the original and the conjugated systems (1) and (51).

Now integrating (48) we find $\tilde{k}_s(\bar{\rho})$ and then s_1 by formula (47):

$$s_1 = \frac{\tilde{\Omega}_s(\tilde{k}_1, \rho_1)}{\tilde{k}_1}, \tag{56}$$

where $\tilde{k}_1 = \tilde{k}_s(\rho_1)$. Analogously to the trailing edge case, the values ρ_2, v_2 enter the expression (56) via the constant C in the characteristic velocity (32) and the initial condition for $\tilde{k}_s(\bar{\rho})$ in (48).

5. Example: fully nonlinear ion-acoustic waves in collisionless plasma

As an example of effective evaluation of the dispersive shock boundaries in a non-integrable system we make use of the classical system describing finite-amplitude ion-acoustic waves in two-temperature ($T_e \gg T_i$) collisionless plasma (see, for instance, [11])

$$\begin{aligned}&\partial_t\rho + \partial_x(\rho v) = 0,\\&\partial_t v + v\partial_x v + \partial_x\phi = 0,\\&\partial^2_{xx}\phi = e^{\phi} - \rho.\end{aligned} \tag{57}$$

Here ρ and v are ion density and velocity and ϕ is the electric potential; all dependent variables are dimensionless. The system (57) possesses all the general properties described in Section 1. In the dispersionless limit $\phi = \ln\rho$ and, therefore, $c_s = 1$ in (3), while the linear dispersion relation has the form

$$\omega_0(k, \bar{\rho}, \bar{v}) = k\big[\bar{v} + (1 + k^2/\bar{\rho})^{-1/2}\big]. \tag{58}$$

The dispersive shock transition condition (5) then takes the form $v_2 - \ln\rho_2 = v_1 - \ln\rho_1$. Without loss of generality we put $v_1 = 0$, $\rho_1 = 1$. Then the relationship (29) between $\bar{\rho}$ and $\bar{v}$ in the degenerate modulation system (25)–(27) assumes the form $\bar{v} = \ln\bar{\rho}$. As a result, we get all the necessary ingredients (32) for the basic differential equation (33):

$$\begin{aligned}&V(\bar{\rho}) = \ln\bar{\rho} + 1,\\&\Omega_0(k, \bar{\rho}) = k\big[\ln\bar{\rho} + (1 + k^2/\bar{\rho})^{-1/2}\big].\end{aligned} \tag{59}$$

Eq. (33) then, after elementary transformations, assumes the form with separated variables

$$\bar{\rho}\frac{d\alpha}{d\bar{\rho}} = -\frac{(1+\alpha)^2\alpha}{2(1+\alpha+\alpha^2)}, \quad \alpha(1) = 1, \tag{60}$$

where $\alpha = (1 + k_0^2/\bar{\rho})^{-1/2}$. Integrating (60) we get

$$\ln\bar{\rho} + 2\ln\alpha + \frac{2}{1+\alpha} - 1 = 0. \tag{61}$$

Now, using (34) we obtain a simple implicit formula determining velocity of the trailing edge s_2 in terms of

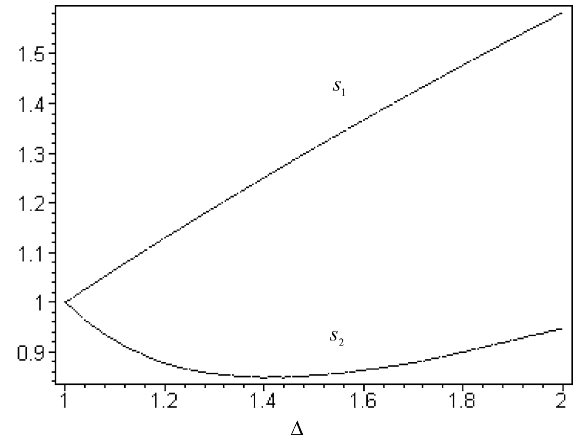


Fig. 2. Boundaries of the dispersive shock $s_{1,2}$ versus density jump Δ: 1—leading edge, 2—trailing edge.

the density jump $\Delta = \rho_2/\rho_1 = \rho_2$,

$$\ln \Delta + \frac{2}{3}\ln(s_2 - \ln \Delta) = \frac{(s_2 - \ln \Delta)^{1/3} - 1}{(s_2 - \ln \Delta)^{1/3} + 1}. \qquad (62)$$

