The geometry of dissipative evolution equations: the porous medium equation

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Abstract

We show that the porous medium equation has a gradient flow structure which is both physically and mathematically natural. In order to convince the reader that it is mathematically natural, we show the time asymptotic behavior can be easily understood in this framework. We use the intuition and the calculus of Riemannian geometry to quantify this asymptotic behavior.

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1 The porous medium equation as a gradient flow

1.1 The porous medium equation

The porous medium equation is given by

$$\frac{\partial \rho}{\partial t} - \nabla^2 \rho^m \; = \; 0. \tag{1}$$

Throughout the article, $\rho \geq 0$ should be thought of as a (time–dependent) density function on the whole N–dimensional space $I\!R^N$. Here, $\frac{\partial \rho}{\partial t}$ denotes the (partial) derivative w. r. t. time $t \in [0, \infty)$, ∇ denotes the gradient with respect to the spatial variables $x \in I\!R^N$, $\nabla \cdot$ the divergence w. r. t. x and ∇^2 the Laplacian w. r. t. x. In section 2, we will give a brief derivation of (1) from assumptions on the physics of a gas flow through a porous medium. We restrict our attention to the case where the exponent satisfies $m \geq 1 - \frac{1}{N}$ and $m > \frac{N}{N+2}$; the reason for these restrictions will become apparent in the sequel.

The porous medium equation is a parabolic equation, more precisely: a diffusion equation for ρ. In case of $m > 1$, the diffusion degenerates for $\rho = 0$. This for instance has the effect of preserving a compact support and hence is called "slow diffusion". The case $m \leq 1$ is called fast diffusion. In a weak setting, which will be intruduced in section 5, the Cauchy problem for (1) is well–posed. Therefore, (1) defines an evolution of densities on $I\!R^N$, in other words: a semi group on the space of densities on $I\!R^N$. We will show that this semi group has the structure of a gradient flow.

1.2 Abstract gradient flow

We claim that the porous medium equation can be interpreted as a gradient flow. Let us first introduce the notion of a gradient flow in the generality we need. The mathematical structure required to make sense of a gradient flow is

- a differentiable manifold $\mathcal{M}$,
- a metric tensor g on $\mathcal{M}$, which makes $(\mathcal{M}, g)$ a Riemannian manifold,
- and a function(al) E on $\mathcal{M}$.

We call the dynamical system in $\mathcal{M}$ given by the autonomous differential equation

$$\frac{d\rho}{dt} \;=\; -\mathrm{grad}\, E_{|\rho} \tag{2}$$

the gradient flow of E on $(\mathcal{M}, g)$. Observe that the metric tensor g is a necessary ingredient to the notion. It converts the differential $\mathrm{diff}\, E$ of E, which is a cotangent vector field, into the gradient $\mathrm{grad}\, E$ of E, which is a tangent vector field:

$$g(\mathrm{grad}\, E, s) \;=\; \mathrm{diff}E.s \quad \text{for all vector fields } s \text{ on } \mathcal{M}. \tag{3}$$

Hence (2) can be expanded into

$$g_\rho(\frac{d\rho}{dt}, s) \;+\; \mathrm{diff}E_{|\rho}.s \;=\; 0 \quad \text{for all vector fields } s \text{ along } \rho. \tag{4}$$

We point out that the basic property of a gradient flow is that the energy is decreasing along trajectories:

$$\frac{d}{dt}\, E(\rho) \;=\; \mathrm{diff}E_{|\rho}.\frac{d\rho}{dt} \;\stackrel{(4)}{=}\; -g_\rho(\frac{d\rho}{dt}, \frac{d\rho}{dt}). \tag{5}$$

1.3 Two interpretations of the porous medium equation as gradient flow

It is actually well known that the porous medium equation can be interpreted as a gradient flow. We will introduce this "traditional" gradient flow interpretation in this section. Parallel to this, we will introduce a *new* gradient flow interpretation. In the following two sections, we will try to convince the reader that our new way of interpreting the porous medium equation is more natural than the traditional way.

The evolution defined by (1) preserves non negativity of ρ and its mass $\int \rho$. In both approaches, the manifold is accordingly given by

$$\mathcal{M} \;=\; \left\{ \text{non negative functions } \rho \text{ on } \mathbb{R}^N \text{ with } \textstyle\int \rho = 1 \right\}.$$

We will be deliberately sloppy about the differential structure of the manifold and think of the tangent space as follows

$$T_\rho\mathcal{M} \;=\; \left\{ \text{functions } s \text{ on } \mathbb{R}^N \text{ with } \textstyle\int s = 0 \right\}.$$

We now come to the metric tensor. Both approaches are based on an identification of the tangent vector space

$$T_\rho\mathcal{M} \;\cong\; \left\{ \text{functions } p \text{ on } \mathbb{R}^N \right\} / \sim . \tag{6}$$

where the identification is defined via the elliptic equation

$$-\nabla^2 p \;=\; s \quad \text{for the traditional approach} \tag{7}$$

and

$$-\nabla\cdot(\rho\nabla p) \;=\; s \quad \text{for the new approach.} \tag{8}$$

The "$\sim$" in (6) is to indicate that we identify p's which only differ by an additive constant. Now, the metric tensor is defined by

$$g_\rho(s_1,s_2) \;=\; \int \nabla p_1\cdot\nabla p_2 \quad \text{for the traditional approach}$$

and

$$g_\rho(s_1,s_2) \;=\; \int \rho\nabla p_1\cdot\nabla p_2 \quad \text{for the new approach,} \tag{9}$$

where p_i is related to s_i via (7) resp. (8). For further reference, we notice that this implies

$$g_\rho(s_1, s_2) \;=\; \int s_1\, p_2 \quad \text{for both approaches.} \tag{10}$$

Finally the functional: It is given by

$$E(\rho) \;=\; \tfrac{1}{m+1}\int \rho^{m+1} \quad \text{for the traditional approach}$$

and

$$E(\rho) \;=\; \left\{\begin{array}{ll} \tfrac{1}{m-1}\int \rho^m & \text{for } m \neq 1 \\ \int \rho\,\ln\rho & \text{for } m = 1 \end{array}\right\} \quad \text{for the new approach.} \tag{11}$$

Observe that the differential of the functional is given by

$$\operatorname{diff} E(\rho).s \;=\; \int \rho^m\, s \quad \text{for the traditional approach} \tag{12}$$

and

$$\operatorname{diff} E(\rho).s \;=\; \left\{\begin{array}{ll} \int \tfrac{m}{m-1}\rho^{m-1}\, s & \text{for } m \neq 1 \\ \int (\ln\rho + 1)\, s & \text{for } m = 1 \end{array}\right\} \quad \text{for the new approach.} \tag{13}$$

We now have to show that the porous medium equation indeed coincides with the gradient flow of E on $(\mathcal{M}, g)$ for both approaches. First the traditional approach: According to (10) and (12), the identity (4) takes on the form

$$\int \frac{\partial\rho}{\partial t}\, p \;+\; \int \rho^m\, s \;=\; 0,$$

where p is related to s via $-\nabla^2 p = s$. We substitute s accordingly and obtain

$$\int \frac{\partial\rho}{\partial t}\, p \;-\; \int \rho^m\, \nabla^2 p \;=\; 0,$$

and after integration by parts

$$\int (\frac{\partial\rho}{\partial t} \;-\; \nabla^2\rho^m)\, p \;=\; 0.$$

Since p is arbitrary, we recover the porous medium equation.

Now the new approach: According to (10) and (13), the identity (4) takes on the form

$$\left\{ \begin{array}{l} \int \frac{\partial \rho}{\partial t}\, p \;+\; \frac{m}{m-1}\int \rho^{m-1}\, s \;=\; 0 \quad \text{for } m \neq 1 \\ \int \frac{\partial \rho}{\partial t}\, p \;+\; \int (\ln \rho + 1)\, s \;=\; 0 \quad \text{for } m = 1 \end{array} \right\},$$

where p is related to s via $-\nabla \cdot (\rho \nabla p) = s$. We substitute s accordingly and obtain

$$\left\{ \begin{array}{l} \int \frac{\partial \rho}{\partial t}\, p \;-\; \int \frac{m}{m-1}\, \rho^{m-1}\, \nabla \cdot (\rho \nabla p) \;=\; 0 \quad \text{for } m \neq 1 \\ \int \frac{\partial \rho}{\partial t}\, p \;-\; \int (\ln \rho + 1)\, \nabla \cdot (\rho \nabla p) \;=\; 0 \quad \text{for } m = 1 \end{array} \right\}.$$

We obtain after integration by parts

$$\int (\frac{\partial \rho}{\partial t} \;-\; \nabla^2 \rho^m)\, p \;=\; 0.$$

Also here, we recover the porous medium equation.

In case of the traditional approach, g does not depend on ρ and therefore is a scalar product on the space of functions s with mean value zero. In fact, it is the homogeneous part of the H^{-1}–scalar product. Hence, in the traditional approach, the Riemannian space $(\mathcal{M}, g)$ carries the structure of a convex subspace of a Euclidean space. On the other hand, the new approach is genuinely Riemannian. Hence we must bring forth good reasons for considering the more complicated, new structure. We will attempt to do this in the next two sections.

2 A physical argument in favor of the new gradient flow

We give a brief physical derivation of the porous medium equation. The function ρ describes the mass density of a gas in a porous medium. The first assumption is conservation of mass, expressed in the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, u) \;=\; 0, \tag{14}$$

where the vector field u on $I\!R^N$ describes the (average) velocity of the gas. The second assumption is Darcy's law

$$u \;=\; -M.\nabla p,$$

where the function p on $I\!R^N$ describes the pressure of the gas and the matrix M describes the mobility of the gas in the porous medium. M depends on the permeability of the medium and the viscosity of the gas. We assume that the permeability is isotropic and homogeneous, so that $K = \mathrm{id}$ by an appropriate non–dimensionalization:

$$u \;=\; -\nabla p. \tag{15}$$

The third assumption comes from thermodynamics:

$$p \;=\; \frac{\delta E}{\delta \rho}, \tag{16}$$

where E denotes the free energy and $\frac{\delta E}{\delta \rho}$ its functional derivative with respect to ρ. In case of a free energy of the form

$$E \;=\; \int e(\rho),$$

where the function $z \mapsto e(z)$ describes how the free energy density e depends on the density ρ, (16) reads

$$p \;=\; e'(\rho). \tag{17}$$

Hence (14), (15) and (17) combine to

$$\frac{\partial \rho}{\partial t} - \nabla^2 \pi(\rho) \;=\; 0, \tag{18}$$

where the function $z \mapsto \pi(z)$ describes how the osmotic pressure π depends on the density ρ and is related to $z \mapsto e(z)$ via

$$\pi(z) \;=\; z\, e'(z) - e(z). \tag{19}$$

From (19) we see that (18) turns into the porous medium equation (1), that is,

$$\pi(z) \;=\; z^m \tag{20}$$

if and only if

$$e(z) \;=\; \left\{ \begin{array}{ll} \frac{1}{m-1}\, z^m & \text{for } m \neq 1 \\ z\,\ln z & \text{for } m = 1 \end{array} \right\}. \tag{21}$$

Hence, *only in the new formulation does E have a physical meaning.*

Also the metric tensor g of the new formulation has a physical meaning. For this we observe that the definition (9) of g in the new approach can be reformulated as

$$g_\rho(s,s) \;=\; \inf\left\{ \int \rho\,|u|^2 \;\middle|\; \begin{array}{l} \text{for all vector fields } u \text{ on } I\!R^N \\ \text{with } s + \nabla\cdot(\rho\, u) = 0 \end{array} \right\}. \tag{22}$$

Indeed, the minimizer u of the quadratic variational problem in (22) satisfies

$$\int v\cdot u \quad \text{for all vector fields } v \text{ on } I\!R^N \text{ with } \nabla\cdot v = 0,$$

so that there exists a function p on $I\!R^N$ such that

$$u \;=\; -\nabla p.$$

We now observe that the quantity $\int \rho\,|u|^2$ in (22) has a physical meaning: It is the rate of dissipation of kinetic energy by friction when the gas moves with velocity u through the pores of the porous medium. Hence $g_\rho(s,s)$ measures the minimal rate of dissipation of kinetic energy by friction required to produce the rate of change s of the density ρ. This allows for a nice physical interpretation of (5), that is

$$\frac{d}{dt}\,E(\rho) \;=\; -g_\rho(\frac{d\rho}{dt},\frac{d\rho}{dt}).$$

The right hand side is the rate of change of the free energy, the left hand side is the rate of dissipation of kinetic energy by friction; the dynamics are such that both quantities are equal. In general terms: The merit of the right gradient flow formulation of a dissipative evolution equation is that it separates energetics and kinetics: The energetics endow the state space $\mathcal{M}$ with a functional E, the kinetics endow the state space with a (Riemannian) geometry via the metric tensor g.

3 A mathematical argument in favor of new gradient flow

3.1 Self similar solutions and asymptotic behaviour

It is well–known that the long–time asymptotics of the porous medium equation is described by the Barenblatt solution. Let us make this more precise: The porous medium equation allows for a self–similar solution of the form

$$\rho_*(t,x) \;=\; \frac{1}{t^{N\alpha}}\,\hat\rho_*(\frac{x}{t^\alpha})$$

where the profile $\hat\rho_*$ is given implicitly in the "pressure variable" (for a motivation of this wording see the previous section)

$$e'(\hat\rho_*(y)) \;=\; \left\{\begin{array}{lcll} \frac{m}{m-1}\,\hat\rho_*(y)^{m-1} & = & \max\{\lambda-\alpha\,\frac12\,|y|^2,0\} & \text{for } m>1 \\ \ln\hat\rho_*(y)+1 & = & \lambda-\alpha\,\frac12\,|y|^2 & \text{for } m=1 \\ \frac{m}{m-1}\,\hat\rho_*(y)^{m-1} & = & \lambda-\alpha\,\frac12\,|y|^2 & \text{for } m<1 \end{array}\right\}.$$

Here

$$\alpha = \frac{1}{N\,(m-1)+2}$$

and λ is such that

$$\int \hat\rho_* = 1.$$

These solutions were discovered by Barenblatt and Prattle [4, 31].

The Barenblatt solution describes the long–time asymptotics of an arbitrary solution ρ in the following sense: Rescale time and space according to

$$x \;=\; t^\alpha\,y \quad\text{and}\quad t \;=\; \exp(\tau).$$

In terms of density functions, this means: pass from ρ to $\hat\rho$ given by

$$\rho(t,x) \;=\; \frac{1}{t^{N\alpha}}\,\hat\rho(\ln t,\frac{x}{t^\alpha})$$

Then $\hat\rho$ approaches the profile $\hat\rho^*$ of the Barenblatt solution for large times. In case of $m>1$, Friedman, Kamin and Vazquez (in [16] and [23]) have proved that the profiles converge uniformly

$$\lim_{\tau\uparrow\infty}\|\hat\rho-\hat\rho_*\|_{L^\infty(\boldsymbol{R}^N)} \;=\; 0\,.$$

Their proof is based on a C^α–a priori estimates for the solution of the porous medium equation by Caffarelli and Friedman [8].

3.2 A new asymptotic result

Hoping to convince the reader of our new approach, we will derive a new and more quantitative asymptotic result using it. Our arguments are based on a simple Riemannian calculus applied to the infinite dimensional $(\mathcal{M}, g)$. From now on, the notation g and E pertains solely to the new approach, that is, it is defined like in (9) and (11). Next to the metric tensor g, which we also denote by $\langle \cdot, \cdot \rangle$, and its induced norm $|\cdot|$, we will need a few notions from Riemannian geometry, like the gradient $\mathrm{grad} F$, the Hessian $\mathrm{Hess} F$ of a function F on $\mathcal{M}$, the latter being defined via the covariant derivative, see for instance [30, Section 2.1.3], and the induced distance d, see [30, Section 5.3].

The three key ingredients for our asymptotic result are

- $\hat\rho$ satisfies
$$\frac{d\hat\rho}{d\tau} \;=\; -\mathrm{grad}\, F_{|\hat\rho}. \tag{23}$$
  In words, $\hat\rho$ evolves according to the gradient flow on the same Riemannian manifold $(\mathcal{M}, g)$ of an augmented functional F given by
$$F(\hat\rho) \;=\; E(\hat\rho) \;+\; \alpha\, M(\hat\rho),$$
  where M denotes the second moment of the density $\hat\rho$
$$M(\hat\rho) \;=\; \int \tfrac{1}{2}\, |y|^2\, \hat\rho(y)\, dy.$$

- $\hat\rho_*$ satisfies
$$F(\hat\rho) - F(\hat\rho_*) \;\geq\; 0 \quad \text{for all } \hat\rho \in \mathcal{M}. \tag{24}$$
  In words, $\hat\rho_*$ is a minimizer of F on $\mathcal{M}$. Hence it it is also a stationary point of F, that is
$$0 \;=\; -\mathrm{grad}\, F_{|\hat\rho_*}. \tag{25}$$

- F satisfies

$$\mathrm{Hess}\, F_{|\hat\rho} \;\geq\; \alpha\, \mathrm{id} \quad \text{for all } \hat\rho \in \mathcal{M}$$

in the sense of

$$\langle s, \mathrm{Hess}\, F_{|\hat\rho} s\rangle \;\geq\; \alpha\, |s|^2 \quad \text{for all } s \in T_{\hat\rho}\mathcal{M} \text{ and } \hat\rho \in \mathcal{M}. \tag{26}$$

In words, F is uniformly strictly convex on $(\mathcal{M}, g)$. This is a consequence of

$$\mathrm{Hess}\, E_{|\hat\rho} \;\geq\; 0 \quad \text{and} \quad \mathrm{Hess}\, M_{|\hat\rho} \;=\; \mathrm{id} \quad \text{for all } \hat\rho \in \mathcal{M}. \tag{27}$$

We will check (23) and (24) in subsection 3.4; (27) will be established in subsection 4.4. The condition $m \geq 1 - \frac{1}{N}$ is the one which ensures that E is convex on $(\mathcal{M}, g)$. The condition $m \geq \frac{N}{N+2}$ ensures that $E(\hat\rho_*)$ and $M(\hat\rho_*)$ are well–defined and finite.

As we will see in subsection 3.5, (23), (25) and (26) yield by formal but basic Riemannian calculus

$$\frac{d}{d\tau}\left(\exp(2\,\alpha\,\tau)\,|\mathrm{grad}\, F_{|\hat\rho}|^2\right) \;\leq\; 0, \tag{28}$$

$$\frac{d}{d\tau}\left(\exp(2\,\alpha\,\tau)\,(F(\hat\rho) - F(\hat\rho_*))\right) \;\leq\; 0, \tag{29}$$

$$\frac{d}{d\tau}\left(\exp(2\,\alpha\,\tau)\, d(\hat\rho, \hat\rho_*)^2\right) \;\leq\; 0. \tag{30}$$

We consider these three inequalities the main result of this paper. Observe that (28), (29) and (30) express a single fact in different form. The single fact being: $\hat\rho$ converges to $\hat\rho_*$ with rate α. More precisely, α is an exponential rate with respect to τ or a polynomial rate with respect to t. The different forms being: $|\mathrm{grad}\, F_{|\hat\rho}|$ in (28) measures how far $\hat\rho$ is from being a stationary point of F, $F(\hat\rho) - F(\hat\rho_*)$ in (29) is measuring how far $\hat\rho$ is from being a minimizer of F and finally $d(\hat\rho, \hat\rho_*)$ in (30) is measuring how far $\hat\rho$ is from $\hat\rho_*$.

In subsection 3.4, we will identify $|\mathrm{grad} F_{|\hat\rho}|^2$ as the functional

$$|\mathrm{grad} F_{|\hat\rho}|^2 \;=\; \int \hat\rho\, |\nabla p|^2 \quad \text{where} \quad p(y) \;=\; e'(\hat\rho(y)) + \alpha\, \frac{1}{2}|y|^2$$

and e is the energy density given in (21). In subsection 4.3, we will identify the induced metric d with the Wasserstein metric, that is

$$d(\hat\rho_0,\hat\rho_1)^2 \;=\; \inf_{\hat\rho_1\,=\,\Phi\#\hat\rho_0} \int \hat\rho_0\,|\mathrm{id}-\Phi|^2,$$

where $\Phi\#\hat\rho_0$ denotes the push forward of the density $\hat\rho_0$ under the transformation Φ of $I\!R^N$. By carefully mimicking the formal Riemannian calculus from subsection 3.5, we will make the above results rigorous in Theorem 1 in section 5. A relationship, not in the above concise form though, between the porous medium equation, its self similar solution and the Wasserstein metric was discovered by the author in [29].

In the linear case $m = 1$, above results are known to the Fokker–Planck community in a different form. In this case,

$$\begin{aligned}
|\mathrm{grad}F_{|\hat\rho}|^2 &= \int \hat\rho\,|\nabla p|^2\\
&= \int \hat\rho\,|\frac{1}{\hat\rho}\,\nabla\hat\rho+\alpha\,y|^2\\
&= \int \frac{1}{\hat\rho}\,|\nabla\hat\rho|^2 - 2\,\alpha\,N\int\hat\rho+\alpha^2\int\hat\rho|y|^2\\
&= \int \frac{1}{\hat\rho}\,|\nabla\hat\rho|^2 - 2\,\alpha\,N + 2\,\alpha^2\,M(\hat\rho).
\end{aligned}$$

In particular, $0 = |\mathrm{grad}F_{|\hat\rho_*}|^2 = \int \frac{1}{\hat\rho_*}\,|\nabla\hat\rho_*|^2 - 2\,\alpha\,N + \alpha^2\,M(\hat\rho_*)$, so that

$$|\mathrm{grad}F_{|\hat\rho}|^2 \;=\; \int\frac{1}{\hat\rho}\,|\nabla\hat\rho|^2 - \int\frac{1}{\hat\rho_*}\,|\nabla\hat\rho_*|^2 \quad \text{if} \quad M(\hat\rho) \;=\; M(\hat\rho_*).$$

The quantity $\int\frac{1}{\hat\rho}\,|\nabla\hat\rho|^2$ is called the "Fisher information functional". Also in this case

$$F(\hat\rho)-F(\hat\rho_*) \;=\; \int\hat\rho\,\ln\hat\rho - \int\hat\rho_*\,\ln\hat\rho_* \quad \text{if} \quad M(\hat\rho) \;=\; M(\hat\rho_*)$$

and the quantity $\int\hat\rho\,\ln\hat\rho$ is called the "entropy functional". The decay of the Fisher information functional and the entropy functional expressed in (28) resp. (29) for $m = 1$ seems to be due to McKean [26] and Toscani [10]. Recently and independently of our work, these ideas for (28) resp.

(29) have been extended to the case $m \neq 1$ by Carrillo & Toscani [11] (for $m > 1$) and Dolbeault & del Pino [13] (for $m < 1$). Forerunners in this Liapunov–functional based approach were also Newman [27] and Ralston [33]. *The novelty of our above results is their formulation, interpretation and proof in framework of Riemannian geometry*, which make the approach more transparent and the calculations seem less arbitrary.

3.3 The asymptotic result expressed in a more traditional framework

Convergence with rate α in a more traditional way can be derived from (29) with help of

- the inequalities

$$F(\hat\rho) - F(\hat\rho_*) \left\{ \begin{array}{ll} \geq & H(\hat\rho, \hat\rho_*) \quad \text{for } m > 1 \\ = & H(\hat\rho, \hat\rho_*) \quad \text{for } m \leq 1 \end{array} \right\}, \tag{31}$$

  where

$$H(\hat\rho_1, \hat\rho_0) \;=\; \int \{ e(\hat\rho_1) - e(\hat\rho_0) - e'(\hat\rho_0)\,(\hat\rho_1 - \hat\rho_0) \} \;\geq\; 0.$$

  Here the e is the energy density, see (21), and in the case $m < 1$, we set $H(\hat\rho_1, \hat\rho_0) = +\infty$ if $\hat\rho_0$ vanishes on a set of positive measure.

- the estimate for $m \leq 2$

$$\int |\hat\rho_1 - \hat\rho_0| \;\leq\; C \left(\int \hat\rho_0^{2-m} \right)^{\frac{1}{2}} \; H(\hat\rho_1, \hat\rho_0)^{\frac{1}{2}}, \tag{32}$$

  where C is a constant which only depends on m.

It is conceivable that convergence of rate α in stronger traditional norms can be derived from (28), (29) and (30). But this is not the focus of this paper.

The inequality (31) will be established in subsection 3.4, the non negativity of H follows immediately from the convexity of e. In the case of $m = 1$, we have

$$e(z_1) - e(z_0) - e'(z_0)\,(z_1 - z_0) \;=\; \frac{z}{z_0}\,\ln\frac{z}{z_0}\, z_0 - (z_1 - z_0),$$

so that

$$H(\hat\rho_1,\hat\rho_0) \;=\; \int \frac{\hat\rho_1}{\hat\rho_0}\,\ln\frac{\hat\rho_1}{\hat\rho_0}\,\hat\rho_0. \tag{33}$$

Therefore, $H(\hat\rho_1,\hat\rho_0)$ is also called the relative entropy of $\hat\rho_1$ w. r. t. $\hat\rho_0$. The estimate (32) is known to the Fokker–Planck community under the name of Csiszar–Kullback inequality [25].

