A SHORT PROOF OF LEGENDRE'S CONJECTURE

FRANK VEGA

Abstract. It was proved for every natural number n, there is always a prime number between the integers n^2 and $(n+1)^2$. Therefore, it was demonstrated the Legendre's conjecture as a consequence of using a modification to the sieve of Legendre.

1. Introduction

In Number Theory, the Legendre's conjecture is one of the open problems of Mathematics and belongs to Landau's Problems [3]. This conjecture tells us about the existence of at least one prime between the integers n^2 and $(n+1)^2$ [5]. Although it is not known if there always exists a prime between n^2 and $(n+1)^2$, Chen has shown that a number which is a prime or a number which is the product of two primes does always satisfy this inequality [2].

2. Demonstration

2.1. **The sieve of Legendre.** In mathematics, the prime-counting function is the function counting the number of prime numbers less than or equal to some real number x [3]. It is denoted by $\pi(x)$. An elaborate way of finding $\pi(x)$ is due to the sieve of Legendre [1]: given x, if $p_1, p_2, ..., p_k$ are distinct prime numbers, then the number of integers less than or equal to x which are divisible by no p_i is

$$\lfloor x \rfloor - \sum_i (\lfloor \frac{x}{p_i} \rfloor) + \sum_{i<j} (\lfloor \frac{x}{p_i \times p_j} \rfloor) - \sum_{i<j<k} (\lfloor \frac{x}{p_i \times p_j \times p_k} \rfloor) ... \tag{1}$$

where $\lfloor ... \rfloor$ denotes the floor function. This number is therefore equal to

$$\pi(x) - \pi(\sqrt{x}) + 1 \tag{2}$$

when the numbers $p_1, p_2, ..., p_k$ are the prime numbers less than or equal to $\sqrt{x}$. If it is taken the integers n^2 and $(n+1)^2$, then it can build the Equation

$$((n+1)^2 - n^2) - (\sum_i (\lfloor \frac{(n+1)^2}{p_i} \rfloor) - \sum_i (\lfloor \frac{n^2}{p_i} \rfloor)) ... \tag{3}$$

where the numbers $p_1, p_2, ..., p_k$ are the prime numbers less than or equal to $n+1$. The equation (3) will be the result of subtracting the sieve the Legendre by the integer $(n+1)^2$ with n^2 for the prime numbers less than or equal to $n+1$.

2000 *Mathematics Subject Classification.* Primary 11-XX, 11Axx, 11A41.
Key words and phrases. Primes, Legendre's conjecture, sieve of Legendre.

Theorem 2.1. *For every integer $n \geq 3$, the Equation (3) is always greater than or equal to* 2.

It is already known $(n+1)^2 = n^2 + 2 \times n + 1$, then $(n+1)^2 - n^2 = 2 \times n + 1$. Moreover, for each integer m, it has $\lfloor \frac{(n+1)^2}{m} \rfloor \geq \lfloor \frac{n^2}{m} \rfloor + \lfloor \frac{2 \times n+1}{m} \rfloor$ by properties of the floor function [4]. Therefore, it always complies $\lfloor \frac{(n+1)^2}{m} \rfloor - \lfloor \frac{n^2}{m} \rfloor \geq \lfloor \frac{2 \times n+1}{m} \rfloor$ for each integer m. For all these reasons, the equation (3) is greater than or equal to the following result

$$2 \times n + 1 - \sum_i (\lfloor \frac{2 \times n + 1}{p_i} \rfloor) + \sum_{i<j} (\lfloor \frac{2 \times n + 1}{p_i \times p_j} \rfloor) - \sum_{i<j<k} (\lfloor \frac{2 \times n + 1}{p_i \times p_j \times p_k} \rfloor) ... \tag{4}$$

where the numbers $p_1, p_2, ..., p_k$ are the prime numbers less than or equal to $n+1$. This equation (4) is therefore equal to

$$\pi(2 \times n + 1) - \pi(n+1) + 1 \tag{5}$$

due to apply the properties of the sieve of Legendre with the prime numbers $p_1, p_2, ..., p_k$ less than or equal to $n+1$.

Bertrand's postulate, proven first by Chebyshev, states that there always exists at least one prime number p with $m < p < 2 \times m - 2$, for any natural number $m > 3$ [6]. Then, there is always a prime number p with $n + 1 < p < 2 \times n$, for any natural number $n + 1 > 3$.

Then, the Equation (5) is greater than or equal to 2 for $n \geq 3$ and so the Equation (4). For that reason, the Equation (3) is greater than or equal to 2 for $n \geq 3$.

2.2. **Main Proof.**

Theorem 2.2. *The number of primes between the integers n^2 and $(n+1)^2$ is greater than or equal to Equation (3) minus* 1.

The number of primes between the integers n^2 and $(n+1)^2$ is equal to

$$\pi((n+1)^2) - \pi(n^2) \tag{6}$$

However, the subtracting of the sieve of Legendre by the integer $(n+1)^2$ with n^2 for the prime numbers less than or equal to $n+1$ is equal to

$$\pi((n+1)^2) - \pi(n^2) + 1 \tag{7}$$

But, this is equal to equation (3), then it was proved this Theorem and this proof will help us to demonstrate the final result.

Theorem 2.3. *The Legendre's conjecture is true.*

It has been checked the Legendre's conjecture for $1 \leq n < 3$. By applying the Theorem 2.2 and 2.1, it is possible to obtain with $n \geq 3$ the following result: the amount of primes between the integers n^2 and $(n+1)^2$ is greater than or equal to 1. The final conclusion is this one: the Legendre's conjecture is true for every natural number $n \geq 1$.

3. Conclusion

The result of this paper tells us in some way about the distribution of primes. Besides, it was proved one of the open problems of Landau, and therefore, it was resolved one of the most important conjectures in mathematics. By developing another open problems such as others Landau's problems and the Riemann hypothesis might help us to make a bigger approximation of how the primes are distributed or maybe to get a more easy way for finding primes.

References

1. Alina, C. C. and Murty, M. R, *An introduction to sieve methods and their applications*, London Mathematical Society Student Texts, Cambridge University Press, **66**, 2006.
2. Chen, J. R, *On the Distribution of Almost Primes in an Interval*, Sci. Sinica **18**, 611–627, 1975.
3. Dickson, L. E, *History of the Theory of Numbers, Vol. 1: Divisibility and Primality*, New York, Dover, 2005.
4. Dudley, U, *Elementary number theory*, Second Edition, W. H. Freeman and Co., 1978.
5. Guy, R. K, *Unsolved Problems in Number Theory*, 3rd ed. New York, Springer-Verlag, 2004.
6. Hardy, G. H. and Wright, E. M, *An Introduction to the Theory of Numbers*, Fifth Edition, Clarendon Press, Oxford, 1979.

Datys, Havana, Cuba

E-mail address: vega.frank@gmail.com