The leading edge is handled in the same way. The solitary wave dispersion relation (49) is obtained from (59) by the change (55) and has the form

$$\tilde{\Omega}_s(\tilde{k}_s, \bar{\rho}) = \tilde{k}_s\big[\ln \bar{\rho} + (1 - \tilde{k}_s^2/\bar{\rho})^{-1/2}\big]. \qquad (63)$$

Then, integrating (48) we obtain $\tilde{k}_s(\bar{\rho})$ (it is convenient to introduce $\tilde{\alpha} = (1 - \tilde{k}_s^2/\bar{\rho})^{-1/2}$ instead of $\tilde{k}_s$, cf. (60)) and substituting it into (56) eventually get for the leading edge

$$\frac{1 - s_1}{1 + s_1} + 2\ln s_1 = \ln \Delta. \qquad (64)$$

Both curves $s_1(\Delta)$ and $s_2(\Delta)$ are presented in Fig. 2 and demonstrate complete agreement with results of direct numerical simulation of the decay of an initial discontinuity for the system (57) obtained in [6]. From the theoretical point of view this agreement can be viewed a strong indication of validity of using the modulation theory in non-integrable initial value problems where rigorous derivation of the Whitham asymptotics is not available.

The weakly nonlinear asymptotics of (62) and (64) for $\eta \equiv \Delta - 1 \ll 1$:

$$s_2 \approx 1 - \eta, \qquad s_1 \approx 1 + \frac{2}{3}\eta, \qquad (65)$$

corresponds to the boundaries in the well-known exact analytic solution for the modulation KdV system found by Gurevich and Pitaevskii [5], which is another confirmation of validity of our approach. Curiously, as is clearly seen from Fig. 2, the fully nonlinear dynamics of the leading (solitary wave) edge s_1 is quite well approximated by the weakly nonlinear asymptotics (65) while the speed of the trailing (vanishing amplitude) edge s_2 demonstrates significant qualitative and quantitative deviations from its weakly nonlinear analog even for quite moderate values of the initial jump.

In conclusion we note that the method of obtaining the dispersive shock boundaries proposed in this Letter is consistent with the concept of 'local Riemann invariant transport' for the solutions of the Whitham systems proposed in [12] but is considerably more transparent and effective.

Acknowledgements

Authors are grateful to V. El and O. Green for fruitful discussions.

References

[1] R.Z. Sagdeev, in: M.A. Leontovich (Ed.), in: Reviews of Plasma Physics, Vol. 4, Consultants Bureau, New York, 1968, p. 20.
[2] P.D. Lax, C.D. Levermore, Commun. Pure Appl. Math. 36 (1983) 253;
P.D. Lax, C.D. Levermore, Commun. Pure Appl. Math. 36 (1983) 571;
P.D. Lax, C.D. Levermore, Commun. Pure Appl. Math. 36 (1983) 809.
[3] S. Venakides, Commun. Pure Appl. Math. 43 (1990) 335.
[4] G.B. Whitham, Proc. R. Soc. A 283 (1965) 238.
[5] A.V. Gurevich, L.P. Pitaevsky, Sov. Phys. JETP 38 (1974) 291.
[6] A.V. Gurevich, A.P. Meshcherkin, Sov. Phys. JETP 60 (1984) 732.
[7] D.J. Kaup, Prog. Theor. Phys. 54 (1976) 396.
[8] A.E. Green, P.M. Naghdi, J. Fluid Mech. 78 (1976) 237.
[9] S.L. Gavrilyuk, V.M. Teshukov, Contin. Mech. Termodyn. 13 (2001) 365.
[10] M. Wesner, C. Herden, R. Pankrath, D. Kip, Phys. Rev. E 64 (2001) 036613.
[11] V.I. Karpman, Nonlinear Waves in Dispersive Media, Pergamon, Oxford, 1975.
[12] A.V. Tyurina, G.A. El, J. Exp. Theor. Phys. 88 (1999) 615.
[13] G.A. El, Undular bore transition in fully nonlinear dispersive wave dynamics, in preparation.

[14] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[15] B.A. Dubrovin, S.P. Novikov, Russian Math. Surveys 44 (1989) 35.
[16] A.V. Gurevich, A.L. Krylov, Sov. Phys. JETP 65 (1987) 944.
[17] G.A. El, R.H.J. Grimshaw, M.V. Pavlov, Stud. Appl. Math. 106 (2001) 157.