

Products and Ratios of Characteristic Polynomials of Random Hermitian Matrices

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Abstract

We present new and streamlined proofs of various formulae for products and ratios of characteristic polynomials of random Hermitian matrices that have appeared recently in the literature.

1 Introduction

In random matrix theory, unitary ensembles of $N \times N$ matrices $\{H\}$ play a central role [16]. Such ensembles are described by a measure $d\alpha$ with finite moments $\int_{\mathbb{R}} |x|^k d\alpha(x) < \infty$, $k = 0, 1, 2, \dots$, and the distribution function for the eigenvalues $\{x_i = x_i(H)\}$ of matrices H in the ensembles has the form

$$dP_{\alpha,N}(x) = \frac{1}{Z_N} \Delta(x)^2 d\alpha(x) \quad (1.1)$$

where $d\alpha(x) = \prod_{i=1}^N d\alpha(x_i)$, $\Delta(x) = \prod_{N \geq i > j \geq 1} (x_i - x_j)$ is the Vandermonde determinant for the x_i 's, and $Z_N = \int \cdots \int \Delta(x)^2 d\alpha(x)$ is the normalization constant. The special case $d\alpha(x) = e^{-x^2} dx$ is known as the Gaussian Unitary Ensemble (GUE). For symmetric functions $f(x) = f(x_1, \dots, x_N)$ of the x_i 's,

$$\langle f \rangle_{\alpha} \equiv \frac{1}{Z_N} \int \cdots \int f(x) \Delta(x)^2 d\alpha(x) \quad (1.2)$$

denotes the average of f with respect to $dP_{\alpha,N}$.

Recently there has been considerable interest in the averages of products and ratios of the characteristic polynomials $D_N[\mu, H] = \prod_{i=1}^N (\mu - x_i(H))$ of random matrices with respect to various ensembles. Such averages are used, in particular, in making predictions about the moments of the Riemann-zeta function, see [15, 14, 13] (circular ensembles) and [3] (unitary ensembles). Many other uses are described, for example, in [1], [12] and [11].

By (1.2), for unitary ensembles, such averages have the form

$$\left\langle \frac{\prod_{j=1}^K D_N[\mu_j, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \frac{1}{Z_N} \int \cdots \int \frac{\prod_{j=1}^K \prod_{i=1}^N (\mu_j - x_i)}{\prod_{j=1}^M \prod_{i=1}^N (\epsilon_j - x_i)} \Delta(x)^2 d\alpha(x). \quad (1.3)$$

In this paper we consider certain explicit determinantal formulae for (1.3) – see (2.6), (2.24), (2.36), (3.3), (3.12) below. Formula (2.6) is due to Brezin and Hikami [3] (see also [17], and when all the μ_j 's are equal, see [10]), whereas (2.24), (2.36), (3.3) and (3.12) are due to Fyodorov and Strahov [12, 11]. The papers [12, 11] also contain a discussion of the history of these formulae. The formulae (3.3) and (3.12) are particularly useful in proving universality results for the ratios (1.3) in the Dyson limit as $N \rightarrow \infty$ (see [11]). For a discussion of other universality results, particularly the work of Brezin-Hikami and Fyodorov in special cases, we again refer the reader to [11]. The asymptotic analysis in [11] is based on the reformulation of the orthogonal polynomial problem as a Riemann-Hilbert problem by Fokas, Its and Kitaev [9]. The Riemann-Hilbert problem is then analyzed asymptotically using the non-commutative steepest-descent method introduced by Deift and Zhou [8], and further developed with Venakides in [7] to allow for fully non-linear oscillations, and in [6], [5].

Our goal in this paper is to give new, streamlined proofs of (2.6)-(3.12), using only the properties of orthogonal polynomials and a minimum of combinatorics. Along the way we will also need an integral version of the classical Binet-Cauchy formula due to C.Andréief dating back to 1883 (see Lemma 2.1 below).

Let $\pi_j(z) = x^j + \dots$ denote the j th monic orthogonal polynomial with respect to the measure $d\alpha$,

$$\int_{\mathbb{R}} \pi_j(x)\pi_k(x)d\alpha(x) = c_j c_k \delta_{jk}, \quad j, k \geq 0, \quad (1.4)$$

where the norming constants c_j 's are positive. The key observation in our approach is that for $K = 1$ and $M = 0$ in (1.3)

$$\langle D_N[\mu, H] \rangle_{\alpha} = \pi_N(\mu) \quad (1.5)$$

(see [18]). In our words, the orthogonal polynomial $\pi_N(\mu)$ with respect to $d\alpha$ is also precisely the average polynomial $\prod_{i=1}^N (\mu - x_i)$ with respect to $dP_{\alpha, N}$. Formula (1.5) appears already in the work of Heine in the 1880's (see [18]). Set

$$d\alpha^{[\ell, m]}(t) \equiv \frac{\prod_{j=1}^{\ell} (\mu_j - t)}{\prod_{j=1}^m (\epsilon_j - t)} d\alpha(t), \quad \ell, m \geq 0, \quad (1.6)$$

($d\alpha^{[0, 0]}(t) \equiv d\alpha(t)$), and let $\pi_j^{[\ell, m]}(t)$ denote the j th monic orthogonal polynomial with respect to $d\alpha^{[\ell, m]}$. With this notation we see immediately from (1.3), (1.5) that $\langle \frac{\prod_{j=1}^K D_N[\mu_j, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \rangle_{\alpha}$ is proportional to $\pi_N^{[K-1, M]}(\mu_K)$. Using a classical determinantal formula of Christoffel (see [18]) for $\pi_N^{[\ell, 0]}(\mu)$ and a more recent formula of Uvarov [19] for $\pi_N^{[0, m]}(\mu)$, we are then led (see Section 2. Formulae of Christoffel-Uvarov type) to (2.6), (2.24) and (2.36) in a rather straightforward way. Formula (3.3) appears to have a different character from (2.6), (2.24), (2.36), and relies on Lemma 2.1 mentioned above, which computes the integral of the product of two determinants: formula (3.12) follows (see Section 3. Formulae of two-point function type) by combining (3.3) with (2.6) and (2.36). In [11] the authors present a variety of additional formulae for $\langle \frac{\prod_{j=1}^K D_N[\mu_j, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \rangle_{\alpha}$ for cases of K and M not covered by (2.6)-(3.12): we leave it to the interested reader to verify that the method of this paper can also be used to derive these formulae in a straightforward manner.