Let us now establish (32). Since $m \le 2$ implies

$$e''(w) \;=\; m\,w^{m-2} \;\ge\; m \quad \text{for } w\in[0,1]$$

we have

$$\begin{aligned}
&\tfrac{1}{m-1}\,w^m - \tfrac{1}{m-1} - \tfrac{m}{m-1}\,w^{m-1}\,(w-1)\\
&\quad=\; e(w)-e(1)-e'(1)\,(w-1)\\
&\quad\ge\; \tfrac12 \inf_{(0,1)} e''\,(w-1)^2\\
&\quad=\; \tfrac{m}{2}\,(w-1)^2 \quad \text{for all } w\in[0,1].
\end{aligned} \tag{34}$$

We observe that since

$$\int(\hat\rho_1-\hat\rho_0) \;=\; 0,$$

we have

$$\int|\hat\rho_1-\hat\rho_0| \;=\; 2\int_{\{\hat\rho_1<\hat\rho_0\}}|\hat\rho_1-\hat\rho_0|.$$

On the other hand, setting

$$u \;=\; \left\{\begin{array}{cc} \dfrac{\hat\rho_1}{\hat\rho_0} & \text{if } \hat\rho_1<\hat\rho_0 \\ 1 & \text{else} \end{array}\right\} \;\in\; [0,1],$$

we have

$$\begin{aligned}
&\int_{\{\hat\rho_1<\hat\rho_0\}}|\hat\rho_1-\hat\rho_0|\\
&\quad=\; \int \hat\rho_0\,|u-1|\\
&\quad\le\; \left(\int \hat\rho_0^{2-m}\int \hat\rho_0^{m}\,(u-1)^2\right)^{\frac12}
\end{aligned}$$

$$\overset{(34)}{\leq} \quad \left(\int \hat\rho_0^{2-m}\, \tfrac{2}{m} \int \hat\rho_0^m \left\{\tfrac{1}{m-1}\, u^m - \tfrac{1}{m-1} - \tfrac{m}{m-1}\, u^{m-1}\,(u-1)\right\}\right)^{\frac12}$$

$$= \quad \left(\int \hat\rho_0^{2-m}\, \tfrac{2}{m} \int_{\{\hat\rho_1<\hat\rho_0\}} \left\{\tfrac{1}{m-1}\, \hat\rho_1^m - \tfrac{1}{m-1}\, \hat\rho_0^m - \tfrac{m}{m-1}\, \hat\rho_0^{m-1}\,(\hat\rho_0-\hat\rho_1)\right\}\right)^{\frac12}.$$

3.4 Verification of key ingredients to asymptotic result

Let us now check (23). It is left to the reader to verify that $\hat\rho$ satisfies the equation

$$\frac{\partial\hat\rho}{\partial\tau} \;-\; \nabla_y^2\hat\rho^m - \alpha\,\nabla_y\cdot(\hat\rho\, y) \;=\; 0. \tag{35}$$

¿From now on, we drop the subscript y. We observe that the differential of F is given by

$$\text{diff}\, F(\hat\rho).s \;=\; \left\{ \begin{array}{ll} \int (\frac{m}{m-1}\hat\rho^{m-1} + \alpha\,\frac12|y|^2)\, s & \text{for } m\neq 1 \\ \int (\ln\hat\rho + 1 + \alpha\,\frac12|y|^2)\, s & \text{for } m=1 \end{array} \right\}.$$

According to this and (10), the identity (4) takes on the form

$$\begin{array}{ll} \int \frac{\partial\hat\rho}{\partial\tau}\, p \;+\; \int (\frac{m}{m-1}\hat\rho^{m-1} + \alpha\,\frac12|y|^2)\, s \;=\; 0 & \text{for } m\neq 1, \\ \int \frac{\partial\hat\rho}{\partial\tau}\, p \;+\; \int (\ln\hat\rho + 1 + \alpha\,\frac12|y|^2)\, s \;=\; 0 & \text{for } m=1, \end{array}$$

where p is related to s via $-\nabla\cdot(\hat\rho\nabla p) = s$. We substitute s accordingly and obtain after an integration by part

$$\begin{array}{ll} \int \{\frac{\partial\hat\rho}{\partial\tau} \;-\; \nabla\cdot[\hat\rho\,\nabla\,(\frac{m}{m-1}\hat\rho^{m-1} + \alpha\,\frac12|y|^2)]\}\, p \;=\; 0 & \text{for } m\neq 1, \\ \int \{\frac{\partial\hat\rho}{\partial\tau} \;-\; \nabla\cdot[\hat\rho\,\nabla\,(\ln\hat\rho + 1 + \alpha\,\frac12|y|^2)]\}\, p \;=\; 0 & \text{for } m=1. \end{array}$$

Hence (4) can be rewritten as

$$\begin{array}{rcll} \frac{\partial\hat\rho}{\partial\tau} \;-\; \nabla\cdot[\hat\rho\,\nabla\,(\frac{m}{m-1}\hat\rho^{m-1} + \alpha\,\frac12|y|^2)] & = & 0 & \text{for } m\neq 1, \\ \frac{\partial\hat\rho}{\partial\tau} \;-\; \nabla\cdot[\hat\rho\,\nabla\,(\ln\hat\rho + 1 + \alpha\,\frac12|y|^2)] & = & 0 & \text{for } m=1, \end{array}$$

which turns into (35).

Let us now check that in the notation of (31),

$$F(\hat\rho)-F(\hat\rho_*)\left\{\begin{array}{ll}\ge\ H(\hat\rho,\hat\rho_*) & \text{for } m>1\\ =\ H(\hat\rho,\hat\rho_*) & \text{for } m\le 1\end{array}\right\}. \tag{36}$$

This validates both (24) and (31). In order to show (36) in case of $m\neq 1$, we observe that by definition of $H(\hat\rho,\hat\rho_*)$

$$E(\hat\rho)\ =\ E(\hat\rho_*)+H(\hat\rho,\hat\rho_*)+\int\tfrac{m}{m-1}\,\hat\rho_*^{m-1}(\hat\rho-\hat\rho_*),$$

so that by definition of F,

$$F(\hat\rho)\ =\ F(\hat\rho_*)+H(\hat\rho,\hat\rho_*)+\int(\tfrac{m}{m-1}\,\hat\rho_*^{m-1}+\alpha\,\tfrac12|y|^2)\,(\hat\rho-\hat\rho_*).$$

In case of $m<1$, we have by definition of $\hat\rho_*$

$$\tfrac{m}{m-1}\,\hat\rho_*^{m-1}+\alpha\,\tfrac12|y|^2\ =\ \lambda,$$

so that we obtain

$$F(\hat\rho)\ =\ F(\hat\rho_*)+H(\hat\rho,\hat\rho_*)+\lambda\int(\hat\rho-\hat\rho_*)\ =\ F(\hat\rho_*)+H(\hat\rho,\hat\rho_*).$$

In case of $m>1$, we have by definition of $\hat\rho_*$,

$$(\tfrac{m}{m-1}\,\hat\rho_*^{m-1}+\alpha\,\tfrac12|y|^2)\,(\hat\rho-\hat\rho_*)\ \ge\ \lambda\,(\hat\rho-\hat\rho_*)\quad\text{for all } y\in I\!R^N. \tag{37}$$

Indeed, if y is such that $\lambda-\alpha\,\frac12|y|^2\ge 0$ then $\frac{m}{m-1}\,\hat\rho_*^{m-1}=\lambda-\alpha\,\frac12|y|^2$ and the inequality (37) turns into an equality. On the other hand, if y is such that $\lambda-\alpha\,\frac12|y|^2\le 0$, $\hat\rho_*=0$ and the above inequality turns into $\alpha\,\frac12\,|y|^2\,\hat\rho\ge\lambda\hat\rho$, which is true since $\hat\rho\ge 0$. Hence we obtain in this case only an inequality

$$F(\hat\rho)\ \ge\ F(\hat\rho_*)+H(\hat\rho,\hat\rho_*)+\lambda\int(\hat\rho-\hat\rho_*)\ =\ F(\hat\rho_*)+H(\hat\rho,\hat\rho_*).$$

The identity (36) in case of $m=1$ is also quite obvious. From definition of F and $\hat\rho_*$ we obtain

$$F(\hat\rho)\ =\ \int(\ln\hat\rho+\alpha\,\tfrac12|y|^2)\,\hat\rho\ =\ \int(\ln\hat\rho+\lambda-\ln\hat\rho_*)\,\hat\rho\ \stackrel{(33)}{=}\ H(\hat\rho,\hat\rho_*)+\lambda.$$

In particular $F(\hat\rho_*) = \lambda$, so that

$$F(\hat\rho) - F(\hat\rho_*) \;=\; H(\hat\rho, \hat\rho_*).$$

As announced, we will now argue that

$$|\mathrm{grad} F_{|\hat\rho}|^2 \;=\; \int \hat\rho\, |\nabla p|^2 \quad \text{where} \quad p(y) \;=\; e'(\hat\rho(y)) + \alpha\, \frac{1}{2}|y|^2.$$

Indeed, we have by the abstract definition (3) of the gradient

$$\frac{1}{2} g_{\hat\rho}(\mathrm{grad} F_{|\hat\rho}, \mathrm{grad} F_{|\hat\rho}) \;=\; \sup_{s \in T_{\hat\rho}\mathcal{M}} \left\{ dF_{|\hat\rho}.s - \frac{1}{2} g_{\hat\rho}(s,s) \right\}.$$

By definition of our functional F,

$$dF_{|\hat\rho}.s \;=\; \int p\, s \quad \text{with } p \text{ as above.}$$

By definition of our inner product

$$\int p\, s - \frac{1}{2} g_{\hat\rho}(s,s) \;=\; \int \hat\rho\, \nabla p \cdot \nabla q - \int \hat\rho\, \frac{1}{2} |\nabla q|^2,$$

if $s \in T_{\hat\rho}\mathcal{M}$ and the function q on $\mathbb{R}^N$ are related by

$$-\nabla \cdot (\hat\rho\, \nabla q) \;=\; s.$$

Hence

$$\begin{aligned} \frac{1}{2} g_{\hat\rho}(\mathrm{grad} F_{|\hat\rho}, \mathrm{grad} F_{|\hat\rho}) \;&=\; \sup_{\text{function } p \text{ on } \mathbb{R}^N} \left\{ \int \hat\rho\, \nabla p \cdot \nabla q - \int \hat\rho\, \frac{1}{2} |\nabla q|^2 \right\} \\ &=\; \int \hat\rho\, \frac{1}{2} |\nabla p|^2. \end{aligned}$$

3.5 Derivation of asymptotic result by formal Riemannian calculus

Let us now show how (23), (25) and (26) imply (28), (29) and (30) by formal Riemannian calculus. For this, we forget about where our structure $(\mathcal{M}, g)$

and F came from and work exclusively within the abstract framework. The derivation of (28) is easiest:

$$\begin{aligned}\frac{d}{d\tau}\,|\mathrm{grad}F_{|\hat\rho}|^2 &= 2\,\langle \mathrm{grad}F_{|\hat\rho}, \frac{D}{D\tau}\,\mathrm{grad}F_{|\hat\rho}\rangle \\ &= 2\,\langle \mathrm{grad}F_{|\hat\rho}, \mathrm{Hess}F_{|\hat\rho}\,\frac{d}{d\tau}\,\rho\rangle \\ &\stackrel{(23)}{=} -2\,\langle \mathrm{grad}F_{|\hat\rho}, \mathrm{Hess}F_{|\hat\rho}\,\mathrm{grad}F_{|\hat\rho}\rangle \\ &\stackrel{(26)}{\leq} -2\,\alpha\,|\mathrm{grad}F_{|\hat\rho}|^2. \end{aligned} \tag{38}$$

Here $\frac{D}{D\tau}$ denotes the covariant derivative along the curve $\hat\rho$. The first equality comes from the fundamental property of the covariant derivative [30, Section 2.1.2]; the second equality follows from the definition of the Hessian [30, Section 2.1.3].

We now tackle (29) and (30). There are different ways to derive (29) and (30) from (23), (25) and (26) by Riemannian calculus. We choose the one we are able to make rigorous in section 5. We need the following auxiliary result. We recall the definition of the induced metric $d(\hat\rho_0,\hat\rho_1)^2$ as the infimum of the energy (modulo a factor 2) $\int_0^1 |\frac{d\tilde{\hat\rho}}{d\sigma}|^2\,d\sigma$ over all curves $[0,1]\ni\sigma\mapsto\tilde{\hat\rho}(\sigma)\in\mathcal{M}$ which connect $\hat\rho_0$ to $\hat\rho_1$:

$$d(\hat\rho_0,\hat\rho_1)^2 \;=\; \Big\{ \int_0^1 |\tfrac{d\tilde{\hat\rho}}{d\sigma}|^2\,d\sigma \;\Big|\; \begin{array}{l}[0,1]\ni\sigma\mapsto\tilde{\hat\rho}(\sigma)\in\mathcal{M} \\ \text{with } \tilde{\hat\rho}(0)=\hat\rho_0,\ \tilde{\hat\rho}(1)=\hat\rho_1\end{array} \Big\}\,.$$

Let $[0,1]\ni\sigma\mapsto\tilde{\hat\rho}(\sigma)$ denote a curve of least energy between $\hat\rho_0$ and $\hat\rho_1$, that is,

$$d(\hat\rho_0,\hat\rho_1)^2 \;=\; \int_0^1 |\frac{d\tilde{\hat\rho}}{d\sigma}|^2\,d\sigma. \tag{39}$$

In particular, $[0,1]\ni\sigma\mapsto\tilde{\hat\rho}(\sigma)$ is a geodesic, that is

$$\frac{D}{d\sigma}\,\frac{d\tilde{\hat\rho}}{d\sigma} \;=\; 0, \tag{40}$$

which implies

$$\frac{d}{d\sigma}\,|\frac{d\tilde{\hat\rho}}{d\sigma}|^2 \;=\; 2\,\langle\frac{d\hat\rho}{d\sigma},\frac{D}{d\sigma}\,\frac{d\tilde{\hat\rho}}{d\sigma}\rangle \;=\; 0. \tag{41}$$

The auxiliary result we claim is

$$F(\hat\rho_1) - F(\hat\rho_0) \;\geq\; \langle \frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=0}, \mathrm{grad}F_{|\hat\rho_0}\rangle + \alpha\,\frac{1}{2} d(\hat\rho_0,\hat\rho_1)^2. \tag{42}$$

Indeed, this is a consequence of

$$\frac{d}{d\sigma}\,F(\tilde{\hat\rho}) \;=\; \langle \frac{d\tilde{\hat\rho}}{d\sigma}, \mathrm{grad}F_{|\tilde{\hat\rho}}\rangle$$

and

$$\begin{aligned}
\frac{d^2}{d\sigma^2}\,F(\tilde{\hat\rho}) \quad &= \quad \langle \frac{D}{d\sigma}\,\frac{d\tilde{\hat\rho}}{d\sigma}, \mathrm{grad}F_{|\tilde{\hat\rho}}\rangle + \langle \frac{d\tilde{\hat\rho}}{d\sigma}, \frac{D}{d\sigma}\,\mathrm{grad}F_{|\tilde{\hat\rho}}\rangle \\
&\overset{(40)}{=} \quad \langle \frac{d\tilde{\hat\rho}}{d\sigma}, \mathrm{Hess}F_{|\tilde{\hat\rho}}\,\frac{d\tilde{\hat\rho}}{d\sigma}\rangle \\
&\overset{(26)}{\geq} \quad \alpha\,|\frac{d\tilde{\hat\rho}}{d\sigma}|^2 \\
&\overset{(39),(41)}{=} \quad \alpha\,d(\hat\rho_0,\hat\rho_1)^2.
\end{aligned}$$

By symmetry, we also have

$$F(\hat\rho_0) - F(\hat\rho_1) \;\geq\; -\langle \frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=1}, \mathrm{grad}F_{|\hat\rho_1}\rangle + \alpha\,\frac{1}{2} d(\hat\rho_0,\hat\rho_1)^2. \tag{43}$$

Adding (42) and (43) yields

$$\langle \frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=1}, \mathrm{grad}F_{|\hat\rho_1}\rangle - \langle \frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=0}, \mathrm{grad}F_{|\hat\rho_0}\rangle \;\geq\; \alpha\,d(\hat\rho_0,\hat\rho_1)^2. \tag{44}$$

For later reference, we note that (42) also implies

$$\begin{aligned}
F(\rho_1) - F(\rho_0) \quad &\geq \quad -|\frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=0}|\,|\mathrm{grad}F_{|\hat\rho_0}| \\
&\overset{(39),(41)}{=} \quad d(\hat\rho_0,\hat\rho_1)\,|\mathrm{grad}F_{|\hat\rho_0}|,
\end{aligned}$$

hence by symmetry,

$$|F(\rho_1) - F(\rho_0)| \;\leq\; d(\hat\rho_0,\hat\rho_1)\,\max\{|\mathrm{grad}F_{|\hat\rho_0}|, |\mathrm{grad}F_{|\hat\rho_1}|\}. \tag{45}$$

We now derive (30) by formal Riemannian calculus. Because of (25), $\hat\rho_0(\tau) = \hat\rho_*$ defines a (stationary) solution of (23). Hence it suffices to show the contraction property

$$\frac{d^+}{d\tau}\, d(\hat\rho_1,\hat\rho_0)^2 \;\le\; 2\,\alpha\, d(\hat\rho_1,\hat\rho_0)^2 \tag{46}$$

for two solutions $\hat\rho_i$ of (23). Here, $\frac{d^+}{d\tau}$ denotes

$$\frac{d^+}{d\tau}_{|\tau_0} f \;=\; \limsup_{\tau\downarrow\tau_0} \frac{f(\tau)-f(\tau_0)}{\tau-\tau_0}.$$

We fix a τ_0. For any τ, let $[0,1]\ni\sigma\mapsto\tilde{\hat\rho}(\tau,\sigma)\in\mathcal{M}$ be a curve between $\tilde{\hat\rho}(\tau,0)=\hat\rho_0(\tau)$ and $\tilde{\hat\rho}(\tau,1)=\hat\rho_1(\tau)$. We may arrange for that it is the curve of least energy for $\tau=\tau_0$ and depends smoothly on τ, so that

$$d(\hat\rho_1,\hat\rho_0)^2 \left\{ \begin{array}{lll} = & \displaystyle\int_0^1 |\frac{\partial\tilde{\hat\rho}}{\partial\sigma}|^2\, d\sigma & \text{for } \tau=\tau_0 \\ \le & \displaystyle\int_0^1 |\frac{\partial\tilde{\hat\rho}}{\partial\sigma}|^2\, d\sigma & \text{for any } \tau \end{array} \right\}. \tag{47}$$

Hence for $\tau=\tau_0$,

$$\begin{aligned}
\frac{d^+}{d\tau}_{|\tau_0} d(\hat\rho_1,\hat\rho_0)^2 &\overset{(47)}{\le} \frac{d}{d\tau}_{|\tau_0}\int_0^1 |\frac{\partial\tilde{\hat\rho}}{\partial\sigma}|^2\, d\sigma \\
&= 2\int_0^1 \langle\frac{\partial\tilde{\hat\rho}}{\partial\sigma}, \frac{D}{\partial\tau}_{|\tau_0}\frac{\partial\tilde{\hat\rho}}{\partial\sigma}\rangle\, d\sigma \\
&= 2\int_0^1 \langle\frac{\partial\tilde{\hat\rho}}{\partial\sigma}, \frac{D}{\partial\sigma}\frac{\partial\tilde{\hat\rho}}{\partial\tau}_{|\tau_0}\rangle\, d\sigma \\
&= 2\int_0^1 \left\{\frac{d}{d\sigma}\langle\frac{\partial\tilde{\hat\rho}}{\partial\sigma}, \frac{\partial\tilde{\hat\rho}}{\partial\tau}_{|\tau_0}\rangle - \langle\frac{D}{\partial\sigma}\frac{\partial\tilde{\hat\rho}}{\partial\sigma}, \frac{\partial\tilde{\hat\rho}}{\partial\tau}_{|\tau_0}\rangle\right\} d\sigma \\
&\overset{(40)}{=} 2\int_0^1 \frac{d}{d\sigma}\langle\frac{\partial\tilde{\hat\rho}}{\partial\sigma}, \frac{\partial\tilde{\hat\rho}}{\partial\tau}_{|\tau_0}\rangle\, d\sigma \\
&= 2\left(\langle\frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=1}, \frac{d\hat\rho_1}{d\tau}\rangle - \langle\frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=0}, \frac{d\hat\rho_0}{d\tau}\rangle\right) \\
&\overset{(23)}{=} -2\left(\langle\frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=1}, \mathrm{grad}F_{|\hat\rho_1}\rangle - \langle\frac{d\tilde{\hat\rho}}{d\sigma}_{|\sigma=0}, \mathrm{grad}F_{|\hat\rho_0}\rangle\right) \\
&\overset{(44)}{\le} -2\,\alpha\, d(\hat\rho_0,\hat\rho_1)^2,
\end{aligned}$$

which establishes (46).

We finally show how to get (29) by formal Riemannian calculus. We need the ingredient that

$$\lim_{\tau\uparrow\infty}(F(\hat\rho)-F(\hat\rho_*)) \;=\; 0, \tag{48}$$

which in a finite dimensional context would immediately follow from (30) in the weakened form of

$$\lim_{\tau\uparrow\infty} d(\hat\rho,\hat\rho_*) \;=\; 0, \tag{49}$$

In our infinite dimensional context, we obtain (48) from (49) and from (28) in the weakened form of

$$\lim_{\tau\uparrow\infty} |\mathrm{grad}\, F_{|\hat\rho}| \;=\; 0 \tag{50}$$

via the interpolation inequality

$$|F(\hat\rho)-F(\hat\rho_*)| \;\le\; |\mathrm{grad}\, F_{|\hat\rho}|\, d(\hat\rho,\hat\rho_*) \quad \text{for all } \hat\rho\in\mathcal{M},$$

which we obtain from (45), using $\mathrm{grad} F_{|\hat\rho_*} \stackrel{(25)}{=} 0$.

We now derive (29) in form of

$$\frac{d}{d\tau}\,(F(\hat\rho)-F(\hat\rho_*)) \;\le\; -2\,\alpha\,(F(\hat\rho)-F(\hat\rho_*)).$$

We first observe that

$$\frac{d}{d\tau}\,(F(\hat\rho)-F(\hat\rho_*)) \;=\; \langle \mathrm{grad}\, F_{|\hat\rho}, \frac{d\hat\rho}{d\tau}\rangle \;\stackrel{(23)}{=}\; -|\mathrm{grad}\, F_{|\hat\rho}|^2. \tag{51}$$

Hence we get

$$\begin{aligned}
\frac{d}{d\tau}\,(F(\hat\rho)-F(\hat\rho_*)) \;&\stackrel{(51)}{=}\; -|\mathrm{grad}\, F_{|\hat\rho}|^2\\
&\stackrel{(50)}{=}\; \int_\tau^\infty \frac{d}{d\tau}\,|\mathrm{grad}\, F_{|\hat\rho}|^2\, d\tau\\
&\stackrel{(38)}{\le}\; -2\,\alpha \int_\tau^\infty |\mathrm{grad}\, F_{|\hat\rho}|^2\, d\tau\\
&\stackrel{(51)}{=}\; 2\,\alpha \int_\tau^\infty \frac{d}{d\tau}\,(F(\hat\rho)-F(\hat\rho_*))\, d\tau\\
&\stackrel{(48)}{=}\; -2\,\alpha\,(F(\hat\rho)-F(\hat\rho_*)).
\end{aligned}$$

4 The geometry of $(\mathcal{M}, g)$

The best way to understand the geometry of $(\mathcal{M}, g)$ is: It is induced by a flat Riemannian space $(\mathcal{M}^*, g^*)$ via a submersion Π. The intuition behind this is the following: The porous medium equation describes the diffusion of gas particles through a porous medium. $\mathcal{M}$ describes the state via the particle densities ρ — an Eulerian description. $\mathcal{M}^*$ will describe the state via the particle coordinates or flow map Φ — a Lagrangian description.

4.1 The isometric submersion Π

We fix a $\rho_0 \in \mathcal{M}$. We start by introducing the manifold $\mathcal{M}^*$ and the submersion $\Pi\colon \mathcal{M}^* \to \mathcal{M}$. The manifold is the set of all diffeomorphisms of $I\!R^N$:

$$\mathcal{M}^* \;=\; \left\{ \text{diffeomorphisms } \Phi \text{ of } I\!R^N \right\}.$$

And $\rho = \Pi(\Phi)$ is given by the push forward of the reference density ρ_0 under the map Φ. More precisely,

$$\int \rho\, \zeta \;=\; \int \rho_0\, \zeta \circ \Phi \quad \text{for all functions } \zeta \text{ on } I\!R^N. \tag{52}$$

We also use the notation

$$\rho \;=\; \Phi \# \rho_0.$$

We now endow $\mathcal{M}^*$ with a metric tensor g^*. Again, we will be sloppy about the differential structure of $\mathcal{M}^*$ and think of the tangent space as the space of all vector fields on $I\!R^N$

$$T_\Phi \mathcal{M}^* \;=\; \left\{ \text{vector fields } v \text{ on } I\!R^N \right\},$$

which we endow with the scalar product

$$g^*_\Phi(v_1, v_2) \;=\; \int \rho_0\, v_1 \cdot v_2.$$

In other words, $(\mathcal{M}^*, g^*)$ carries the geometry of the ambient L^2–space with weight ρ_0. In particular $(\mathcal{M}^*, g^*)$ is *flat*.

We now argue that Π is an isometric submersion from $(\mathcal{M}^*, g^*)$ into $(\mathcal{M}, g)$; for the notion of isometric submersion, see for instance [30, Chapter 1.1]. We have to show: For any $\Phi \in \mathcal{M}$, the tangential

$$T_\Phi \Pi \colon T_\Phi \mathcal{M}^* \to T_\rho \mathcal{M} \tag{53}$$

of Π at Φ has the property

$$g_\rho(s,s) \;=\; \inf_{T_\Phi \Pi . v = s} g_\Phi(v,v) \quad \text{for all } s \in T_\rho \mathcal{M}, \tag{54}$$

where $\rho = \Pi(\Phi)$. We observe that (54) implies that $T_\Phi \Pi$ is an isometry when restricted to the orthogonal complement $(\ker T_\Phi \Pi)^-$ of its kernel $\ker T_\Phi \Pi \subset T_\Phi \mathcal{M}^*$. In the language of differential geometry: A tangent vector in $\ker T_\Phi \Pi$ is called 'vertical', a tangent vector in $(\ker T_\Phi \Pi)^-$ is called 'horizontal'.

In order to establish property (54), we give a characterization of the tangential $T_\Phi \Pi$ and the spaces $\ker T_\Phi \Pi$ and $(\ker T_\Phi \Pi)^-$. It is convenient to do so in terms of the identification (8) of tangent vectors $s \in T_\rho \mathcal{M}$ with potentials p and the following identification of tangent vectors $v \in T_\Phi \mathcal{M}^*$ with velocity fields u:

$$T_\Phi \mathcal{M}^* \;\cong\; \left\{ \text{vector fields } u \text{ on } I\!\!R^N \right\} \tag{55}$$

via

$$v \;=\; u \circ \Phi.$$

We observe that in terms of u, the metric tensor g^* assumes the form

$$g^*_\Phi(v_1, v_2) \;=\; \int \rho\, u_1 \cdot u_2, \quad \text{where } \rho = \Pi(\Phi). \tag{56}$$

We now show that in terms of p and u,

- $T_\Phi \Pi . u$ is the function p on $I\!\!R^N$ (determined up to additive constants) which solves
$$-\nabla \cdot (\rho \nabla p) \;=\; -\nabla \cdot (\rho\, u). \tag{57}$$
- $u \in \ker T_\Phi \Pi$ if and only if the vector field u on $I\!\!R^N$ satisfies
$$\nabla \cdot (\rho\, u) \;=\; 0. \tag{58}$$
- $u \in (\ker T_\Phi \Pi)^-$ if and only if the vector field u on $I\!\!R^N$ satisfies
$$u \;=\; \nabla p \quad \text{for some function } p \text{ on } I\!\!R^N. \tag{59}$$

The line (58) follows immediately from (57). The line (59) follows from (57) easily: Because of (56) and (58), $u \in (\ker T_\Phi \Pi)^-$ means

$$\int w \cdot u \;=\; 0 \quad \text{for all vector fields } w \text{ on } I\!R^N \text{ with } \nabla \cdot w = 0,$$

which implies (59) by elementary vector calculus.

It remains to establish (57), which we will do in the variational form

$$\int \rho \, \nabla p \cdot \nabla \zeta \;=\; \int \rho \, u \cdot \nabla \zeta \quad \text{for all functions } \zeta \text{ on } I\!R^N. \tag{60}$$

Obviously,

$$\frac{\partial \tilde\Phi}{\partial \sigma}(\sigma) \;=\; u \circ \tilde\Phi(\sigma), \quad \tilde\Phi(0) \;=\; \Phi \tag{61}$$

defines a curve $\sigma \mapsto \tilde\Phi(\sigma) \in \mathcal{M}^*$ which for $\sigma = 0$ passes through Φ and has tangent u there. Now consider the image $\sigma \mapsto \tilde\rho(\sigma)$ of this curve under Π. It suffices to show that its tangent p at $\sigma = 0$ satisfies (60). Indeed, by definition of Π,

$$\int \tilde\rho(\sigma) \, \zeta \;=\; \int \rho_0 \, (\zeta \circ \tilde\Phi(\sigma)) \quad \text{for all functions } \zeta \text{ on } I\!R^N, \tag{62}$$

which we differentiate w. r. t. σ and evaluate at $\sigma = 0$,

$$\begin{aligned} \int \frac{\partial \tilde\rho}{\partial \sigma}_{|\sigma=0} \zeta \;&\overset{(61)}{=}\; \int \rho_0 \, (\nabla \zeta \circ \Phi) \cdot (u \circ \Phi) \\ &=\; \int \rho \, \nabla \zeta \cdot u \quad \text{for all functions } \zeta \text{ on } I\!R^N. \end{aligned}$$

On the other hand, we have by definition (8) of p, after integration by parts,

$$\int \rho \, \nabla \zeta \cdot \nabla p \;=\; \int \frac{\partial \tilde\rho}{\partial \sigma}_{|\sigma=0} \zeta \quad \text{for all functions } \zeta \text{ on } I\!R^N.$$

This establishes (60).