Remark 1.1. As is well-known (see e.g., [18]), each measure $d\alpha$ gives rise to a tridiagonal operator

$$J = J(d\alpha) = \begin{pmatrix} a_1 & b_1 & 0 & & \\ b_1 & a_2 & b_2 & & \\ 0 & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_i > 0 \quad (1.7)$$

with generalized eigenfunctions given by the orthonormal polynomials

$$p_j(x) = c_j^{-1} \pi_j(x), \quad j = 0, 1, \dots, \quad (1.8)$$

i.e.

$$b_{j-1} p_{j-1}(x) + a_j p_j(x) + b_j p_{j+1}(x) = x p_j(x), \quad j \geq 1 \quad (1.9)$$

where $b_0 \equiv 0$. Conversely, modulo certain essential self-adjointness issues, $d\alpha$ is the spectral measure for J in the cyclic subspace generated by J and the vector $e_1 = (1, 0, 0, \dots)^T$ (see, e.g., [4]). It follows that the transformation of measures

$$d\alpha \rightarrow d\alpha^{[\ell, m]} \quad (1.10)$$

leads to the transformation of operators

$$J(d\alpha) \rightarrow J(d\alpha^{[\ell, m]}). \quad (1.11)$$

For appropriate choices of μ_1, \dots, μ_m and $\epsilon_1, \dots, \epsilon_\ell$, such transformations corresponding to removing m points from the spectrum of $J(d\alpha)$ and inserting ℓ points: in the spectral theory literature, such transformations are known as Darboux transformations. The formulae in this paper clearly provide formulae for the generalized eigenfunctions $p_j^{[\ell, m]}(x)$ of the Darboux-transformed operator $J(d\alpha^{[\ell, m]})$, as well as the matrix entries, $a_j^{[\ell, m]}$ and $b_j^{[\ell, m]}$, in terms of the corresponding objects for $J(d\alpha)$. Again we leave the details to the reader. Here the elementary formulae

$$b_n^2(d\alpha) = \frac{n+1}{n+2} \frac{Z_n(d\alpha) Z_{n+2}(d\alpha)}{(Z_{n+1}(d\alpha))^2}, \quad a_n(d\alpha) = \frac{d}{dt} \Big|_{t=0} \log \frac{Z_n(d\alpha_t)}{Z_{n+1}(d\alpha_t)} \quad (1.12)$$

where $d\alpha_t(x) = e^{tx} d\alpha(x)$, are useful.

Technical Remark 1.2. Formulae (2.6)-(3.12) clearly do not make sense for all values of the parameters. In ***all the calculations that follow, we will assume that $d\alpha$ has compact support, $\text{support}(d\alpha) = [-Q, Q]$, say, and that the μ_i 's and ϵ_j 's are distinct real numbers greater than Q*** : under these assumptions, $d\alpha^{[\ell, m]}(t)$ becomes, in particular, a bona-fide measure, etc. By analytic continuation one sees that the formulae remain true for complex values of $\{\mu_i\}$ and $\{\epsilon_j\}$, as long as they remain distinct. Furthermore, if the μ_i 's and ϵ_j 's are distinct, and $\text{Im}(\epsilon_j) \neq 0$ for all j , then we can let $Q \rightarrow \infty$ and so the formulae are true for measures $d\alpha$ with unbounded support. Finally we can, for example, let $\mu_j \rightarrow \mu_k$ for some $j \neq k$, which leads to formulae involving derivatives of the π_j 's, etc.

2 Formulae of Christoffel-Uvarov type

We use the notations $d\alpha$, π_j , $d\alpha^{[\ell, m]}$, $\pi_j^{[\ell, m]}$, ... of Section 1. In addition, in all the calculations that follow we assume that $d\alpha$, $\{\mu_j\}$, $\{\epsilon_k\}$ satisfy the conditions described in Technical Remark 1.2 above: the natural analytical continuation of the formulae obtained to complex values of the parameters, and the limit $Q \rightarrow \infty$, is left to the reader.

The following result of Christoffel (see [18]) plays a basic role in what follows.

Lemma 2.1. Consider the measure $d\alpha^{[\ell,0]}(t) = \prod_{j=1}^{\ell} (\mu_j - t) d\alpha(t)$, where $\ell = 1, 2, \dots$. Then the n^{th} monic orthogonal polynomial $\pi_n^{[\ell,0]}(t)$ associated with the new measure $d\alpha^{[\ell,0]}(t)$ can be expressed as follows:

$$\pi_n^{[\ell,0]}(t) = \frac{1}{(t - \mu_1) \dots (t - \mu_\ell)} \frac{\begin{vmatrix} \pi_n(\mu_1) & \dots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \vdots \\ \pi_n(\mu_\ell) & \dots & \pi_{n+\ell}(\mu_\ell) \\ \pi_n(t) & \dots & \pi_{n+\ell}(t) \end{vmatrix}}{\begin{vmatrix} \pi_n(\mu_1) & \dots & \pi_{n+\ell-1}(\mu_1) \\ \vdots & & \vdots \\ \pi_n(\mu_\ell) & \dots & \pi_{n+\ell-1}(\mu_\ell) \end{vmatrix}}. \quad (2.1)$$

Proof. Set

$$q_n^{[\ell,0]}(t) = \begin{vmatrix} \pi_n(\mu_1) & \dots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \vdots \\ \pi_n(\mu_\ell) & \dots & \pi_{n+\ell}(\mu_\ell) \\ \pi_n(t) & \dots & \pi_{n+\ell}(t) \end{vmatrix}. \quad (2.2)$$

We note that $q_n^{[\ell,0]}(t)$ satisfies the condition $\int t^j q_n^{[\ell,0]}(t) d\alpha(t) = 0$ for all $j \in \{0, \dots, n-1\}$. Also $q_n^{[\ell,0]}(\mu_j) = 0$, $j = 1, \dots, \ell$, and so $\frac{q_n^{[\ell,0]}(t)}{(\mu_1 - t) \dots (\mu_\ell - t)}$ is a polynomial of degree at most n . Now observe that

$$\int t^j \left[\frac{q_n^{[\ell,0]}(t)}{(\mu_1 - t) \dots (\mu_\ell - t)} \right] d\alpha^{[\ell,0]}(t) = 0, \quad 0 \leq j < n \quad (2.3)$$

which means that $q_n^{[\ell,0]}(t)$ divided by the product $(\mu_1 - t) \dots (\mu_\ell - t)$ is proportional to the n^{th} monic orthogonal polynomial $\pi_n^{[\ell,0]}(t)$ associated with the new measure $d\alpha^{[\ell,0]}(t)$. Now $q_n^{[\ell,0]}(t)$ cannot vanish for any $t = \mu_{\ell+1} > Q$, $\mu_{\ell+1} \notin \{\mu_1, \dots, \mu_\ell\}$. Indeed, if $q_n^{[\ell,0]}(\mu_{\ell+1}) = 0$, then there exist $\{\alpha_i\}_{i=0}^{\ell}$, not all zero, such that $p(t) \equiv \sum_{i=0}^{\ell} \alpha_i \pi_{n+i}(t)$ vanishes at $\{\mu_i\}_{i=1}^{\ell+1}$. Thus $\tilde{p}(t) \equiv p(t) / \prod_{i=1}^{\ell+1} (\mu_i - t)$ is a polynomial of order $< n$, and as above, $\tilde{p}(t)$ is orthogonal to t^j , $0 \leq j < n$, with respect to the measure $d\alpha^{[\ell+1,0]}(t)$. Thus $\tilde{p}(t) \equiv 0$ and hence $\alpha_0 = \dots = \alpha_\ell = 0$, which is a contradiction. Replacing ℓ by $\ell - 1$, we conclude that

$$\begin{vmatrix} \pi_n(\mu_1) & \dots & \pi_{n+\ell-1}(\mu_1) \\ \vdots & & \vdots \\ \pi_n(\mu_\ell) & \dots & \pi_{n+\ell-1}(\mu_\ell) \end{vmatrix} \neq 0. \quad (2.4)$$

Taking the limit $t \rightarrow \infty$ and noting that the coefficient of the highest degree of $\pi_n^{[\ell,0]}(t)$ should be equal to 1, we find the coefficient of proportionality and establish formula (2.1). \square

Representation (2.1) for the monic orthogonal polynomials associated with the measure $d\alpha^{[\ell,0]}(t)$ immediately leads to the following result:

Corollary 2.2. *The product of monic orthogonal polynomials $\prod_{j=0}^{\ell} \pi_n^{[j,0]}(\mu_{j+1})$ defined with respect to the different measures $d\alpha^{[j,0]}(t) \equiv (\mu_j - t) \cdots (\mu_1 - t)d\alpha(t)$ is given by the formula*

$$\prod_{j=0}^{\ell} \pi_n^{[j,0]}(\mu_{j+1}) = \frac{1}{\Delta(\mu)} \begin{vmatrix} \pi_n(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \vdots \\ \pi_n(\mu_{\ell+1}) & \cdots & \pi_{n+\ell}(\mu_{\ell+1}) \end{vmatrix} \quad (2.5)$$

where $\Delta(\mu) = \prod_{\ell+1 \geq i > j \geq 1} (\mu_i - \mu_j)$.

We observe that Corollary (2.2) gives the identity for the average of products of random characteristic polynomials obtained first by Brezin and Hikami [3].

Theorem 2.3. *Let $D_N[\mu, H]$ be the characteristic polynomial of the Hermitian matrix H . The following identity is valid:*

$$\left\langle \prod_{j=1}^L D_N[\mu_j, H] \right\rangle_{\alpha} = \frac{1}{\Delta(\mu)} \begin{vmatrix} \pi_N(\mu_1) & \cdots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \vdots \\ \pi_N(\mu_L) & \cdots & \pi_{N+L-1}(\mu_L) \end{vmatrix} \quad (2.6)$$

where the average is defined by (1.2).

Proof. To prove formula (2.6) we use the representation for the monic orthogonal polynomials in the case $L = 1$ given in (1.5),

$$\pi_N(\mu) = \frac{1}{Z_N} \int \cdots \int \prod_{i=1}^N (\mu - x_i) \Delta^2(x) d\alpha(x). \quad (2.7)$$

Let $Z_N^{[\ell,0]}$ be defined by

$$Z_N^{[\ell,0]} = \int \cdots \int \Delta^2(x) d\alpha^{[\ell,0]}(x), \quad \ell = 1, 2, \dots \quad (2.8)$$

where $d\alpha^{[\ell,0]}(x) = \prod_{i=1}^N d\alpha^{[\ell,0]}(x_i)$. With this notation, we have

$$\left\langle \prod_{j=1}^L D_N[\mu_j, H] \right\rangle_{\alpha} = \frac{Z_N^{[L,0]}}{Z_N} = \frac{Z_N^{[L,0]}}{Z_N^{[L-1,0]}} \frac{Z_N^{[L-1,0]}}{Z_N^{[L-2,0]}} \cdots \frac{Z_N^{[1,0]}}{Z_N}. \quad (2.9)$$

Equation (2.7) implies that $\pi_n^{[\ell-1,0]}(\mu_{\ell})$ can be represented as the ratio $Z_N^{[\ell,0]}/Z_N^{[\ell-1,0]}$, where $\pi_N^{[0,0]}(\mu) \equiv \pi_N(\mu)$, and $Z_N^{[0,0]} \equiv Z_N$. Thus we obtain

$$\left\langle \prod_{j=1}^L D_N[\mu_j, H] \right\rangle_{\alpha} = \prod_{j=0}^{L-1} \pi_N^{[j,0]}(\mu_{j+1}) \quad (2.10)$$

The above equation together with Corollary (2.2) proves formula (2.6). \square

Remark 2.4. Notice (see equations (2.7) and (2.10)) that the average of products of characteristic polynomials can be rewritten as a product of averages. Namely,

$$\left\langle \prod_{j=1}^L D_N[\mu_j, H] \right\rangle_{\alpha} = \prod_{j=1}^L \langle D_N[\mu_j, H] \rangle_{\alpha^{[j-1,0]}} \quad (2.11)$$

where $\langle \dots \rangle_{\alpha^{[j-1,0]}}$ means the average defined by equation (1.2) but with respect to the new measure $d\alpha^{[j-1,0]}(x)$, and $d\alpha(x) \equiv d\alpha^{[0,0]}(x)$.