In view of the definition (9) and of (56), the identity (54) turns into

$$\int \rho \, |\nabla p|^2 \;=\; \inf_{T_\Phi \Pi . u = p} \int \rho \, |u|^2,$$

which now is an immediate consequence of the characterization (60) of $T_\Phi \Pi . u$.

4.2 A property of the map Π

The map Π has the following important property. Let $\sigma \mapsto \Phi(\sigma)$ be a geodesic on $(\mathcal{M}^*, g^*)$. Then

$$\frac{d\Phi}{d\sigma}(0) \;\in\; (\ker T_{\Phi(0)}\Pi)^{-} \quad \text{implies} \quad \frac{d\Phi}{d\sigma}(\sigma) \;\in\; (\ker T_{\Phi(\sigma)}\Pi)^{-} \quad \text{for all } \sigma. \tag{63}$$

Let us establish this property. Since $(\mathcal{M}^*, g^*)$ carries the geometry of the ambient Euclidean L^2–space with weight ρ_0, the geodesic equation is

$$\frac{\partial^2\Phi}{\partial\sigma^2} \;=\; 0.$$

We express the geodesic equation in terms of the tangent field $\sigma \mapsto u(\sigma)$ given by by

$$\frac{\partial\Phi}{\partial\sigma} \;=\; u\circ\Phi. \tag{64}$$

Since

$$\begin{aligned}
\frac{\partial^2\Phi}{\partial\sigma^2} &\overset{(64)}{=} \frac{\partial}{\partial\sigma}(u\circ\Phi)\\
&= \frac{\partial u}{\partial\sigma}\circ\Phi + (\mathrm{D}u\circ\Phi).\frac{\partial\Phi}{\partial\sigma}\\
&\overset{(64)}{=} \left(\frac{\partial u}{\partial\sigma} + \mathrm{D}u.u\right)\circ\Phi,
\end{aligned}$$

where $\mathrm{D}u$ denotes the Jacobian of u w. r. t. the spatial variables, the geodesic equation reads

$$\frac{\partial u}{\partial\sigma} + \mathrm{D}u.u \;=\; 0. \tag{65}$$

According to (59), the left hand side of (63) means that there exists a function p_0 on $I\!R^N$ such that $u(0) = \nabla p_0$. Now let the function $\tilde{p}(\sigma, x)$ solve the Hamilton–Jacobi equation

$$\frac{\partial\tilde{p}}{\partial\sigma} + \frac{1}{2}\,|\nabla\tilde{p}|^2 \;=\; 0 \quad \text{with initial data } p_0.$$

Then its spatial gradient $\tilde{u} = \nabla\tilde{p}$ solves

$$\frac{\partial\tilde{u}}{\partial\sigma} + \mathrm{D}\tilde{u}.\tilde{u} \;=\; 0 \quad \text{with initial data } \nabla p_0.$$

Hence $\tilde{u}$ and u solve the same evolution equation with identical initial data. Therefore, $\tilde{u}$ and u coincide. In particular,

$$u(\sigma) \;=\; \nabla\tilde{p}(\sigma) \quad \text{for all } \sigma,$$

which according to (59) entails the right hand side of (63).

4.3 Identification of geodesics and the induced distance

We will first characterize geodesics and the induced distance on $(\mathcal{M}, g)$ in terms of geodesics and the induced distance on $(\mathcal{M}^*, g^*)$. For this, we forget about where our structure $\Pi\colon (\mathcal{M}^*, g^*) \to (\mathcal{M}, g)$ came from and work exclusively within the abstract framework of a Riemannian submersion with the additional property established in the previous subsection.

The first observation states how the 'energy' of curves transforms under Π: Let $\sigma \mapsto \Phi(\sigma)$ be a curve on $\mathcal{M}^*$. Consider its image $\sigma \mapsto \rho(\sigma)$ on $\mathcal{M}$ under Π, that is: $\rho(\sigma) = \Pi(\Phi(\sigma))$. Then

$$\begin{aligned} \int g_\rho(\frac{d\rho}{d\sigma}, \frac{d\rho}{d\sigma})\, d\sigma \;&\leq\; \int g^*_\Phi(\frac{d\Phi}{d\sigma}, \frac{d\Phi}{d\sigma})\, d\sigma \\ \text{with equality if} \qquad &\frac{d\Phi}{d\sigma} \in (\mathrm{ker} T_\Phi\Pi)^-. \end{aligned} \tag{66}$$

Indeed, we have $\frac{d\rho}{d\sigma} = T_\Phi\Pi.\frac{d\Phi}{d\sigma}$, and therefore, according to (54),

$$g_\rho(\frac{d\rho}{d\sigma}, \frac{d\rho}{d\sigma}) \;\leq\; g^*_\Phi(\frac{d\Phi}{d\sigma}, \frac{d\Phi}{d\sigma}),$$

with equality if $\frac{d\Phi}{d\sigma} \in (\mathrm{ker} T_\Phi\Pi)^-$.

The second observation states what happens to geodesics under Π.

$$\left.\begin{array}{l}\text{If } \sigma \mapsto \Phi(\sigma) \text{ is a geodesic on } (\mathcal{M}^*, g^*) \text{ with } \frac{d\Phi}{d\sigma} \in (\mathrm{ker} T_\Phi\Pi)^-, \\ \text{then its image } \sigma \mapsto \rho(\sigma) \text{ is a geodesic on } (\mathcal{M}, g).\end{array}\right\} \tag{67}$$

$$\left.\begin{array}{l}\text{If } \sigma \mapsto \rho(\sigma) \text{ is a geodesic on } (\mathcal{M}, g) \text{ with } \rho(0) = \rho_0, \text{ then there} \\ \text{exists a geodesic } \sigma \mapsto \Phi(\sigma) \text{ on } (\mathcal{M}^*, g^*) \text{ with } \Phi(0) = \mathrm{id}, \frac{d\Phi}{d\sigma} \in \\ (\mathrm{ker} T_\Phi\Pi)^- \text{ and such that } \sigma \mapsto \rho(\sigma) \text{ is its image under } \Pi.\end{array}\right\} \tag{68}$$

Indeed, for (67), it suffices to show that $\sigma \mapsto \rho(\sigma)$ has lowest energy among all small variations on small σ–intervals, using the fact that $\sigma \mapsto \Phi(\sigma)$ has the same property. Let $(\epsilon,\sigma) \mapsto \tilde\rho(\epsilon,\sigma)$ be a given variation of $\sigma \mapsto \rho(\sigma)$, that is: $\tilde\rho(0,\sigma) = \rho(\sigma)$. Since $T_\phi\Pi$ is an isomorphism of $(\mathrm{ker}T_\phi\Pi)^-$ onto $T_{\Pi(\Phi)}\mathcal{M}$, and since $\frac{d\Phi}{d\sigma} \in (\mathrm{ker}T_\Phi\Pi)^-$, we can 'lift' the variation $(\epsilon,\sigma) \mapsto \tilde\rho(\epsilon,\sigma)$ to a variation $(\epsilon,\sigma) \mapsto \tilde\Phi(\epsilon,\sigma)$ of $\sigma \mapsto \Phi(\sigma)$, that is: $\Pi(\tilde\Phi(\epsilon,\sigma)) = \tilde\rho(\epsilon,\sigma)$, with $\frac{d\tilde\Phi}{d\sigma} \in (\mathrm{ker}T_{\tilde\Phi}\Pi)^-$. Therefore the energy of $\sigma \mapsto \rho(\sigma)$ does not exceed the energy of the variation $\sigma \mapsto \tilde\rho(\epsilon,\sigma)$ for any ϵ:

$$\begin{aligned}\int g_\rho(\frac{d\rho}{d\sigma},\frac{d\rho}{d\sigma})\,d\sigma &\overset{(66)}{=} \int g^*_\Phi(\frac{d\Phi}{d\sigma},\frac{d\Phi}{d\sigma})\,d\sigma \\ &\leq \int g^*_{\tilde\Phi}(\frac{d\tilde\Phi}{d\sigma},\frac{d\tilde\Phi}{d\sigma})\,d\sigma \\ &\overset{(66)}{=} \int g_{\tilde\rho}(\frac{d\tilde\rho}{d\sigma},\frac{d\tilde\rho}{d\sigma})\,d\sigma.\end{aligned}$$

The argument for (68) goes as follows: Let $\sigma \mapsto \rho(\sigma)$ be a geodesic on $(\mathcal{M},g)$ with $\rho(0)=\rho_0$. Let $\sigma \mapsto \Phi(\sigma)$ be the geodesic on $(\mathcal{M}^*,g^*)$ with $\Phi(0)=\mathrm{id}$ and

$$T_{\Phi(0)}\Pi.\frac{d\Phi}{d\sigma}(0) \;=\; \frac{d\rho}{d\sigma}(0) \quad\text{and}\quad \frac{d\Phi}{d\sigma}(0) \;\in\; (\mathrm{ker}T_{\Phi(0)}\Pi)^-.$$

According to the previous subsection, namely (63), the last property is preserved along the geodesic:

$$\frac{d\Phi}{d\sigma}(\sigma) \;\in\; (\mathrm{ker}T_{\Phi(\sigma)}\Pi)^- \quad\text{for all }\sigma.$$

By (67), this implies that the image under Π, $\sigma \mapsto \Pi(\Phi(\sigma))$, is a geodesic on $(\mathcal{M},g)$. By construction, it has the same initial data as $\sigma \mapsto \rho(\sigma)$. Hence both geodesics coincide:

$$\Pi(\Phi(\sigma)) \;=\; \rho(\sigma) \quad\text{for all }\sigma.$$

The third observation states what happens to the induced distance under Π. Let d^* denote the induced distance on $(\mathcal{M}^*,g^*)$ and d the one on $(\mathcal{M},g)$. Let $\rho \in \mathcal{M}$ be arbitrary; then

- For all Φ with $\Pi(\Phi)=\rho$,

$$d(\rho_0,\rho)^2 \;\leq\; d^*(\mathrm{id},\Phi)^2. \tag{69}$$

- There exists a Φ with $\Pi(\Phi)=\rho$ and

$$d(\rho_0,\rho)^2 \;\ge\; d^*(\mathrm{id},\Phi)^2. \tag{70}$$

We observe that (69) and (70) imply

$$d(\rho_0,\rho)^2 \;=\; \inf_{\Pi(\Phi)=\rho} d^*(\mathrm{id},\Phi)^2. \tag{71}$$

Let us start with (69). Indeed, let $[0,1]\ni\sigma\mapsto\tilde\Phi(\sigma)$ be any curve on $\mathcal{M}^*$ with $\tilde\Phi(0)=\mathrm{id}$ and $\tilde\Phi(1)=\Phi$. Consider its image $[0,1]\ni\sigma\mapsto\tilde\rho(\sigma)$ under Π. By assumption, $\tilde\rho(1)=\rho$, and by definition of Π, $\tilde\rho(0)=\rho_0$. Therefore

$$\begin{aligned} d(\rho_0,\rho)^2 &\le \int_0^1 g_{\tilde\rho}(\frac{d\tilde\rho}{d\sigma},\frac{d\tilde\rho}{d\sigma})\,d\sigma \\ &\stackrel{(66)}{\le} \int_0^1 g^*_{\tilde\Phi}(\frac{d\tilde\Phi}{d\sigma},\frac{d\tilde\Phi}{d\sigma})\,d\sigma. \end{aligned}$$

Since $\tilde\Phi$ was an arbitrary curve connecting id to Φ, this inequality yields (69). Now let $[0,1]\ni\sigma\mapsto\tilde\rho(\sigma)$ be a curve connecting ρ_0 to ρ with minimal energy. According to (68), there exists a curve $[0,1]\ni\sigma\mapsto\tilde\Phi(\sigma)$ on $(\mathcal{M}^*,g^*)$ such that

$$\frac{d\tilde\Phi}{d\sigma} \;\in\; (\mathrm{ker}T_{\tilde\Phi}\Pi)^-,\quad \tilde\Phi(0)\;=\;\mathrm{id}\quad\text{and}\quad \Pi(\tilde\Phi(\sigma))\;=\;\tilde\rho(\sigma).$$

In particular, $\Phi:=\tilde\Phi(1)$ satisfies $\Pi(\Phi)=\rho$, and we have

$$\begin{aligned} d(\rho_0,\rho)^2 &= \int_0^1 g_{\tilde\rho}(\frac{d\tilde\rho}{d\sigma},\frac{d\tilde\rho}{d\sigma})\,d\sigma \\ &\stackrel{(66)}{=} \int_0^1 g^*_{\tilde\Phi}(\frac{d\tilde\Phi}{d\sigma},\frac{d\tilde\Phi}{d\sigma})\,d\sigma \\ &\ge d^*(\mathrm{id},\Phi)^2. \end{aligned}$$

This establishes (70).

We will now use the above characterization of geodesics on $(\mathcal{M},g)$ in terms of geodesics on $(\mathcal{M}^*,g^*)$ and our good understanding of the latter to identify the former. Since $(\mathcal{M}^*,g^*)$ carries the geometry of the ambient L^2–space with weight ρ_0, geodesics $\sigma\mapsto\Phi(\sigma)$ are characterized by

$$\frac{\partial^2\Phi}{\partial\sigma^2} \;=\; 0. \tag{72}$$

Now let $\sigma \mapsto \rho(\sigma)$ be a geodesic on $(\mathcal{M}, g)$ with the initial data

$$\rho(0) \;=\; \rho_0 \quad \text{and} \quad \frac{d\rho}{d\sigma}(0) \;=\; s. \tag{73}$$

We represent the tangent vector $s \in T_{\rho_0}\mathcal{M}$ by

$$-\nabla \cdot (\rho_0 \, \nabla p) \;=\; s. \tag{74}$$

According to (68), there exists a geodesic $\sigma \mapsto \Phi(\sigma)$ on $(\mathcal{M}^*, g^*)$ with $\Phi(0) =$ id,$\frac{d\Phi}{d\sigma} \in (\ker T_\Phi \Pi)^-$ and such that $\sigma \mapsto \rho(\sigma)$ is its image under Π, that is

$$\rho(\sigma) \;=\; \Phi(\sigma)\#\rho_0 \quad \text{for all } \sigma.$$

Since in particular, $\Phi(0) = \mathrm{id}$ and

$$T_{\mathrm{id}}\Pi.\frac{d\Phi}{d\sigma}(0) \;=\; s \quad \text{and} \quad \frac{d\Phi}{d\sigma}(0) \;\in\; (\ker T_{\mathrm{id}}\Pi)^-,$$

we must have by our characterization of $T_{\mathrm{id}}\Pi$

$$\frac{\partial\Phi}{\partial\sigma}(0) \;=\; \nabla p.$$

Together with (72), we infer that $\Phi(\sigma)$ is of the form

$$\Phi(\sigma) \;=\; \nabla(\frac{1}{2}|y|^2 + \sigma \, p).$$

Therefore, we have characterized our geodesic $\sigma \mapsto \rho(\sigma)$ with initial data (73) as

$$\rho(\sigma) \;=\; \Big[\nabla(\frac{1}{2}|y|^2 + \sigma \, p)\Big] \#\rho_0, \tag{75}$$

where p is related to s by (74).

We will now use the above characterization of the induced distance on $(\mathcal{M}, g)$ in terms of the induced distance on $(\mathcal{M}^*, g^*)$ and our good understanding of the latter to identify the former. Since $(\mathcal{M}^*, g^*)$ carries the geometry of the ambient L^2–space with weight ρ_0, d^* is given by

$$d^*(\Phi_0, \Phi_1)^2 \;=\; \int \rho_0 \, |\Phi_0 - \Phi_1|^2.$$

We therefore obtain from (71) that

$$d(\rho_0, \rho)^2 \;=\; \inf_{\rho = \Phi\#\rho_0} \int \rho_0 \, |\mathrm{id} - \Phi|^2 \tag{76}$$

Hence we have identified the induced distance on $\mathcal{M}$ with what is called the Wasserstein distance, which we formally introduce in section 5.

4.4 Computation of the Hessians Hess$E$ and Hess$M$

The Hessian Hess$F$ of a function F on a Riemannian manifold $(\mathcal{M}, g)$ can be computed by taking second derivatives of F along geodesics. More precisely, if $\sigma \mapsto \rho(\sigma)$ is a geodesic on $(\mathcal{M}, g)$ with

$$\rho(0) \;=\; \rho_0 \quad \text{and} \quad \frac{d\rho}{d\sigma}(0) \;=\; s,$$

then

$$g_{\rho_0}(s, \mathrm{Hess}F_{|\rho_0} s) \;=\; \frac{d^2}{d\sigma^2} F(\rho(\sigma))_{|\sigma=0}. \tag{77}$$

As always, we represent the tangent vector $s \in T_{\rho_0}\mathcal{M}$ by

$$-\nabla \cdot (\rho_0 \, \nabla p) \;=\; s.$$

In the previous subsection, we characterized the geodesic $\sigma \mapsto \rho(\sigma)$ as

$$\rho(\sigma) \;=\; \nabla\varphi(\sigma) \# \rho_0, \tag{78}$$

where the function $\varphi(\sigma)$ on $I\!R^N$ is given by

$$\varphi(\sigma, y) = \frac{1}{2}|y|^2 + \sigma \, p(y). \tag{79}$$

Hence convexity of a function F on the Riemannian manifold $(\mathcal{M}, g)$ reduces to McCann's 'displacement convexity' [9]. McCann introduced, established and used this notion for our energy functional E to prove uniqueness for a variational problem on $\mathcal{M}$ — without referral to the Riemannian structure. As a guideline for our rigorous arguments in the next section, it will be convenient to explicitly find Hess$E_{|\rho_0}$ (as opposed to just showing that it is positive semi definite). Therefore the calculation which now follows deviate a bit from McCann's.

We observe that (78) can be reformulated as

$$\det \mathrm{D}^2\varphi(\sigma) \, (\rho(\sigma) \circ \nabla\varphi(\sigma)) \;=\; \rho_0,$$

so that

$$E(\rho(\sigma)) \;=\; \int e\left(\frac{\rho_0}{\det \mathrm{D}^2\varphi(\sigma)}\right) \det \mathrm{D}^2\varphi(\sigma), \tag{80}$$

where $\mathrm{D}^2\varphi(\sigma)$ denotes the $N \times N$–matrix of second spatial derivatives of $\varphi(\sigma)$ and e the energy density, as defined in (21).

Guided by the above, we consider a curve $\sigma \mapsto A(\sigma)$ in the space of symmetric and positive definite $N \times N$–matrices and a positive number $z > 0$. Let us recall that the energy density e and the osmotic pressure π (defined in (20)) are related by

$$\pi(z) \;=\; z\, e'(z) - e(z).$$

Therefore, we have

$$\begin{aligned}
\frac{d}{d\sigma}\left[e\left(\frac{z}{\det A}\right)\,\det A\right] &= -\pi\left(\frac{z}{\det A}\right)\,\frac{d}{d\sigma}\det A,\\
\frac{d^2}{d\sigma^2}\left[e\left(\frac{z}{\det A}\right)\,\det A\right] &= \pi'\left(\frac{z}{\det A}\right)\,\frac{z}{(\det A)^2}\,(\frac{d}{d\sigma}\det A)^2\\
&\quad -\pi\left(\frac{z}{\det A}\right)\,\frac{d^2}{d\sigma^2}\det A.
\end{aligned}$$

By elementary linear algebra,

$$\begin{aligned}
\frac{d}{d\sigma}\det A &= \mathrm{tr}(A^{-1}\,\frac{\partial A}{\partial\sigma})\det A,\\
\frac{d^2}{d\sigma^2}\det A &= -\mathrm{tr}\left(A^{-1}\,\frac{\partial A}{\partial\sigma}\right)^2\,\det A + \left(\mathrm{tr}(A^{-1}\,\frac{\partial A}{\partial\sigma})\right)^2\,\det A\\
&\quad + \mathrm{tr}(A^{-1}\,\frac{d^2A}{d\sigma^2})\det A.
\end{aligned}$$

Hence if the curve $\sigma \mapsto A(\sigma)$ additionally satisfies $\frac{d^2A}{d\sigma^2} = 0$, we obtain

$$\begin{aligned}
\frac{d^2}{d\sigma^2}\left[e\left(\frac{z}{\det A}\right)\,\det A\right] &= (w\,\pi'(w) - \pi(w))\,\left(\mathrm{tr}\,(A^{-1}\,B)\right)^2\,\det A\\
&\quad + \pi(w)\,\mathrm{tr}\,\left(A^{-1}\,B\right)^2\,\det A,
\end{aligned}$$

where we have used the abbreviations

$$B \;:=\; \frac{dA}{d\sigma} \quad\text{and}\quad w \;:=\; \frac{z}{\det A}.$$

Since

$$(A^{-1}\,B)^2 \;=\; A^{-1/2}\,C^2\,A^{1/2} \quad\text{with}\quad C \;:=\; A^{-1/2}\,B\,A^{-1/2},$$

where C is a symmetric matrix, we have

$$\mathrm{tr}\,\left(A^{-1}\,B\right)^2 \;=\; \mathrm{tr}\,C^2 \;\geq\; \frac{1}{N}\,(\mathrm{tr}C)^2 \;=\; \frac{1}{N}\,\left(\mathrm{tr}\,(A^{-1}\,B)\right)^2. \tag{81}$$

Since $\pi(w) = w^m \geq 0$, we therefore obtain

$$\frac{d^2}{d\sigma^2}\,\left[e\left(\frac{z}{\det A}\right)\,\det A\right] \;\geq\; (w\,\pi'(w) - (1-\frac{1}{N})\,\pi(w))\,\left(\mathrm{tr}\,(A^{-1}\,B)\right)^2\,\det A.$$

Since $w\,\pi'(w) - (1-\frac{1}{N})\,\pi(w) = (m-(1-\frac{1}{N}))\,w^m \geq 0$ by our assumption $m \geq 1-\frac{1}{N}$, this implies

$$\frac{d^2}{d\sigma^2}\,\left[e\left(\frac{z}{\det A}\right)\,\det A\right] \;\geq\; 0. \tag{82}$$

For later reference, we notice that if in addition $A(0) = \mathrm{id}$,

$$\frac{d^2}{d\sigma^2}_{|\sigma=0}\,\left[e\left(\frac{z}{\det A}\right)\,\det A\right] \;=\; (z\,\pi'(z) - \pi(z))\,\left(\mathrm{tr}\,B\right)^2 + \pi(z)\,\mathrm{tr}\,B^2. \tag{83}$$

Now consider $\mathrm{D}^2\varphi(\sigma)$. We observe that 1) $\mathrm{D}^2\varphi(\sigma)$ is symmetric, 2) $\mathrm{D}^2\varphi(\sigma)$ is positive definite (for sufficiently small σ) since $\mathrm{D}^2\varphi(0) = \mathrm{id}$, 3) $\frac{\partial^2}{\partial\sigma^2}\mathrm{D}^2\varphi(\sigma) = 0$ because of (79). Hence we may apply the above to $A(\sigma) = \mathrm{D}^2\varphi(\sigma)$ and obtain

$$\begin{aligned}\frac{d^2}{d\sigma^2}\,E(\rho(\sigma)) \;&\overset{(80)}{=}\; \int \frac{\partial^2}{\partial\sigma^2}\,\left[e\left(\frac{\rho_0}{\det\mathrm{D}\Phi}\right)\,\det\mathrm{D}\Phi\right]\\ &\overset{(82)}{\geq}\; 0.\end{aligned}$$

Since $\frac{\partial}{\partial\sigma}_{|\sigma=0}\mathrm{D}^2\varphi \overset{(79)}{=} \mathrm{D}^2 p$ and $\mathrm{D}^2\varphi(0) = 0$, we also get

$$\begin{aligned}\frac{d^2}{d\sigma^2}_{|\sigma=0}\,E(\rho(\sigma)) \;&=\; \int \frac{\partial^2}{\partial\sigma^2}_{|\sigma=0}\,\left[e\left(\frac{\rho_0}{\det\mathrm{D}^2\varphi}\right)\,\det\mathrm{D}^2\varphi\right]\\ &\overset{(83)}{=}\; \int \left\{(\pi'(\rho_0)\,\rho_0 - \pi(\rho_0))\,(\nabla^2 p)^2 + \pi(\rho_0)\,\mathrm{tr}\,(\mathrm{D}^2 p)^2\right\}.\end{aligned}$$

We conclude by

$$\begin{aligned}g_{\rho_0}(s, \mathrm{Hess}E_{|\rho_0}\,s) \;&\overset{(77)}{=}\; \frac{d^2}{d\sigma^2}_{|\sigma=0}\,E(\rho(\sigma))\\ &=\; \int \left\{(\pi'(\rho_0)\,\rho_0 - \pi(\rho_0))\,(\nabla^2 p)^2 + \pi(\rho_0)\,\mathrm{tr}\,(\mathrm{D}^2 p)^2\right\}\\ &\geq\; 0.\end{aligned} \tag{84}$$

This establishes the first part of (27). Let us point out again that it is the condition $m \geq 1 - \frac{1}{N}$ which ensures that E is convex on $(\mathcal{M}, g)$.

Let us now identify $\mathrm{Hess} M_{|\rho_0}$. It follows immediately from (78) that

$$M(\rho(\sigma)) \;=\; \int \rho_0 \, \frac{1}{2} |\nabla \varphi(\sigma)|^2.$$

Hence we obtain

$$\begin{aligned} g_{\rho_0}(s, \mathrm{Hess} M_{|\rho_0} s) &\overset{(77)}{=} \frac{d^2}{d\sigma^2}_{|\sigma=0} M(\rho(\sigma)) \\ &\overset{(78)}{=} \int \rho_0 \, |\frac{\partial \nabla \varphi}{\partial \sigma}_{|\sigma=0}|^2 \\ &\overset{(79)}{=} \int \rho_0 \, |\nabla p|^2 \\ &\overset{(9)}{=} g_{\rho_0}(s, s). \end{aligned} \tag{85}$$

This establishes the second part of (27).