The formula of Christoffel (equation (2.1)) enables us to construct the orthogonal polynomials associated with the measure $d\alpha^{[\ell,0]}(t) = \prod_{j=1}^{\ell} (\mu_j - t) d\alpha(t)$ in terms of the orthogonal polynomials associated with the measure $d\alpha(t)$. Now we derive a formula due to Uvarov [19] expressing the monic orthogonal polynomials $\pi_n^{[0,m]}(t)$ associated with the measure $d\alpha^{[0,m]}(t) = \prod_{j=1}^m (\epsilon_j - t)^{-1} d\alpha(t)$, again in terms of the monic orthogonal polynomials $\pi_n(t)$ associated with the measure $d\alpha(t)$.

Lemma 2.5. *Suppose $0 \leq m \leq n$. The monic orthogonal polynomials $\pi_n^{[0,m]}(t)$ associated with the measure $d\alpha^{[0,m]}(t)$ can be expressed as ratios of determinants,*

$$\pi_n^{[0,m]}(t) = \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_n(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \dots & h_n(\epsilon_m) \\ \pi_{n-m}(t) & \dots & \pi_n(t) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_{n-1}(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \dots & h_{n-1}(\epsilon_m) \end{vmatrix}}. \quad (2.12)$$

Here the $h_k(\epsilon_j)$'s are the Cauchy transformations of the monic orthogonal polynomials $\pi_k(t)$,

$$h_k(\epsilon_j) = \frac{1}{2\pi i} \int \frac{\pi_k(t) d\alpha(t)}{t - \epsilon_j}. \quad (2.13)$$

Proof. Set

$$q_n^{[0,m]}(t) = \begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_n(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \dots & h_n(\epsilon_m) \\ \pi_{n-m}(t) & \dots & \pi_n(t) \end{vmatrix}. \quad (2.14)$$

Now $q_n^{[0,m]}(t)$ is proportional to the n^{th} monic orthogonal polynomial $\pi_n^{[0,m]}(t)$ with respect to the measure $d\alpha^{[0,m]}(t)$. Indeed, first observe that

$$\int \frac{q_n^{[0,m]}(t)}{t - \epsilon_j} d\alpha(t) = 0, \quad j = 1, \dots, m. \quad (2.15)$$

Also, for $0 \leq k < n$,

$$\frac{t^k}{\prod_{\ell=1}^m (\epsilon_\ell - t)} = \sum_{\ell=1}^m \frac{\beta_\ell}{\epsilon_\ell - t} + p(t) \quad (2.16)$$

for suitable constants $\{\beta_\ell\}$ and for some polynomial of degree $< n - m$. But for $0 \leq k < n$,

$$\int t^k q_n^{[0,m]}(t) d\alpha^{[0,m]}(t) = - \sum_{\ell=1}^m \beta_\ell \int \frac{q_n^{[0,m]}(t)}{t - \epsilon_\ell} d\alpha(t) + \int p(t) q_n^{[0,m]}(t) d\alpha(t). \quad (2.17)$$

The terms in the sum are zero by (2.15) and the final integral is zero by the construction (2.14) of $q_n^{[0,m]}(t)$ and the fact that $\deg p(t) < n - m$. Thus $q_n^{[0,m]}(t)$ is proportional to $\pi_n^{[0,m]}(t)$. An argument similar to the proof in Lemma 2.1 that

$$\begin{vmatrix} \pi_n(\mu_1) & \dots & \pi_{n+\ell-1}(\mu_1) \\ \vdots & & \vdots \\ \pi_n(\mu_\ell) & \dots & \pi_{n+\ell-1}(\mu_\ell) \end{vmatrix} \neq 0, \quad (2.18)$$

shows that the denominator in (2.12) does not vanish. Letting $t \rightarrow \infty$ in (2.14), and matching leading terms, we prove Lemma 2.5. \square

Remark 2.6. In [19], Uvarov obtains formulae for $\pi_n^{[0,m]}(t)$ of type (2.12) also in the case $m > n$. These formulae can be used to obtain analogues of (2.24) and (2.36) below in the case $M > N$.

Remark 2.7. As noted in [12, 11], the Cauchy transformations $h_k(\epsilon)$ of the π_k 's occur explicitly, together with the π_k 's, in the solution of the Fokas-Its-Kitaev Riemann-Hilbert problem for orthogonal polynomials [9].

Lemma (2.5) implies the following analogue of the Christoffel formula for the Cauchy transforms of monic orthogonal polynomials.

Corollary 2.8. *Let $h_k^{[0,m]}(\epsilon)$ be the Cauchy transform of the monic polynomial $\pi_k^{[0,m]}(t)$ with respect to the measure $d\alpha^{[0,m]}(t)$,*

$$h_k^{[0,m]}(\epsilon) = \frac{1}{2\pi i} \int \frac{\pi_k^{[0,m]}(t)}{t - \epsilon} d\alpha^{[0,m]}(t). \quad (2.19)$$

Let also $0 \leq m \leq n$. Then $h_n^{[0,m]}(\epsilon)$ has a representation similar to that for the monic orthogonal polynomials $\pi_n^{[l,0]}(t)$ (equation (2.1)),

$$h_n^{[0,m]}(\epsilon) = \frac{(-1)^m}{(\epsilon - \epsilon_m) \dots (\epsilon - \epsilon_1)} \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_n(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \dots & h_n(\epsilon_m) \\ h_{n-m}(\epsilon) & \dots & h_n(\epsilon) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_{n-1}(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \dots & h_{n-1}(\epsilon_m) \end{vmatrix}}. \quad (2.20)$$

Proof. The above representation follows from formula (2.12) and from the fact that

$$\frac{1}{(t - \epsilon_{m+1}) \dots (t - \epsilon_1)} = \sum_{j=1}^{m+1} \frac{1}{t - \epsilon_j} \prod_{k \neq j} \frac{1}{\epsilon_j - \epsilon_k}. \quad (2.21)$$

Indeed we find from formula (2.12) that $h_n^{[0,m]}(\epsilon)$ is the ratio of the determinants. The elements of the last row of the determinant in the numerator are the integrals

$$\frac{1}{2\pi i} \int \frac{\pi_{n-k}(t) d\alpha(t)}{(t - \epsilon)(t - \epsilon_m) \dots (t - \epsilon_1)}, \quad 0 \leq k \leq m.$$

Using identity (2.21) and noting that the only term

$$\frac{1}{t - \epsilon} \frac{1}{(\epsilon - \epsilon_m) \dots (\epsilon - \epsilon_1)}. \quad (2.22)$$

of the sum (2.21) contributes to the determinant, (2.20) follows. \square

Equation (2.20) immediately implies the following analogy of (2.5) for the $h_k^{[0,m]}$'s.

Corollary 2.9. *Let $0 \leq m \leq n$. Then the product of the Cauchy transforms of monic orthogonal polynomials with respect to the measures $d\alpha^{[0,j]}(t)$, $0 \leq j \leq m$ can be written as a determinant,*

$$\prod_{j=0}^m h_{n-m+j}^{[0,j]}(\epsilon_{j+1}) = \frac{(-1)^{\frac{m(m+1)}{2}}}{\Delta(\epsilon)} \begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_n(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_{m+1}) & \dots & h_n(\epsilon_{m+1}) \end{vmatrix}. \quad (2.23)$$

Now we derive the identity for the average of the product of inverse random characteristic polynomials.

Theorem 2.10. *Suppose $1 \leq M \leq N$ and let $\gamma_n = -\frac{2\pi i}{c_n^2}$, where c_n is the normality constant defined by equation (1.4). Then we have the following formula*

$$\left\langle \prod_{j=1}^M D_N^{-1}[\epsilon_j, H] \right\rangle_{\alpha} = (-1)^{\frac{M(M-1)}{2}} \frac{\prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\epsilon)} \begin{vmatrix} h_{N-M}(\epsilon_1) & \dots & h_{N-1}(\epsilon_1) \\ \vdots & & \vdots \\ h_{N-M}(\epsilon_M) & \dots & h_{N-1}(\epsilon_M) \end{vmatrix}. \quad (2.24)$$

Proof. When $M = 1$, we use the identity (2.21) together with (2.7) and the relation (see, e.g., [18])

$$\gamma_{n-1} = -2\pi i n \frac{Z_{n-1}}{Z_n} \quad (2.25)$$

to obtain

$$\langle D_N^{-1}[\epsilon, H] \rangle_{\alpha} = \gamma_{N-1} h_{N-1}(\epsilon). \quad (2.26)$$

We rewrite the average in equation (2.24) as follows:

$$\left\langle \prod_{j=1}^M D_N^{-1}[\epsilon_j, H] \right\rangle_{\alpha} = \frac{Z_N^{[0,M]}}{Z_{N-1}^{[0,M-1]}} \frac{Z_{N-1}^{[0,M-1]}}{Z_{N-2}^{[0,M-2]}} \cdots \frac{Z_{N-M}^{[0,0]}}{Z_N^{[0,0]}} \quad (2.27)$$

where

$$Z_N^{[0,M]} = \int \cdots \int \Delta^2(x) d\alpha^{[0,M]}(x), \quad (2.28)$$

$Z_N^{[0,0]} \equiv Z_N$ and $d\alpha^{[0,0]}(x) = d\alpha(x)$. The following relation can be observed from equations (2.26) and (2.25):

$$\frac{Z_{N-K}^{[0,m]}}{Z_{N-K-1}^{[0,m-1]}} = -2\pi i(N-K) h_{N-K-1}^{[0,m-1]}(\epsilon_m). \quad (2.29)$$

Inserting this relation in (2.27) we find

$$\left\langle \prod_{j=1}^M D_N^{-1}[\epsilon_j, H] \right\rangle_{\alpha} = \prod_{j=1}^M \gamma_{N-j} h_{N-j}^{[0,M-j]}(\epsilon_{M-j+1}). \quad (2.30)$$

Our result (2.24) immediately follows from the above equation and formula (2.23). \square

We now repeat the above considerations for the case

$$d\alpha^{[\ell,m]}(t) = \frac{(\mu_1 - t) \cdots (\mu_\ell - t)}{(\epsilon_1 - t) \cdots (\epsilon_m - t)} d\alpha(t). \quad (2.31)$$

The first result is a Christoffel type formula for the measure (2.31), which is due to Uvarov [19]:

Lemma 2.11. *Suppose $0 \leq m \leq n$. Then the monic orthogonal polynomials $\pi_n^{[\ell,m]}(t)$'s with respect to the measure $d\alpha^{\ell,m}(t)$ have the following representation:*

$$\pi_n^{[\ell,m]}(t) = \frac{1}{(t - \mu_\ell) \cdots (t - \mu_1)} \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \cdots & h_{n+\ell}(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \cdots & h_{n+\ell}(\epsilon_m) \\ \pi_{n-m}(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \vdots \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell) \\ \pi_{n-m}(t) & \cdots & \pi_{n+\ell}(t) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \cdots & h_{n+\ell}(\epsilon_1) \\ \vdots & & \vdots \\ h_{n-m}(\epsilon_m) & \cdots & h_{n+\ell}(\epsilon_m) \\ \pi_{n-m}(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \vdots \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell) \end{vmatrix}}. \quad (2.32)$$

Proof. As in the previous cases we define $q_n^{[\ell,m]}(t)$ to be the determinant in the numerator of (2.32). Observe that

$$q_n^{[\ell,m]}(\mu_1) = \dots = q_n^{[\ell,m]}(\mu_\ell) = 0 \quad (2.33)$$

and that

$$\int \frac{q_n^{[\ell,m]}(t)d\alpha(t)}{\epsilon_1 - t} = \dots = \int \frac{q_n^{[\ell,m]}(t)d\alpha(t)}{\epsilon_m - t} = 0. \quad (2.34)$$

The next steps are the same as in the proofs of Lemma (2.1) and Lemma (2.5). \square

Corollary 2.12.

$$\left\langle \prod_{j=1}^K D_N[\mu_j, H] \right\rangle_{\alpha^{[0,M]}} = \frac{1}{\Delta(\mu)} \frac{\begin{vmatrix} h_{N-M}(\epsilon_1) & \dots & h_{N+K-1}(\epsilon_1) \\ \vdots & & \vdots \\ h_{N-M}(\epsilon_M) & \dots & h_{N+K-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \dots & \pi_{N+K-1}(\mu_1) \\ \vdots & & \vdots \\ \pi_{N-M}(\mu_K) & \dots & \pi_{N+K-1}(\mu_K) \end{vmatrix}}{\begin{vmatrix} h_{N-M}(\epsilon_1) & \dots & h_N(\epsilon_1) \\ \vdots & & \vdots \\ h_{N-M}(\epsilon_M) & \dots & h_N(\epsilon_M) \end{vmatrix}}. \quad (2.35)$$

Proof. Identity (2.35) follows from equations (2.10) and (2.32) once we note that equation (2.32) can be rewritten in a similar manner as equation (2.5). \square

Finally we generalize Theorem (2.3) and Theorem (2.10) and obtain a formula for the average of ratios of characteristic polynomials.