4.5 Formula for the sectional curvature

In this section, we derive a formula for the sectional curvature (see for instance [30, Section 2.2.3] or [17, 3.7 Definition]) of $(\mathcal{M}, g)$. This is *not* required for our program, but might be enlightening. O'Neill discovered a simple relation between the sectional curvature of two Riemannian manifolds $(\mathcal{M}^*, g^*)$ and $(\mathcal{M}, g)$ under an isometric submersion $\Pi\colon (\mathcal{M}^*, g^*) \to (\mathcal{M}, g)$, see for instance [30, 2.7 Exercises, exercise 21] or [17, 3.61 Theorem]. If $(\mathcal{M}^*, g^*)$ is flat, as in our case, one so obtains a formula for the sectional curvature of $(\mathcal{M}, g)$. In order to state this formula, we need some notation: For any vector field $\mathbf{u}$ on $\mathcal{M}^*$ we define the vector field $\mathbf{u^v}$ on $\mathcal{M}^*$ via

$$\mathbf{u^v}(\Phi) \quad \text{is the } g^*_\Phi\text{–orthogonal projection of } \mathbf{u}(\Phi) \text{ onto } \ker T_\Phi \Pi.$$

For two vector fields $\mathbf{u_1}, \mathbf{u_2}$ on $\mathcal{M}^*$, let $[\mathbf{u_1}, \mathbf{u_2}]$ denote their bracket; $[\mathbf{u_1}, \mathbf{u_2}]$ is itself a vector field on $\mathcal{M}^*$, see for instance [17, Chapter 1 C]. Now let $\mathbf{p_1}, \mathbf{p_2}$ be two vector fields on $\mathcal{M}$ and let $\mathbf{u_1}, \mathbf{u_2}$ be two vector fields on $\mathcal{M}^*$ related to $\mathbf{p_1}, \mathbf{p_2}$ via

$$\mathbf{u_i}(\Phi) \;\in\; (\ker T_\Phi \Pi)^- \quad \text{and} \quad T_\Phi \Pi . \mathbf{u_i}(\Phi) \;=\; \mathbf{p_i}(\Pi(\Phi)) \quad \text{for all } \Phi \in \mathcal{M}^*. \tag{86}$$

Then, according to O'Neill, for fixed $\Phi \in \mathcal{M}^*$,

$$[\mathbf{u_1}, \mathbf{u_2}]^{\mathbf{v}}(\Phi) \quad \text{depends only on} \quad \mathbf{p_1}(\rho), \mathbf{p_2}(\rho) \quad \text{where } \rho = \Pi(\Phi)$$

and the sectional curvature K_ρ of $(\mathcal{M}, g)$ in ρ is given by

$$K_\rho(\mathbf{p_1}(\rho), \mathbf{p_2}(\rho)) \det \begin{pmatrix} g_\rho(\mathbf{p_1}(\rho), \mathbf{p_1}(\rho)) & g_\rho(\mathbf{p_1}(\rho), \mathbf{p_2}(\rho)) \\ g_\rho(\mathbf{p_2}(\rho), \mathbf{p_1}(\rho)) & g_\rho(\mathbf{p_2}(\rho), \mathbf{p_2}(\rho)) \end{pmatrix}$$
$$= \frac{3}{4} g^*_\Phi([\mathbf{u_1}, \mathbf{u_2}]^{\mathbf{v}}(\Phi), [\mathbf{u_1}, \mathbf{u_2}]^{\mathbf{v}}(\Phi)). \tag{87}$$

Let us now derive what (87) means in our concrete case. Again, it is convenient to do so in terms of the identification (8) of elements of the tangent space $T_\rho\mathcal{M}$ with functions p on $I\!R^N$ and the identification (55) of elements of the tangent space $T_\Phi\mathcal{M}^*$ with vector fields u on $I\!R^N$. In this sense, for any two functions p_1, p_2 on $I\!R^N$, (87) turns into

$$K_\rho(p_1, p_2) \det \begin{pmatrix} g_\rho(p_1, p_1) & g_\rho(p_1, p_2) \\ g_\rho(p_2, p_1) & g_\rho(p_2, p_2) \end{pmatrix} \;=\; \frac{3}{4} \int \rho\, |u|^2, \tag{88}$$

where the vector field u on $I\!R^N$ and the function p on $I\!R^N$ are given by

$$u \;=\; \nabla p - [\nabla p_1, \nabla p_2] \quad \text{and} \quad \nabla \cdot [\rho\, (\nabla p - [\nabla p_1, \nabla p_2])] \;=\; 0. \tag{89}$$

Here $[u_1, u_2]$ denotes the bracket of the vector fields u_1 and u_2 on $I\!R^N$, that is

$$[u_1, u_2] \;=\; \mathrm{D}u_2.u_1 - \mathrm{D}u_1.u_2.$$

Let us now argue why (87) turns into (88). To this purpose, consider the vector fields $\mathbf{p_1}, \mathbf{p_2}$ on $\mathcal{M}$ and the vector fields $\mathbf{u_1}, \mathbf{u_2}$ on $\mathcal{M}^*$ given by

$$\mathbf{p_i}(\rho) \;=\; p_i \quad \text{for all } \rho \in \mathcal{M} \quad \text{and} \quad \mathbf{u_i}(\Phi) \;=\; \nabla p_i \quad \text{for all } \Phi \in \mathcal{M}^*. \tag{90}$$

According to (57) and (59), $\mathbf{p_i}$ and $\mathbf{u_i}$ are related as in (86). Let us argue that in case of "constant" vector fields $\mathbf{u_i}$ on $\mathcal{M}^*$ like in (90), that is

$$\mathbf{u_i}(\Phi) \;=\; u_i \quad \text{for all } \Phi \in \mathcal{M}^*,$$

also the bracket is a constant vector field on $\mathcal{M}^*$, that is,

$$[\mathbf{u_1}, \mathbf{u_2}](\Phi) \;=\; [u_1, u_2] \quad \text{for all } \Phi \in \mathcal{M}^*, \tag{91}$$

with the understanding that the r. h. s. bracket is the bracket for vector fields on $\mathbb{R}^N$. Indeed, for given vector field v on $\mathbb{R}^N$, consider the function(al) $\mathbf{V}$ on $\mathcal{M}^*$ defined via

$$\mathbf{V}(\Phi) \;=\; \int v \cdot \Phi \quad \text{for all } \Phi \in \mathcal{M}^*.$$

According to the identification (55), we have for any vector field $\mathbf{u}$ on $\mathcal{M}^*$ that

$$(\operatorname{diff}\mathbf{V}.\mathbf{u})(\Phi) \;=\; \int v \cdot (\mathbf{u}(\Phi) \circ \Phi) \quad \text{for all } \Phi \in \mathcal{M}^*. \tag{92}$$

In particular for our constant vector fields $\mathbf{u_1}, \mathbf{u_2}$

$$(\operatorname{diff}\mathbf{V}.\mathbf{u_i})(\Phi) \;=\; \int v \cdot (u_i \circ \Phi) \quad \text{for all } \Phi \in \mathcal{M}^*,$$

and therefore, again using (55),

$$(\operatorname{diff}(\operatorname{diff}\mathbf{V}.\mathbf{u_i}).\mathbf{u_j}))(\Phi) \;=\; \int v \cdot ((\mathrm{D}u_i.u_j) \circ \Phi) \quad \text{for all } \Phi \in \mathcal{M}^*,$$

so that by definition of $[\mathbf{u_1}, \mathbf{u_2}]$ and $[u_1, u_2]$ we obtain

$$\begin{aligned}
&(\operatorname{diff}\mathbf{V}.[\mathbf{u_1}, \mathbf{u_2}])(\Phi) \\
&\quad= (\operatorname{diff}(\operatorname{diff}\mathbf{V}.\mathbf{u_2}).\mathbf{u_1}) - \operatorname{diff}(\operatorname{diff}\mathbf{V}.\mathbf{u_1}).\mathbf{u_2}))(\Phi) \\
&\quad= \int v \cdot ((\mathrm{D}u_2.u_1 - \mathrm{D}u_1.u_2) \circ \Phi) \\
&\quad= \int v \cdot ([u_1, u_2] \circ \Phi) \quad \text{for all } \Phi \in \mathcal{M}^*.
\end{aligned} \tag{93}$$

On the other hand, (92) implies

$$(\operatorname{diff}\mathbf{V}.[\mathbf{u_1}, \mathbf{u_2}])(\Phi) \;=\; \int v \cdot ([\mathbf{u_1}, \mathbf{u_2}](\Phi) \circ \Phi) \quad \text{for all } \Phi \in \mathcal{M}^*. \tag{94}$$

Since v was an arbitrary vector field on $\mathbb{R}^N$, we infer from (93) and (94) that

$$[\mathbf{u_1}, \mathbf{u_2}](\Phi) \circ \Phi \;=\; [u_1, u_2] \circ \Phi \quad \text{for all } \Phi \in \mathcal{M}^*,$$

which implies (91).

According to (91) and the definition (90), we have

$$[\mathbf{u_1}, \mathbf{u_2}](\Phi) \;=\; [\nabla p_1, \nabla p_2] \quad \text{for all } \Phi \in \mathcal{M}^*.$$

It follows from (58), (59) that

$$[\mathbf{u_1}, \mathbf{u_2}]^{\mathbf{v}}(\Phi) \;=\; u \quad \text{for all } \Phi \in \mathcal{M}^*, \tag{95}$$

where u is given by (89). We therefore obtain

$$g^*_\Phi([\mathbf{u_1}, \mathbf{u_2}]^{\mathbf{v}}(\Phi), [\mathbf{u_1}, \mathbf{u_2}]^{\mathbf{v}}(\Phi)) \;\overset{(95)}{=}\; g^*_\Phi(u, u) \;\overset{(56)}{=}\; \int \rho\,|u|^2.$$

This shows that (87) turns into (88).

We may learn two things from (88):

- $(\mathcal{M}, g)$ is a space of non negative curvature,
- $(\mathcal{M}, g)$ is flat for $N = 1$ and non flat for $N > 1$.

The first point is immediate from (88). Also the flatness of $(\mathcal{M}, g)$ for $N = 1$ follows immediately from (88). The fact that $(\mathcal{M}, g)$ is not flat for $N > 1$ can be seen as follows: Let the density function ρ on $I\!R^N$ and the functions p_1, p_2 on $I\!R^N$ be given, assume that $\rho > 0$ on all of $I\!R^N$. Then according to (88), $K_\rho(p_1, p_2) = 0$ if and only if $u = 0$ in $I\!R^N$, which according to (89) is true if and only if $[\nabla p_1, \nabla p_2]$ is a gradient. By elementary vector calculus, this is true if and only if $\mathrm{D}[\nabla p_1, \nabla p_2]$ is pointwise symmetric. But since

$$\mathrm{D}[\nabla p_1, \nabla p_2] - (\mathrm{D}[\nabla p_1, \nabla p_2])^t \;=\; 2\,\Big(\mathrm{D}^2 p_1.\mathrm{D}^2 p_2 - \mathrm{D}^2 p_2.\mathrm{D}^2 p_1\Big),$$

$\mathrm{D}[\nabla p_1, \nabla p_2]$ is pointwise symmetric if and only if $\mathrm{D}^2 p_1$ and $\mathrm{D}^2 p_2$ pointwise commute. Since there are symmetric matrices which do not commute, it is clear that we can construct many functions p_1, p_2 on $I\!R^N$ such that $K_\rho(p_1, p_2) > 0$.

We remark that the geometry of $(\mathcal{M}, g)$ is "orthogonal" to the geometry of Arnold's group of volume preserving diffeomorphisms [2]. The geometry of this group is of interest, since the geodesic equation is the Lagrangian formulation of the Euler equations for an incompressible, inviscid fluid. The geometry and its pathologies is well–studied, see [14], [34], [3] and [6]. Let us make more precise what we mean when we say that their geometry is orthogonal to ours. To this purpose, we replace $I\!R^N$ by the N–dimensional torus T^N and let ρ_0 be the uniform density on T^N. Then $\Pi^{-1}(\{\rho_0\})$ is the space of volume preserving transformations. Hence they study the geometry of the "kernel" $\Pi^{-1}(\{\rho_0\})$ of Π, endowed with the Riemannian structure induced from $(\mathcal{M}^*, g^*)$. We on the other hand study the geometry of the "image" $\Pi(\mathcal{M}^*)$ of Π, endowed with the Riemannian structure induced from $(\mathcal{M}^*, g^*)$. Since $(\mathcal{M}^*, g^*)$ is flat, this is also reflected by the fact that the curvature of $\Pi(\mathcal{M}^*)$ is positive, whereas the curvature of $\Pi^{-1}(\{\rho_0\})$ is mostly negative.

4.6 A natural time discretization

Also this section is *not* required for our program but makes the connection to earlier work of the author.

Let us return to the abstract gradient flow setting from subsection 1.2. The dynamical system (2), that is

$$\frac{d\rho}{dt} \;=\; -\mathrm{grad}E_{|\rho}, \tag{96}$$

has a natural time–discretization, which we will introduce now: Let $h > 0$ (the time step size) be given. Consider the algorithm

$$\left.\begin{array}{l} \rho^{(k)} \text{ minimizes} \\ \quad \frac{1}{2\,h}\, d(\rho^{(k-1)},\rho)^2 + E(\rho) \\ \text{among all } \rho \in \mathcal{M} \end{array}\right\}, \tag{97}$$

where d denotes the induced distance on $(\mathcal{M}, g)$. Let us now argue why (97) is a discretization of (96).

To this purpose, we derive the first variation of the minimization problem in (97). Let $[0,1] \ni \sigma \mapsto \tilde\rho^{(k)} \in \mathcal{M}$ denote a curve of least energy connecting $\rho^{(k-1)}$ to $\rho^{(k)}$. Consider a variation $\rho^{(k)}_\epsilon$ of $\rho^{(k)}$, that is: a curve $\epsilon \mapsto \rho^{(k)}_\epsilon \in \mathcal{M}$ which passes through $\rho^{(k)}$ for $\epsilon = 0$. Let $[0,1] \ni \sigma \mapsto \tilde\rho^{(k)}_\epsilon \in \mathcal{M}$ be a curve connecting $\rho^{(k-1)}$ to $\rho^{(k)}_\epsilon$ which coincides with $[0,1] \ni \sigma \mapsto \tilde\rho^{(k)} \in \mathcal{M}$ for $\epsilon = 0$. We have by (97) and the definition of the induced distance that

$$\frac{1}{2\,h}\int_0^1 |\frac{d\tilde\rho^{(k)}}{d\sigma}|^2 d\sigma + E(\rho^{(k)}) \;=\; \frac{1}{2\,h}\, d(\rho^{(k-1)},\rho^{(k)})^2 + E(\rho^{(k)})$$

$$
\begin{aligned}
&\leq \frac{1}{2\,h}\, d(\rho^{(k-1)}, \rho^{(k)}_\epsilon)^2 + E(\rho^{(k)}_\epsilon) \\
&\leq \frac{1}{2\,h} \int_0^1 |\frac{d\tilde{\rho}^{(k)}_\epsilon}{d\sigma}|^2 d\sigma + E(\rho^{(k)}_\epsilon),
\end{aligned}
$$

so that

$$
\begin{aligned}
0 &= \frac{d}{d\epsilon}_{|\epsilon=0} \left(\frac{1}{2\,h} \int_0^1 |\frac{d\tilde{\rho}^{(k)}_\epsilon}{d\sigma}|^2 d\sigma + E(\rho^{(k)}_\epsilon) \right) \\
&= \frac{1}{h} \int_0^1 \langle \frac{d\tilde{\rho}^{(k)}}{d\sigma}, \frac{D}{d\epsilon}_{|\epsilon=0} \frac{d\tilde{\rho}^{(k)}_\epsilon}{d\sigma} \rangle \, d\sigma + \mathrm{diff} E_{|\rho^{(k)}} . \frac{d\rho^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \\
&= \frac{1}{h} \int_0^1 \langle \frac{d\tilde{\rho}^{(k)}}{d\sigma}, \frac{D}{d\sigma} \frac{d\tilde{\rho}^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle \, d\sigma + \langle \mathrm{grad} E_{|\rho^{(k)}}, \frac{d\rho^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle \\
&= \frac{1}{h} \int_0^1 \left\{ \frac{d}{d\sigma} \langle \frac{d\tilde{\rho}^{(k)}}{d\sigma}, \frac{d\tilde{\rho}^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle - \langle \frac{D}{d\sigma} \frac{d\tilde{\rho}^{(k)}}{d\sigma}, \frac{d\tilde{\rho}^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle \right\} d\sigma \\
&\quad + \langle \mathrm{grad} E_{|\rho^{(k)}}, \frac{d\rho^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle \\
&= \frac{1}{h} \langle \frac{d\tilde{\rho}^{(k)}}{d\sigma}_{|\sigma=1}, \frac{d\rho^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle + \langle \mathrm{grad} E_{|\rho^{(k)}}, \frac{d\rho^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0} \rangle .
\end{aligned}
$$

Since $\frac{d\rho^{(k)}_\epsilon}{d\epsilon}_{|\epsilon=0}$ varies freely in $T_\rho \mathcal{M}$, we obtain

$$
\frac{1}{h} \frac{d\tilde{\rho}^{(k)}}{d\sigma}_{|\sigma=1} + \mathrm{grad} E_{|\rho^{(k)}} \; = \; 0, \tag{98}
$$

which is the first variation. Hence in a Euclidean setting, the first variation of (97) coincides with the implicit Euler method for (96). Now consider the interpolation

$$
\rho(t) \; = \; \tilde{\rho}^{(k)}(\frac{t-(k-1)\,h}{h}) \quad \text{for } t \in [(k-1)\,h, k\,h].
$$

Then the curve $[0,\infty) \ni t \mapsto \rho(t) \in \mathcal{M}$ is continuous and piecewise differentiable with

$$
\rho(k\,h) \; = \; \rho^{(k)} \quad \text{and} \quad \frac{d\rho}{dt}(k\,h-) \; = \; \frac{1}{h} \frac{d\tilde{\rho}^{(k)}}{d\sigma}_{|\sigma=1} .
$$

Together with the first variation (98), we obtain

$$\frac{d\rho}{dt}(k\,h-) \;=\; -\mathrm{grad}E_{|\rho(k\,h)}.$$

This visualizes that (97) is a discretization of (96).

In our non–smooth infinite–dimensional setting, this time discretization provides a connection between the the partial differential equation (1) on one side and the Wasserstein metric d (76) and the energy functional E (11) on the other side, without referring to the shaky differential structure of $(\mathcal{M}, g)$. Indeed, In [28] (see also [24]), we rigorously proved convergence of the scheme for $m = 2, N = 2$, in [22], we proved convergence for $m = 1$ and arbitrary N. Kinderlehrer and Walkington work on numerical schemes for (1) based on the time discretization (97).

5 Rigorous results

5.1 Weak solutions of the porous medium equation

In case of $m > 1$ and for non–zero initial data with compact support, there is no (classically) differentiable solution of the porous medium equation. We therefore must work with the notion of a weak solution. The well–established existence and uniqueness theory for weak solutions is based on the traditional gradient flow approach, as presented in the subsection 1.3. In particular, existence is based on the identity (5) in subsection 1.2, which in the traditional approach reads as

$$\frac{d}{dt}\int \tfrac{1}{m+1}\,\rho(t)^{m+1} \;=\; -\int |\nabla\rho^m(t)|^2.$$

This identity yields the essential a priori estimates. Uniqueness is based on the convexity of the functional in the traditional approach, which leads to a contraction property of the semi group in the induced norm (remember that in the traditional approach, the space carries the geometry of a convex subset of a euclidean function space, so that intrinsic convexity of the functional reduces to ordinary convexity). Here that means that if ρ_0, ρ_1 are solutions, then

$$\frac{d}{dt}\int |\nabla p(t)| \;=\; \int (\rho_1(t) - \rho_0(t))\,(\rho_1(t)^m - \rho_0(t)^m) \;\le\; 0,$$

where, in the spirit of (7),

$$-\nabla^2 p(t) \;=\; \rho_1(t) - \rho_0(t).$$

Definition 1 *Let ρ_0 be a measurable and non negative function on $I\!R^N$ with*

$$\int \tfrac{1}{m+1}\,\rho_0^{m+1} \;<\; \infty.$$

Let ρ be a measurable and non negative function on $[0,\infty) \times I\!R^N$ with

$$\operatorname{ess}\sup_{t\in[0,\infty)} \int \tfrac{1}{m+1}\,\rho(t)^{m+1} \;<\; \infty.$$

Then ρ^m is locally integrable, more precisely

$$\operatorname{ess}\sup_{t\in[0,\infty)} \int (\rho^m(t))^{(m+1)^*} \;<\; \infty,$$

where $(m+1)^*$ denotes the dual exponent to $m+1$. Assume further that ρ^m has a distributional spatial gradient $\nabla\rho^m$ satisfying

$$\int_0^\infty \int |\nabla\rho^m(t)|^2\,dt \;<\; \infty.$$

Then ρ is called a weak solution of the porous medium equation with initial data ρ_0 if

$$\int_{(0,\infty)\times \boldsymbol{R}^N} \left\{-\rho\,\frac{\partial\zeta}{\partial t} + \nabla\rho^m\cdot\nabla\zeta\right\} \;=\; \int \rho_0\,\zeta(0)$$
$$\text{for all } \zeta \;\in C_0^\infty((-\infty,\infty)\times I\!R^N).$$

5.2 The Wasserstein metric

In subsection 4.3, we formally derived that the induced metric d on $(\mathcal{M}, g)$ is given by

$$d(\rho_0,\rho_1)^2 \;=\; \inf_{\rho_1=\Phi\#\rho_0} \int \rho_0\,|\mathrm{id}-\Phi|^2.$$

Morally speaking, this is the Wasserstein metric. The precise definition of the Wasserstein metric relaxes the above variational problem. This is done by embedding the set of one–to–one transformations Φ with $\rho_1 = \Phi\#\rho_0$ into the set of all probability measures μ on $I\!R^N \times I\!R^N$ with marginals given by the Lebesgue densities ρ_0, ρ_1 via

$$\mu \;=\; (\mathrm{id} \times \Phi)\#\rho_0, \tag{99}$$

that is

$$\int \zeta(x_0, x_1)\, \mu(dx_0 dx_1) \;=\; \int \rho_0(x_0)\, \zeta(x_0, \Phi(x_0))\, dx_0 \quad \text{for all } \zeta \in C_0^\infty(I\!R^N \times I\!R^N).$$

Then

$$\int \rho_0\, |\mathrm{id} - \Phi|^2 \;=\; \int |y_0 - y_1|^2\, \mu(dy_0, dy_1),$$

and it is the latter functional one minimizes on the set of all probability measures μ with marginals $\rho_0\, dy_0$ and $\rho_1\, dy_1$. The relaxed variational problem is one version of the Monge–Kantorowicz mass transference problems; the μ's are called "transference plans", the function $|y_0 - y_1|^2$ is the "cost" of transferring a unit mass from y_0 to y_1.

Definition 2 *For two non negative Borel measures μ_0 and μ_1 of equal mass, we introduce*

$$\begin{aligned} P(\mu_0, \mu_1) \\ = \Big\{ \;& \textit{non negative Borel measure } \mu \textit{ on } I\!R^N \times I\!R^N \;\Big| \\ & \int \zeta(y_0)\, \mu(dy_0\, dy_1) = \int \zeta(y_0)\, \mu_0(dy_0) \quad \textit{and} \\ & \int \zeta(y_1)\, \mu(dy_0\, dy_1) = \int \zeta(y_1)\, \mu_1(dy_1) \quad \textit{for all } \zeta \in C_0^\infty(I\!R^N) \Big\}. \end{aligned}$$

$d(\mu_0, \mu_1)^2$ is defined as

$$d(\mu_0, \mu_1)^2 \;=\; \inf_{\mu \in P(\mu_0, \mu_1)} \int |y_1 - y_0|^2\, \mu(dy_0\, dy_1)\,. \tag{100}$$

If μ_0 and μ_1 have Lebesgue densities $\mu_0 = \rho_0\, dy_0$ resp. $\mu_1 = \rho_1\, dy_1$, we also write

$$d(\rho_0, \rho_1)^2 \;=\; d(\mu_0, \mu_1)^2.$$

The space $P(\mu_0,\mu_1)$ always contains the product measure $\mu_0\times\mu_1$. Hence $d(\mu_0,\mu_1)\in[0,\infty]$ is well defined. If the second moments of μ_0 and μ_1 are finite, then the transference plan $\mu_0\times\mu_1$ has finite cost, so that $d(\mu_0,\mu_1)\in[0,\infty)$. It is then an easy exercise in soft methods that the variational problem (100) admits a minimizer of finite cost. It has been known to the probabilists for a long time that d indeed defines a metric on the space of probability measures on $I\!R^N$ with finite second moments. This distance function is popular in probability theory since it metrizes the topology of weak–* convergence (up to second moments). We found the few results on d we need in [32] or [21]. We summarize them in the following Lemma.

Lemma 1 *Let $\{\mu_{0,\nu}\}_{\nu\uparrow\infty}$ and $\{\mu_{1,\nu}\}_{\nu\uparrow\infty}$ be two sequences of non negative Borel measures on $I\!R^N$. We assume that the masses of $\mu_{0,\nu}$ and $\mu_{1,\nu}$ are finite and equal and that there exist two non negative Borel measures μ_0 and μ_1 on $I\!R^N$ of finite mass such that*

$$\int\zeta\,d\mu_i \;=\; \lim_{\nu\uparrow\infty}\int\zeta\,d\mu_{i,\nu}\quad \textit{for all } \zeta\in C_0^\infty(I\!R^N)\quad \textit{and } i=0,1.$$

Then

$$d(\mu_0,\mu_1)^2\;\le\;\liminf_{\nu\uparrow\infty} d(\mu_{0,\nu},\mu_{1,\nu})^2.$$

If in addition

$$\int\frac{1}{2}|y|^2\,d\mu_i\;=\;\lim_{\nu\uparrow\infty}\int\frac{1}{2}|y|^2\,d\mu_{i,\nu}\quad \textit{for } i=0,1.$$

then

$$d(\mu_0,\mu_1)^2\;=\;\lim_{\nu\uparrow\infty} d(\mu_{0,\nu},\mu_{1,\nu})^2.$$

The variational problem in (100) has recently received some attention by analysts. If the measures μ_0 and μ_1 have bounded support and Lebesgue densities $\mu_0=\rho_0\,dy_0$ resp. $\mu_1=\rho_1\,dy_1$, Brenier [5] has shown uniqueness of the minimizing transference plan μ and proved that the support of μ is the graph of the gradient of a (generically non smooth) convex function, more precisely,

$$\mu\;=\;(\mathrm{id}\times\nabla\varphi)\#\rho_0. \tag{101}$$

A glance back to (99) then shows that the initial relaxation from one–to–one transformations Φ to transference plans μ is non essential and just of technical convenience. In particular, (101) yields that

$$\rho_1 \;=\; \nabla\varphi\#\rho_0. \qquad (102)$$

We also invite the reader to compare (102) with (78) in subsection 4.4.

Caffarelli [7] and Gangbo & McCann [18, 19, 20] have extended Brenier's result to more general strictly convex cost functions. The case of cost functions of degenerate convexity [15] and concave cost functions [20] is qualitatively different.

5.3 The statement of the rigorous result

Theorem 1 *Let m satisfy $m > \frac{N}{N+2}$ and $m \geq 1 - \frac{1}{N}$. Let ρ be a weak solution of the porous medium equation with initial data ρ_0 in the sense of Definition 1. We assume that additionally*

$$\int \rho_0 \;=\; 1 \quad \textit{and} \quad \int \rho_0 \, \frac{1}{2}|x|^2 \;<\; \infty.$$

We consider the function $\hat\rho$ on $(-\infty,\infty)\times I\!R^N$ given by

$$\rho(t,x) \;=\; \frac{1}{t^{N\alpha}}\,\hat\rho(\ln t, \frac{x}{t^\alpha}),$$

where $\alpha = \frac{1}{(m-1)\,N+2}$. Then, in a distributional sense,

$$\begin{aligned} \frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\,|\mathrm{grad}F_{|\hat\rho(\tau)}|^2\right] \;&\leq\; 0,\\ \frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\,(F(\hat\rho(\tau)) - F(\hat\rho_*))\right] \;&\leq\; 0,\\ \frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\,d(\hat\rho(\tau),\hat\rho_*)^2\right] \;&\leq\; 0, \end{aligned}$$

with the understanding that the quantities in the square brackets are finite for $\tau > -\infty$. The precise meaning of $|\mathrm{grad}F_{\hat\rho}|^2$ and $F(\hat\rho) - F(\hat\rho_*)$ is given in (104) resp. (105), d denotes the Wasserstein distance as in Definition 2.

Let us now explain what we understand by $|\mathrm{grad} F_{\hat\rho}|^2$ and $F(\hat\rho) - F(\hat\rho_*)$ in Theorem 1. In subsection (3.4), we have identified $|\mathrm{grad} F_{\hat\rho}|^2$ as

$$|\mathrm{grad} F_{|\hat\rho}|^2 \;=\; \int \hat\rho\,|\nabla p|^2 \quad \text{where} \quad p(y) \;=\; e'(\hat\rho(y)) + \alpha\,\frac{1}{2}|y|^2.$$

We observe that thanks to the fundamental relationship $z\,e''(z) = \pi'(z)$ between energy density (21) and osmotic pressure (20), we have

$$\frac{1}{\hat\rho}\,\nabla\pi(\hat\rho) \;=\; \hat\rho\,\nabla e'(\hat\rho)$$

and thus

$$\int \frac{1}{\hat\rho}\,|\nabla\pi(\hat\rho) + \alpha\,\hat\rho\,y|^2 \;=\; \int \hat\rho\,|\nabla p|^2, \tag{103}$$

provided $\hat\rho$ is locally bounded away from zero. Observe that even if this is not the case, the l. h. s. of (103) is well defined as a number in $[0,\infty]$, since $\nabla\pi(\hat\rho)$ vanishes almost everywhere on the set where $\hat\rho$ vanishes. It is this weak formulation

$$|\mathrm{grad} F_{\hat\rho}|^2 \;=\; \int \frac{1}{\hat\rho}\,|\nabla\pi(\hat\rho) + \alpha\,\hat\rho\,y|^2, \tag{104}$$

we use in Theorem 1.

By $F(\hat\rho) - F(\hat\rho_*)$ we understand

$$\begin{aligned} &F(\hat\rho) - F(\hat\rho_*) \qquad\qquad (105)\\ &= \left\{ \begin{array}{ll} \left(\int e(\hat\rho) + \alpha \int \hat\rho\,\frac{1}{2}|y|^2\right) - \left(\int e(\hat\rho_*) + \alpha\int \hat\rho_*\,\frac{1}{2}|y|^2\right) & \text{for } m > 1, \\ \int \{e(\hat\rho) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho - \hat\rho_*)\} & \text{for } m \le 1 \end{array} \right\}. \end{aligned}$$

The second line is inspired by the identity (31) in subsection 3.3. We point out that in both lines, the integrands are non negative; hence the number $F(\hat\rho) - F(\hat\rho_*) \in [0,\infty]$ is well defined.

The main technical difficulty in mimicking the Riemannian calculus is the possible lack of regularity of solutions of the porous medium equation. Our approach is to mimic the Riemannian calculus in a completely smooth setting (Proposition 1) and then to use an approximation argument (in the proof of Theorem 1).

Proposition 1 *Let e and π be smooth functions on $(0,\infty)$ related by*

$$\pi(z) \;=\; z\,e'(z) - e(z) \quad \text{and thus} \quad \pi'(z) \;=\; z\,e''(z) \tag{106}$$

and satisfying

$$\pi(z) \;\geq\; 0 \quad \text{and} \quad z\,\pi'(z) - (1-\frac{1}{N})\,\pi(z) \;\geq\; 0, \tag{107}$$

$$lim_{z\downarrow 0} e'(z) \;=\; -\infty \quad \text{and} \quad lim_{z\downarrow 0} e(z) \;=\; 0. \tag{108}$$

Let the open $\Omega \subset I\!R^N$ satisfy

$$\Omega \text{ is convex} \quad \text{and} \quad \partial\Omega \text{ is smooth.}$$

Let the function $\hat\rho$ of be a smooth and positive function on $(-\infty,\infty)\times\overline{\Omega}$ which solves

$$\frac{\partial\hat\rho}{\partial\tau} - \nabla\cdot(\hat\rho\,\nabla p) \;=\; 0 \quad in\,(-\infty,\infty)\times\Omega, \tag{109}$$

$$\hat\rho\,\nabla p\cdot\nu \;=\; 0 \quad on\,(-\infty,\infty)\times\partial\Omega, \tag{110}$$

where

$$p \;=\; e'(\hat\rho) + \alpha\,\frac{1}{2}|y|^2$$

for some fixed $\alpha \geq 0$. We observe that the evolution equation (109,110) conserves mass. Thanks to (108), there exists a smooth stationary solution $\hat\rho_*$ of (109,110) with the same mass, it is given by

$$e'(\hat\rho_*(y)) + \alpha\,\frac{1}{2}|y|^2 \;=\; \lambda \quad \text{and} \quad \int_\Omega \hat\rho_* \;=\; \int_\Omega \hat\rho(\tau). \tag{111}$$

Then $\hat\rho$ and $\hat\rho_*$ satisfy

$$\frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\int_\Omega \hat\rho(\tau)\,|\nabla p(\tau)|^2\right] \;\leq\; 0, \tag{112}$$

$$\frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\,(F(\hat\rho(\tau)) - F(\hat\rho_*))\right] \;\leq\; 0, \tag{113}$$

$$\frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\,d(\hat\rho_*,\hat\rho(\tau))^2\right] \;\leq\; 0, \tag{114}$$

where

$$F(\hat\rho) \;=\; E(\hat\rho) + \alpha\,M(\hat\rho) \;=\; \int_\Omega e(\hat\rho) + \alpha\int_\Omega \hat\rho\,\frac{1}{2}|y|^2.$$

5.4 Proof of the Proposition

We start with the proof of (112). At the center of our attention is

$$p(y) \;=\; e'(\hat\rho(y)) + \alpha\,\frac{1}{2}|y|^2,$$

that is, $p \cong \mathrm{grad}F_{|\hat\rho} = -\frac{d\hat\rho}{d\tau}$. We have

$$\begin{aligned}
\frac{d}{d\tau}\int_\Omega \hat\rho\,|\nabla p|^2 \quad &= \quad \int_\Omega \Big\{\hat\rho\,2\,\nabla p\cdot\nabla\partial_\tau p + \partial_\tau\hat\rho\,|\nabla p|^2\Big\} \\
&\overset{(109)}{=} \quad 2\,\int_\Omega \Big\{\hat\rho\,2\,\nabla p\cdot\nabla\partial_\tau p + \nabla\cdot(\hat\rho\nabla p)\,\frac{1}{2}|\nabla p|^2)\Big\} \\
&= \quad 2\,\int_\Omega \hat\rho\,\nabla p\cdot\Big[\nabla\partial_\tau p - \nabla(\frac{1}{2}|\nabla p|^2)\Big] + \int_{\partial\Omega}\hat\rho\,\nabla p\cdot\nu\,|\nabla p|^2 \\
&\overset{(110)}{=} \quad 2\,\int_\Omega \hat\rho\,\nabla p\cdot\Big[\nabla\partial_\tau p - \nabla(\frac{1}{2}|\nabla p|^2)\Big] \\
&= \quad 2\,\int_\Omega \hat\rho\,\nabla p\cdot\Big[\nabla\partial_\tau p - \mathrm{D}^2 p.\nabla p\Big]\,,
\end{aligned}$$

which mimics $\frac{d}{d\tau}|\mathrm{grad}F_{|\hat\rho}|^2 = 2\,\langle \mathrm{grad}F_{|\hat\rho}, \frac{D}{d\tau}\mathrm{grad}F_{|\hat\rho}\rangle$. We now split p into

$$p \;=\; p_1 + \alpha\,p_2 \quad \text{where} \quad p_1(y) \;=\; e'(\hat\rho(y)) \quad \text{and} \quad p_2(y) \;=\; \frac{1}{2}|y|^2,$$

that is, $p_1 \cong \mathrm{grad}E_{|\hat\rho}$ and $p_2 \cong \mathrm{grad}M_{|\hat\rho}$. We will show that

$$\begin{aligned}
&-\int_\Omega \hat\rho\,\nabla p\cdot\Big[\nabla\partial_\tau p_1 - \mathrm{D}^2 p_1.\nabla p\Big] \\
&\quad = \;\int_\Omega \Big\{(\pi'(\hat\rho)\,\hat\rho - \pi(\hat\rho))\,(\nabla^2 p)^2 + \pi(\hat\rho)\,\mathrm{tr}\,(\mathrm{D}^2 p)^2\Big\} \\
&\qquad + \int_{\partial\Omega}\pi(\hat\rho)\,\nabla p\cdot II.\nabla p,
\end{aligned} \tag{115}$$

where $v\cdot II.v$ denotes the second fundamental form of $\partial\Omega$. This mimics

$$-\langle \mathrm{grad}F_{|\hat\rho}, \frac{D}{d\tau}\mathrm{grad}E_{|\hat\rho}\rangle \;=\; \langle \mathrm{grad}F_{|\hat\rho}, \mathrm{Hess}E_{|\hat\rho}\,\mathrm{grad}F_{|\hat\rho}\rangle,$$

as can be seen from (84). On the other hand, it is obvious that

$$\int_\Omega \hat\rho\,\nabla p\cdot\Big[\nabla\partial_\tau p_2 - \mathrm{D}^2 p_2.\nabla p\Big] \;=\; -\int_\Omega \hat\rho\,|\nabla p|^2, \tag{116}$$

which mimics

$$\langle \mathrm{grad} F_{|\hat\rho}, \frac{D}{d\tau}\mathrm{grad} M_{|\hat\rho}\rangle \;=\; -\langle \mathrm{grad} F_{|\hat\rho}, \mathrm{Hess} M_{|\hat\rho}\, \mathrm{grad} F_{|\hat\rho}\rangle \;=\; -|\mathrm{grad} F_{|\hat\rho}|^2,$$

as can be seen from (85).

Let us establish the identity (115). For this, we write the first part of the integrand of the l. h. s. in (115) as follows:

$$\begin{aligned}
&-\hat\rho\,\nabla p\cdot\nabla\partial_\tau p_1\\
&\quad= \quad -\hat\rho\,\nabla p\cdot\nabla[e''(\hat\rho)\,\partial_\tau\hat\rho]\\
&\quad\overset{(109)}{=} \quad -\hat\rho\,\nabla p\cdot\nabla[e''(\hat\rho)\,\nabla\cdot(\hat\rho\nabla p)]\\
&\quad= \quad -\nabla p\cdot\nabla[\hat\rho\, e''(\hat\rho)\,\nabla\cdot(\hat\rho\nabla p)] + \nabla p\cdot\nabla\hat\rho\, e''(\hat\rho)\,\nabla\cdot(\hat\rho\nabla p)\\
&\quad\overset{(106)}{=} \quad -\nabla p\cdot\nabla[\pi'(\hat\rho)\,\nabla\cdot(\hat\rho\nabla p)] + \nabla p\cdot\nabla\hat\rho\, e''(\hat\rho)\,\nabla\cdot(\hat\rho\nabla p).
\end{aligned}$$

Thanks to the formula

$$\begin{aligned}
\nabla\cdot[\nabla p\cdot\nabla p_1\,\hat\rho\,\nabla p] &= \nabla(\nabla p\cdot\nabla p_1)\cdot(\hat\rho\,\nabla p) + \nabla p\cdot\nabla p_1\,\nabla\cdot(\hat\rho\,\nabla p)\\
&= \hat\rho\,\nabla p\cdot \mathrm{D}^2 p_1.\nabla p + \hat\rho\,\nabla p_1\cdot\mathrm{D}^2 p.\nabla p\\
&\quad + \nabla p\cdot\nabla p_1\,\nabla\cdot(\hat\rho\,\nabla p),
\end{aligned}$$

which we rearrange to

$$\begin{aligned}
\hat\rho\,\nabla p\cdot\mathrm{D}^2 p_1.\nabla p &= -\Big\{\nabla p\cdot\nabla p_1\,\nabla\cdot(\hat\rho\,\nabla p) + \hat\rho\,\nabla p_1\cdot\mathrm{D}^2 p.\nabla p\Big\}\\
&\quad + \nabla\cdot[\nabla p\cdot\nabla p_1\,\hat\rho\,\nabla p]\,,
\end{aligned}$$

we have for the second part of the integral of the l. h. s. in (115):

$$\begin{aligned}
&\int_\Omega \hat\rho\,\nabla p\cdot\mathrm{D}^2 p_1.\nabla p\\
&\quad= \quad -\int_\Omega\Big\{\nabla\cdot(\hat\rho\,\nabla p)\,\nabla p\cdot\nabla p_1 + \hat\rho\,\nabla p_1\cdot\mathrm{D}^2 p.\nabla p\Big\}\\
&\qquad\quad + \int_{\partial\Omega}\nabla p\cdot\nabla p_1\,\hat\rho\,\nabla p\cdot\nu\\
&\quad\overset{(110)}{=} \quad -\int_\Omega\Big\{\nabla\cdot(\hat\rho\,\nabla p)\,\nabla p\cdot\nabla p_1 + \hat\rho\,\nabla p_1\cdot\mathrm{D}^2 p.\nabla p\Big\}
\end{aligned}$$

and rewrite the integrand as

$$\begin{aligned}
&\nabla \cdot (\hat\rho\, \nabla p)\, \nabla p \cdot \nabla p_1 + \hat\rho\, \nabla p_1 \cdot \mathrm{D}^2 p.\nabla p \\
&\overset{(106)}{=} \quad \nabla \cdot (\hat\rho\, \nabla p)\, e''(\hat\rho)\, \nabla p \cdot \nabla \hat\rho\, + \nabla[\pi(\hat\rho)] \cdot \mathrm{D}^2 p.\nabla p.
\end{aligned}$$

Hence we obtain for the whole integral of the l. h. s. in (115):

$$\begin{aligned}
&\int_\Omega \hat\rho\, \nabla p \cdot \left[\nabla \partial_\tau p_1 + \mathrm{D}^2 p_1.\nabla p\right] \\
&\quad = \; -\int_\Omega \left\{\nabla p \cdot \nabla[\pi'(\hat\rho)\, \nabla \cdot (\hat\rho\, \nabla p)] + \nabla[\pi(\hat\rho)] \cdot \mathrm{D}^2 p.\nabla p\right\}.
\end{aligned}$$

A further integration by parts yields

$$\begin{aligned}
&\int_\Omega \hat\rho\, \nabla p \cdot \left[\nabla \partial_\tau p_1 + \mathrm{D}^2 p_1.\nabla p\right] \\
&\quad = \; \int_\Omega \left\{\nabla^2 p\; \pi'(\hat\rho)\, \nabla \cdot (\hat\rho\, \nabla p) + \pi(\hat\rho)\, \nabla \cdot (\mathrm{D}^2 p.\nabla p)\right\} \\
&\qquad - \int_{\partial\Omega} \left\{\nabla p \cdot \nu\, \pi'(\hat\rho)\, \nabla \cdot (\hat\rho\, \nabla p) + \pi(\hat\rho)\; \nu \cdot (\mathrm{D}^2 p.\nabla p)\right\} \qquad (117)
\end{aligned}$$

Let us consider the boundary integral in (117). The Neumann boundary condition (110) means that ∇p is a tangential vector field on $\partial\Omega$. Differentiating the Neumann boundary condition along this tangential vector field yields

$$\nu \cdot \mathrm{D}^2 p.\nabla p + \nabla p \cdot II.\nabla p \;=\; 0 \quad \text{on } \partial\Omega. \qquad (118)$$

We use the identity (118) to substitute $\nu \cdot \mathrm{D}^2 p.\nabla p$ in (117) and obtain

$$\begin{aligned}
&\int_\Omega \hat\rho\, \nabla p \cdot \left[\nabla \partial_\tau p_1 + \mathrm{D}^2 p_1.\nabla p\right] \\
&\quad = \; \int_\Omega \left\{\nabla^2 p\, \pi'(\hat\rho)\, \nabla \cdot (\hat\rho\, \nabla p) + \pi(\hat\rho)\, \nabla \cdot (\mathrm{D}^2 p.\nabla p)\right\} \\
&\qquad + \int_{\partial\Omega} \pi(\hat\rho)\, \nabla p \cdot II.\nabla p. \qquad (119)
\end{aligned}$$

We reconsider the first part of the bulk integrand on the r. h. s. in (119):

$$\begin{aligned}
\pi'(\hat\rho)\, \nabla \cdot (\hat\rho\, \nabla p)\, \nabla^2 p \;&=\; \pi'(\hat\rho)\, \hat\rho\, (\nabla^2 p)^2 + \pi'(\hat\rho)\, \nabla \hat\rho \cdot \nabla p\, \nabla^2 p \\
&=\; \pi'(\hat\rho)\, \hat\rho\, (\nabla^2 p)^2 + \nabla[\pi(\hat\rho)] \cdot \nabla p\, \nabla^2 p
\end{aligned}$$

and perform a last integration by parts

$$\begin{aligned}
&\int_\Omega \nabla[\pi(\hat\rho)]\cdot\nabla p\,\nabla^2 p \\
&\quad = \quad -\int_\Omega \pi(\hat\rho)\,\nabla\cdot(\nabla^2 p\,\nabla p) + \int_{\partial\Omega}\pi(\hat\rho)\,\nu\cdot\nabla p\,\nabla^2 p \\
&\quad \overset{(110)}{=} \quad -\int_\Omega \pi(\hat\rho)\,\nabla\cdot(\nabla^2 p\,\nabla p).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\int_\Omega \hat\rho\,\nabla p\cdot\Big[\nabla\partial_\tau p_1 + \mathrm{D}^2 p_1.\nabla p\Big] \\
&\quad = \int_\Omega \Big\{\pi'(\hat\rho)\,\hat\rho\,(\nabla^2 p)^2 + \pi(\hat\rho)\,\Big[-\nabla\cdot(\nabla^2 p\,\nabla p) + \nabla\cdot(\mathrm{D}^2 p.\nabla p)\Big]\Big\} \\
&\qquad + \int_{\partial\Omega}\pi(\hat\rho)\,\nabla p\cdot II.\nabla p.
\end{aligned}$$

We conclude the proof of identity (115) by evoking the formula

$$-\nabla\cdot(\nabla^2 p\,\nabla p) + \nabla\cdot(\mathrm{D}^2 p.\nabla p) \;=\; \mathrm{tr}(\mathrm{D}^2 p)^2 - (\nabla^2 p)^2.$$

In order to conclude

$$\frac{d}{d\tau}\int_\Omega \hat\rho\,|\nabla p|^2 \;\le\; -2\,\alpha\int_\Omega \hat\rho\,|\nabla p|^2,$$

and thereby the proof of (112), it remains to show that the right hand side of (115) is non negative, that is: $\langle s, \mathrm{Hess}E_{|\hat\rho} s\rangle \ge 0$. Here we use our assumptions on π and Ω. The integral over Ω is non negative, since its integrand is non negative:

$$\begin{aligned}
&(\pi'(\hat\rho)\,\hat\rho - \pi(\hat\rho))\,(\nabla^2 p)^2 + \pi(\hat\rho)\,\mathrm{tr}\,(\mathrm{D}^2 p)^2 \\
&\quad \overset{(107)}{\ge} \quad (\pi'(\hat\rho)\,\hat\rho - \pi(\hat\rho))\,(\nabla^2 p)^2 + \pi(\hat\rho)\,\frac{1}{N}\,(\nabla^2 p)^2 \\
&\quad \overset{(107)}{\ge} \quad 0,
\end{aligned}$$

where we have used $\mathrm{tr}C^2 \ge \frac{1}{N}(\mathrm{tr}C)^2$ for a symmetric $N\times N$–matrix C as in (81) of subsection 4.4. The integral over $\partial\Omega$ is non negative since its integrand is non negative: Our assumption (107) on π implies that the first factor $\pi(\hat\rho)$

is non negative; the convexity of Ω implies that the second fundamental form II of $\partial\Omega$ is positive semi definite, hence also the second factor $\nabla p \cdot II.\nabla p$ is non negative.

Let us now tackle (114). Following the lines of subsection 3.5, we start by deriving an auxiliary result. Let $\hat\rho_0, \hat\rho_1$ be smooth and positive functions on $\overline{\Omega}$. We think of $\hat\rho_i$ $(i = 1, 2)$ as being extended on $I\!R^N$ by zero so that according to (108),

$$F(\hat\rho_i) \;=\; E(\hat\rho_i) + \alpha\, M(\hat\rho_i) \;=\; \int e(\hat\rho_i) + \alpha \int \hat\rho_i\, \frac{1}{2}|y|^2.$$

Let μ denote an optimal transference plan in the definition of $d(\hat\rho_0, \hat\rho_1)^2$. We consider $p_i \cong \mathrm{grad} F_{|\hat\rho_i}$, that is,

$$p_i(y) \;=\; e'(\hat\rho_i(y)) + \alpha\, \frac{1}{2}|y|^2.$$

The auxiliary result states that

$$F(\hat\rho_1) - F(\hat\rho_0) \;\geq\; \int \nabla p_0(y_0) \cdot (y_1 - y_0)\, \mu(dy_0 dy_1) + \alpha\, \frac{1}{2} d(\hat\rho_0, \hat\rho_1)^2. \qquad (120)$$

The integral is well defined, since p_0 is smooth on $\overline{\Omega}$ and μ is supported on $\overline{\Omega} \times \overline{\Omega}$. In order to derive and interpret inequality (120), we need the curve $[0,1] \ni \sigma \mapsto \hat\rho_\sigma$ of least energy between $\hat\rho_0$ and $\hat\rho_1$. In this sense, (120) mimics (42) in subsection 3.5, that is,

$$F(\hat\rho_1) - F(\hat\rho_0) \;\geq\; \langle \mathrm{grad} F_{|\hat\rho_0}, \frac{d\hat\rho}{d\sigma}_{|\sigma=0} \rangle + \alpha\, \frac{1}{2} d(\hat\rho_0, \hat\rho_1)^2.$$

In terms of $[0,1] \ni \sigma \mapsto \hat\rho_\sigma$, inequality (120) obviously is a consequence of

$$\frac{d^+}{d\sigma}_{|\sigma=0} F(\hat\rho_\sigma) \;\geq\; \int \nabla p_0(y_0) \cdot (y_1 - y_0)\, \mu(dy_0 dy_1) \qquad (121)$$

and

$$\frac{d^2}{d\sigma^2} F(\hat\rho_\sigma) \;\geq\; \alpha\, d(\hat\rho_0, \hat\rho_1)^2. \qquad (122)$$

The latter obviously splits into

$$\frac{d^2}{d\sigma^2}\, E(\hat\rho_\sigma) \;\ge\; 0 \quad \text{and} \tag{123}$$
$$\frac{d^2}{d\sigma^2}\, M(\hat\rho_\sigma) \;=\; d(\hat\rho_0,\hat\rho_1)^2. \tag{124}$$

For these statements to make sense, we need the existence of a (weak) curve $[0,1]\ni\sigma\mapsto\hat\rho_\sigma$ of least energy between $\hat\rho_0$ and $\hat\rho_1$. It is provided by results of McCann [9], which rely on earlier work by Brenier [5]. Let us state these results: According to Brenier [5], there exists a convex function φ_1 on $I\!\!R^N$ such that

$$\mu \;=\; (\mathrm{id}\times\nabla\varphi_1)\#\hat\rho_0 \quad \text{and in particular} \quad \hat\rho_1 \;=\; \nabla\varphi_1\#\hat\rho_0. \tag{125}$$

According to McCann [9, Proposition 1.3 (ii)],

$$\hat\rho_\sigma \;=\; \nabla\varphi_\sigma\#\hat\rho_0 \quad \text{where} \quad \varphi_\sigma(y) \;=\; (1-\sigma)\,\frac{1}{2}|y|^2+\sigma\,\varphi_1(y)$$

defines a non negative and integrable function $\hat\rho_\sigma$ on $I\!\!R^N$. A glance back to (75) in subsection 4.3 will convince the reader of our interpretation of $\sigma\mapsto\hat\rho_\sigma$ as a geodesic — which by construction is the curve of least energy between $\hat\rho_0$ and $\hat\rho_1$. We observe that in terms of μ,

$$\int \hat\rho_\sigma\,\zeta \;=\; \int \zeta(\sigma\,y_1+(1-\sigma)\,y_0)\,\mu(dy_0dy_1) \quad \text{for all } \zeta\in C_0^\infty(I\!\!R^N). \tag{126}$$

Furthermore, McCann shows in [9, Theorem 4.4] that the transformation formula (80) in subsection 4.4 can be made rigorous: For all $\sigma\in(0,1)$ we have

$$\int e(\hat\rho_\sigma(y))\,dy \;=\; \int e\left(\frac{\hat\rho_0(y_0)}{\det \mathrm{D}^2\varphi_\sigma(y_0)}\right)\,\det \mathrm{D}^2\varphi_\sigma(y_0)\,dy_0,. \tag{127}$$

We recall that a convex function φ has a gradient $\nabla\varphi$ and a Hessian $\mathrm{D}^2\varphi$ in the sense that for almost every y_0,

$$\begin{aligned}\varphi(y)\\ =\;\; &\varphi(y_0)+(y-y_0)\cdot\nabla\varphi(y_0)+(y-y_0)\cdot \mathrm{D}^2\varphi(y_0).(y-y_0)+o((y-y_0)^2).\end{aligned}$$

A proof of this result of Alexandrov can be found in [12, Theorem A.2.]. The symmetric and positive semi definite matrix $\mathrm{D}^2\varphi_\sigma(y_0)$ in (127) is to be understood in this sense. We observe that $\mathrm{D}^2\varphi_\sigma(y_0) = \sigma\,\mathrm{D}^2\varphi(y_0) + (1-\sigma)\,\mathrm{id}$ is positive definite for $\sigma < 1$. Hence the division by $\det \mathrm{D}^2\varphi(y_0)$ in (127) causes no problem.

Let us start with (121). We note that our assumptions on e and π imply the convexity of $[0,\infty) \ni z \mapsto e(z)$. Indeed,

$$z^2\, e''(z) \;\overset{(106)}{=}\; z\,\pi'(z) \;\overset{(107)}{\geq}\; z\,(1-\frac{1}{N})\,\pi(z) \;\overset{(107)}{\geq}\; 0 \quad \text{for all } z > 0.$$

Therefore, we have

$$e(z) - e(z_0) \;\geq\; e'(z_0)\,(z - z_0) \quad \text{for all } z \geq 0 \text{ and } z_0 > 0.$$

We thus obtain

$$E(\hat\rho_\sigma) - E(\hat\rho_0) \;\geq\; \int e'(\hat\rho_0)\,(\hat\rho_\sigma - \hat\rho_0).$$

Trivially,

$$M(\hat\rho_\sigma) - M(\hat\rho_0) \;=\; \int \frac{1}{2}|y|^2\,(\hat\rho_\sigma(y) - \hat\rho_0(y))\,dy,$$

so that by definition of p_0

$$\begin{aligned} &\frac{1}{\sigma}\,(F(\hat\rho_\sigma) - F(\hat\rho_0)) && (128)\\ &\quad\geq \int p_0\,\frac{1}{\sigma}\,(\hat\rho_\sigma - \hat\rho_0)\\ &\quad\overset{(126)}{=} \int \frac{1}{\sigma}\,(p_0(\sigma\,y_1 + (1-\sigma)\,y_0) - p_0(y_0))\,\mu(dy_0 dy_1). && (129)\end{aligned}$$

We observe that 1.) the μ–integral is supported on $\overline{\Omega}\times\overline{\Omega}$, 2.) for all $(y_0, y_1) \in \overline{\Omega}\times\overline{\Omega}$ we have, as a consequence of the convexity of $\overline{\Omega}$, $\sigma\,y_1 + (1-\sigma)\,y_0 \in \overline{\Omega}$, 3.) p_0 is smooth in $\overline{\Omega}$. This implies that

$$\lim_{\sigma\downarrow 0}\frac{1}{\sigma}\,(p_0(\sigma\,y_1 + (1-\sigma)\,y_0) - p_0(y_0)) \;=\; \nabla p_0(y_0)\cdot(y_1 - y_0)$$

uniformly in (y_0, y_1) in the support of μ.