Theorem 2.13. *Suppose $0 \leq M \leq N$. Then the average of ratios of characteristic polynomials of $N \times N$ Hermitian matrices H is given by the following formula:*

$$\left\langle \frac{\prod_{j=1}^K D_N[\mu_j, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \frac{(-1)^{\frac{M(M-1)}{2}} \prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\mu)\Delta(\epsilon)} \begin{vmatrix} h_{N-M}(\epsilon_1) & \dots & h_{N+K-1}(\epsilon_1) \\ \vdots & & \vdots \\ h_{N-M}(\epsilon_M) & \dots & h_{N+K-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \dots & \pi_{N+K-1}(\mu_1) \\ \vdots & & \vdots \\ \pi_{N-M}(\mu_K) & \dots & \pi_{N+K-1}(\mu_K) \end{vmatrix}. \quad (2.36)$$

Proof. Let $\alpha^{[0,0]} \equiv \alpha$, $\mathcal{Z}_n^{[0,0]} \equiv Z_n$. Then we have

$$\left\langle \frac{\prod_{j=1}^K D_N[\mu_j, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \frac{Z_N^{[K,M]}}{Z_N^{[0,0]}} = \frac{Z_N^{[K,M]} Z_N^{[0,M]}}{Z_N^{[0,M]} Z_N^{[0,0]}} \quad (2.37)$$

i.e.

$$\left\langle \frac{\prod_{j=1}^K D_N[\mu_j, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \right\rangle_\alpha = \left\langle \prod_{j=1}^K D_N[\mu_j, H] \right\rangle_{\alpha^{[0, M]}} \left\langle \prod_{j=1}^M D_N^{-1}[\epsilon_j, H] \right\rangle_\alpha. \quad (2.38)$$

We use Corollary (2.12) and Theorem (2.10) to obtain formula (2.36). \square

Remark 2.14. Observe that formulae (2.6), (2.24) do not follow immediately as special cases of (2.36): some further algebraic manipulation is required. Similarly, the process of adding and removing zeros is clearly reciprocal. More precisely, given $\epsilon_1, \dots, \epsilon_\ell$, we can construct the polynomials $\pi_n^{[0, \ell]}(t; d\alpha^{[0, \ell]})$ associated with the measure $d\alpha^{[0, \ell]}(t) = (\prod_{i=1}^\ell (\epsilon_i - t)^{-1}) dt$ by (2.12): We can then construct $\pi_n^{[\ell, 0]}(t; d(\alpha^{[0, \ell]})^{[\ell, 0]})$ with $\mu_i = \epsilon_i$, inserting $\pi_n^{[0, \ell]}(t; d\alpha^{[0, \ell]})$ for $\pi_n(t)$ on the right-hand-side of (2.1). We should find that $\pi_n^{[\ell, 0]}(t; d(\alpha^{[0, \ell]})^{[\ell, 0]}) = \pi_n(t; d\alpha)$. However, again, this relation is not immediately clear, and requires further algebraic manipulation.

3 Formulae of two-point function type

The following integral version of the Binet-Cauchy formula is due to Andréief [2], and plays a basic role in our calculations.

Lemma 3.1. *Let $(X, d\mu)$ be a measure space and suppose $f_i, g_j \in L^2(X, d\mu)$ for $1 \leq i, j \leq k$. Then*

$$\begin{aligned} & \int_X \cdots \int_X \det(f_i(x_j))_{1 \leq i, j \leq k} \det(g_i(x_j))_{1 \leq i, j \leq k} d\mu(x_1) \cdots d\mu(x_k) \\ &= k! \det \left(\int_X f_i(x) g_j(x) d\mu(x) \right)_{1 \leq i, j \leq k}. \end{aligned} \quad (3.1)$$

Proof. Set $c_{ij} = \int_X f_i(x) g_j(x) d\mu(x)$. Then

$$\begin{aligned} & \int_X \cdots \int_X \det(f_i(x_j))_{1 \leq i, j \leq k} \det(g_i(x_j))_{1 \leq i, j \leq k} d\mu(x_1) \cdots d\mu(x_k) \\ &= \sum_{\sigma, \tau \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) c_{\sigma(1)\tau(1)} \cdots c_{\sigma(k)\tau(k)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) \sum_{\tau} \operatorname{sgn}(\tau \circ \sigma) c_{\sigma(1)\tau \circ \sigma(1)} \cdots c_{\sigma(k)\tau \circ \sigma(k)} \\ &= \sum_{\sigma} (\operatorname{sgn}(\sigma))^2 \sum_{\tau} \operatorname{sgn}(\tau) c_{1\tau(1)} \cdots c_{k\tau(k)} \\ &= k! \det(c_{ij})_{1 \leq i, j \leq k} \end{aligned} \quad (3.2)$$

as desired. In (3.2) we used $\operatorname{sgn}(\tau \circ \sigma) = (\operatorname{sgn} \tau)(\operatorname{sgn} \sigma)$ and the fact that $c_{\sigma(1)\tau \circ \sigma(1)} \cdots c_{\sigma(k)\tau \circ \sigma(k)} = c_{1\tau(1)} \cdots c_{k\tau(k)}$ for all σ . \square

Theorem 3.2. *Let $K \geq 1$. Then the following identity is valid:*

$$\left\langle \prod_{j=1}^K D_N[\lambda_j, H] D_N[\mu_j, H] \right\rangle_{\alpha} = \frac{C_{N,K}}{\Delta(\lambda)\Delta(\mu)} \det(W_{I,N+K}(\lambda_i, \mu_j))_{1 \leq i, j \leq K} \quad (3.3)$$

where

$$W_{I,N+K}(x, y) = \frac{\pi_{N+K}(x)\pi_{N+K-1}(y) - \pi_{N+K}(y)\pi_{N+K-1}(x)}{x - y} \quad (3.4)$$

and

$$C_{N,K} = \frac{\prod_{\ell=N}^{N+K-1} c_{\ell}^2}{(c_{N+K-1})^{2K}} \quad (3.5)$$

where c_{ℓ} is again the norming constant for π_{ℓ} given in (1.4).