Therefore, the passage to the limit $\sigma \downarrow 0$ in the inequality (129) yields (121).

Convexity of E along geodesics as expressed in (123) can be derived from the representation (127) by copying the arguments given in subsection 3.5. The argument for the strict convexity of M along geodesics as quantified in (124) is simpler: According to (126),

$$M(\hat\rho_\sigma) \;=\; \int \frac{1}{2}|y|^2 \hat\rho_\sigma(y)\,dy \;=\; \int \frac{1}{2}|\sigma\, y_1 + (1-\sigma)\, y_0|^2\, \mu(dy_0, dy_1),$$

and therefore

$$\frac{d^2}{d\sigma^2}\, M(\hat\rho_\sigma) \;=\; \int |y_1 - y_0|^2\, \mu(dy_0, dy_1) \;=\; d(\hat\rho_0, \hat\rho_1)^2.$$

Now that we have established our auxiliary result (120), we observe that by symmetry, we also have

$$F(\hat\rho_0) - F(\hat\rho_1) \;\geq\; -\int \nabla p_1(y_1)\cdot(y_1 - y_0)\,\mu(dy_0 dy_1) + \alpha\,\frac{1}{2} d(\hat\rho_0, \hat\rho_1)^2. \tag{130}$$

Adding (120) and (130) yields

$$\int (\nabla p_1(y_1) - \nabla p_0(y_0))\cdot(y_1 - y_0)\,\mu(dy_0 dy_1) \;\geq\; \alpha\, d(\hat\rho_0, \hat\rho_1)^2. \tag{131}$$

Furthermore, we obtain from (120), dropping the $\alpha\,\frac{1}{2} d(\hat\rho_0, \hat\rho_1)^2$–term,

$$\begin{aligned} & F(\hat\rho_1) - F(\hat\rho_0) \\ & \quad \geq \; \int \nabla p_0(y_0)\cdot(y_1 - y_0)\,\mu(dy_0 dy_1) \\ & \quad \geq \; -\Big(\int |\nabla p_0(y_0)|^2\,\mu(dy_0 dy_1)\Big)^{\frac{1}{2}} \Big(\int |y_1 - y_0|^2\,\mu(dy_0 dy_1)\Big)^{\frac{1}{2}} \\ & \quad = \; -\Big(\int_\Omega \hat\rho_0\, |\nabla p_0|^2\Big)^{\frac{1}{2}}\, d(\hat\rho_0, \hat\rho_1), \end{aligned}$$

and thus by symmetry

$$\begin{aligned} & |F(\hat\rho_1) - F(\hat\rho_0)| \\ & \quad \leq \; \max\Big\{\Big(\int_\Omega \hat\rho_0\, |\nabla p_0|^2\Big)^{\frac{1}{2}}, \Big(\int_\Omega \hat\rho_1\, |\nabla p_1|^2\Big)^{\frac{1}{2}}\Big\}\, d(\hat\rho_0, \hat\rho_1). \end{aligned} \tag{132}$$

We now are in the position to prove that for two smooth and positive solutions $\hat\rho_0$ and $\hat\rho_1$ of (109,110) we have

$$\frac{d^+}{d\tau}\, d(\hat\rho_0,\hat\rho_1)^2 \;\le\; -2\,\alpha\, d(\hat\rho_0,\hat\rho_1)^2. \tag{133}$$

Since $\hat\rho_0(\tau) \;=\; \hat\rho_*$ defines a (stationary) smooth and positive solution of (109,110), this proves (114). In order to prove (133), we consider the smooth velocity fields

$$u_i \;=\; -\nabla p_i \quad\text{where}\quad p_i(y) \;=\; e'(\hat\rho_i(y)) + \alpha\,\frac{1}{2}\,|y|^2. \tag{134}$$

Since $\hat\rho_i$ satisfies (109,110), we have

$$\begin{aligned}\frac{\partial\hat\rho_i}{\partial\tau} + \nabla\cdot(\hat\rho_i\,u_i) \;&=\; 0 \quad \text{in } [0,\infty)\times\Omega,\\ u_i\cdot\nu \;&=\; 0 \quad \text{on } [0,\infty)\times\partial\Omega.\end{aligned}$$

Let us fix a time τ_0 and show that the last two lines imply that

$$\begin{aligned}&\frac{d^+}{d\tau}_{|\tau=\tau_0}\, d(\hat\rho_0(\tau),\hat\rho_1(\tau))^2\\ &\le\; 2\int (u_1(\tau_0,y_1)-u_0(\tau_0,y_0))\cdot(y_1-y_0)\,\mu(\tau_0,dy_0dy_1),\end{aligned} \tag{135}$$

where $\mu(\tau_0)$ is an optimal transference plan in the definition of the Wasserstein metric $d(\hat\rho_0(\tau_0),\hat\rho_1(\tau_0))^2$. Obviously, (135) together with the definition (134) of the velocities and the inequality (131) imply (133). In order to prove (135), we observe that

$$\frac{\partial\Phi_i}{\partial\tau}(\tau) \;=\; u_i(\tau)\circ\Phi_i(\tau) \quad\text{and}\quad \Phi_i(\tau_0) \;=\; \mathrm{id};$$

defines a family $\{\Phi_i(\tau)\}_\tau$ of diffeomorphism of Ω which are such that for any $\tau\in[0,\infty]$

$$\hat\rho_i(\tau) \;=\; \Phi_i(\tau)\#\hat\rho_i(\tau_0).$$

Therefore,

$$\mu(\tau) \;=\; (\Phi_0(\tau)\times\Phi_1(\tau))\#\mu(\tau_0)$$

defines an admissible transference plan in the definition of $d(\hat\rho_0(\tau),\hat\rho_1(\tau))^2$ (observe that $\mu(\tau_0)$ is supported in $\overline{\Omega}\times\overline{\Omega}$, where $\Phi_0(\tau)\times\Phi_1(\tau)$ is defined). Hence for all $\tau>\tau_0$,

$$\begin{aligned}
&\frac{1}{\tau-\tau_0}\,\Big(d(\hat\rho_0(\tau),\hat\rho_1(\tau))^2-d(\hat\rho_0(\tau_0),\hat\rho_1(\tau_0))^2\Big)\\
&\quad\le\ \frac{1}{\tau-\tau_0}\,\Big(\int|y_1-y_0|^2\,\mu(\tau,dy_0dy_1)-\int|y_1-y_0|^2\,\mu(\tau_0,dy_0dy_1)\Big)\\
&\quad=\ \int\frac{1}{\tau-\tau_0}\,\Big(|\Phi_1(\tau,y_1)-\Phi_0(\tau,y_0)|^2-|y_1-y_0|^2\Big)\,\mu(\tau_0,dy_0dy_1)
\end{aligned}\tag{136}$$

We observe that by definition of Φ_i

$$\begin{aligned}
&\lim_{\tau\downarrow\tau_0}\frac{1}{\tau-\tau_0}\,\Big(|\Phi_1(\tau,y_1)-\Phi_0(\tau,y_0)|^2-|y_1-y_0|^2\Big)\\
&\quad=\ 2\,(u_1(\tau_0,y_1)-u_0(\tau_0,y_0))\cdot(y_1-y_0)\\
&\qquad\text{uniformly in } (y_0,y_1) \text{ in the support of } \mu(\tau_0).
\end{aligned}$$

Therefore, the passage to the limit $\tau\downarrow\tau_0$ in the inequality (136) yields (135).

We finally address (113). As in our formal calculus in subsection 3.5, we need to know beforehand that

$$\lim_{\tau\uparrow\infty}(F(\hat\rho)-F(\hat\rho_*))\ =\ 0.\tag{137}$$

As in subsection 3.5, we obtain (137) from (112) and (114) in the weakened form of

$$\lim_{\tau\uparrow\infty}\int_\Omega\hat\rho\,|\nabla p|^2\ =\ 0\quad\text{resp.}\quad\lim_{\tau\uparrow\infty}d(\hat\rho,\hat\rho^*)^2\ =\ 0$$

via the interpolation

$$|F(\hat\rho)-F(\hat\rho_*)|\ \le\ \Big(\int_\Omega\hat\rho\,|\nabla p|^2\Big)^{\frac12}\ d(\hat\rho,\hat\rho_*),$$

which follows from (132) and (111). The other ingredient is $\frac{d}{d\tau}\,(F(\hat\rho)-F(\hat\rho_*))=-|\mathrm{grad}F_{|\hat\rho}|^2$, that is,

$$\frac{d}{d\tau}\,(F(\hat\rho)-F(\hat\rho_*))\quad=\quad\frac{d}{d\tau}\,F(\hat\rho)$$

$$
\begin{aligned}
&= \int_\Omega p\,\frac{\partial\rho}{\partial\tau} \\
&\overset{(109)}{=} -\int_\Omega p\,\nabla\cdot(\hat\rho(-\nabla p)) \\
&\overset{(110)}{=} -\int_\Omega \hat\rho\,|\nabla p|^2.
\end{aligned}
$$

The rest of the argument follows the few corresponding lines in subsection 3.5.

5.5 Proof of the Theorem, part I

Deriving Theorem 1 from Proposition 1 is an uninspiring exercise in approximation arguments. For convenience, we set again

$$
\begin{aligned}
e(z) &= \left\{ \begin{array}{cc} \frac{1}{m-1}\,z^m & \text{for } m \neq 1 \\ z\,\ln z & \text{for } m = 1 \end{array} \right\}, \\
\pi(z) &= z^m, \\
\psi(z) &= \tfrac{1}{m+1}\,z^{m+1}
\end{aligned}
$$

and observe that these functions are related by

$$
z\,e''(z) \;=\; \pi'(z) \quad \text{and} \quad \psi'(z) \;=\; \pi(z).
$$

We divide the statement of the theorem into two parts. The first part is to show that for a. e. τ_0 and a. e. τ with $\tau > \tau_0$ we have

$$
\begin{aligned}
&\exp(2\,\alpha\,\tau)\int \frac{1}{\hat\rho(\tau)}\,|\nabla\pi(\hat\rho(\tau)) + \alpha\,\hat\rho(\tau)\,y|^2 \\
&\quad\le\; \exp(2\,\alpha\,\tau_0)\int \frac{1}{\hat\rho_0}\,|\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\,y|^2, \qquad (138)\\
&\exp(2\,\alpha\,\tau)\left\{\left(\int e(\hat\rho(\tau)) + \alpha\int\hat\rho(\tau)\,\frac{1}{2}|y|^2\right) - \left(\int e(\hat\rho_*) + \alpha\int\hat\rho_*\,\frac{1}{2}|y|^2\right)\right\} \\
&\quad\le\; \exp(2\,\alpha\,\tau_0)\left\{\left(\int e(\hat\rho_0) + \alpha\int\hat\rho_0\,\frac{1}{2}|y|^2\right) - \left(\int e(\hat\rho_*) + \alpha\int\hat\rho_*\,\frac{1}{2}|y|^2\right)\right\} \\
&\qquad\quad \text{for } m > 1, \qquad (139)\\
&\exp(2\,\alpha\,\tau)\int \{e(\hat\rho(\tau)) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho(\tau) - \hat\rho_*)\}
\end{aligned}
$$

$$\begin{aligned} &\leq \ \exp(2\,\alpha\,\tau_0) \int \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\} \\ &\qquad \text{for } m \leq 1, \qquad (140) \end{aligned}$$

$$\exp(2\,\alpha\,\tau)\, d(\hat\rho(\tau), \hat\rho_*)^2 \ \leq \ \exp(2\,\alpha\,\tau_0)\, d(\hat\rho_0, \hat\rho_*)^2, \qquad (141)$$

where $\hat\rho_0 = \hat\rho(\tau_0)$. We start by observing that for a. e. $\tau_0 = \ln(t_0)$ we have

$$\int \hat\rho_0 \ = \ 1 \quad \text{and} \quad \int \psi(\rho_0) \ < \ \infty,$$
$$\pi(\hat\rho_0) \ \in \ L^1_{loc}(I\!R^N) \quad \text{and} \quad \nabla\pi(\hat\rho_0) \ \in \ L^2(I\!R^N),$$

where $\rho_0 = \rho(t_0)$. We fix such a $\tau_0 = \ln(t_0)$. W. l. o. g. we may assume that the r. h. s. of (138), (139), (140) and (141) are finite, that is

$$\begin{aligned} \int \frac{1}{\hat\rho_0}\, |\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\, y|^2 \ &< \ \infty, \\ \int e(\hat\rho_0) \ &< \ \infty \quad \text{for } m > 1, \\ \int \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\} \ &< \ \infty \quad \text{for } m \leq 1, \\ \int \hat\rho_0\, \frac{1}{2}|y|^2 \ &< \ \infty, \end{aligned}$$

the latter being a consequence of the assumption that $d(\hat\rho_*, \hat\rho_0)^2 < \infty$ and the fact that $\int \hat\rho_*\, \frac{1}{2}|y|^2 < \infty$.

We now approximate $\hat\rho_0$. We will construct functions $\hat\rho_{0,\nu}$ and $R_\nu < \infty$ such that

$$\begin{aligned} &\Omega_\nu \ = \ \{|y| < R_\nu\}, \\ &\hat\rho_{0,\nu} \quad \text{is smooth and positive on } \overline{\Omega}_\nu \end{aligned}$$

with

$$R_\nu \ \stackrel{\nu\uparrow\infty}{\longrightarrow} \ \infty,$$

$$\left\{ \begin{array}{cc} \hat\rho_{0,\nu} & \text{on } \Omega_\nu \\ 0 & \text{else} \end{array} \right\} \ \stackrel{\nu\uparrow\infty}{\longrightarrow} \ \hat\rho_0 \quad \text{in } L^1(I\!R^N), \qquad (142)$$

$$\int_{\Omega_\nu} \frac{1}{\hat\rho_{0,\nu}}\, |\nabla\pi(\hat\rho_{0,\nu}) + \alpha\,\hat\rho_{0,\nu}\, y|^2 \ \stackrel{\nu\uparrow\infty}{\longrightarrow} \ \int \frac{1}{\hat\rho_0}\, |\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\, y|^2, \qquad (143)$$

$$\int_{\Omega_\nu} e(\hat\rho_{0,\nu}) \ \stackrel{\nu\uparrow\infty}{\longrightarrow} \ \int e(\hat\rho_0) \quad \text{for } m > 1, \qquad (144)$$

$$\int_{\Omega_\nu} \{e(\hat\rho_{0,\nu}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\nu} - \hat\rho_*)\}$$
$$\stackrel{\nu\uparrow\infty}{\longrightarrow} \quad \int \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\} \quad \text{for } m \le 1, \tag{145}$$

$$\int_{\Omega_\nu} \hat\rho_{0,\nu}\,\frac{1}{2}|y|^2 \quad \stackrel{\nu\uparrow\infty}{\longrightarrow} \quad \int \hat\rho_0\,\frac{1}{2}|y|^2, \tag{146}$$

and such that $\rho_{0,\nu}$ and $\check\Omega_\nu$ (defined by $\rho_{0,\nu}(t,x) = \frac{1}{t_0^{N\alpha}}\,\hat\rho_{0,\nu}(\frac{x}{t_0^\alpha})$ and $\check\Omega_\nu = t_0^{N\alpha}\Omega_\nu$) satisfy

$$\left\{\begin{array}{cc} \rho_{0,\nu} & \text{on } \check\Omega_\nu \\ 0 & \text{else} \end{array}\right\} \quad \stackrel{\nu\uparrow\infty}{\longrightarrow} \quad \rho_0 \quad \text{in } L^1(I\!R^N), \tag{147}$$

$$\int_{\check\Omega_\nu} \psi(\rho_{0,\nu}) \quad \stackrel{\nu\uparrow\infty}{\longrightarrow} \quad \int \psi(\rho_0). \tag{148}$$

We construct $\hat\rho_{0,\nu}$ in three steps. The first step: For $R < \infty$ we set

$$\Omega_R \;=\; \{|y| < R\}, \quad \check\Omega_R \;=\; \{|x| < t_0^{N\alpha}\,R\}.$$

and observe that

$$\left\{\begin{array}{cc} \hat\rho_0 & \text{on } \Omega_R \\ 0 & \text{else} \end{array}\right\} \quad \stackrel{R\uparrow\infty}{\longrightarrow} \quad \hat\rho_0 \quad \text{in } L^1(I\!R^N),$$
$$\left\{\begin{array}{cc} \rho_0 & \text{on } \check\Omega_R \\ 0 & \text{else} \end{array}\right\} \quad \stackrel{R\uparrow\infty}{\longrightarrow} \quad \rho_0 \quad \text{in } L^1(I\!R^N).$$

By monotone convergence,

$$\int_{\Omega_R} \frac{1}{\hat\rho_0}\,|\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\,y|^2 \quad \stackrel{R\uparrow\infty}{\longrightarrow} \quad \int \frac{1}{\hat\rho_0}\,|\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\,y|^2$$
$$\int_{\Omega_R} e(\hat\rho_0) \quad \stackrel{R\uparrow\infty}{\longrightarrow} \quad \int e(\hat\rho_0) \quad \text{for } m > 1,$$
$$\int_{\Omega_R} \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\}$$
$$\stackrel{R\uparrow\infty}{\longrightarrow} \quad \int \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\} \quad \text{for } m \le 1,$$

$$\begin{aligned}\int_{\Omega_R} \hat\rho_0 \, \frac{1}{2}|y|^2 \quad &\stackrel{R\uparrow\infty}{\longrightarrow} \quad \int \hat\rho_0 \, \frac{1}{2}|y|^2, \\ \int_{\check\Omega_R} \psi(\rho_0) \quad &\stackrel{R\uparrow\infty}{\longrightarrow} \quad \int \psi(\rho_0).\end{aligned}$$

The second step: We fix $R < \infty$ an set for $\delta > 0$

$$\begin{aligned}\hat\rho_{0,\delta} \quad &= \quad \min\{\max\{\hat\rho_0, \delta\}, \frac{1}{\delta}\} \quad \text{and hence} \\ \rho_{0,\delta} \quad &= \quad \min\{\max\{\rho_0, \frac{\delta}{t_0^{N\alpha}}\}, \frac{1}{\delta\, t_0^{N\alpha}}\}.\end{aligned}$$

Obviously

$$\begin{aligned}\hat\rho_{0,\delta} \ &\stackrel{\delta\downarrow 0}{\longrightarrow} \ \hat\rho_0 \qquad \text{in } L^1(\Omega_R), \\ \rho_{0,\delta} \ &\stackrel{\delta\downarrow 0}{\longrightarrow} \ \rho_0 \qquad \text{in } L^1(\check\Omega_R).\end{aligned}$$

Since

$$\nabla\pi(\hat\rho_{0,\delta}) \ = \ \left\{ \begin{array}{cc} \nabla\pi(\hat\rho_0) & \text{if } \hat\rho_0 \in (\delta, \frac{1}{\delta}) \\ 0 & \text{else} \end{array} \right\},$$

we have

$$\frac{1}{\hat\rho_{0,\delta}}\, |\nabla\pi(\hat\rho_{0,\delta}) + \alpha\, \hat\rho_{0,\delta}\, y|^2 \ \stackrel{\delta\downarrow 0}{\longrightarrow} \ \frac{1}{\hat\rho_0}\, |\nabla\pi(\hat\rho_0) + \alpha\, \hat\rho_0\, y|^2 \quad \text{a. e. in } \Omega_R.$$

We also have for sufficiently small δ

$$\begin{aligned}\frac{1}{\hat\rho_{0,\delta}}\, |\nabla\pi(\hat\rho_{0,\delta}) + \alpha\, \hat\rho_{0,\delta}\, y|^2 \quad &\leq \quad \frac{2}{\hat\rho_{0,\delta}}\, |\nabla\pi(\hat\rho_{0,\delta})|^2 + 2\,\alpha^2\, \hat\rho_{0,\delta}\, |y|^2 \\ &\leq \quad \frac{2}{\hat\rho_0}\, |\nabla\pi(\hat\rho_0)|^2 + 2\,\alpha^2\, \max\{\hat\rho_0, 1\}\, |y|^2\end{aligned}$$

with

$$\begin{aligned}&\left(\int_{\Omega_R} \frac{1}{\hat\rho_0}\, |\nabla\pi(\hat\rho_0)|^2\right)^{\frac{1}{2}} \\ &\quad \leq \quad \left(\int_{\Omega_R} \frac{1}{\hat\rho_0}\, |\nabla\pi(\hat\rho_0) + \alpha\, \hat\rho_0\, y|^2\right)^{\frac{1}{2}} + \alpha\, \left(\int_{\Omega_R} \hat\rho_0\, |y|^2\right)^{\frac{1}{2}} \ < \ \infty, \\ &\int_{\Omega_R} \max\{\hat\rho_0, 1\}\, |y|^2 \ < \ \infty.\end{aligned}$$

Therefore by dominated convergence

$$\int_{\Omega_R} \frac{1}{\hat\rho_{0,\delta}}\,|\nabla\pi(\hat\rho_{0,\delta}) + \alpha\,\hat\rho_{0,\delta}\,y|^2 \;\stackrel{\delta\downarrow 0}{\longrightarrow}\; \int_{\Omega_R} \frac{1}{\hat\rho_0}\,|\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\,y|^2.$$

In case of $m > 1$ we have for $\delta \ll 1$

$$0 \;\le\; e(\hat\rho_{0,\delta}) \;\le\; \max\{e(\hat\rho_0), 1\}.$$

Since $\max\{e(\hat\rho_0), 1\}$ is integrable on Ω_R, we obtain by dominated convergence

$$\int_{\Omega_R} e(\hat\rho_{0,\delta}) \stackrel{\delta\downarrow 0}{\longrightarrow} \int_{\Omega_R} e(\hat\rho_0).$$

In case of $m \le 1$, we observe that for

$$z_\delta \;=\; \min\{\max\{z,\delta\}, \frac{1}{\delta}\} \quad\text{and}\quad \delta \;\le\; z_* \;\le\; \frac{1}{\delta}$$

we have

$$0 \;\le\; e(z_\delta) - e(z_*) - e'(z_*)\,(z_\delta - z_*) \;\le\; e(z) - e(z_*) - e'(z_*)\,(z - z_*).$$

Since

$$\delta \;\le\; \hat\rho_* \;\le\; \frac{1}{\delta} \quad \text{on } \Omega_R$$

for $\delta \ll 1$, we have

$$\begin{aligned} &\int_{\Omega_R} \{e(\hat\rho_{0,\delta}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\delta} - \hat\rho_*)\} \\ &\quad\le\; \int_{\Omega_R} \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\}\,. \end{aligned}$$

On the other hand, we have by Fatou's lemma

$$\begin{aligned} &\int_{\Omega_R} \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\} \\ &\quad\le\; \liminf_{\delta\downarrow 0} \int_{\Omega_R} \{e(\hat\rho_{0,\delta}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\delta} - \hat\rho_*)\}\,, \end{aligned}$$

so that we obtain

$$\begin{aligned} &\int_{\Omega_R} \{e(\hat\rho_{0,\delta}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\delta} - \hat\rho_*)\} \\ &\quad\stackrel{\delta\downarrow 0}{\longrightarrow}\; \int_{\Omega_R} \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\}\,. \end{aligned}$$

Finally, we have for $\delta \ll 1$

$$0 \;\leq\; \psi(\rho_{0,\delta}) \;\leq\; \max\{\psi(\rho_0), 1\}.$$

Since $\max\{\psi(\rho_0), 1\}$ is integrable on $\check{\Omega}_R$, we obtain by dominated convergence

$$\int_{\check{\Omega}_R} \psi(\rho_{0,\delta}) \stackrel{\delta\downarrow 0}{\longrightarrow} \int_{\check{\Omega}_R} \psi(\rho_0).$$

The third step: Fix $R < \infty$ and $\delta > 0$. Since

$$\begin{aligned}
\hat{\rho}_{0,\delta} \;&\in\; [\delta, \frac{1}{\delta}] \qquad \text{on } \Omega_R \quad \text{and} \\
\int_{\Omega_R} |\nabla \pi(\hat{\rho}_{0,\delta})|^2 \;&\leq\; \frac{1}{\delta} \int_{\Omega_R} \frac{1}{\hat{\rho}_{0,\delta}} |\nabla \pi(\hat{\rho}_{0,\delta})|^2 \\
&\leq\; \frac{2}{\delta} \int_{\Omega_R} \frac{1}{\hat{\rho}_{0,\delta}} |\nabla \pi(\hat{\rho}_{0,\delta}) + \alpha\, \hat{\rho}_{0,\delta}\, y|^2 + \frac{2}{\delta}\, \alpha^2 \int_{\Omega_R} \hat{\rho}_{0,\delta}\, |y|^2 \\
&<\; \infty,
\end{aligned}$$

there exists for $\epsilon > 0$ a function $\hat{\rho}_{0,\delta,\epsilon}$ with

$$\hat{\rho}_{0,\delta,\epsilon} \quad \text{is smooth and } [\delta, \tfrac{1}{\delta}]\text{–valued on } \Omega_R$$

and

$$\begin{aligned}
\hat{\rho}_{0,\delta,\epsilon} \;&\stackrel{\epsilon\downarrow 0}{\longrightarrow}\; \hat{\rho}_{0,\delta} \quad \text{in } L^1(\Omega_R), \\
\nabla \pi(\hat{\rho}_{0,\delta,\epsilon}) \;&\stackrel{\epsilon\downarrow 0}{\longrightarrow}\; \nabla \pi(\hat{\rho}_{0,\delta}) \quad \text{in } L^2(\Omega_R).
\end{aligned}$$

This obviously implies

$$\begin{aligned}
\int_{\Omega_R} \frac{1}{\hat{\rho}_{0,\delta,\epsilon}} |\nabla \pi(\hat{\rho}_{0,\delta,\epsilon}) + \alpha\, \hat{\rho}_{0,\delta,\epsilon}\, y|^2 \;&\stackrel{\epsilon\downarrow 0}{\longrightarrow}\; \int_{\Omega_R} \frac{1}{\hat{\rho}_{0,\delta}} |\nabla \pi(\hat{\rho}_{0,\delta}) + \alpha\, \hat{\rho}_{0,\delta}\, y|^2 \\
\int_{\Omega_R} e(\hat{\rho}_{0,\delta,\epsilon}) \;&\stackrel{\epsilon\downarrow 0}{\longrightarrow}\; \int_{\Omega_R} e(\hat{\rho}_{0,\delta}) \quad \text{for } m > 1,
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_R} \{e(\hat{\rho}_{0,\delta,\epsilon}) - e(\hat{\rho}_*) - e'(\hat{\rho}_*)\,(\hat{\rho}_{0,\delta,\epsilon} - \hat{\rho}_*)\} \\
&\quad \stackrel{\epsilon\downarrow 0}{\longrightarrow} \int_{\Omega_R} \{e(\hat{\rho}_{0,\delta}) - e(\hat{\rho}_*) - e'(\hat{\rho}_*)\,(\hat{\rho}_{0,\delta} - \hat{\rho}_*)\} \quad \text{for } m \leq 1,
\end{aligned}$$

$$\int_{\Omega_R} \hat\rho_{0,\delta,\epsilon}\,\frac{1}{2}|y|^2 \quad \stackrel{\epsilon\downarrow 0}{\longrightarrow} \quad \int_{\Omega_R} \hat\rho_{0,\delta}\,\frac{1}{2}|y|^2,$$

and

$$\begin{aligned} \rho_{0,\delta,\epsilon} \quad &\stackrel{\epsilon\downarrow 0}{\longrightarrow} \quad \rho_0 \quad \text{in } L^1(\check\Omega_R), \\ \int_{\check\Omega_R} \psi(\rho_{0,\delta,\epsilon}) \quad &\stackrel{\epsilon\downarrow 0}{\longrightarrow} \quad \int_{\check\Omega_R} \psi(\rho_{0,\delta}). \end{aligned}$$

Collecting all these results, we obtain

$$\begin{aligned} \lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \left\{ \begin{array}{cc} \hat\rho_{0,\delta,\epsilon} & \text{on } \Omega_R \\ 0 & \text{else} \end{array} \right\} \quad &= \quad \hat\rho_0 \quad \text{in } L^1(I\!\!R^N), \\ \lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \int_{\Omega_R} \frac{1}{\hat\rho_{0,\delta,\epsilon}}\,|\nabla\pi(\hat\rho_{0,\delta,\epsilon}) + \alpha\,\hat\rho_{0,\delta,\epsilon}\,y|^2 \quad &= \quad \int \frac{1}{\hat\rho_0}\,|\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\,y|^2 \\ \lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \int_{\Omega_R} e(\hat\rho_{0,\delta,\epsilon}) \quad &= \quad \int e(\hat\rho_0) \quad \text{for } m > 1, \end{aligned}$$

$$\begin{aligned} &\lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \int \{e(\hat\rho_{0,\delta,\epsilon}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\delta,\epsilon} - \hat\rho_*)\} \\ &= \quad \int \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0 - \hat\rho_*)\} \quad \text{for } m \le 1, \end{aligned}$$

$$\lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \int_{\Omega_R} \hat\rho_{0,\delta,\epsilon}\,\frac{1}{2}|y|^2 \quad = \quad \int \hat\rho_0\,\frac{1}{2}|y|^2,$$

and

$$\begin{aligned} \lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \left\{ \begin{array}{cc} \rho_{0,\delta,\epsilon} & \text{on } \check\Omega_R \\ 0 & \text{else} \end{array} \right\} \quad &= \quad \rho_0 \quad \text{in } L^1(I\!\!R^N), \\ \lim_{R\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \int_{\check\Omega_R} \psi(\rho_{0,\delta,\epsilon}) \quad &= \quad \int \psi(\rho_0). \end{aligned}$$

Choosing appropriate sequences $R_\nu \stackrel{\nu\uparrow\infty}{\to} \infty$, $\delta_\nu \stackrel{\nu\uparrow\infty}{\to} 0$ and $\epsilon_\nu \stackrel{\nu\uparrow\infty}{\to} 0$ gives the desired $\Omega_\nu = \Omega_{R_\nu}$ and $\hat\rho_{0,\nu} = \hat\rho_{0,\delta_\nu,\epsilon_\nu}$.

We have to approximate π in the degenerate case, that is, for $m > 1$. We start by constructing a smooth π_1 with

$$\left. \begin{array}{rcl} \pi_1(z) & > & 0 \\ z\,\pi_1'(z) - (1 - \frac{1}{N})\,\pi_1(z) & > & 0 \end{array} \right\} \quad \text{for all } z > 0, \tag{149}$$

and

$$\begin{array}{rcll} \pi_1' & = & const & \text{on } [0, \frac{1}{2}], \\ \pi_1' & = & \pi' & \text{on } [2, \infty), \\ \pi_1' & \geq & \pi' & \text{on } [0, \infty). \end{array}$$

Indeed, consider

$$\pi_1(z) \;=\; \left\{ \begin{array}{cl} m\,z & \text{for } z \leq 1 \\ z^m + (m-1) & \text{for } z \geq 1 \end{array} \right\},$$

which is differentiable with

$$\pi_1'(z) \;=\; \left\{ \begin{array}{cl} m & \text{for } z \leq 1 \\ m\,z & \text{for } z \geq 1 \end{array} \right\}.$$

It is easy to check that this π_1 satisfies (149) and

$$\begin{array}{rcll} \pi_1' & = & const & \text{on } [0, 1], \\ \pi_1' & = & \pi' & \text{on } [1, \infty), \\ \pi_1' & \geq & \pi' & \text{on } [0, \infty). \end{array}$$

It remains to smoothen π_1 in a neighborhood of $z = 1$. We now set

$$\pi_\delta(z) \;=\; \delta^m\,\pi_1(\frac{z}{\delta}).$$

Then we have by rescaling

$$\left. \begin{array}{rcl} \pi_\delta(z) & > & 0 \\ z\,\pi_\delta'(z) - (1 - \frac{1}{N})\,\pi_\delta(z) & > & 0 \end{array} \right\} \quad \text{for all } z > 0, \tag{150}$$

and

$$\left. \begin{array}{rcll} \pi_\delta' & = & const & \text{on } [0, \frac{\delta}{2}], \\ \pi_\delta' & = & \pi' & \text{on } [2\,\delta, \infty), \\ \pi_\delta' & \geq & \pi' & \text{on } [0, \infty), \end{array} \right\} \tag{151}$$

while

$$\pi_\delta \;\stackrel{\delta\downarrow 0}{\longrightarrow}\; \pi \quad \text{uniformly in } [0, \infty). \tag{152}$$

We define e_δ by

$$\pi_\delta(z) \;=\; z\,e_\delta'(z) - e_\delta(z) \;=\; z^2\,\frac{d}{dz}[\frac{1}{z}\,e_\delta(z)] \quad \text{and} \quad e_\delta'(1) \;=\; e'(1)$$

and observe that this automatically implies

$$z\, e_\delta''(z) \;=\; \pi_\delta'(z), \tag{153}$$

and, thanks to the first line in (151),

$$\lim_{z\downarrow 0} e_\delta'(z) \;=\; -\infty \quad \text{and} \quad \lim_{z\downarrow 0} e_\delta(z) \;=\; 0.$$

Since also $e(0) = 0$, we infer that

$$e_\delta \;\stackrel{\delta\downarrow 0}{\longrightarrow}\; e \quad \text{uniformly in } [0,\infty).$$

Since for fixed $\nu < \infty$, $\hat\rho_{0,\nu}$ is bounded away from zero on the bounded set Ω_ν, we have we may choose a sequence of positive numbers $\delta_\nu \stackrel{\nu\uparrow\infty}{\to} 0$ such that for $(\pi_\nu, e_\nu) = (\pi_{\delta_\nu}, e_{\delta_\nu})$ we have

$$\pi_\nu'(\hat\rho_{0,\nu}) \quad = \quad \pi'(\hat\rho_{0,\nu}) \quad \text{on } \Omega_\nu \text{ for } \nu \gg 1, \tag{154}$$

$$\sup_{z\in(0,\infty)} |e_\nu(z) - e(z)|\,|\Omega_\nu| \quad \stackrel{\delta\downarrow 0}{\longrightarrow} \quad 0. \tag{155}$$

Whenever it is notationally convenient, we write $(\pi_\nu, e_\nu) = (\pi, e)$ in the case of $m \le 1$.

We now consider the solution of

$$\frac{\partial\hat\rho_\nu}{\partial\tau} - \nabla^2\pi_\nu(\hat\rho_\nu) - \alpha\,\nabla\cdot(\hat\rho_\nu\, y) \;=\; 0 \quad \text{in } (\tau_0,\infty)\times\Omega_\nu, \tag{156}$$

$$(\nabla\pi_\nu(\hat\rho_\nu) + \alpha\,\hat\rho_\nu\, y)\cdot n \;=\; 0 \quad \text{on } (\tau_0,\infty)\times\partial\Omega_\nu, \tag{157}$$

$$\hat\rho_\nu \;=\; \hat\rho_{0,\nu} \quad \text{on } \{\tau_0\}\times\Omega_\nu. \tag{158}$$

Since $\hat\rho_{0,\nu}$ is bounded on Ω_ν,

$$\overline{\hat\rho}(\tau,y) \;=\; C\,\exp(\alpha\,\tau\,N)$$

is a supersolution for (156,157,158) for $C \gg 1$. Since $\hat\rho_{0,\nu}$ is bounded away from zero on Ω_ν,

$$e_\nu'(\underline{\hat\rho}(\tau,y)) + \alpha\,\frac{1}{2}|y|^2 \;=\; -C$$

defines a subsolution for (156,157,158) for $C \gg 1$. Since for both $m > 1$ and $m \leq 1$, $\lim_{z \downarrow 0} e_\nu'(z) = -\infty$, this subsolution is bounded away from 0 in $(-\infty,\infty)\times\Omega$. By the comparison principle for (156,157), $\hat\rho_\nu$ is bounded away from 0 and ∞ on bounded subsets of $(-\infty,\infty)\times\Omega$. Since in addition, $\hat\rho_{0,\nu}$ is smooth and positive on $\overline{\Omega}$, also $\hat\rho_\nu$ is smooth and positive on $(-\infty,\infty)\times\overline{\Omega}$ by standard linear parabolic theory. Hence, thanks to the relationship of π_ν and e_ν expressed in (153), $\hat\rho_\nu$ is a smooth and positive solution of

$$\begin{aligned} \frac{\partial\hat\rho_\nu}{\partial\tau} - \nabla\cdot(\hat\rho_\nu\,\nabla p_\nu) &= 0 \quad \text{in } (\tau_0,\infty)\times\Omega_\nu, \\ \hat\rho_\nu\,\nabla p_\nu\cdot n &= 0 \quad \text{on } (\tau_0,\infty)\times\partial\Omega_\nu, \end{aligned}$$

where

$$p_\nu \;=\; e_\nu'(\hat\rho_\nu) + \alpha\,\frac{1}{2}|y|^2.$$

Hence we may apply Proposition 1 and obtain

$$\frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\int_{\Omega_\nu}\hat\rho_\nu(\tau)\,|\nabla p_\nu(\tau)|^2\right] \;\leq\; 0, \tag{159}$$

$$\begin{aligned} \frac{d}{d\tau}\Big[\exp(2\,\alpha\,\tau)\Big(\Big(\int_{\Omega_\nu} e_\nu(\hat\rho_\nu(\tau)) + \alpha\int_{\Omega_\nu}\hat\rho_\nu(\tau)\,\frac{1}{2}|y|^2\Big) \\ -\Big(\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) + \alpha\int_{\Omega_\nu}\hat\rho_{*,\nu}\,\frac{1}{2}|y|^2\Big)\Big)\Big] \;\leq\; 0, \end{aligned} \tag{160}$$

$$\frac{d}{d\tau}\left[\exp(2\,\alpha\,\tau)\,d(\hat\rho_{*,\nu},\hat\rho_\nu(\tau))^2\right] \;\leq\; 0,$$

where $\hat\rho_{*,\nu}$ is defined via

$$e_\nu'(\hat\rho_{*,\nu}(y)) + \alpha\,\frac{1}{2}|y|^2 \;=\; \lambda \quad\text{and}\quad \int_{\Omega_\nu}\hat\rho_{*,\nu} \;=\; \int_{\Omega_\nu}\hat\rho_{0,\nu} \;=\; \int_{\Omega_\nu}\hat\rho_\nu(\tau).$$

We again use the relationship between e_ν and π_ν in (153) to reexpress (159) and the definition of $\hat\rho_{*,\nu}$ to reexpress (160) (at least in case of $m \leq 1$): For all $\tau \in (\tau_0,\infty)$

$$\begin{aligned} &\exp(2\,\alpha\,\tau)\int_{\Omega_\nu}\frac{1}{\hat\rho_\nu(\tau)}\,|\nabla\pi_\nu(\hat\rho_\nu(\tau)) + \alpha\,\hat\rho_\nu(\tau)\,y|^2 \\ &\leq\; \exp(2\,\alpha\,\tau_0)\int_{\Omega_\nu}\frac{1}{\hat\rho_{0,\nu}}\,|\nabla\pi_\nu(\hat\rho_{0,\nu}) + \alpha\,\hat\rho_{0,\nu}\,y|^2, \end{aligned} \tag{161}$$

$$
\begin{aligned}
&\exp(2\,\alpha\,\tau)\Big\{\Big(\int_{\Omega_\nu} e_\nu(\hat\rho_\nu(\tau)) + \alpha\int_{\Omega_\nu}\hat\rho_\nu(\tau)\,\frac{1}{2}|y|^2\Big)\\
&\qquad\qquad -\Big(\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) + \alpha\int_{\Omega_\nu}\hat\rho_{*,\nu}\,\frac{1}{2}|y|^2\Big)\Big\}\\
&\quad\le\ \exp(2\,\alpha\,\tau_0)\Big\{\Big(\int_{\Omega_\nu} e_\nu(\hat\rho_{0,\nu}) + \alpha\int_{\Omega_\nu}\hat\rho_{0,\nu}\,\frac{1}{2}|y|^2\Big)\\
&\qquad\qquad -\Big(\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) + \alpha\int_{\Omega_\nu}\hat\rho_{*,\nu}\,\frac{1}{2}|y|^2\Big)\Big\}\quad \text{for } m>1, \qquad (162)\\
&\exp(2\,\alpha\,\tau)\int_{\Omega_\nu}\{e_\nu(\hat\rho_\nu(\tau)) - e_\nu(\hat\rho_{*,\nu}) - e_\nu'(\hat\rho_{*,\nu})\,(\hat\rho_\nu(\tau)-\hat\rho_{*,\nu})\}\\
&\quad\le\ \exp(2\,\alpha\,\tau_0)\int_{\Omega_\nu}\{e_\nu(\hat\rho_{0,\nu}) - e_\nu(\hat\rho_{*,\nu}) - e_\nu'(\hat\rho_{*,\nu})\,(\hat\rho_{0,\nu}-\hat\rho_{*,\nu})\}\\
&\qquad\quad \text{for } m\le 1, \qquad (163)\\
&\exp(2\,\alpha\,\tau)\,d(\hat\rho_\nu(\tau),\hat\rho_{*,\nu})^2\ \le\ \exp(2\,\alpha\,\tau_0)\,d(\hat\rho_{0,\nu},\hat\rho_{*,\nu})^2. \qquad (164)
\end{aligned}
$$

We will recover (138), (139), (140) and (141) from the above in the limit $\nu\uparrow\infty$.

The first goal is to identify the limit of $\hat\rho_\nu$ with $\hat\rho$. To this purpose, we consider the ρ_ν related to $\hat\rho_\nu$ via the usual transformation

$$
\rho_\nu(t,x)\ =\ \frac{1}{t^{N\alpha}}\,\hat\rho_\nu(\ln t,\frac{x}{t^\alpha}).
$$

We now derive the energy estimate (in the traditional sense). From (156,157) we infer the identity

$$
\frac{d}{d\tau}\int_{\Omega_\nu}\psi(\hat\rho_\nu)\ =\ -\int_{\Omega_\nu}\nabla\psi'(\hat\rho_\nu)\cdot(\nabla\pi_\nu(\hat\rho_\nu)+\alpha\,\hat\rho_\nu\,y)\,.
$$

Since

$$
\begin{array}{ccccc}
\psi''(z)\,\pi_\nu'(z) & \stackrel{(151)}{\ge} & \psi''(z)\,\pi'(z) & = & (\pi'(z))^2,\\
z\,\psi''(z) & = & m\,\psi'(z), & &
\end{array}
$$

we obtain the inequality

$$
\begin{aligned}
&\frac{d}{d\tau}\int_{\Omega_\nu}\psi(\hat\rho_\nu)\\
&\quad\le\ -\int_{\Omega_\nu}\Big\{|\nabla\pi(\hat\rho_\nu)|^2+\alpha\,m\,\nabla\psi(\hat\rho_\nu)\cdot y\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\Omega_\nu} |\nabla \pi(\hat\rho_\nu)|^2 + \alpha\, m\, N \int_{\Omega_\nu} \psi(\hat\rho_\nu) - \alpha\, m \int_{\partial\Omega_\nu} \psi(\hat\rho_\nu)\, y\cdot n \\
&\le -\int_{\Omega_\nu} |\nabla \pi(\hat\rho_\nu)|^2 + \alpha\, m\, N \int_{\Omega_\nu} \psi(\hat\rho_\nu),
\end{aligned}
$$

where we have used in the last line that the integrand of the boundary integral is non negative (remember that Ω_ν is a ball with center 0). We reformulate the inequality as

$$
\frac{d}{d\tau}\left[\exp(-\alpha\, m\, N\,\tau) \int_{\Omega_\nu} \psi(\hat\rho_\nu)\right] + \exp(-\alpha\, m\, N\,\tau) \int_{\Omega_\nu} |\nabla \pi(\hat\rho_\nu)|^2 \;\le\; 0,
$$

which translates into the usual energy estimate

$$
\frac{d}{dt}\left[\int_{\check\Omega_\nu} \psi(\rho_\nu)\right] + \int_{\check\Omega_\nu} |\nabla \pi(\rho_\nu)|^2 \;\le\; 0,
$$

where $\check\Omega_\nu = \check\Omega_\nu(t) = t^{N\alpha}\,\Omega_\nu$. Since $\int_{\check\Omega_\nu} \psi(\rho_{0,\nu})$ is bounded for $\nu \uparrow \infty$ by construction of $\rho_{0,\nu}$ (see (148)), we have

$$
\begin{aligned}
\sup_{t\in(t_0,\infty)} \int_{\check\Omega_\nu} \psi(\rho_\nu(t)) &\qquad \text{is bounded for } \nu\uparrow\infty, \qquad (165)\\
\int_{t_0}^{\infty} \int_{\check\Omega_\nu} |\nabla \pi_\nu(\rho_\nu(t))|^2\, dt &\qquad \text{is bounded for } \nu\uparrow\infty.
\end{aligned}
$$

Of course, (156,158) translates into

$$
\begin{aligned}
\frac{\partial \rho_\nu}{\partial t} - \nabla^2 \pi_\nu(\rho_\nu) &= 0 \quad \text{in } (t_0,\infty)\times\check\Omega_\nu,\\
\rho_\nu &= \rho_{0,\nu} \quad \text{on } \{t_0\}\times\check\Omega_\nu.
\end{aligned}
$$

Together with (147) and (152), we obtain by standard techniques for porous–medium type equations (see for instance [1])

$$
\left\{\begin{array}{cc} \rho_\nu & \text{on } (t_0,\infty)\times\check\Omega_\nu \\ 0 & \text{else} \end{array}\right\} \stackrel{\nu\uparrow\infty}{\longrightarrow} \rho \quad \text{in } L^1_{loc}((t_0,\infty)\times I\!R^N).
$$

In particular we have for a. e. $\tau \in (\tau_0,\infty)$

$$
\left\{\begin{array}{cc} \hat\rho_\nu(\tau) & \text{on } \check\Omega_\nu \\ 0 & \text{else} \end{array}\right\} \stackrel{\nu\uparrow\infty}{\longrightarrow} \hat\rho(\tau) \quad \left\{\begin{array}{c} \text{in } L^1_{loc}(I\!R^N) \text{ and} \\ \text{a. e. in } I\!R^N \text{ for a subsequence} \end{array}\right\}. \qquad (166)
$$

We now pass to the limit in the inequalities (161,162,163, 164). We first investigate the convergence of $\hat\rho_{*,\nu}$ to $\hat\rho_*$. We consider the case of $m \le 1$. According to (142) we have

$$\int_{\Omega_\nu} \hat\rho_{0,\nu} \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int \hat\rho_0 \;=\; 1, \tag{167}$$

so that $\hat\rho_{*,\nu}$ satisfies

$$e'(\hat\rho_{*,\nu}(y)) + \alpha\,\frac{1}{2}|y|^2 \;=\; \lambda \quad \text{for } y\in\Omega_\nu \quad \text{and} \quad \int_{\Omega_\nu} \hat\rho_{*,\nu} \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; 1,$$

whereas $\hat\rho_*$ is characterized by

$$e'(\hat\rho_*(y)) + \alpha\,\frac{1}{2}|y|^2 \;=\; \lambda \quad \text{for } y\in I\!R^N \quad \text{and} \quad \int \hat\rho_* \;=\; 1.$$

It is therefore a matter of elementary analysis to show

$$\left\{\begin{array}{cc} \hat\rho_{*,\nu} & \text{on } \Omega_\nu \\ 0 & \text{else}\end{array}\right\} \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \hat\rho_* \;\left\{\begin{array}{c} \text{in } L^1(I\!R^N) \text{ and} \\ \text{a. e. in } I\!R^N \text{ for a subsequence}\end{array}\right\}, \tag{168}$$

$$\int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int \hat\rho_*\,\frac{1}{2}|y|^2, \tag{169}$$

$$\lim_{\nu\uparrow\infty}\Big\{ \Big(\int_{\Omega_\nu} e(\hat\rho_{*,\nu}) + \alpha\int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2\Big) \tag{170}$$

$$-\Big(\int_{\Omega_\nu} e(\hat\rho_*) + \alpha\int_{\Omega_\nu} \hat\rho_*\,\frac{1}{2}|y|^2\Big)\Big\} \;=\; 0, \tag{171}$$

keeping in mind that $m > \frac{N}{N+2}$ ensures that $\int|e(\hat\rho_*)|, \int\hat\rho_*\,\frac{1}{2}|y|^2 < \infty$.

We now consider the case $m > 1$. We observe that $\hat\rho_{*,\nu}$ and $\hat\rho_*$ are also characterized by

$$\left.\begin{array}{l} \hat\rho_{*,\nu} \text{ minimizes} \\ \quad \displaystyle\int_{\Omega_\nu} e_\nu(\hat\rho) + \alpha\int_{\Omega_\nu} \hat\rho\,\frac{1}{2}|y|^2 \\ \text{among all } \hat\rho \;\ge\; 0 \text{ with } \int_{\Omega_\nu}\hat\rho \;=\; \int_{\Omega_\nu}\hat\rho_{0,\nu} \end{array}\right\} \tag{172}$$

resp.

$$\left.\begin{array}{l} \hat\rho_* \text{ minimizes} \\ \quad \displaystyle\int e(\hat\rho) + \alpha\int \hat\rho\,\frac{1}{2}|y|^2 \\ \text{among all } \hat\rho \;\ge\; 0 \text{ with } \int\hat\rho \;=\; 1 \end{array}\right\}. \tag{173}$$

We will employ a variational argument to conclude

$$\left\{ \begin{array}{cc} \hat\rho_{*,\nu} & \text{on } \Omega_\nu \\ 0 & \text{else} \end{array} \right\} \stackrel{\nu\uparrow\infty}{\longrightarrow} \hat\rho_* \quad \text{in } L^1(I\!R^N), \tag{174}$$

$$\int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \stackrel{\nu\uparrow\infty}{\longrightarrow} \int \hat\rho_*\,\frac{1}{2}|y|^2, \tag{175}$$

$$\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) \stackrel{\nu\uparrow\infty}{\longrightarrow} \int e(\hat\rho_*). \tag{176}$$

Indeed, consider

$$\hat\rho_\nu \;=\; a_\nu\,\hat\rho_* \quad \text{with} \quad a_\nu \;=\; \frac{\int_{\Omega_\nu}\hat\rho_{0,\nu}}{\int_{\Omega_\nu}\hat\rho_*},$$

that is, a_ν is chosen such that $\hat\rho_\nu$ is admissible in (172). Hence

$$\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) + \alpha\int_{\Omega_\nu}\hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \;\le\; \int_{\Omega_\nu} e_\nu(\hat\rho_\nu) + \alpha\int_{\Omega_\nu}\hat\rho_\nu\,\frac{1}{2}|y|^2. \tag{177}$$

We would now like to pass to the limit in the r. h. s. of (177). We start by observing that according to (167),

$$a_\nu \stackrel{\nu\uparrow\infty}{\longrightarrow} 1. \tag{178}$$

According to (155), $\lim_{\nu\uparrow\infty}\int_{\Omega_\nu}|e_\nu(\hat\rho_\nu)-e(\hat\rho_\nu)| = 0$; according to (178) and monotone convergence,

$$\int_{\Omega_\nu} e(\hat\rho_\nu) \;=\; a_\nu^m\int_{\Omega_\nu} e(\hat\rho_*) \stackrel{\nu\uparrow\infty}{\longrightarrow} \int e(\hat\rho_*),$$

so that

$$\int_{\Omega_\nu} e_\nu(\hat\rho_\nu) \stackrel{\nu\uparrow\infty}{\longrightarrow} \int e(\hat\rho_*).$$

According to (178) and monotone convergence,

$$\int_{\Omega_\nu}\hat\rho_\nu\,\frac{1}{2}\,|y|^2 \;=\; a_\nu\int_{\Omega_\nu}\hat\rho_*\,\frac{1}{2}\,|y|^2 \stackrel{\nu\uparrow\infty}{\longrightarrow} \int\hat\rho_*\,\frac{1}{2}\,|y|^2.$$

Together we obtain

$$\int_{\Omega_\nu} e_\nu(\hat\rho_\nu) + \alpha\int_{\Omega_\nu}\hat\rho_\nu\,\frac{1}{2}\,|y|^2 \stackrel{\nu\uparrow\infty}{\longrightarrow} \int e(\hat\rho_*) + \alpha\int\hat\rho_*\,\frac{1}{2}\,|y|^2. \tag{179}$$

In particular, the r. h. s. of (177) is bounded, hence is the l. h. s.. This implies due to $e \geq 0$ and (155)

$$\int_{\Omega_\nu} e(\hat\rho_{*,\nu}) \quad \text{and} \quad \int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}\,|y|^2 \quad \text{are bounded for } \nu\uparrow\infty.$$

Since e has superlinear growth, there exists a $\hat\rho$ with

$$\left\{\begin{array}{cc} \hat\rho_{*,\nu} & \text{on } \Omega_\nu \\ 0 & \text{else} \end{array}\right\} \stackrel{\nu\uparrow\infty}{\longrightarrow} \hat\rho \quad \text{weakly in } L^1(I\!R^N), \tag{180}$$

in particular by (167),

$$\int \hat\rho \;=\; \lim_{\nu\uparrow\infty}\int_{\Omega_\nu}\hat\rho_{*,\nu} \;=\; \lim_{\nu\uparrow\infty}\int_{\Omega_\nu}\hat\rho_{0,\nu} \;=\; 1. \tag{181}$$

According to (155), $\lim_{\nu\uparrow\infty}\int_{\Omega_\nu}|e_\nu(\hat\rho_{*,\nu}) - e(\hat\rho_{*,\nu})| = 0$; moreover, $e \geq 0$ is convex, so that (180) is sufficient to ensure

$$\int e(\hat\rho) \;\leq\; \liminf_{\nu\uparrow\infty}\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}). \tag{182}$$

By monotone convergence and (180),

$$\begin{aligned} \int \hat\rho\,\frac{1}{2}|y|^2 \;&=\; \lim_{R\uparrow\infty}\int_{\{|y|\leq R\}}\hat\rho\,\frac{1}{2}|y|^2 \\ &=\; \lim_{R\uparrow\infty}\lim_{\nu\uparrow\infty}\int_{\{|y|\leq R\}}\hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \\ &\leq\; \liminf_{\nu\uparrow\infty}\int_{\Omega_\nu}\hat\rho_{*,\nu}\,\frac{1}{2}|y|^2. \end{aligned} \tag{183}$$

Now by inequality (177) and the convergence expressed in (179), (182), (183),

$$\int e(\hat\rho) + \alpha\int\hat\rho\,\frac{1}{2}|y|^2 \;\leq\; \int e(\hat\rho_*) + \alpha\int\hat\rho_*\,\frac{1}{2}|y|^2.$$

Line (181) ensures that $\hat\rho$ is admissible in (173) and thus also a minimizer. Since e is strictly convex, the minimizers coincide

$$\hat\rho \;=\; \hat\rho_*,$$

so that (182) and (183) turn into

$$\begin{aligned} \int e(\hat\rho_*) &\le \liminf_{\nu\uparrow\infty} \int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}), \\ \int \hat\rho_* \,\frac{1}{2}|y|^2 &\le \liminf_{\nu\uparrow\infty} \int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2. \end{aligned}$$

According to (177) and (179), we also have

$$\int e(\hat\rho_*) + \alpha \int \hat\rho_*\,\frac{1}{2}|y|^2 \;\ge\; \limsup_{\nu\uparrow\infty} \left(\int e_\nu(\hat\rho_{*,\nu}) + \alpha \int \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \right).$$

This establishes (175) and (176). Thanks to the strict convexity of e, (175) and (176) ensure that the weak convergence (180) turns into the desired strong convergence (174).