Proof. Let $p_j(x) = c_j^{-1}\pi_j(x)$, $j \geq 0$, denote the orthonormal polynomials with respect to $d\alpha$. From (1.2) we obtain

$$\left\langle \prod_{j=1}^K D_N[\lambda_j, H] D_N[\mu_j, H] \right\rangle_{\alpha} = \frac{1}{Z_N \Delta(\lambda)\Delta(\mu)} \int \cdots \int \Delta(x, \lambda)\Delta(x, \mu) d\alpha(x). \quad (3.6)$$

Adding columns, we see that the Vandermonde determinant $\Delta(x, \lambda)$ has the form

$$\begin{vmatrix} \pi_0(x_1) & \pi_1(x_1) & \cdots & \pi_{N+K-1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0(x_N) & \pi_1(x_N) & \cdots & \pi_{N+K-1}(x_N) \\ \pi_0(\lambda_1) & \pi_1(\lambda_1) & \cdots & \pi_{N+K-1}(\lambda_1) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0(\lambda_K) & \pi_1(\lambda_K) & \cdots & \pi_{N+K-1}(\lambda_K) \end{vmatrix} \quad (3.7)$$

and similarly for $\Delta(x, \mu)$. Here $\pi_j(t) = \pi_j^{[0,0]}(t)$. The determinant $\Delta(x, \lambda)$ can be evaluated by a Lagrange expansion of the form

$$\sum_{0 \leq i_1 < i_2 < \cdots < i_K \leq N+K-1} \sigma_{i_1, \dots, i_K} \begin{vmatrix} \pi_{i_1}(\lambda_1) & \cdots & \pi_{i_K}(\lambda_1) \\ \vdots & \ddots & \vdots \\ \pi_{i_1}(\lambda_K) & \cdots & \pi_{i_K}(\lambda_K) \end{vmatrix} \begin{vmatrix} \pi_{j_1}(x_1) & \cdots & \pi_{j_N}(x_1) \\ \vdots & \ddots & \vdots \\ \pi_{j_1}(x_N) & \cdots & \pi_{j_N}(x_N) \end{vmatrix} \quad (3.8)$$

where $\sigma_{i_1, \dots, i_K} = \pm 1$ is an appropriate signature and $\{(j_1, \dots, j_N) : 0 \leq j_1 < j_2 < \cdots < j_N \leq N + K - 1\}$ is the complement of $\{i_1, \dots, i_K\}$ in $\{0, 1, \dots, N + K - 1\}$. Multiplying (3.8) by a similar expansion for $\Delta(x, \mu)$, and inserting in (3.6), we obtain a sum of terms of the form

$$\int \cdots \int \begin{vmatrix} \pi_{j_1}(x_1) & \cdots & \pi_{j_K}(x_1) \\ \vdots & \ddots & \vdots \\ \pi_{j_1}(x_N) & \cdots & \pi_{j_K}(x_N) \end{vmatrix} \begin{vmatrix} \pi_{j'_1}(x_1) & \cdots & \pi_{j'_N}(x_1) \\ \vdots & \ddots & \vdots \\ \pi_{j'_1}(x_N) & \cdots & \pi_{j'_N}(x_N) \end{vmatrix} d\alpha(x) \quad (3.9)$$

which is equal by Lemma 3.1 to $N! \det\left(\int \pi_{j'_i}(x)\pi_{j_k}(x)d\alpha(x)\right)_{1\leq i,k\leq N} = N! \det(\delta_{j'_i j_k} c_{j_k}^2)_{1\leq i,k\leq N}$.
From this we see that

$$\begin{aligned}
& \left\langle \prod_{j=1}^K D_N[\lambda_j, H] D_N[\mu_j, H] \right\rangle_{\alpha} \\
&= \frac{N!}{Z_N \Delta(\lambda) \Delta(\mu)} \sum_{0\leq i_1 < \dots < i_K \leq N+K-1} \sigma_{i_1, \dots, i_K}^2 \begin{vmatrix} \pi_{i_1}(\lambda_1) & \dots & \pi_{i_K}(\lambda_1) \\ \vdots & & \vdots \\ \pi_{i_1}(\lambda_K) & \dots & \pi_{i_K}(\lambda_K) \end{vmatrix} \\
& \quad \times \prod_{k=1}^N c_{j_k}^2 \begin{vmatrix} \pi_{i_1}(\mu_1) & \dots & \pi_{i_K}(\mu_1) \\ \vdots & & \vdots \\ \pi_{i_1}(\mu_K) & \dots & \pi_{i_K}(\mu_K) \end{vmatrix} \tag{3.10} \\
&= \frac{N! \prod_{q=N}^{N+K-1} c_q^2}{Z_N \Delta(x, \lambda) \Delta(x, \mu)} \sum_{0\leq i_1 < \dots < i_K \leq N+K-1} \det(p_{i_j}(\lambda_k))_{1\leq j,k\leq K} \det(p_{i_j}(\mu_k))_{1\leq j,k\leq K} \\
&= \frac{N! \prod_{q=N}^{N+K-1} c_q^2}{Z_N \Delta(x, \lambda) \Delta(x, \mu)} \det\left(\sum_{0\leq i\leq N+K-1} p_i(\lambda_j) p_i(\mu_k)\right)_{1\leq j,k\leq K}
\end{aligned}$$

where the last line follows by applying Lemma 3.1 to the discrete measure $d\mu = \sum_{i=0}^{N+K-1} \delta_i$.
But by the Christoffel-Darboux formula

$$\sum_{0\leq i\leq N+K-1} p_i(\lambda_j) p_i(\mu_k) = \frac{\pi_{N+K}(\lambda_j) \pi_{N+K-1}(\mu_k) - \pi_{N+K}(\mu_k) \pi_{N+K-1}(\lambda_j)}{\lambda_j - \mu_k} \tag{3.11}$$

which then implies (3.3) as $Z_N = N! \prod_{\ell=0}^{N-1} c_{\ell}^2$ (see, e.g. [18]). \square

Theorem 3.3. *Suppose $1 \leq K \leq N$. Then the following identity is valid:*

$$\left\langle \prod_{j=1}^K \frac{D_N[\mu_j, H]}{D_N[\epsilon_j, H]} \right\rangle_{\alpha} = (-1)^{K(K-1)/2} \gamma_{N-1}^K \frac{\Delta(\epsilon, \mu)}{\Delta^2(\epsilon) \Delta^2(\mu)} \det(W_{II, N}(\epsilon_i, \mu_j))_{1\leq i, j\leq K} \tag{3.12}$$

where

$$W_{II, N}(x, y) = \frac{h_N(\epsilon) \pi_{N-1}(\mu) - h_{N-1}(\epsilon) \pi_N(\mu)}{\epsilon - \mu} \tag{3.13}$$

and again $h_k(\epsilon) = \frac{1}{2\pi i} \int \frac{\pi_k(t) d\alpha(t)}{t - \epsilon}$ is the Cauchy transform of $\pi_k(t)$ and $\gamma_{N-1} = -2\pi i / C_{N-1}^2$.

Observe first that by linearity

$$\begin{aligned}
& \left| \begin{array}{ccc} h_{N-M}(\epsilon_1) & \cdots & h_{N+L-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \cdots & h_{N+L-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_L) & \cdots & \pi_{N+L-1}(\mu_L) \end{array} \right| = \int \cdots \int \frac{d\alpha(\lambda)}{(2\pi i)^M \prod_{j=1}^M (\lambda_j - \epsilon_j)} \\
& \times \left| \begin{array}{ccc} \pi_{N-M}(\lambda_1) & \cdots & \pi_{N+L-1}(\lambda_1) \\ \vdots & & \\ \pi_{N-M}(\lambda_M) & \cdots & \pi_{N+L-1}(\lambda_M) \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_L) & \cdots & \pi_{N+L-1}(\mu_L) \end{array} \right|. \tag{3.14}
\end{aligned}$$

Inserting (2.36) on the left-hand-side, and using (2.5) to re-express the integrand on the right-hand-side, we obtain the following result, which is of independent interest. The result expresses averages of ratios of characteristic polynomials in terms of averages of products of such polynomials.

Proposition 3.4. *Let $1 \leq M \leq N$. Then*

$$\begin{aligned}
& \left\langle \frac{\prod_{j=1}^L D_N[\mu_i, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \frac{(-1)^{M(M-1)/2} \prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\mu)\Delta(\epsilon)} \\
& \times \int \cdots \int \frac{d\alpha(\lambda)}{(2\pi i)^M \prod_{j=1}^M (\lambda_j - \epsilon_j)} \Delta(\lambda, \mu) \left\langle \prod_{j=1}^M D_{N-M}[\lambda_j, H] \prod_{j=1}^L D_{N-M}[\mu_j, H] \right\rangle_{\alpha}. \tag{3.15}
\end{aligned}$$

Proof of Theorem 3.2. For $M = L = K \leq N$, by (3.15) and (3.3),

$$\begin{aligned}
& \frac{\Delta(\mu)\Delta(\epsilon)}{(-1)^{K(K-1)/2} \prod_{j=N-K}^{N-1} \gamma_j} \left\langle \frac{\prod_{j=1}^K D_N[\mu_i, H]}{\prod_{j=1}^K D_N[\epsilon_j, H]} \right\rangle_{\alpha} \\
& = \int \cdots \int \frac{d\alpha(\lambda)}{(2\pi i)^M \prod_{j=1}^M (\lambda_j - \epsilon_j)} C_{N-K, K} \prod_{i=1}^K \prod_{j=1}^K (\mu_i - \lambda_j) \det(W_{I, N}(\lambda_i, \mu_j))_{1 \leq i, j \leq K}. \tag{3.16}
\end{aligned}$$

But

$$\begin{aligned}
& \frac{1}{2\pi i} \int \frac{d\alpha(\lambda_j)}{\lambda_j - \epsilon_j} \prod_{i=1}^K (\mu_i - \lambda_j) \frac{\pi_N(\lambda_j)\pi_{N-1}(\mu_k) - \pi_{N-1}(\lambda_j)\pi_N(\mu_k)}{\lambda_j - \mu_k} \\
&= \frac{1}{2\pi i} \int d\alpha(\lambda_j) \left(1 - \frac{\mu_1 - \epsilon_j}{\lambda_j - \epsilon_j}\right) \left(\prod_{\substack{i=2 \\ i \neq k}}^K (\mu_i - \lambda_j)\right) (\pi_N(\lambda_j)\pi_{N-1}(\mu_k) - \pi_{N-1}(\lambda_j)\pi_N(\mu_k)) \\
&= -\frac{1}{2\pi i} \int d\alpha(\lambda_j) \frac{\mu_1 - \epsilon_j}{\lambda_j - \epsilon_j} \left(\prod_{\substack{i=2 \\ i \neq k}}^K (\mu_i - \lambda_j)\right) (\pi_N(\lambda_j)\pi_{N-1}(\mu_k) - \pi_{N-1}(\lambda_j)\pi_N(\mu_k))
\end{aligned} \tag{3.17}$$

as $\int d\alpha(\lambda_j)\lambda_j^\ell \pi_{N-1}(\lambda_j) = \int d\alpha(\lambda_j)\lambda_j^\ell \pi_N(\lambda_j) = 0$ for $0 \leq \ell \leq K-2 < N-1$. Continuing in this way, the integral reduces to $\prod_{i=1}^K (\mu_i - \epsilon_j) W_{II,N}(\epsilon_i, \mu_k)$. Thus we find

$$\frac{\Delta(\mu)\Delta(\epsilon)}{(-1)^{K(K-1)/2} \prod_{j=N-K}^{N-1} \gamma_j} \left\langle \frac{\prod_{j=1}^K D_N[\mu_i, H]}{\prod_{j=1}^K D_N[\epsilon_j, H]} \right\rangle_\alpha = \frac{\Delta(\epsilon, \mu)}{\Delta(\epsilon)\Delta(\mu)} \det(W_{I,N+K}(\lambda_i, \mu_k))_{1 \leq i, k \leq K} \tag{3.18}$$

and (3.12) follows. \square

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