We now investigate the convergence of the r. h. s. of the inequalities (161), (162), (163) and (164). According to (154), (143) turns into

$$\int_{\Omega_\nu} \frac{1}{\hat\rho_{0,\nu}}\,|\nabla \pi_\nu(\hat\rho_{0,\nu}) + \alpha\,\hat\rho_{0,\nu}\,y|^2 \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int \frac{1}{\hat\rho_0}\,|\nabla\pi(\hat\rho_0) + \alpha\,\hat\rho_0\,y|^2.$$

Let us now consider the case of $m > 1$. According to (155), (144) turns into

$$\int_{\Omega_\nu} e_\nu(\hat\rho_{0,\nu}) \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int e(\hat\rho_0).$$

Together with (146), (175) and (176), we obtain as desired

$$\begin{aligned} &\left\{ \left(\int_{\Omega_\nu} e_\nu(\hat\rho_{0,\nu}) + \alpha \int_{\Omega_\nu} \hat\rho_{0,\nu}\,\frac{1}{2}|y|^2 \right) - \left(\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) + \alpha \int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \right) \right\} \\ &\stackrel{\nu\uparrow\infty}{\longrightarrow} \;\left\{ \left(\int e(\hat\rho_0) + \alpha \int \hat\rho_0\,\frac{1}{2}|y|^2 \right) - \left(\int e(\hat\rho_*) + \alpha \int \hat\rho_*\,\frac{1}{2}|y|^2 \right) \right\}. \end{aligned}$$

We now address the case of $m \le 1$. By definition of $\hat\rho_{*,\nu}$ and $\hat\rho_*$ we have

$$\begin{aligned} &\left(\int_{\Omega_\nu} \{ e(\hat\rho_{0,\nu}) - e(\hat\rho_{*,\nu}) - e'(\hat\rho_{*,\nu})\,(\hat\rho_{0,\nu}) - \hat\rho_{*,\nu}) \} \right) \\ &\qquad - \left(\int_{\Omega_\nu} \{ e(\hat\rho_{0,\nu}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\nu}) - \hat\rho_*) \} \right) \\ &= \qquad \left(\int_{\Omega_\nu} e(\hat\rho_{*,\nu}) + \alpha \int_{\Omega_\nu} \hat\rho_{*,\nu}\,\frac{1}{2}|y|^2 \right) \end{aligned}$$

$$
\begin{aligned}
& -\Big(\int_{\Omega_\nu} e(\hat\rho_*) + \alpha \int_{\Omega_\nu} \hat\rho_* \frac{1}{2}|y|^2\Big) \\
& +\lambda \int_{\Omega_\nu} (\hat\rho_{0,\nu} - \hat\rho_*) \\
\stackrel{(170),(142),\nu\uparrow\infty}{\longrightarrow} \quad & \lambda \int (\hat\rho_0 - \hat\rho_*) \;=\; 0.
\end{aligned}
$$

Hence (145) turns into

$$
\begin{aligned}
& \int_{\Omega_\nu} \{e(\hat\rho_{0,\nu}) - e(\hat\rho_{*,\nu}) - e'(\hat\rho_{*,\nu})\,(\hat\rho_{0,\nu}) - \hat\rho_{*,\nu})\} \\
& \quad \stackrel{\nu\uparrow\infty}{\longrightarrow} \quad \int \{e(\hat\rho_0) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_0) - \hat\rho_*)\}
\end{aligned}
$$

Finally, since $\int_{\Omega_\nu} \hat\rho_{0,\nu} = \int_{\Omega_\nu} \hat\rho_{*,\nu}$ by construction of $\hat\rho_{*,\nu}$, (142), (146) resp. (168), (169) (for $m \le 1$) or (174), (175) (for $m > 1$) imply by Lemma 1

$$
d(\hat\rho_{*,\nu}, \hat\rho_{0,\nu})^2 \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; d(\hat\rho_*, \hat\rho_0)^2.
$$

We finally address the convergence of the l. h. s. of the inequalities (161), (162), (163) and (164). This will be an exercise in lower semi–continuity arguments. Let us start by showing that for a. e. $\tau \in (\tau_0, \infty)$,

$$
\int \frac{1}{\hat\rho(\tau)}\,|\nabla\pi(\hat\rho(\tau)) + \alpha\,\hat\rho(\tau)\,y|^2 \;\le\; \liminf_{\nu\uparrow\infty} \int_{\Omega_\nu} \frac{1}{\hat\rho_\nu(\tau)}\,|\nabla\pi_\nu(\hat\rho_\nu(\tau)) + \alpha\,\hat\rho_\nu(\tau)\,y|^2. \tag{184}
$$

Because of the second statement in (166) and (152), we have pointwise almost everywhere convergence of $\pi_\nu(\hat\rho_\nu(\tau))$ to $\pi(\hat\rho(\tau))$ for a subsequence. Since ψ dominates π_ν uniformly in $\nu \uparrow \infty$, and the latter quantity is controlled according to (165), this pointwise almost everywhere convergence improves to

$$
\left\{ \begin{array}{cc} \pi(\hat\rho_\nu(\tau)) & \text{on } \Omega_\nu \\ 0 & \text{else} \end{array} \right\} \stackrel{\nu\uparrow\infty}{\longrightarrow} \pi(\hat\rho(\tau)) \quad \text{in } L^1_{loc}(I\!\!R^N) \tag{185}
$$

for a. e. $\tau \in (\tau_0, \infty)$. We set for convenience

$$
f_\nu(\tau) \;=\; \nabla\pi_\nu(\hat\rho_\nu(\tau)) + \alpha\,\hat\rho_\nu(\tau)\,y \quad \text{and} \quad f(\tau) \;=\; \nabla\pi(\hat\rho(\tau)) + \alpha\,\hat\rho(\tau)\,y.
$$

We restrict ourselves to one of the almost every τ with $f(\tau) \in L^1_{loc}(I\!\!R^N)$ (keep in mind that our notion of weak solution presumes $\nabla\pi(\hat\rho) \in L^1_{loc}((-\infty,\infty) \times$

$I\!R^N$)). We observe that the first statement in (166) and (185) imply

$$\int_{\Omega_\nu} \hat\rho_\nu(\tau)\,\zeta \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int \hat\rho(\tau)\,\zeta \quad \text{for all } \zeta\in C_0^\infty(I\!R^N),$$
$$\int_{\Omega_\nu} f_\nu(\tau)\cdot\xi \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int f(\tau)\cdot\xi \quad \text{for all } \xi\in C_0^\infty(I\!R^N).$$

It is obvious that this implies (184) in form of

$$\int \frac{1}{\hat\rho(\tau)}\,|f(\tau)|^2 \;\le\; \liminf_{\nu\uparrow\infty}\int_{\Omega_\nu}\frac{1}{\hat\rho_\nu(\tau)}\,|f_\nu(\tau)|^2,$$

once we established the identity

$$\int \frac{1}{\rho}\frac{1}{2}|f|^2 \;=\; \sup_{\xi\in C_0^\infty(\boldsymbol{R}^N)}\left\{\int f\cdot\xi - \int \rho\,\frac{1}{2}|\xi|^2\right\}, \tag{186}$$

which we claim is true for all non negative $\rho\in L^1_{loc}(I\!R^N)$ and vector valued $f\in L^1_{loc}(I\!R^N)$ with the understanding that

$$\frac{1}{\rho}\frac{1}{2}|f|^2 \;=\; \left\{\begin{array}{cl} 0 & \text{if } \rho=0 \text{ and } f=0\\ +\infty & \text{if } \rho=0 \text{ and } f\neq 0\end{array}\right\}.$$

Let us now prove (186). The $\ge$–part is an immediate consequence of the Cauchy–Schwarz inequality:

$$\begin{aligned}\int f\cdot\xi - \int\rho\,\frac{1}{2}|\xi|^2 \;&\le\; \left(\int\rho\,|\xi|^2\int\frac{1}{\rho}\,|f|^2\right)^{\frac{1}{2}} - \int\rho\,\frac{1}{2}|\xi|^2\\ &\le\; \int\rho\,\frac{1}{2}|\xi|^2 + \int\frac{1}{\rho}\frac{1}{2}|f|^2 - \int\rho\,\frac{1}{2}|\xi|^2\\ &=\; \int\frac{1}{\rho}\frac{1}{2}|f|^2.\end{aligned}$$

The $\le$–part can be seen as follows: We would like to set $\xi=\frac{1}{\rho}f$ and hence need an approximation argument. For $R<\infty$, we consider

$$\rho_R(y) \;=\; \left\{\begin{array}{cc}\rho(y)+\frac{1}{R} & \text{if } |y|\le R\\ 0 & \text{else}\end{array}\right\},$$
$$f_R(y) \;=\; \left\{\begin{array}{cc} f(y) & \text{if } |f(y)|\le R \text{ and } |y|\le R\\ 0 & \text{else}\end{array}\right\}.$$

By monotone convergence, we have

$$\int \frac{1}{\rho}\,\frac{1}{2}|f|^2 \;=\; \lim_{R\uparrow\infty}\int_{\{|y|\le R\}} \frac{\rho_R-\frac{1}{2}\rho}{\rho_R^2}\,|f_R|^2.$$

Set

$$\xi_R(y) \;=\; \left\{\begin{array}{cc} \frac{1}{\rho_R(y)}\,f_R(y) & \text{if } |y|\le R \\ 0 & \text{else} \end{array}\right\}$$

and let $\xi_{R,\epsilon}\in C_0^\infty(I\!R^N)$ be a mollification of ξ_R. For fixed $R<\infty$, the $\xi_{R,\epsilon}$ stay uniformly bounded, have uniformly bounded support and converge pointwise a. e. to ξ_R for $\epsilon\downarrow 0$. We therefore have by dominated convergence

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\left\{\int f\cdot\xi_{R,\epsilon}-\int\rho\,\frac{1}{2}|\xi_{R,\epsilon}|^2\right\} &= \int f\cdot\xi_R-\int\rho\,\frac{1}{2}|\xi_R|^2 \\
&= \int_{\{|y|\le R\}} f\cdot\frac{f_R}{\rho_R}-\int_{\{|y|\le R\}}\rho\,\frac{1}{2}|\frac{f_R}{\rho_R}|^2 \\
&= \int_{\{|y|\le R\}}\frac{\rho_R-\frac{1}{2}\rho}{\rho_R^2}\,|f_R|^2.
\end{aligned}$$

Putting both approximations together, we see

$$\int\frac{1}{\rho}\,\frac{1}{2}|f|^2 \;=\; \lim_{R\uparrow\infty}\;\lim_{\epsilon\downarrow 0}\left\{\int f\cdot\xi_{R,\epsilon}-\int\rho\,\frac{1}{2}|\xi_{R,\epsilon}|^2\right\},$$

which proves the $\le$–part in (186).

Let us now consider the case of $m>1$. Fix one of the almost all $\tau\in(\tau_0,\infty)$ with

$$\hat\rho_\nu(\tau)\;\stackrel{\nu\uparrow\infty}{\longrightarrow}\;\hat\rho(\tau)\quad\text{a. e. in } I\!R^N. \tag{187}$$

(the second statement in (166). Since $e\ge 0$ is continuous and thanks to (155), we obtain by Fatou's lemma

$$\int e(\hat\rho(\tau))\;\le\;\liminf_{\nu\uparrow\infty}\int_{\Omega_\nu} e_\nu(\hat\rho_\nu(\tau)).$$

We immediately obtain from (187) and Fatou's lemma

$$\int\hat\rho(\tau)\,\frac{1}{2}|y|^2\;\le\;\liminf_{\nu\uparrow\infty}\int_{\Omega_\nu}\hat\rho_\nu(\tau)\,\frac{1}{2}|y|^2.$$

Together with (175) and (176), we obtain

$$\begin{aligned}
&\Big\{ \Big(\int e(\hat\rho(\tau)) + \alpha \int \hat\rho(\tau)\, \frac{1}{2}|y|^2 \Big) \\
&\qquad - \Big(\int e(\hat\rho_*) + \alpha \int \hat\rho_*\, \frac{1}{2}|y|^2 \Big) \Big\} \\
&\le \liminf_{\nu\uparrow\infty} \Big\{ \Big(\int_{\Omega_\nu} e_\nu(\hat\rho_\nu(\tau)) + \alpha \int_{\Omega_\nu} \hat\rho_\nu(\tau)\, \frac{1}{2}|y|^2 \Big) \\
&\qquad - \Big(\int_{\Omega_\nu} e_\nu(\hat\rho_{*,\nu}) + \alpha \int_{\Omega_\nu} \hat\rho_{*,\nu}\, \frac{1}{2}|y|^2 \Big) \Big\}.
\end{aligned}$$

Now for the case of $m \le 1$. Again, we fix one of the almost all $\tau \in (\tau_0, \infty)$ with

$$\hat\rho_\nu(\tau) \stackrel{\nu\uparrow\infty}{\longrightarrow} \hat\rho(\tau) \quad \text{a. e. in } I\!R^N.$$

Since we also have by (168)

$$\hat\rho_{\nu,*} \stackrel{\nu\uparrow\infty}{\longrightarrow} \hat\rho_* \; > \; 0 \quad \text{a. e. in } I\!R^N,$$

we obtain from the continuity of e on $[0, \infty)$ and the continuity of e' on $(0, \infty)$ that

$$\begin{aligned}
&e(\hat\rho_\nu(\tau)) - e(\hat\rho_{\nu,*}) - e'(\hat\rho_{\nu,*})\,(\hat\rho_\nu(\tau) - \hat\rho_{\nu,*}) \\
&\stackrel{\nu\uparrow\infty}{\longrightarrow} \; e(\hat\rho(\tau)) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho(\tau) - \hat\rho_*) \quad \text{a. e. in } I\!R^N.
\end{aligned}$$

Since $e(z) - e(z_*) - e'(z_*)\,(z - z_*) \ge 0$ by the convexity of e, we get by Fatou's lemma

$$\begin{aligned}
&\int \{ e(\hat\rho(\tau)) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho(\tau) - \hat\rho_*) \} \\
&\le \liminf_{\nu\uparrow\infty} \int \{ e(\hat\rho_\nu(\tau)) - e(\hat\rho_{\nu,*}) - e'(\hat\rho_{\nu,*})\,(\hat\rho_\nu(\tau) - \hat\rho_{\nu,*}) \}\,.
\end{aligned}$$

Finally, we observe that (166) implies that for a. e. $\tau \in (\tau_0, \infty)$

$$\int_{\Omega_\nu} \hat\rho_\nu(\tau)\,\zeta \stackrel{\nu\uparrow\infty}{\longrightarrow} \int \hat\rho(\tau)\,\zeta \quad \text{for all } \zeta \in C_0^\infty(I\!R^N).$$

We recall that the evolution (156), (157) preserves mass:

$$\int_{\Omega_\nu} \hat\rho_\nu(\tau) \; = \; \int_{\Omega_\nu} \hat\rho_{0,\nu} \; = \; \int_{\Omega_\nu} \hat\rho_{*,\nu},$$

Together with (174) for $m > 1$ resp. (168) for $m \leq 1$, this implies by Lemma 1 that

$$d(\hat\rho(\tau), \hat\rho_*)^2 \;\leq\; \liminf_{\nu\uparrow\infty} d(\hat\rho_\nu(\tau), \hat\rho_{*,\nu})^2.$$

This achieves the passage to the limit $\nu \uparrow \infty$ in the l. h. s. of the inequalities (161,162,163,164) and thereby the proof of (138), (139), (140) and (141).

5.6 Proof of the Theorem, part II

The second part of the proof of the theorem is to show that for a. e. $\tau \in (0, \infty)$,

$$\exp(2\,\alpha\,\tau) \int \frac{1}{\hat\rho(\tau)}\,|\nabla\pi(\hat\rho(\tau)) + \alpha\,\hat\rho(\tau)\,y|^2 \;\;\leq\;\; \alpha^2 \int \rho_0\,|x|^2, \qquad (188)$$

$$\begin{aligned} \exp(2\,\alpha\,\tau) &\left\{\left(\int e(\hat\rho(\tau)) + \alpha \int \hat\rho(\tau)\,\frac{1}{2}|y|^2\right) - \left(\int e(\hat\rho_*) + \alpha \int \hat\rho_*\,\frac{1}{2}|y|^2\right)\right\} \\ &\leq\; \alpha \int \rho_0\,\frac{1}{2}|x|^2 \quad \text{for } m > 1, \qquad (189) \end{aligned}$$

$$\begin{aligned} \exp(2\,\alpha\,\tau) &\int \{e(\hat\rho(\tau)) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho(\tau) - \hat\rho_*)\} \\ &\leq\; \alpha \int \rho_0\,\frac{1}{2}|x|^2 \quad \text{for } m \leq 1, \qquad (190) \end{aligned}$$

$$\exp(2\,\alpha\,\tau)\,d(\hat\rho(\tau), \hat\rho_*)^2 \;\;\leq\;\; \int \rho_0\,|x|^2. \qquad (191)$$

To this purpose, we approximate the initial data ρ by ρ_ν's with the additional regularity expressed by

$$\begin{aligned} \int \frac{1}{\rho_{0,\nu}}\,|\nabla\pi(\rho_{0,\nu})|^2 &\;<\; \infty, \\ \int |e(\rho_{0,\nu})| &\;<\; \infty, \end{aligned}$$

in the following sense

$$\rho_{0,\nu} \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \rho_0 \quad \text{in } L^1(I\!R^N) \quad \text{and} \quad \int \rho_{0,\nu} \;=\; \int \rho_0, \qquad (192)$$

$$\int \psi(\rho_{0,\nu}) \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int \psi(\rho_0) < \infty, \qquad (193)$$

$$\int \rho_{0,\nu}\,\frac{1}{2}|x|^2 \;\stackrel{\nu\uparrow\infty}{\longrightarrow}\; \int \rho_0\,\frac{1}{2}|x|^2 < \infty.$$

This can be done in two steps: The first step is to mollify ρ_0 into a $\rho_{0,\epsilon}$. The second step consists in passing to

$$\rho_{0,\epsilon,\delta} \;=\; a_{\epsilon,\delta}\, \max\{\rho_{0,\epsilon}, \delta\, \rho_*\},$$

where ρ_* denotes a fixed smooth and positive function with $\int \frac{1}{\rho_*}\,|\nabla\pi(\rho_*)|^2 < \infty$, $\int |e(\rho_*)| < \infty$, $\int \rho_*\, \frac{1}{2}|x|^2 < \infty$ and $\int \psi(\rho_*) < \infty$, and $a_{\epsilon,\delta}$ is chosen such that $\int \rho_{0,\epsilon,\delta} = 1$. This time, we leave the details to the reader.

Now for $\delta > 0$, let $\rho_\nu^{(\delta)}$ denote the solution of the porous medium equation (in the sense of Definition 1) for $t \geq \delta$ with $\rho_\nu^{(\delta)}(\delta) = \rho_{0,\nu}$. Let $\hat\rho_\nu^{(\delta)}$ and $\hat\rho_{0,\nu}^{(\delta)}$ be related to $\rho_\nu^{(\delta)}$ and $\rho_{0,\nu}$ via the usual transformation. By the first part of this proof we have for a. e. $\tau \in (\ln\delta, \infty)$

$$\begin{aligned} &\exp(2\,\alpha\,\tau) \int \frac{1}{\hat\rho_\nu^{(\delta)}(\tau)}\,|\nabla\pi(\hat\rho_\nu^{(\delta)}(\tau)) + \alpha\,\hat\rho_\nu^{(\delta)}(\tau)\,y|^2 \\ &\quad\leq\; \exp(2\,\alpha\,\ln\delta) \int \frac{1}{\hat\rho_{0,\nu}^{(\delta)}}\,|\nabla\pi(\hat\rho_{0,\nu}^{(\delta)}) + \alpha\,\hat\rho_{0,\nu}^{(\delta)}\,y|^2, \end{aligned} \tag{194}$$

$$\begin{aligned} &\exp(2\,\alpha\,\tau)\,\left\{\left(\int e(\hat\rho_\nu^{(\delta)}(\tau)) + \alpha \int \hat\rho_\nu^{(\delta)}(\tau)\,\frac{1}{2}|y|^2\right) - F(\hat\rho_*)\right\} \\ &\quad\leq\; \exp(2\,\alpha\,\ln\delta)\,\left\{\left(\int e(\hat\rho_{0,\nu}^{(\delta)}) + \alpha \int \hat\rho_{0,\nu}^{(\delta)}\,\frac{1}{2}|y|^2\right) - F(\hat\rho_*)\right\} \\ &\quad\quad \text{for } m > 1, \end{aligned} \tag{195}$$

$$\begin{aligned} &\exp(2\,\alpha\,\tau) \int \left\{e(\hat\rho_\nu^{(\delta)}(\tau)) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_\nu^{(\delta)}(\tau) - \hat\rho_*)\right\} \\ &\quad\leq\; \exp(2\,\alpha\,\ln\delta) \int \left\{e(\hat\rho_{0,\nu}^{(\delta)}) - e(\hat\rho_*) - e'(\hat\rho_*)\,(\hat\rho_{0,\nu}^{(\delta)} - \hat\rho_*)\right\} \\ &\quad\quad \text{for } m \leq 1, \end{aligned} \tag{196}$$

$$\exp(2\,\alpha\,\tau)\,d(\hat\rho_\nu^{(\delta)}(\tau), \hat\rho_*)^2 \;\;\leq\;\; \exp(2\,\alpha\,\ln\delta)\,d(\hat\rho_{0,\nu}^{(\delta)}, \hat\rho_*)^2, \tag{197}$$

We pass first to the limit $\delta \downarrow 0$ and then to the limit $\nu \uparrow \infty$. We consider the right hand side of (194), (195), (196) and (197) first. Our strategy is to express these terms in terms of $\rho_{0,\nu}$ and then pass to the limits. We obtain

$$\exp(2\,\alpha\,\ln\delta) \int \frac{1}{\hat\rho_{0,\nu}^{(\delta)}}\;|\nabla\pi(\hat\rho_{0,\nu}^{(\delta)}) + \alpha\,\hat\rho_{0,\nu}^{(\delta)}\,y|^2$$

$$
\begin{aligned}
&= \int \frac{1}{\rho_{0,\nu}} \, |\delta \, \nabla \pi(\rho_{0,\nu}) + \alpha \, \rho_{0,\nu} \, x|^2 \\
&\overset{\delta \downarrow 0}{\longrightarrow} \;\; \alpha^2 \int \rho_{0,\nu} \, |x|^2 \quad \text{for fixed } \nu < \infty \\
&\overset{\nu \uparrow \infty}{\longrightarrow} \;\; \alpha^2 \int \rho_0 \, |x|^2.
\end{aligned}
$$

In case of $m > 1$, we have

$$
\begin{aligned}
&\exp(2\,\alpha\,\ln\delta) \left\{ \left(\int e(\hat\rho_{0,\nu}^{(\delta)}) + \alpha \int \hat\rho_{0,\nu}^{(\delta)} \, \frac{1}{2} |y|^2 \right) - F(\hat\rho_*) \right\} \\
&\quad = \;\; \delta \int e(\rho_{0,\nu}) + \alpha \int \rho_{0,\nu} \, \frac{1}{2} \, |x|^2 - \delta^{2\,\alpha} \, F(\hat\rho_*) \\
&\quad \overset{\delta \downarrow 0}{\longrightarrow} \;\; \alpha \int \rho_{0,\nu} \, \frac{1}{2} \, |x|^2 \quad \text{for fixed } \nu < \infty \\
&\quad \overset{\nu \uparrow \infty}{\longrightarrow} \;\; \alpha \int \rho_0 \, \frac{1}{2} \, |x|^2.
\end{aligned}
$$

In case of $m \le 1$, we have by definition of $\hat\rho_*$

$$
\begin{aligned}
&\int \left\{ e(\hat\rho_{0,\nu}^{(\delta)}) - e(\hat\rho_*) - e'(\hat\rho_*) \, (\hat\rho_{0,\nu}^{(\delta)} - \hat\rho_*) \right\} \\
&\quad = \;\; \left(\int e(\hat\rho_{\nu}^{(\delta)}(\tau)) + \alpha \int \hat\rho_{\nu}^{(\delta)}(\tau) \, \frac{1}{2} |y|^2 \right) - F(\hat\rho_*),
\end{aligned}
$$

and then the same argument as in the case $m > 1$ applies. Finally,

$$
\exp(2\,\alpha\,\ln\delta) \, d(\hat\rho_{0,\nu}^{(\delta)}, \hat\rho_*)^2 \;=\; d(\hat\rho_{0,\nu}, \rho_*(\delta))^2,
$$

where $\rho_*(\delta, x) = \frac{1}{\delta^{N\alpha}} \hat\rho_*(\frac{x}{\delta^\alpha})$. Since $\rho_*(\delta)$ converges to the Dirac measure at the origin, μ_0, in the sense of

$$
\begin{aligned}
\int \rho_*(\delta) \, \zeta \;\; &\overset{\delta \downarrow 0}{\longrightarrow} \;\; \int \zeta \, d\mu_0 \quad \text{for all } \zeta \in C_0^\infty(I\!\!R^N), \\
\int \rho_*(\delta) \, \frac{1}{2} |x|^2 \;\; &\overset{\delta \downarrow 0}{\longrightarrow} \;\; \int \frac{1}{2} |x|^2 \, d\mu_0,
\end{aligned}
$$

we obtain by Lemma 1 that

$$
d(\hat\rho_{0,\nu}, \rho_*(\delta))^2 \;\overset{\delta \downarrow 0}{\longrightarrow}\; d(\hat\rho_{0,\nu}, \mu_0)^2.
$$

It follows immediately from Definition 2 that

$$
d(\hat\rho_{0,\nu}, \mu_0)^2 \;=\; \int \rho_{0,\nu} \, |x|^2,
$$

so that also here we obtain

$$\lim_{\nu\uparrow\infty}\lim_{\delta\downarrow 0}\exp(2\,\alpha\,\ln\delta)\,d(\hat\rho_{0,\nu}^{(\delta)}),\hat\rho_*)^2 \;=\; \int\rho_0\,|x|^2.$$

Altogether, we see that the r. h. s. of (194), (195), (196) and (197) converge to the r. h. s. of (188), (189), (190) and (191).

We now consider the l. h. s. of the inequalities (194), (195), (196) and (197) in the limit $\nu\uparrow\infty$ and $\delta\downarrow 0$. From (192) and (193), one can deduce by standard techniques for the porous–medium type equations (see for instance [1]), that

$$\lim_{\substack{\nu\uparrow\infty\\ \delta\downarrow 0}}\rho_\nu^{(\delta)} \;=\; \rho \quad \text{in } L^1((0,\infty)\times I\!R^N),$$

with no restriction on the relation between δ and ν. By the lower semicontinuity arguments from the first part of the proof we see the l. h. s. of (188), (189), (190) and (191) are estimated by the limes superior of the l. h. s. of (194), (195), (196) and (197). This achieves the second part of the proof, that is, the proof of (188), (189), (190) and (191).

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